SECOND CHERN CLASS AND FUJIKI CONSTANTS OF HYPERKÄHLER MANIFOLDS

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Abstract. We study characteristic classes on hyperkähler manifolds with a view towards the Verbitsky component. The case of the second Chern class leads to a conditional upper bound on the second Betti number in terms of the Riemann–Roch polynomial, which is also valid for singular examples. We discuss the general structure of characteristic classes and the Riemann–Roch polynomial on hyperkähler manifolds using among other things Rozansky–Witten theory.

1. Introduction

In the study of smooth projective varieties with trivial canonical bundle, irreducible compact hyperkähler manifolds take up a prominent place, partly due to the scarcity of examples. It is therefore natural to study a priori topological restrictions that such varieties must obey. There are several results in this direction, for example [11, 14, 16, 25–27].

Given an irreducible hyperkähler manifold X of dimension $2n$, its cohomology $H^*(X, \mathbf{R})$ is equipped with the Beauville–Bogomolov–Fujiki form q_X. Moreover, $H^*(X, \mathbf{R})$ is naturally a module under the Looijenga–Lunts–Verbitsky (LLV) Lie algebra $\mathfrak{g}(X)_{\mathbf{R}}$ [10, 17, 28]. This leads to a decomposition of $H^*(X, \mathbf{R})$ into irreducible representations. Arguably, the most important one is the Verbitsky component $\mathrm{SH}(X, \mathbf{R}) \subset H^*(X, \mathbf{R})$, which is the subalgebra generated by $H^2(X, \mathbf{R})$.

A natural question that arises is how much information this subalgebra encodes on the full cohomology. For example, one could ask which Chern classes of sheaves and, in particular, characteristic classes are contained inside the Verbitsky component.

One case we consider here is that of the second Chern class $c_2 \coloneqq c_2(X) \in H^4(X, \mathbf{R})$. Maybe a priori counter-intuitively, it is not always contained in the Verbitsky component, see for example [18, Lem. 1.5] for the case of the Hilbert scheme of n points on a K3 surface with $n > 3$. Note that c_2 lies in the Verbitsky component if and only if it is a multiple of the class $\mathfrak{q} \in H^4(X, \mathbf{Q})$, the dual of the Beauville–Bogomolov–Fujiki form.

We answer completely the question when c_2 lies inside the Verbitsky component using the Riemann–Roch polynomial of X. Recall that for a class $\alpha \in H^{4k}(X, \mathbf{R})$ which remains of type $(2k, 2k)$ on all small deformations of X, there exists a number $C(\alpha)$, called the *generalized Fujiki constant* of α, such that

$$\forall \beta \in H^2(X, \mathbf{R}) \quad C(\alpha) \cdot q_X(\beta)^{n-k} = \int_X \alpha \cdot \beta^{2n-2k}. \tag{1}$$

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Let td be the Todd class of X and let td_{2k} be its degree $2k$ part. The Riemann–Roch polynomial of X is defined as

$$\begin{aligned}\mathrm{RR}_X(q) \coloneqq \sum_{i=0}^{n} \frac{C(\mathrm{td}_{2n-2i})}{(2i)!} q^i = \frac{C(1)}{(2n)!} q^n + \frac{C(\mathrm{td}_2)}{(2n-2)!} q^{n-1} + \cdots + \frac{C(\mathrm{td}_{2n})}{1} \\ =: A_0 q^n + A_1 q^{n-1} + A_2 q^{n-2} + \cdots + A_n.\end{aligned}$$

The Hirzebruch–Riemann–Roch theorem, whence the name, together with the property of the generalized Fujiki constants assert that this polynomial satisfies

$$\mathrm{RR}_X(q_X(c_1(L))) = \chi(X, L)$$

for all line bundles $L \in \mathrm{Pic}(X)$. In particular, we have $A_n = n + 1$.

The following is the main result which, additionally, yields an upper bound on the second Betti number $b_2(X)$ under some conditions.

Theorem 1.1. *Let X be a hyperkähler manifold of dimension $2n$ with second Betti number $b \coloneqq b_2(X)$ and consider its Riemann–Roch polynomial*

$$\mathrm{RR}_X(q) = A_0 q^n + A_1 q^{n-1} + A_2 q^{n-2} + \cdots$$

If the first three coefficients satisfy the condition

$$2nA_0A_2 < (n-1)A_1^2, \tag{2}$$

then we have the inequality

$$b_2(X) \leq \frac{1}{1 - \dfrac{2nA_0A_2}{(n-1)A_1^2}} - (2n-2), \tag{3}$$

and equality holds if and only if $c_2 \in \mathrm{Sym}^2 H^2(X, \mathbf{R})$. If the condition (2) does not hold, then c_2 is not contained in the Verbitsky component.

We show in Corollary 2.11 that the above conditions are also necessary and sufficient for $\mathrm{td}^{1/2}_{2n-2} \in H^{4n-4}(X, \mathbf{R})$, i.e. the degree $2n-2$ component of the square root of the Todd class, to be contained in the Verbitsky component.

Inequality (2) is equivalent to the condition that the generalized Fujiki constant $C(\mathrm{ch}_4)$ is positive, or equivalently, that

$$C(c_2^2) > 2C(c_4). \tag{4}$$

This is satisfied if the polynomial RR_X has n distinct real roots, see Remark 2.8.

Among known smooth hyperkähler manifolds, there are only two types of Riemann–Roch polynomials: the $\mathrm{K3}^{[n]}$-type and the Kum_n-type (OG_6 and OG_{10} fall into these two types, see [24]). On the other hand, Theorem 1.1 can be generalized to singular irreducible symplectic varieties of dimension 4 and this gives rise to many more examples. We check that the inequality (4) is satisfied for all known smooth examples, as well as for many singular examples, in Sections 2 and 3 respectively.

In Section 4, we give an account of all generalized Fujiki constants for the known examples of smooth hyperkähler manifolds. In particular, we prove that when X is of OG_6 or OG_{10}-deformation type, all Chern classes c_{2i} satisfy

$$c_{2i} \in \mathrm{SH}(X, \mathbf{R})$$

and, thus, all characteristic classes of X lie in the Verbitsky component. This easily leads to the determination of the generalized Fujiki constants for all characteristic classes on these manifolds.

In the final section, we further discuss generalized Fujiki constants and Riemann–Roch polynomials using Rozansky–Witten theory. We present a conceptual proof for the fact that the polynomial

$$\mathrm{RR}_{X,1/2}(q) \coloneqq \sum_{i=0}^{n} \frac{C(\mathrm{td}^{1/2}_{2n-2i})}{(2i)!} q^i$$

factorizes as an n-th power using the Wheeling Theorem and discuss how this method could be used in general to analyze the Riemann–Roch polynomial. This leads to conjectural relations between the generalized Fujiki constants. We mention here the degree four case which yields a precise value of $C(\mathrm{ch}_4)$. For another instance of these conjectural relations, see Conjecture 5.4.

Conjecture 1.2. *Let X be a hyperkähler manifold of dimension $2n > 2$. We have*

$$\frac{C(\mathrm{ch}_4)}{C(1)} = \frac{5(n+1)}{(2n-1)(2n-3)}.$$

Note that, in particular, Conjecture 1.2 would imply (4). We prove in Proposition 5.5 that the conjecture holds true if the Riemann–Roch polynomial satisfies certain expectations on its shape such as [14, Conj. 1.3 (3)] or Conjecture 5.1. We present a possible strategy towards proving these conjectures.

We want to remark that we expect the inequality (4) to hold true pointwise on the level of forms for the right representative of ch_4 and therefore be of local nature. In contrast, Conjecture 1.2 is of global nature. The distinction between these two expectations will occur frequently in the paper.

If proven true, Conjecture 1.2 would imply that for hyperkähler fourfolds there are exactly two possible sets of values that the generalized Fujiki constants can take, see Proposition 5.6. As a consequence, we obtain the following.

Corollary 1.3. *Assuming Conjecture 1.2 in dimension 4, the Betti numbers of a hyperkähler fourfold are one of the following:*

- $b_2 = 5, b_3 = 0, b_4 = 96$;
- $b_2 = 6, b_3 = 4, b_4 = 102$;
- $b_2 = 7, b_3 = 8, b_4 = 108$;
- $b_2 = 23, b_3 = 0, b_4 = 276$.

Hence, Conjecture 1.2 would reduce the number of possible Hodge diamonds and LLV decompositions of hyperkähler fourfolds to four. The two known cases are the ones where c_2 lies in the Verbitsky component. In the case $b_2 = 7$, there are 80 trivial representations of the LLV algebra in $H^{2,2}$, whereas there are 81 trivial representations when the second Betti number is smaller than seven.

In the upcoming work [5, Thm. 9.3], the authors obtained a similar result under a different assumption. We remark that the condition in our Conjecture 1.2 is stronger but makes no explicit assumption on the lattice $H^2(X, \mathbf{Z})$. It focuses only on numerical properties of the Riemann–Roch polynomial.

Relation to other work. While working on further results related to the topic of the paper we learned about the recent preprint of Justin Sawon [26] who independently obtained the same bound on the second Betti number as in Theorem 1.1. The pointwise conjectural relations in Section 5 are in the same flavor as the ones in [26, Sec. 2].

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2. The inequality

We prove Theorem 1.1 in this section. Let X be a hyperkähler manifold of complex dimension $2n$ with $n \geq 2$. We first recall the following result by Fujiki [9] and Huybrechts [13].

Theorem 2.1 (Fujiki, Huybrechts)**.** *Let $\alpha \in H^{4k}(X, \mathbf{R})$ be a class that remains of type $(2k, 2k)$ on all small deformations of X (for example, all characteristic classes satisfy this condition). Then there exists a constant $C(\alpha) \in \mathbf{R}$, called the* generalized Fujiki constant of *α, such that*

$$\forall \beta \in H^2(X, \mathbf{R}) \quad C(\alpha) \cdot q_X(\beta)^{n-k} = \int_X \alpha \cdot \beta^{2n-2k}.$$

Remark 2.2. The term *Fujiki constant* is reserved for the value $C(1) = C(1_X)$. There is also the notion of *small Fujiki constant* c_X: it differs from $C(1)$ by a constant multiple

$$C(1) = \frac{(2n)!}{2^n n!} c_X = (2n-1)!! \cdot c_X.$$

For example, it is known that $c_{\mathrm{K3}^{[n]}} = 1$ and $c_{\mathrm{Kum}_n} = n + 1$.

Denote by $\mathfrak{q} \in \operatorname{Sym}^2 H^2(X, \mathbf{R})$ the dual of the Beauville–Bogomolov–Fujiki form, and by $\mathrm{SH}(X, \mathbf{R}) \subset H^\bullet(X, \mathbf{R})$ the Verbitsky component, which is the subalgebra generated by $H^2(X, \mathbf{R})$. The key step to Theorem 1.1 is the following result.

Proposition 2.3. *We have the following inequality*

$$C(c_2^2) \geq \frac{C(c_2)^2}{C(\mathfrak{q})^2} C(\mathfrak{q}^2), \tag{5}$$

where equality holds if and only if $c_2 \in \operatorname{Sym}^2 H^2(X, \mathbf{R})$.

Proof. We write

$$c_2 = a\mathfrak{q} + z \quad \text{where } a \in \mathbf{R}, z \in \mathrm{SH}(X, \mathbf{R})^\perp.$$

In other words, we project c_2 orthogonally to the Verbitsky component and let $a\mathfrak{q}$ be its image. Then we have

$$C(c_2) = C(a\mathfrak{q}), \quad \text{so } a = \frac{C(c_2)}{C(\mathfrak{q})}.$$

Now we consider the square $c_2^2 = a^2\mathfrak{q}^2 + 2a\mathfrak{q}z + z^2 \in H^8(X, \mathbf{R})$. Since the class z is in $\mathrm{SH}(X, \mathbf{R})^\perp$, it is orthogonal to the image of $\operatorname{Sym}^{2n-2} H^2(X, \mathbf{R})$, so the class $\mathfrak{q}z$ is orthogonal to the image of $\operatorname{Sym}^{2n-4} H^2(X, \mathbf{R})$ and also lies in $\mathrm{SH}(X, \mathbf{R})^\perp$.

On the other hand, for any Kähler class $\omega \in H^2(X, \mathbf{R})$, since z lies $\mathrm{SH}(X, \mathbf{R})^\perp$, the class $z \cdot \omega^{2n-3} \in H^{4n-2}(X, \mathbf{R})$ is orthogonal to the entire $H^2(X, \mathbf{R})$ hence must vanish. So the class

z is primitive of type $(2,2)$ with respect to all Kähler classes on X. By the Hodge–Riemann bilinear relations, for a Kähler class $\omega \in H^2(X, \mathbf{R})$ we have

$$\int_X z^2 \cdot \omega^{2n-4} \geq 0, \quad \text{hence } C(z^2) \geq 0,$$

where equality holds if and only if $z = 0$, i.e. $c_2 \in \operatorname{Sym}^2 H^2(X, \mathbf{R})$. In other words, the projection of z^2 to the Verbitsky component is non-trivial, unless z is itself trivial. Therefore we obtain the desired inequality

$$C(c_2^2) = a^2 C(\mathfrak{q}^2) + C(z^2) \geq a^2 C(\mathfrak{q}^2) = \frac{C(c_2)^2}{C(\mathfrak{q})^2} C(\mathfrak{q}^2),$$

where equality holds if and only if $c_2 \in \operatorname{Sym}^2 H^2(X, \mathbf{R})$. □

We now study the values of the various generalized Fujiki constants that appear in (5).

Proposition 2.4. *Let X be a hyperkähler manifold of dimension $2n$ with second Betti number $b \coloneqq b_2(X)$. For any $\alpha \in H^{4k}(X, \mathbf{R})$ that is of type $(2k, 2k)$ on all small deformations of X, we have*

$$C(\mathfrak{q} \cdot \alpha) = \frac{b + 2n - 2k - 2}{2n - 2k - 1} C(\alpha).$$

In particular, we get

$$C(\mathfrak{q}^k) = \frac{b + 2n - 2k}{1 + 2n - 2k} C(\mathfrak{q}^{k-1}) = \prod_{i=1}^{k} \frac{b + 2n - 2i}{1 + 2n - 2i} \cdot C(1).$$

Proof. Take a basis $(e_1, \ldots, e_b)$ of $H^2(X, \mathbf{R})$ such that

$$\mathfrak{q} = e_1^2 + e_2^2 + e_3^2 - e_4^2 - \cdots - e_b^2.$$

Writing $s_i \coloneqq q_X(e_i) \in \{\pm 1\}$, we have

$$\begin{aligned} C(\mathfrak{q} \cdot \alpha) = \int_X \mathfrak{q} \cdot \alpha \cdot e_1^{2n-2k-2} &= \int_X \alpha \cdot (e_1^{2n-2k} + e_1^{2n-2k-2} e_2^2 + \cdots - e_1^{2n-2k-2} e_b^2) \\ &= C(\alpha) + \sum_{i>1} s_i \int_X \alpha \cdot e_1^{2n-2k-2} e_i^2. \end{aligned}$$

For each term $e_1^{2n-2k-2} e_i^2$, consider the function

$$t \longmapsto \int_X \alpha \cdot (e_1 + t e_i)^{2n-2k} = C(\alpha) \cdot (1 + t^2 s_i)^{n-k},$$

which is a polynomial in t. Comparing the coefficients of t^2, we get

$$\binom{2n-2k}{2} \int_X \alpha \cdot e_1^{2n-2k-2} e_i^2 = C(\alpha) \cdot (n-k) s_i.$$

So we have

$$C(\mathfrak{q} \cdot \alpha) = C(\alpha) + \sum_{i>1} s_i \frac{C(\alpha) s_i}{2n - 2k - 1} = C(\alpha) + (b-1) \frac{C(\alpha)}{2n - 2k - 1} = \frac{b + 2n - 2k - 2}{2n - 2k - 1} C(\alpha),$$

where we used the fact that $s_i^2 = 1$. □

We use the above description to replace $C(\mathfrak{q})$ and $C(\mathfrak{q}^2)$ in (5) and get

$$C(c_2^2) \geq \frac{(2n-1)(b+2n-4)C(c_2)^2}{(2n-3)(b+2n-2)C(1)}. \tag{6}$$

On the other hand, we have the following result by Nieper-Wißkirchen [21], which generalizes the work of Hitchin–Sawon [12]. In particular, it produces linear relations among certain generalized Fujiki constants. We will present a proof of the theorem in Section 5.3.

Theorem 2.5. *Let X be a hyperkähler manifold of dimension $2n$. Consider the following polynomial*

$$\begin{aligned}\mathrm{RR}_{X,1/2}(q) &\coloneqq \sum_{i=0}^{n} \frac{C(\mathrm{td}^{1/2}_{2n-2i})}{(2i)!} q^i \\ &= \frac{C(1)}{(2n)!}q^n + \frac{C(\frac{1}{24}c_2)}{(2n-2)!}q^{n-1} + \frac{C(\frac{7}{5760}c_2^2 - \frac{1}{1440}c_4)}{(2n-4)!}q^{n-2} + \cdots + \frac{C(\mathrm{td}^{1/2}_{2n})}{1}.\end{aligned}$$

There exists a constant r_X such that this polynomial factorizes as

$$\mathrm{RR}_{X,1/2}(q) = C(\mathrm{td}^{1/2}_{2n}) \left(1 + \frac{1}{2r_X} q\right)^n.$$

In [1, Sec. 3], by comparing the first two coefficients, it is shown that

$$r_X = \frac{(2n-1)C(c_2)}{24C(1)} = \frac{(2n-1)2^n n! C(c_2)}{24(2n)! c_X},$$

and

$$C(\mathrm{td}^{1/2}_{2n}) = \frac{C(1)(2r_X)^n}{(2n)!} = c_X \frac{r_X^n}{n!},$$

where c_X is the small Fujiki constant. By comparing the third coefficients, we get the following relation.

Corollary 2.6. *Let X be a hyperkähler manifold of dimension $2n > 2$. Then*

$$7C(c_2^2) - 4C(c_4) = \frac{5(2n-1)C(c_2)^2}{(2n-3)C(1)}.$$

Corollary 2.7. *All generalized Fujiki constants for characteristic classes of degree ≤ 4 are determined by the Riemann–Roch polynomial, or more precisely, by its first three coefficients*

$$\begin{aligned}\mathrm{RR}_X(q) = \sum_{i=0}^{n} \frac{C(\mathrm{td}_{2n-2i})}{(2i)!} q^i &= \frac{C(1)}{(2n)!}q^n + \frac{C(\frac{1}{12}c_2)}{(2n-2)!}q^{n-1} + \frac{C(\frac{1}{240}c_2^2 - \frac{1}{720}c_4)}{(2n-4)!}q^{n-2} + \cdots \\ &= A_0 q^n + A_1 q^{n-1} + A_2 q^{n-2} + \cdots\end{aligned}$$

Proof. Clearly $C(1)$ and $C(c_2)$ appear as coefficients of the Riemann–Roch polynomial so we have

$$C(1) = (2n)! A_0, \quad C(c_2) = 12(2n-2)! A_1.$$

For $C(c_2^2)$ and $C(c_4)$, we already have one linear relation

$$7C(c_2^2) - 4C(c_4) = \frac{5(2n-1)C(c_2)^2}{(2n-3)C(1)} = 720(2n-4)! \frac{(n-1)A_1^2}{nA_0}.$$

The third coefficient gives another one

$$3C(c_2^2) - C(c_4) = 720(2n-4)!A_2,$$

which allows us to uniquely determine their values

$$C(c_2^2) = 144(2n-4)!\left(4A_2 - \frac{(n-1)A_1^2}{nA_0}\right),$$
$$C(c_4) = 144(2n-4)!\left(7A_2 - \frac{3(n-1)A_1^2}{nA_0}\right).$$

Hence we get all four generalized Fujiki constants of degree ≤ 4. ☐

Proof of Theorem 1.1. We replace all generalized Fujiki constants in (6) by the coefficients of the Riemann–Roch polynomial. After some simplifications we get

$$\begin{aligned} 4A_2 &\geq \frac{(n-1)A_1^2}{nA_0}\left(1 + \frac{b+2n-4}{b+2n-2}\right) \\ &= \frac{(n-1)A_1^2}{nA_0}\left(2 - \frac{2}{b+2n-2}\right), \end{aligned} \tag{7}$$

or equivalently,

$$\frac{1}{b+2n-2} \geq 1 - \frac{2nA_0A_2}{(n-1)A_1^2},$$

which yields the assertion. ☐

Remark 2.8. Suppose that the Riemann–Roch polynomial factorizes as a product of linear factors

$$\mathrm{RR}_X(q) = A_0 \prod_i (q + \lambda_i).$$

It was shown in [14] that all the coefficients of $\mathrm{RR}_X(q)$ are positive. Hence the λ_i must all be positive. If, moreover, we assume that the λ_i are not all equal, then condition (2) is satisfied by Cauchy–Schwarz, and the inequality (3) can be written as

$$b_2(X) \leq \frac{n-1}{\dfrac{n\sum \lambda_i^2}{(\sum \lambda_i)^2} - 1} - (2n-2).$$

This is homogeneous with respect to the λ_i and measures in a certain sense the dispersion of the roots.

Remark 2.9. Using the expressions for $C(c_2^2)$ and $C(c_4)$, the condition (2) becomes

$$C(c_2^2) > 2C(c_4),$$

and writing $C(c_2^2) = \mu C(c_4)$ for some $\mu > 2$, the bound (3) becomes

$$b_2(X) \leq 9 - 2n + \frac{10}{\mu - 2}.$$

So we could still get a bound on $b_2(X)$ without even knowing the values for $C(1)$ and $C(c_2)$.

The condition for c_2 to be contained inside the Verbitsky component actually also gives an equivalent condition for $\mathrm{td}_{2n-2}^{1/2}$ to lie inside $\mathrm{SH}(X, \mathbf{R})$, by the following result.

Proposition 2.10. *For a hyperkähler manifold X of dimension $2n$, we have $\mathrm{td}^{1/2}_{2k} \in \mathrm{SH}(X, \mathbf{R})$ if and only if $\mathrm{td}^{1/2}_{2n-2k} \in \mathrm{SH}(X, \mathbf{R})$. Moreover, $\mathrm{td}^{1/2}_{2k} \in \mathrm{SH}(X, \mathbf{R})$ implies $\mathrm{td}^{1/2}_{2k'} \in \mathrm{SH}(X, \mathbf{R})$ for $k' < k < n$.*

Proof. For a class $\alpha \in H^2(X, \mathbf{C})$, denote by $e_\alpha \in \mathfrak{g}(X)_{\mathbf{C}}$ the operator $x \mapsto x \cdot \alpha$. Define h_p to be the holomorphic grading operator that acts on $H^{p,q}(X)$ as $(n-p)\,\mathrm{Id}$ (which is denoted by Π in [14]), and similarly the antiholomorphic grading operator h_q which acts on $H^{p,q}(X)$ as $(n-q)\,\mathrm{Id}$. Recall that for the class σ of a symplectic form, the operator e_σ has the Lefschetz property with respect to the grading given by h_p: there exists a dual Lefschetz operator $\Lambda_\sigma \in \mathfrak{g}(X)_{\mathbf{C}}$, such that together with the operator h_p, we get an $\mathfrak{sl}_2$-triple $(e_\sigma, h_p, \Lambda_\sigma)$ in the LLV algebra. The same result holds if we consider $e_{\overline{\sigma}}$ and h_q.

Jiang [14, Cor. 3.19] showed that there exists a constant $r_\sigma \in \mathbf{R}_{>0}$ such that

$$\Lambda_\sigma(\mathrm{td}^{1/2}_{2k}) = r_\sigma\, \mathrm{td}^{1/2}_{2k-2} \wedge \overline{\sigma}. \tag{8}$$

Furthermore, the operators $e_{\overline{\sigma}}$ and Λ_σ commute for degree reasons. Applying (8) repeatedly, we see that the following holds for all $k < n/2$

$$\Lambda_\sigma^{n-k}(\mathrm{td}^{1/2}_{2n-2k}) = r_\sigma^{n-k}\, \mathrm{td}^{1/2}_{2k} \wedge \overline{\sigma}^{n-k}. \tag{9}$$

On the other hand, Fujiki [9] showed that the operators $e_{\overline{\sigma}}$ and Λ_σ yield isomorphisms

$$e_{\overline{\sigma}}^s \colon H^{l,n-s}(X) \xrightarrow{\sim} H^{l,n+s}(X), \quad \Lambda_\sigma^s \colon H^{n+s,l}(X) \xrightarrow{\sim} H^{n-s,l}(X).$$

Moreover, these isomorphisms are compatible with the decomposition of $H^*(X, \mathbf{C})$ into irreducible $\mathfrak{g}(X)_{\mathbf{C}}$-representations, i.e. for each irreducible representation $V \subset H^*(X, \mathbf{C})$, the isomorphism $e_{\overline{\sigma}}^s$ restricts to an isomorphism

$$e_{\overline{\sigma}}^s \colon H^{l,n-s}(X) \cap V \xrightarrow{\sim} H^{l,n+s}(X) \cap V,$$

and similar for Λ_σ^s. Combining this with (9) yields the first assertion. The second statement also follows from (9) using the same line of arguments. □

Corollary 2.11. *For a hyperkähler manifold X of dimension $2n$, the class $\mathrm{td}^{1/2}_{2n-2}$ lies in the Verbitsky component if and only if the condition* (2) *is satisfied and the equality in* (3) *holds.*

We now examine the bound (3) for the known deformation types of smooth hyperkähler manifolds. There are only two types of Riemann–Roch polynomials

$$\mathrm{RR}_{\mathrm{K3}^{[n]}}(q) = \binom{q/2+n+1}{n}, \quad \mathrm{RR}_{\mathrm{Kum}_n}(q) = (n+1)\binom{q/2+n}{n},$$

see [6, Lem. 5.1] and [21, Lem. 5.2]. Ríos Ortiz showed that O'Grady's sporadic examples satisfy $\mathrm{RR}_{\mathrm{OG}_{10}} = \mathrm{RR}_{\mathrm{K3}^{[5]}}$ and $\mathrm{RR}_{\mathrm{OG}_6} = \mathrm{RR}_{\mathrm{Kum}_3}$ in [24].

Example 2.12 ($\mathrm{K3}^{[n]}$-type)**.** We compute the first three coefficients

$$\begin{aligned} \mathrm{RR}_{\mathrm{K3}^{[n]}}(q) &= \binom{q/2+n+1}{n} \\ &= \frac{1}{2^n n!} q^n + \frac{n+3}{2^n (n-1)!} q^{n-1} + \frac{3n^2+17n+26}{3 \cdot 2^{n+1}(n-2)!} q^{n-2} + \cdots \end{aligned}$$

Then by inserting the values A_0, A_1, A_2 into (3), we get the following upper bound

$$b_2(X) \leq n + 17 + \frac{12}{n+1}.$$

Alternatively, we could also have used Remark (2.8) to obtain the expression. When $n = 2$ or $n = 3$, it evaluates to 23 and is attained by $\mathrm{K3}^{[n]}$; when $n = 5$, it evaluates to 24 and is attained by OG_{10}. In particular, these are exactly the three known deformation types with this Riemann–Roch polynomial for which we have $c_2 \in \operatorname{Sym}^2 H^2(X, \mathbf{R})$.

Example 2.13 (Kum$_n$-type)**.** We compute similarly the first three coefficients

$$\begin{aligned}\mathrm{RR}_{\mathrm{Kum}_n}(q) &= (n+1)\binom{q/2+n}{n} \\ &= \frac{n+1}{2^n n!}q^n + \frac{(n+1)^2}{2^n(n-1)!}q^{n-1} + \frac{(n+1)^2(3n+2)}{3\cdot 2^{n+1}(n-2)!}q^{n-2} + \cdots\end{aligned}$$

and insert these three coefficients into (3). In this case, the upper bound we get is

$$b_2(X) \leq n + 5.$$

When $n = 2$, it is attained by Kum_2; when $n = 3$ it is attained by OG_6. Again, for these two types, we have $c_2 \in \operatorname{Sym}^2 H^2(X, \mathbf{R})$.

Note also that for $n = 2$, the bound $b_2(X) \leq 7$ is much stronger than the general bound $b_2(X) \leq 23$ by Guan.

Another consequence of the inequality is the positivity of the generalized Fujiki constants $C(c_2^2)$ and $C(c_4)$.

Proposition 2.14. *Let X be a hyperkähler manifold of dimension $2n$. The generalized Fujiki constant $C(c_2^2)$ is always positive, and $C(c_4)$ is positive except possibly when $n = 2$ and $b_2(X) = 3, 4, 5$ or when $n = 3$ and $b_2(X) = 3$.*

Proof. From the inequality (6), it is clear that $C(c_2^2)$ is positive. For $C(c_4)$ to be positive, it is equivalent to have

$$\frac{nA_0A_2}{(n-1)A_1^2} > \frac{3}{7}.$$

By (7), we have

$$\frac{nA_0A_2}{(n-1)A_1^2} \geq \frac{1}{4}\left(2 - \frac{2}{b+2n-2}\right).$$

So we want the inequality

$$\frac{1}{4}\left(2 - \frac{2}{b+2n-2}\right) > \frac{3}{7}$$

which is equivalent to $b + 2n > 9$, and is satisfied except when $n = 2$ and $b \leq 5$ or $n = 3$ and $b = 3$. □

Remark 2.15. When $n = 2$, these two generalized Fujiki constants are just Chern numbers, and this is already known by the results of Guan [11].

3. Orbifold examples

Theorem 1.1 can also be generalized to the singular case, at least when $n = 2$. The proof is exactly the same as in Section 2, so we only indicate the key ingredients. We follow the paper by Fu–Menet [7] and the notation therein.

- We consider *primitively irreducible symplectic orbifolds* [7, Def. 3.1]. In dimension 4, such orbifolds only contain isolated singular points.
- Generalized Fujiki constants still exist, as proved by Menet in [19, Lem. 4.6]. Hence we may still define the Riemman–Roch polynomial using the generalized Fujiki constants of the Todd class

$$\mathrm{RR}_X(q) \coloneqq \sum_{i=0}^{n} \frac{C(\mathrm{td}_{2n-2i})}{(2i)!} q^i = A_0 q^n + \cdots + A_n.$$

- Orbifold versions of the Gauss–Bonnet theorem and the Hirzebruch–Riemann–Roch theorem exist in dimension 4 (or more generally, for orbifolds with only isolated singularities), as proved by Blache in [2] (see [7, Thm. 2.12 and Thm. 2.13]): we have

$$\chi_{\mathrm{top}}(X) = \int_X c_4 + \sum_{x \in \mathrm{Sing}(X)} \left(1 - \frac{1}{|G_x|}\right),$$

  and for all $L \in \mathrm{Pic}(X)$,

$$\chi(X, L) = \int_X \mathrm{ch}(L) \cdot \mathrm{td}(X) + \sum_{x \in \mathrm{Sing}(X)} \frac{1}{|G_x|} \sum_{g \in G_x \setminus \{e\}} \frac{1}{\det(\mathrm{Id} - \rho_{x,T_X}(g))}.$$

  Beware that the Riemann–Roch polynomial as defined above no longer gives the correct Euler characteristic, due to the contribution from singular points: instead we have

$$\forall L \in \mathrm{Pic}(X) \quad \chi(X, L) = \mathrm{RR}_X(q_X(L)) + (3 - C(\mathrm{td}_4)).$$

- An orbifold version of the Hitchin–Sawon formula exists: this is [7, Prop. 4.2]. In particular, when $n = 2$, this gives the orbifold version of Corollary 2.6. One would expect that the more general result of Nieper-Wißkirchen should also hold for the singular case.

Using these ingredients and repeating the proof in Section 2, we obtain Theorem 1.1 for primitively irreducible symplectic orbifolds in dimension 4. We apply it to examine the examples listed in [7, Sec. 5]. We will use a_m to denote the number of isolated cyclic quotient singularities of order m.

Remark 3.1. The conceptual reason why Theorem 1.1 remains valid also in the singular case is that this type of result holds pointwise and, therefore, generalizes to orbifolds.

Example 3.2. Let M' be the irreducible symplectic orbifold of dimension 4 with second Betti number $b_2(M') = 16$, also known as a *Nikulin orbifold* (see [7, Sec. 5.11] and [3]). It has 28 isolated quotient singularities of order 2, i.e., $a_2 = 28$. The orbifold M' has topological Euler characteristic $\chi_{\mathrm{top}}(M') = 212$ and Fujiki constant $C(1) = 6$.

Using the orbifold Riemann–Roch and Gauss–Bonnet theorems, we get

$$\begin{aligned}\int_{M'} \mathrm{td}_4 = \int_{M'} \frac{3c_2^2 - c_4}{720} &= \chi(M', \mathcal{O}_{M'}) - \sum_{x\in \mathrm{Sing}(M')} \frac{1}{|G_x|} \sum_{g\in G_x\setminus\{e\}} \frac{1}{\det(\mathrm{Id} - \rho_{x,T_{M'}}(g))}\\ &= 3 - 28\cdot\frac{1}{2}\cdot\frac{1}{16}\\ &= \frac{17}{8},\end{aligned}$$

and

$$\int_{M'} c_4 = \chi_{\mathrm{top}}(M') - \sum_{x\in\mathrm{Sing}(M')}\left(1 - \frac{1}{|G_x|}\right) = 198.$$

Therefore we may compute

$$C(c_2^2) = 576, \quad C(c_4) = 198.$$

The orbifold Hitchin–Sawon formula gives the relation in Corollary 2.6, from which we deduce that $C(c_2) = 36$. Hence we have obtained the Riemann–Roch polynomial of M':

$$\mathrm{RR}_{M'}(q) = \frac{1}{4}q^2 + \frac{3}{2}q + \frac{17}{8}.$$

Note that this polynomial was also computed directly from the geometry of M' by Camere–Garbagnati–Kaputska–Kaputska [3, Thm. 1.3].

Now if we insert the values into (3), we get

$$b_2(X) \leq 16,$$

for any irreducible symplectic orbifold X with the same Riemann–Roch polynomial as M'. The Nikulin orbifold M' attains the upper bound, and we have $c_2(M') \in \mathrm{Sym}^2 H^2(M', \mathbf{R})$. Note that the two roots of $\mathrm{RR}_{M'}(q)$ are $-3 \pm \frac{\sqrt{2}}{2}$, so they are not integers.

Example 3.3. Let K' be the orbifold example in [7, Sec. 5.6] with second Betti number $b_2(K') = 8$ and $a_2 = 36$: we have $\chi_{\mathrm{top}}(K') = 108$ and $C(1) = 8$. Similarly, we compute

$$C(c_2) = 40, \quad C(c_2^2) = 480, \quad C(c_4) = 90,$$

and

$$\mathrm{RR}_{K'}(q) = \frac{1}{3}q^2 + \frac{5}{3}q + \frac{15}{8}.$$

Using (3), we get the bound

$$b_2(X) \leq 8,$$

which again holds for any irreducible symplectic orbifold with the same Riemann–Roch polynomial. So the example K' also attains the upper bound. The two roots are $\frac{-10\pm\sqrt{10}}{4}$.

Note that surprisingly, the Beauville–Bogomolov-Fujiki form of K' is odd and represents the value 1. If we take a line bundle H with $q(c_1(H)) = 1$, after adding the correction term, the Riemann–Roch formula tells us that $\chi(K', H) = 5$, so one could expect that the linear system $|H|$ gives a (rational) finite cover of $\mathbf{P}^4$.

Example 3.4. The following examples are obtained as cyclic quotients of smooth hyperkähler manifolds of $\mathrm{K3}^{[2]}$-type [7, Sec. 5.2, 5.3, and 5.9].

- Case $b_2(M^i_{11}) = 3$ for $i = 1, 2$ with $a_{11} = 5$: we have $\chi_{\mathrm{top}}(M^i_{11}) = 34$ and $C(1) = 33$ for both $i = 1, 2$, so
$$C(c_2) = 30, \quad C(c_2^2) = \frac{828}{11}, \quad C(c_4) = \frac{324}{11},$$
and
$$\mathrm{RR}_{M^i_{11}}(q) = \frac{11}{8}q^2 + \frac{5}{4}q + \frac{3}{11} = \frac{1}{11}\mathrm{RR}_{\mathrm{K3}^{[2]}}(11q).$$
- Case $b_2(M_7) = 5$ with $a_7 = 9$: we have $\chi_{\mathrm{top}}(M_7) = 54$ and $C(1) = 21$, so
$$C(c_2) = 30, \quad C(c_2^2) = \frac{828}{7}, \quad C(c_4) = \frac{324}{7},$$
and
$$\mathrm{RR}_{M_7}(q) = \frac{7}{8}q^2 + \frac{5}{4}q + \frac{3}{7} = \frac{1}{7}\mathrm{RR}_{\mathrm{K3}^{[2]}}(7q).$$
- Case $b_2(M_3) = 11$ with $a_3 = 27$: we have $\chi_{\mathrm{top}}(M_3) = 126$ and $C(1) = 9$, so
$$C(c_2) = 30, \quad C(c_2^2) = 276, \quad C(c_4) = 108,$$
and
$$\mathrm{RR}_{M_3}(q) = \frac{3}{8}q^2 + \frac{5}{4}q + 1 = \frac{1}{3}\mathrm{RR}_{\mathrm{K3}^{[2]}}(3q).$$

In all these cases, the bound we get is $b_2(X) \leq 23$, which is not attained. These are all equal to the bound for $\mathrm{K3}^{[2]}$, due to the fact that the expression in (3) is homogeneous in terms of the roots of $\mathrm{RR}_X(q)$, hence will remain invariant after a change of variables.

In some sense, taking cyclic quotient does not produce genuinely "new" examples or Riemann–Roch polynomials.

Example 3.5. For the following examples, we could not find the values of the Fujiki constant $C(1)$ in the literature. But a bound on b_2 can still be given, due to the observation in Remark 2.9. We will simply write the upper bound obtained as $b_2(X) \leq B$, where X is understood as an irreducible symplectic orbifold with the same Riemann–Roch polynomial.

- Case $b_2(K'_4) = 6$ with $a_2 = 45, a_4 = 2$ and $\chi_{\mathrm{top}}(K'_4) = 69$ [7, Sec. 5.4]: we have
$$C(c_2) = \sqrt{142C(1)}, \quad C(c_2^2) = 330, \quad C(c_4) = 45,$$
and
$$b_2(X) \leq \frac{55}{8} = 6.875.$$
So $b_2(K'_4) = 6$ is the maximal possible but does not attain the bound.
- Case $b_2(K'_3) = 7$ with $a_3 = 12$ and $\chi_{\mathrm{top}}(K'_3) = 108$ [7, Sec. 5.5]: we have
$$C(c_2) = 26\sqrt{C(1)/3}, \quad C(c_2^2) = 540, \quad C(c_4) = 100,$$
and
$$b_2(X) \leq \frac{135}{17} \approx 7.94$$
So $b_2(K'_3) = 7$ is the maximal possible but does not attain the bound.
- Case $b_2(Y_{\mathrm{K3}}(D_3)) = 9$: the description of this example in [7] appears to be incorrect.[1]

[1]Namely, the orbifold is described as the quotient of an $S^{[2]}$ by some symplectic automorphisms forming the dihedral group D_3. But such a quotient would necessarily contain singularities in codimension 2.

- Case $b_2(Y_{\mathrm{K3}}(\mathbf{Z}/4\mathbf{Z})) = 10$ with $a_2 = 10, a_4 = 6$ and $\chi_{\mathrm{top}} = 140$ [8, Table 1]: we have

$$C(c_2) = 8\sqrt{3C(1)}, \quad C(c_2^2) = 486, \quad C(c_4) = \frac{261}{2},$$

and

$$b_2(X) \leq \frac{54}{5} = 10.8.$$

So $b_2(Y_{\mathrm{K3}}(\mathbf{Z}/4\mathbf{Z})) = 10$ is the maximal possible but does not attain the bound.

- Case $b_2\big(Y_{\mathrm{K3}}\big((\mathbf{Z}/2\mathbf{Z})^2\big)\big) = 14$ with $a_2 = 36$ and $\chi_{\mathrm{top}} = 180$ [8, Table 1]: we have

$$C(c_2) = 8\sqrt{3C(1)}, \quad C(c_2^2) = 504, \quad C(c_4) = 162,$$

and

$$b_2(X) \leq 14.$$

So the bound is attained in this example.

Example 3.6 (Kim)**.** This example was studied by Kim in [15, Sec. 7]: let X be a hyperkähler fourfold of Kum_2-type admitting a Lagrangian fibration. We consider its dual Lagrangian fibration $\check{X}$. It is a singular hyperkähler orbifold with only isolated quotient singularities.

However, the analysis in *loc. cit.* of the singularities of $\check{X}$ contains an error: the group action admits 108 fixed points on X, and every other 3 of them are identified after the quotient. So one should have $a_3 = 36$, that is, $\check{X}$ admits 36 isolated cyclic quotient singularities of order 3, instead of just 18 of them as claimed in *loc. cit.* Since $\chi_{\mathrm{top}}(X) = 108$, we may conclude that $\chi(\check{X}) = 108/3 = 36$, which is consistent with the description of the cohomology.

We compute the numerical invariants. By the orbifold Gauss–Bonnet theorem, we have $C(c_4) = \chi_{\mathrm{top}} - a_3 \cdot \frac{2}{3} = 12$. Then by the orbifold Riemann–Roch theorem, we have $\frac{1}{720}\big(3C(c_2^2) - C(c_4)\big) = 3 - a_3 \cdot \frac{1}{3} \cdot \frac{2}{9} = \frac{1}{3}$, hence $C(c_2^2) = 84$. This already gives us the bound on b_2

$$b_2 \leq \frac{10}{\frac{84}{12} - 2} - 2 \cdot 2 + 9 = 7,$$

which is attained by the dual Lagrangian fibration $\check{X}$.

Kim showed that the Fujiki constant $C(1_{\check{X}})$ of the dual Lagrangian fibration $\check{X}$ is $1/C(1_X)$, so $C(1_{\check{X}}) = \frac{1}{9}$ in the dual Kum_2 case. Then by the orbifold Hitchin–Sawon formula, we may compute that $C(c_2) = 2$. Hence the Riemann–Roch polynomial is given by

$$\mathrm{RR}_{\check{\mathrm{Kum}}_2}(q) = \frac{1}{216}q^2 + \frac{1}{12}q + \frac{1}{3} = \frac{1}{9}\mathrm{RR}_{\mathrm{Kum}_2}(\tfrac{1}{3}q).$$

In particular, for a line bundle H with square $q(c_1(H)) = 18$, we can use the Riemann–Roch formula with the correction term to compute $\chi(\check{X}, H) = 6$. So one could expect that the linear system $|H|$ gives a hypersurface (or a cover thereof) in $\mathbf{P}^5$.

4. Generalized Fujiki constants for known smooth examples

In this section, we give an account for the generalized Fujiki constants $C(c_\lambda)$ of characteristic classes $c_\lambda \coloneqq c_2^{\lambda_2} c_4^{\lambda_4} \cdots c_{2n}^{\lambda_{2n}}$ for all known deformation types of hyperkähler manifolds.

4.1. $\mathrm{K3}^{[n]}$ **and** Kum_n**.** The results are classical for the two infinite families. In the $\mathrm{K3}^{[n]}$-case, the method in Ellingsrud–Göttsche–Lehn [6] can be used to compute all the generalized Fujiki constants using a computer for small n. A similar algorithmic method can be used to treat the Kum_n-case, with some slight modifications based on the work of Nieper-Wißkirchen [22, Sec. 4.2.3]. An implementation for these algorithms in `Sage` can be found on the second-named author's webpage. Closed formulae for the values $C(c_{2k})$ for both families were recently established in [4, Thm. 4.2].

4.2. OG_6**.** By Corollary 2.7, the generalized Fujiki constants for characteristic classes of degree ≤ 4 for OG_6 are the same as those for Kum_3, since they share the same Riemann–Roch polynomial. Since the Chern numbers of OG_6 are also known [20, Prop. 6.8], we can obtain all of them:

α	1	c_2	c_4	c_2^2	c_6	c_4c_2	c_2^3
$C(\alpha)$	60	288	480	1920	1920	7680	30720

Alternatively, since for OG_6-type the second Chern class c_2 lies in the Verbitsky component (namely, $c_2(\mathrm{OG}_6) = 2\mathfrak{q}$), Corollary 2.11 shows that the class $\mathrm{td}_4^{1/2}$ also lies in $\mathrm{SH}(X, \mathbf{R})$. Now $\mathrm{td}_4^{1/2}$ is a linear combination of c_2^2 and c_4, so the same may be said for the class c_4. Then we can use Proposition 2.4 to determine that $c_4(\mathrm{OG}_6) = \mathfrak{q}^2$, which then allows us to also compute $C(c_4c_2)$ and $C(c_2^3)$. Finally we can use $C(\mathrm{td}_6) = 4$ to solve the Euler characteristic $C(c_6)$.

Proposition 4.1. *For hyperkähler manifolds of* OG_6*-type, all Chern classes* c_2, c_4, c_6 *lie in the Verbitsky component. We have*

$$c_2(\mathrm{OG}_6) = 2\mathfrak{q}, \quad c_4(\mathrm{OG}_6) = \mathfrak{q}^2, \quad c_6(\mathrm{OG}_6) = \tfrac{1}{2}\mathfrak{q}^3.$$

4.3. OG_{10}**.** The question for OG_{10} might seem difficult at first, as there are many more unknown Fujiki constants to determine. It turns out to be quite easy, due to the following observation.

Proposition 4.2. *For hyperkähler manifolds of* OG_{10}*-type, all Chern classes* $c_2, \ldots, c_{10}$ *lie in the Verbitsky component. We have*

$$c_2(\mathrm{OG}_{10}) = \tfrac{3}{2}\mathfrak{q}, \quad c_4(\mathrm{OG}_{10}) = \tfrac{15}{16}\mathfrak{q}^2, \quad c_6(\mathrm{OG}_{10}) = \tfrac{21}{64}\mathfrak{q}^3,$$
$$c_8(\mathrm{OG}_{10}) = \tfrac{237}{3328}\mathfrak{q}^4, \quad c_{10}(\mathrm{OG}_{10}) = \tfrac{27}{2560}\mathfrak{q}^5.$$

Proof. We use the LLV decomposition of the cohomology obtained in [10, Thm 3.26]

$$H^*(\mathrm{OG}_{10}, \mathbf{Q}) = V_{(5)} \oplus V_{(2,2)} \quad \text{as } \mathfrak{so}(4,22)\text{-modules}.$$

We are interested in the second component, which only contributes to cohomological degree k for $k \in \{6, 8, 10, 12, 14\}$.

For a generic X in the moduli space, the (special) Mumford–Tate algebra is the maximal possible and is isomorphic to $\mathfrak{so}(3,21)$. Using the branching rules, we get the following decompositions of $\mathfrak{so}(3,21)$-modules/Hodge structures (H^{12} and H^{14} are omitted by symmetry)

$$\begin{aligned} H^6(X, \mathbf{Q}) &= \mathrm{SH}^6(X, \mathbf{Q}) \oplus V_{(2)}, \\ H^8(X, \mathbf{Q}) &= \mathrm{SH}^8(X, \mathbf{Q}) \oplus V_{(2,1)} \oplus V_{(1)}, \\ H^{10}(X, \mathbf{Q}) &= \mathrm{SH}^{10}(X, \mathbf{Q}) \oplus V_{(2,2)} \oplus V_{(2)} \oplus V_{(1,1)} \oplus \mathbf{Q}. \end{aligned}$$

In other words, up to multiplying by a non-zero scalar, there is only one Hodge class $\eta \in H^{10}(X, \mathbf{Q})$ that lies in $\mathrm{SH}(X, \mathbf{Q})^{\perp}$ for a generic X. In particular, this means that all the Chern classes $c_2, \ldots, c_{10}$ lie in the Verbitsky component.

For a generic X, the only Hodge classes in the Verbitsky components are multiples of powers of $\mathfrak{q}$, so each Chern class c_{2k} is a multiple of $\mathfrak{q}^k$. We explain how to determine the scalars, starting from smaller k: we use Corollary 2.7 to determine $C(c_2)$ and $C(c_4)$. Since the values of $C(\mathfrak{q}^k)$ are known by Proposition 2.4, we have determined c_2 and c_4. Once all c_{2i} for $i < k$ are known, we study the class $\mathrm{td}^{1/2}_{2k}$, whose generalized Fujiki constant $C(\mathrm{td}^{1/2}_{2k})$ is known by Theorem 2.5 and whose only unknown term is a given multiple of c_{2k}. Therefore we will be able to uniquely determine $C(c_{2k})$ and thus c_{2k} itself. $\square$

It is then straightforward to compute the generalized Fujiki constants, which we include for the reader's convenience.

α	1	c_2	c_4	c_2^2	c_6	c_4c_2	c_2^3	c_8	c_6c_2	c_4^2	$c_4c_2^2$	c_2^4
$C(\alpha)$	945	5040	13500	32400	26460	113400	272160	49770	343980	614250	1474200	3538080

c_{10}	c_8c_2	c_6c_4	$c_6c_2^2$	$c_4^2c_2$	$c_4c_2^3$	c_2^5
176904	1791720	5159700	12383280	22113000	53071200	127370880

Note that the Chern numbers for OG_{10} have already been computed by Cao–Jiang in the appendix of [24].

It is remarkable that the knowledge of the Riemann–Roch polynomial together with the assumption that all Chern classes lie in the Verbitsky component allow us to completely determine the second Betti number as well as all the generalized Fujiki constants, in particular all the Chern numbers including the Euler characteristic $C(c_{2n}) = \int_X c_{2n}$.

5. Further discussions

We see that the Riemann–Roch polynomial RR_X of a hyperkähler manifold X is a very important notion: it puts strong topological restriction on X, namely an upper bound for the second Betti number. We now formulate some conjectures on the shape of such polynomials and discuss some possible ways of studying them.

Recall from Theorem 2.5 that the polynomial $\mathrm{RR}_{X,1/2}$ factors as a n-th power. The proof by Nieper-Wißkirchen [21] uses the machinery of Rozansky–Witten invariants. We will briefly explain the proof, and discuss the possibility of using this method to study the Riemann–Roch polynomial RR_X.

5.1. **Conjectural form of the Riemann–Roch polynomial.** Motivated by the above discussions, we speculate about the general shape of the Riemann–Roch polynomial of certain symplectic varieties.

We make the following conjecture. Similar conjectures have already been formulated by Ríos Ortiz and Jiang in [14, Conj. 1.3].

Conjecture 5.1. *Let X be an irreducible symplectic orbifold of dimension $2n$.*

(1) The Riemann–Roch polynomial RR_X has n distinct negative real roots forming an arithmetic sequence.
(2) If X is smooth, then its Riemann–Roch polynomial $\mathrm{RR}_X(q)$ has even negative integer roots $\lambda_1, \ldots, \lambda_n$ satisfying $\lambda_i - \lambda_{i-1} = 2$.

The second point is a slight strengthening of [14, Conj. 1.3(3)]. Note that it fails already in the case of four-dimensional orbifolds as demonstrated in Section 3 and should necessarily involve the smoothness assumption.

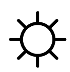
By Remark 2.8, Conjecture 5.1 (1) would imply the inequality (2) and therefore yield the bound on the second Betti number.

5.2. **Rozansky–Witten invariants.** We give a very rough overview of parts of Rozansky–Witten theory that we want to employ. For proofs, details and a general overview we refer mainly to the book [22]. See also [12, 14, 27].

After choosing a symplectic form $\sigma \in H^0(X, \Omega_X^2)$, the *Rozansky–Witten weight system* RW_σ is a ring homomorphism

$$\mathrm{RW}_\sigma \colon B \longrightarrow H^*(X, \mathbf{C}), \tag{10}$$

where B denotes the graph homology space, i.e., the $\mathbf{C}$-algebra spanned by all unitrivalent graphs modulo the antisymmetry and IHX relation. Important graphs are ℓ, the unique univalent graph with two vertices, Θ, the trivalent graph $\ominus$ with two vertices, and the $2k$-wheels w_{2k} which, for example, looks like



for $k = 4$.

Using the $2k$-wheels, we can define the wheeling element

$$\Omega := \exp\left(\sum_{k=1}^{\infty} b_{2k} w_{2k}\right)$$

contained in the completion $\hat{B}$ of B with b_{2k} the modified Bernoulli numbers. We have

- $\mathrm{RW}_\sigma(\ell) = 2\sigma$,
- $\mathrm{RW}_\sigma(\Theta) = b_\Theta \left[\frac{2\bar{\sigma}}{q(\sigma+\bar{\sigma})}\right]$, where $b_\Theta = 48\, r_X = \frac{2(2n-1)C(c_2)}{C(1)}$ [21, Prop. 7],
- $\mathrm{RW}_\sigma(w_{2k}) = -(2k)!\, \mathrm{ch}_{2k}$,
- $\mathrm{RW}_\sigma(\Omega) = \mathrm{td}^{1/2}$.

There is a bilinear product $\langle -, - \rangle$ on the graph homology space defined by summing over all possible ways of gluing all univalent vertices of the graphs under consideration, see [22, Def. 2.39] for a precise account. One form of the Wheeling Theorem is the following [22, Cor. 2.3].

Theorem 5.2. *The map*

$$\langle \Omega, - \rangle \colon x \longmapsto \langle \Omega, x \rangle$$

respects the ring structure on B given by disjoint union.

There is also a bilinear product $\langle -, - \rangle_\sigma$ defined on the cohomology [22, Def. 3.9], which depends on the symplectic form σ chosen. We use the subscript σ to emphasize this dependence. The map RW_σ respects the two bilinear products [22, Prop. 3.4]

$$\mathrm{RW}_\sigma(\langle x, y \rangle) = \langle \mathrm{RW}_\sigma(x), \mathrm{RW}_\sigma(y) \rangle_\sigma.$$

This is the crucial result which allows us to transport relations present inside the graph homology space to the cohomology of X.

Generalized Fujiki constants naturally appear in the study of Rozansky–Witten invariants, which can already be seen in the above formula for $\mathrm{RW}_\sigma(\Theta)$. The key idea for the formula is that $\mathrm{RW}_\sigma(\Theta)$ is a class in $H^{0,2}(X)$, which is generated by $[\overline{\sigma}]$. So we can uniquely determine the class just by a scalar. To determine this number, one could cup the two classes with $\exp(\sigma+\overline{\sigma})$ and compare the integral.

To illustrate this method, we determine the value of $\mathrm{RW}_\sigma(\Theta_2)$, where Θ_2 is the necklace graph with two beads.

Proposition 5.3. *We have*

$$\begin{aligned}\mathrm{RW}_\sigma(\Theta_2) &= -\frac{4\int(c_2^2-2c_4)\exp(\sigma+\overline{\sigma})}{5n(n-1)\int\exp(\sigma+\overline{\sigma})}[\overline{\sigma}]^2\\ &= -\frac{4(2n-1)(2n-3)C(c_2^2-2c_4)}{5C(1)}\left[\frac{2\overline{\sigma}}{q(\sigma+\overline{\sigma})}\right]^2.\end{aligned}$$

Proof. Using the definition of the pairing $\langle -,-\rangle$ on the graph homology space, one can verify that

$$\left\langle w_4,\ell^2\right\rangle = 20\Theta_2.$$

Hence

$$\begin{aligned}\mathrm{RW}_\sigma(\Theta_2) = \mathrm{RW}_\sigma\left(\frac{1}{20}\left\langle w_4,\ell^2\right\rangle\right) &= \frac{1}{20}\left\langle \mathrm{RW}_\sigma(w_4),\mathrm{RW}_\sigma(\ell^2)\right\rangle_\sigma\\ &= \frac{1}{20}\left\langle -24\left(\tfrac{1}{12}c_2^2-\tfrac{1}{6}c_4\right),4\sigma^2\right\rangle_\sigma = -\frac{2}{5}\left\langle c_2^2-2c_4,\sigma^2\right\rangle_\sigma = -\frac{4}{5}\left\langle c_2^2-2c_4,\exp\sigma\right\rangle_\sigma.\end{aligned}$$

Cupping it with $\exp(\sigma+\overline{\sigma})$ and comparing the integral, we get

$$\mathrm{RW}_\sigma(\Theta_2) = -\frac{4\int\langle c_2^2-2c_4,\exp\sigma\rangle_\sigma\exp(\sigma+\overline{\sigma})}{5\int\overline{\sigma}^2\exp(\sigma+\overline{\sigma})}[\overline{\sigma}]^2.$$

For the denominator, we can simplify it as

$$\begin{aligned}\int_X\overline{\sigma}^2\exp(\sigma+\overline{\sigma}) &= \int_X\overline{\sigma}^2\frac{1}{(2n-2)!}(\sigma+\overline{\sigma})^{2n-2}\\ &= \int_X\frac{1}{n!(n-2)!}(\sigma\overline{\sigma})^n\\ &= n(n-1)\int_X\exp(\sigma+\overline{\sigma}).\end{aligned}$$

For the numerator, we use the following equality [22, Lem. 3.4]

$$\int_X\langle\alpha,\exp\sigma\rangle_\sigma\exp(\sigma+\overline{\sigma}) = \int_X\alpha\exp(\sigma+\overline{\sigma}).$$

This shows the first equality that we want to prove.

For the second equality, we note that for a class of type $(2j,2j)$, the Fujiki relations give

$$\int_X\alpha\exp(\sigma+\overline{\sigma}) = \int_X\alpha\cdot\frac{1}{(2n-2j)!}(\sigma+\overline{\sigma})^{2n-2j} = \frac{C(\alpha)}{(2n-2j)!}q(\sigma+\overline{\sigma})^{n-j}.$$

Taking α to be 1_X and $c_2^2-2c_4$ respectively, we get the desired equality. □

In general, for a trivalent graph Γ with $2k$ vertices, there is a number b_Γ independent of the symplectic form σ chosen, such that we have

$$\mathrm{RW}_\sigma(\Gamma) = b_\Gamma \left[\frac{2\overline{\sigma}}{q(\sigma+\overline{\sigma})}\right]^k \in H^{0,2k}(X).$$

For example, we have obtained that

$$b_\Theta = \frac{2(2n-1)C(c_2)}{C(1)}, \quad b_{\Theta_2} = -\frac{4(2n-1)(2n-3)C(c_2^2-2c_4)}{5C(1)}.$$

This is the same notation used by Sawon in [26, 27], although he only used the letter b_Γ for graphs with exactly $2n$ vertices and referred to those as the *Rozansky–Witten invariants* of X. By the properties of the map RW_σ, the values b_Γ are multiplicative with respect to disjoint union.

There is another way to obtain the value of $\mathrm{RW}_\sigma(\Theta_2)$. Namely

$$\mathrm{RW}_\sigma(2\Theta_2) = \mathrm{RW}_\sigma(\langle w_2, w_2\rangle) = 4\langle c_2, c_2\rangle_\sigma$$

where we used the relation $\mathrm{RW}_\sigma(w_2) = 2c_2$. We therefore obtain from Proposition 5.3 the equality

$$\langle c_2, c_2\rangle_\sigma = \frac{b_{\Theta_2}}{2}\left[\frac{2\overline{\sigma}}{q(\sigma+\overline{\sigma})}\right]^2 \in H^4(X, \mathcal{O}_X).$$

We expect that this equality is equivalent to the equality obtained in Corollary 2.6, but have not pursued this further.

5.3. **Proof of Theorem 2.5.** Using the map RW_σ and the Wheeling Theorem, we can obtain a very conceptual proof and see why the polynomial $\mathrm{RR}_{X,1/2}$ factorizes as an n-th power.

Proof of Theorem 2.5. For a class α of degree $(2k, 2k)$ admitting a generalized Fujiki constant, we follow the same method as in the proof of Proposition 5.3 to compute

$$\begin{aligned}
\left\langle \alpha, (2\sigma)^k\right\rangle_\sigma &= 2^k k! \langle \alpha, \exp\sigma\rangle_\sigma \\
&= 2^k k! \frac{\int \langle \alpha, \exp\sigma\rangle_\sigma \exp(\sigma+\overline{\sigma})}{n(n-1)\cdots(n-(k-1))\int \exp(\sigma+\overline{\sigma})}[\overline{\sigma}]^k \\
&= \frac{2^k}{\binom{n}{k}} \frac{\int \alpha\exp(\sigma+\overline{\sigma})}{\int \exp(\sigma+\overline{\sigma})}[\overline{\sigma}]^k \\
&= \frac{2^k}{\binom{n}{k}} \frac{\frac{C(\alpha)}{(2n-2k)!}q(\sigma+\overline{\sigma})^{n-k}}{\frac{C(1)}{(2n)!}q(\sigma+\overline{\sigma})^n}[\overline{\sigma}]^k \\
&= \frac{1}{\binom{n}{k}\frac{C(1)}{(2n)!}} \frac{C(\alpha)}{(2n-2k)!}\left[\frac{2\overline{\sigma}}{q(\sigma+\overline{\sigma})}\right]^k.
\end{aligned} \tag{11}$$

We can take α to be $\mathrm{td}_{2k}^{1/2}$, which gives us

$$\begin{aligned}
\frac{C(1)}{(2n)!}\left\langle \mathrm{td}^{1/2}, (1+2\sigma)^n\right\rangle_\sigma &= \frac{C(1)}{(2n)!}\left\langle \mathrm{td}^{1/2}, \sum_{k=0}^{n}\binom{n}{k}(2\sigma)^k\right\rangle_\sigma \\
&= \sum_{k=0}^{n}\binom{n}{k}\frac{C(1)}{(2n)!}\left\langle \mathrm{td}_{2k}^{1/2}, (2\sigma)^k\right\rangle_\sigma \\
&= \sum_{k=0}^{n}\frac{C(\mathrm{td}_{2k}^{1/2})}{(2n-2k)!}\left[\frac{2\overline{\sigma}}{q(\sigma+\overline{\sigma})}\right]^k \\
&= \mathrm{RR}'_{X,1/2}\left(\left[\frac{2\overline{\sigma}}{q(\sigma+\overline{\sigma})}\right]\right).
\end{aligned}$$

Here $\mathrm{RR}'_{X,1/2}(q) \coloneqq q^n\mathrm{RR}_{X,1/2}(1/q)$ is the polynomial obtained by reversing the coefficients. The polynomial evaluated at the class $\left[\frac{2\overline{\sigma}}{q(\sigma+\overline{\sigma})}\right]$ is an element in the cohomology ring, with terms in various degrees.

On the graph homology side, the Wheeling Theorem provides the relation

$$\langle \Omega, (1+\ell)^n\rangle = \langle \Omega, 1+\ell\rangle^n.$$

Since the Rozansky–Witten invariant RW_σ is a ring homomorphism respecting the bilinear form $\langle -,-\rangle$, we get

$$\left\langle \mathrm{td}^{1/2}, (1+2\sigma)^n\right\rangle_\sigma = \left\langle \mathrm{td}^{1/2}, 1+2\sigma\right\rangle_\sigma^n.$$

Hence the polynomial $\mathrm{RR}_{X,1/2}$ must indeed factorize as an n-th power. □

5.4. **Riemann–Roch polynomial via RW invariants.** Following the idea of the proof of Theorem 2.5, if we want to study the Riemann–Roch polynomial RR_X, we should replace α with td_{2k} in (11): summing over all k we get similarly

$$\frac{C(1)}{(2n)!}\langle \mathrm{td}, (1+2\sigma)^n\rangle_\sigma = \mathrm{RR}'_X\left(\left[\frac{2\overline{\sigma}}{q(\sigma+\overline{\sigma})}\right]\right).$$

So for the same strategy to work, we need to study how the graph homology element

$$\left\langle \Omega^2, (1+\ell)^n\right\rangle$$

might potentially factorize into linear terms. Since the multiplication for the graph homology classes is the disjoint union, this would unfortunately not be possible in general. Below we compute its value for $n\leq 4$:

$$\begin{aligned}
\left\langle \Omega^2, 1+\ell\right\rangle &= 1+\frac{1}{12}\Theta, \\
\left\langle \Omega^2, (1+\ell)^2\right\rangle &= 1+\frac{1}{12}2\Theta+\frac{1}{12^2}(\Theta^2+\Theta_2), \\
\left\langle \Omega^2, (1+\ell)^3\right\rangle &= 1+\frac{1}{12}3\Theta+\frac{1}{12^2}3(\Theta^2+\Theta_2)+\frac{1}{12^3}(\Theta^3+3\Theta\Theta_2), \\
\left\langle \Omega^2, (1+\ell)^4\right\rangle &= 1+\frac{1}{12}4\Theta+\frac{1}{12^2}6(\Theta^2+\Theta_2)+\frac{1}{12^3}4(\Theta^3+3\Theta\Theta_2) \\
&\quad +\frac{1}{12^4}(\Theta^4+6\Theta^2\Theta_2+3\Theta_2^2+\tfrac{144}{25}\Xi-\tfrac{162}{25}\Theta_4),
\end{aligned}$$

where Ξ is the extra graph for $n = 4$. We study the implications on the Riemann–Roch polynomial.

- When $n = 2$, we get
$$\mathrm{RR}_X(q) = \frac{C(1)}{(2\cdot 2)!}\left(q^2 + \frac{1}{12}2b_\Theta q + \frac{1}{12^2}(b_\Theta^2 + b_{\Theta_2})\right).$$
For the polynomial to admit two real roots, the value b_{Θ_2} needs to be negative, or equivalently, the integral $C(\mathrm{ch}_4) = \int_X \mathrm{ch}_4$ needs to be positive. For smooth hyperkähler fourfolds, this indeed holds by the bound of Guan (see [23, Lem. 4.6] or [26, Thm. 7]).
- When $n = 3$, the graph homology class admits a factor $1 + \frac{1}{12}\Theta$, so we also get a factorization for the Riemann–Roch polynomial
$$\mathrm{RR}_X(q) = \frac{C(1)}{(2\cdot 3)!}\left(q + \frac{1}{12}b_\Theta\right)\left(q^2 + \frac{1}{12}2b_\Theta q + \frac{1}{12^2}(b_\Theta^2 + 3b_{\Theta_2})\right).$$
So if b_{Θ_2} is negative, the polynomial will indeed admit three real roots forming an arithmetic sequence, with difference $\frac{1}{12}\sqrt{-3b_{\Theta_2}}$.
- When $n = 4$, the graph homology class becomes more complicated due to the extra graph Ξ. If we expect the Riemann–Roch polynomial to admit four real roots forming an arithmetic sequence, this would lead to the following conjectural relations among certain generalized Fujiki constants.

Conjecture 5.4. *If X is of dimension $2n \geq 8$, then*
$$\frac{C(\mathrm{ch}_4^2 + 120\mathrm{ch}_8)\cdot C(1)}{C(\mathrm{ch}_4)^2} = \frac{(5n+7)(2n-1)(2n-3)}{5(n+1)(2n-5)(2n-7)}.$$

Admitting this relation, we would then get
$$\left\langle \mathrm{ch}_4^2 + 120\mathrm{ch}_8, (2\sigma)^4\right\rangle_\sigma = \left(\tfrac{5}{3(n+1)} + \tfrac{25}{6}\right)\mathrm{RW}_\sigma(\Theta_2^2).$$
On the other hand, based on the computation of Sawon [27], we have
$$\tfrac{1}{384}\left\langle w_4^2, \ell^4\right\rangle = 24\Xi + 48\Theta_4 + \tfrac{25}{4}\Theta_2^2,$$
$$\tfrac{1}{384}\left\langle w_8, \ell^4\right\rangle = 7\Xi + \tfrac{287}{8}\Theta_4.$$
Taking a suitable linear combination and applying RW_σ, we get
$$\left\langle \mathrm{ch}_4^2 + 120\mathrm{ch}_8, (2\sigma)^4\right\rangle_\sigma = \mathrm{RW}_\sigma(8\Xi - 9\Theta_4 + \tfrac{25}{6}\Theta_2^2),$$
so
$$\mathrm{RW}_\sigma(8\Xi - 9\Theta_4) = \tfrac{5}{3(n+1)}\mathrm{RW}_\sigma(\Theta_2^2).$$
Hence we can express the Rozansky–Witten invariant of $\frac{144}{25}\Xi - \frac{162}{25}\Theta_4 = \frac{18}{25}(8\Xi - 9\Theta_4)$ in terms of b_{Θ_2}, so the Riemann–Roch polynomial has the following form
$$\mathrm{RR}_X(q) = \frac{C(1)}{(2\cdot 4)!}\left(q^2 + \tfrac{1}{12}2b_\Theta q + \tfrac{1}{12^2}(b_\Theta^2 + \tfrac{3}{5}b_{\Theta_2})\right)\left(q^2 + \tfrac{1}{12}2b_\Theta q + \tfrac{1}{12^2}(b_\Theta^2 + \tfrac{27}{5}b_{\Theta_2})\right).$$
If b_{Θ_2} is negative, then it indeed admits four roots forming an arithmetic progression with difference $\frac{1}{6}\sqrt{-\frac{3}{5}b_{\Theta_2}}$.

5.5. **Conjectural value for generalized Fujiki constants.** In the above examples we see that the value b_{Θ_2} or equivalently $C(\mathrm{ch}_4)$ governs the differences between the roots of the Riemann–Roch polynomial. We speculate that the roots always form an arithmetic progression with difference 2. This is our main motivation for Conjecture 1.2. Note that the conjectural value for $C(\mathrm{ch}_4)$ also predicts that one should always have $b_{\Theta_2} = -48(n+1)$ by Proposition 5.3. It can also be seen as a weaker version of Conjecture 5.1 (2), for purely algebraic reasons.

Proposition 5.5. *Conjecture 5.1 (2) implies Conjecture 1.2.*

Proof. By assumption, the roots of $\mathrm{RR}_X(q)$ form an arithmetic progression with difference 2, so we have

$$\begin{aligned}\mathrm{RR}_X(q) &= \frac{C(1)}{(2n)!}(q+a)(q+a+2)\cdots(q+a+2n-2)\\ &= \frac{C(1)}{(2n)!}\Big(q^n + (na+n(n-1))q^{n-1}\\ &\qquad + \Big(\tfrac{n(n-1)}{2}a^2 + (n-1)^2na + \tfrac{(3n-1)n(n-1)(n-2)}{6}\Big)q^{n-2} + \dots\Big)\end{aligned}$$

Then by the result of Corollary 2.7, we may deduce the values for $C(c_2^2)$ and $C(c_4)$, and consequently $C(\mathrm{ch}_4)$, which turns out to depend only on $C(1)$ and n, and not on a. ☐

We also explore some consequences of Conjecture 1.2.

Proposition 5.6. *Assuming Conjecture 1.2, for $n = 2$ the following are the only possibilities for the generalized Fujiki constants of a hyperkähler fourfold.*

$C(1)$	$C(c_2)$	$C(c_2^2)$	$C(c_4)$
3	30	828	324
9	54	756	108

Proof. We have the following three relations

$$\begin{aligned}7C(c_2^2) - 4C(c_4) &= 15\tfrac{C(c_2)^2}{C(1)},\\ C(c_2^2) - 2C(c_4) &= 60C(1),\\ 3C(c_2^2) - C(c_4) &= 2160,\end{aligned}$$

from which we may deduce that

$$\begin{aligned}C(c_2) &= 2\sqrt{C(1)^2 + 72C(1)},\\ C(c_2^2) &= -12C(1) + 864,\\ C(c_4) &= -36C(1) + 432.\end{aligned}$$

The top-degree ones are just Chern numbers, and using the relations on Betti numbers by Salamon, we have

$$c_2^2 = 736 + 4b_2 - b_3, \quad c_4 = 48 + 12b_2 - 3b_3.$$

Since b_3 is a multiple of 4, the Chern number c_2^2 must also be a multiple of 4, so we have $C(1) \in \frac{1}{3}\mathbf{Z}$. By the bounds of Guan, we have $-120 \le c_4 \le 324$, hence $\frac{46}{3} \ge C(1) \ge 3$, so we only have a finite number of possibilities left.

By definition, the generalized Fujiki constant $C(c_2)$ should be rational. Using this property we may verify that only the listed two cases are possible, which are realized by $\mathrm{K3}^{[2]}$ and Kum_2 respectively. □

This further reduces the number of possibilities for Betti numbers to 4, as stated in the introduction.

Proof of Corollary 1.3. From Salamon's relations [25] one obtains the formula

$$c_4 = 48 + 12b_2 - 3b_3.$$

By Proposition 5.6 there are only two possible values for c_4 which together with previously obtained bounds from Guan yield the assertion. □

Finally, motivated by the degree 4 case, we conjecture the following behaviour to be true for arbitrary dimensions.

Conjecture 5.7. *For $k_1, \ldots, k_r \in \mathbf{Z}_{>0}$ with $k \coloneqq \sum_i k_i \leq n$ we have*

$$(-1)^k C(\mathrm{ch}_{2k_1} \cdots \mathrm{ch}_{2k_r}) > 0 \quad \textit{as well as} \quad C(c_{2k_1} \cdots c_{2k_r}) > 0.$$

This generalizes the conjectures in [23, Questions 4.7 and 4.8] to products which do not necessarily live in top degree.

The conjectured alternating behaviour of products of Chern characters together with the positivity of products of Chern classes would yield in combination many restrictions and inequalities between these characteristic values. We expect the above positivity to hold pointwise and to be of local nature.

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