DUAL LAGRANGIAN FIBRATIONS

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ABSTRACT. These are notes of a survey talk on fibrations naturally associated with a Lagrangian fibration of a hyperkähler manifold and recent results by Arinkin–Fedorov [AF16], Kim [Ki21], Nagai [Na05], and Saccà [Sa20, Sa21] addressing the possibility of (partially) compactifying them.

1. INTRODUCTION

We consider a Lagrangian fibration $f\colon X \twoheadrightarrow B$ of a projective hyperkähler manifold X and denote by $X_0 \subset X$ the open union of all smooth fibres which comes with a smooth, projective morphism $f_0\colon X_0 \longrightarrow B_0$ over some open subset $B_0 \subset B$. With this situation, two abelian schemes are naturally associated:

$$P_0 \longrightarrow B_0 \text{ and } P_0^\vee \longrightarrow B_0.$$

The first one, $P_0 \simeq \mathrm{Aut}_0(X_0/B_0) \simeq \mathrm{Alb}(X_0/B_0)$, can be described as the relative Albanese scheme or, alternatively, as a relative scheme of automorphisms, while the second one $P_0^\vee \simeq \mathrm{Pic}^0(X_0/B_0)$ is the (identity component of the) relative Picard scheme. The later is the dual abelian scheme of the former. Note that the original $X_0 \longrightarrow B_0$ is a torsor over $P_0 \longrightarrow B_0$, which naturally compactifies to $X \longrightarrow B$.

From here, two parallel stories develop, but the main questions are the same for both:

(1) Can $P_0 \longrightarrow B_0$ and $P_0^\vee \longrightarrow B_0$ be (partially) compactified to a (smooth) symplectic variety with a Lagrangian fibration over B?

(2) Can torsors over $P_0 \longrightarrow B_0$ or over $P_0^\vee \longrightarrow B_0$ be (partially) compactified to (smooth) symplectic varieties with a Lagrangian fibration over B?

We will discuss work of Arinkin–Fedorov [AF16], Kim [Ki21], Nagai [Na05], and Saccà [Sa20, Sa21] addressing these questions.

- In [AF16] a construction is described that partially compactifies $P_0 \longrightarrow B_0$ to a smooth commutative group scheme $P_1 \twoheadrightarrow B_1$ over the open subscheme $B_1 \subset B$ of all points with integral fibres, see Section 6. Furthermore, P_1 acts faithfully on X over B.

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• In [Na05] a smooth commutative group scheme (or, rather, a separated algebraic space) $P_1^\vee \twoheadrightarrow B_1$ is constructed that partially compactifies $P_0^\vee \longrightarrow B_0$ and admits a symplectic structure, see Section 5.[1] The construction extends to the open subset of B of all points with fibres with at least one reduced component.

• In [Ki21], at least for the standard deformation types of compact hyperkähler manifolds, the natural torsor over $P_0^\vee \longrightarrow B_0$ associated with the torsor $X_0 \longrightarrow B_0$ over $P_0 \longrightarrow B_0$ is compactified to a projective but singular hyperkähler variety $X^\vee \longrightarrow B$, see Section 7.

• In the forthcoming work [Sa21], assuming $B_1 = B$, any torsor over the Arinkin–Fedorov partial compactification $P_1 \longrightarrow B_1 = B$ is compactified to a smooth projective hyperkähler manifold with Lagrangian fibration over B, see Section 8.

Apart from [AF16] the existence of the compact hyperkähler manifold X as the starting point of the consideration is crucial.

2. Preparations I: Abelian schemes, Albanese, Picard

2.1. Let us start with a smooth complex projective variety Y (or, more generally, a compact Kähler manifold). Classically, one associates two abelian varieties with Y (or, in the Kähler case, two complex tori): The Albanese variety $\mathrm{Alb}(Y)$ and the Picard variety $\mathrm{Pic}^0(Y)$. The Albanese variety is characterized by its universality property.[2]

Both, Picard and Albanese variety, admit an explicit description as complex tori naturally associate with Hodge structures of weight one:

$$\mathrm{Pic}^0(Y) \simeq H^{0,1}(Y)/H^1(Y,\mathbb{Z}) \text{ and } \mathrm{Alb}(Y) \simeq H^{1,0}(Y)^*/H^1(Y,\mathbb{Z})^*.$$

Note that using this description, there exists a natural isomorphism of integral Hodge structures of weight one

$$H^1(\mathrm{Alb}(Y),\mathbb{Z}) \simeq H^1(Y,\mathbb{Z})$$

which immediately leads to the following classical consequence.

Corollary 2.1. *The Albanese variety* $\mathrm{Alb}(Y)$ *and the Picard variety* $\mathrm{Pic}^0(Y)$ *are dual abelian varieties, i.e. there exist natural isomorphisms*

$$\mathrm{Alb}(Y)^\vee := \mathrm{Pic}^0(\mathrm{Alb}(Y)) \simeq \mathrm{Pic}^0(Y) \textit{ and } \mathrm{Alb}(Y) \simeq \mathrm{Pic}^0(\mathrm{Pic}^0(Y)).$$

Remark 2.2. (i) Although, both abelian varieties $\mathrm{Pic}^0(Y)$ and $\mathrm{Alb}(Y)$ can be constructed purely algebraically, i.e. they can be constructed for varieties over any field, they are very different in nature. The Picard variety is a moduli space of sheaves on Y, while the Albanese

[1] The notation suggests that P_1 and $P_1^\vee$ are dual, but it is not clear whether (and in what sense) this is true.

[2] For any point $y \in Y$ there exists a morphism $\mathrm{alb}_y\colon Y \longrightarrow \mathrm{Alb}(Y)$ with $\mathrm{alb}(y) = 0_{\mathrm{Alb}}$ such that every other morphism $Y \longrightarrow T$, $y \longmapsto 0_T$, to an abelian variety T factorizes through a unique morphism of abelian varieties $\mathrm{Alb}(Y) \longrightarrow T$.

variety is described by its universal property. Once duality has been established, $\mathrm{Alb}(Y)$ can also be viewed as a moduli space of sheaves on $\mathrm{Pic}(Y)$.

(ii) Also the Albanese morphism $\mathrm{alb}_y\colon Y \longrightarrow \mathrm{Alb}(Y)$ can be reconstructed using the description of $\mathrm{Alb}(Y)$ as the dual of $\mathrm{Pic}^0(Y)$. More precisely, consider the Poincaré bundle $\mathcal{P}$ on $Y \times \mathrm{Pic}^0(Y)$ normalized such that $\mathcal{P}|_{y\times \mathrm{Pic}^0(Y)}$ is trivial. Then, $\mathcal{P}$ as a family of line bundles on $\mathrm{Pic}^0(Y)$ parametrized by Y determines the classifying morphism

$$Y \longrightarrow \mathrm{Pic}^0(\mathrm{Pic}^0(Y)) \simeq \mathrm{Alb}(Y),\ \ x \longmapsto \mathcal{P}|_{x\times \mathrm{Pic}^0(Y)},$$

which is nothing but alb_y.

(iii) This point of view has the advantage to work more generally for normal projective (or complete) varieties Y. In this case, $\mathrm{Pic}^0(Y)$ is still an abelian variety (in positive characteristic one needs to pass to the reduction) and the dual abelian variety of $\mathrm{Pic}^0(Y)$ can be taken as the definition of $\mathrm{Alb}(Y)$.

The automorphism group $\mathrm{Aut}(Y)$ acts naturally on $\mathrm{Alb}(Y)$ and $\mathrm{Pic}^0(Y)$. More precisely, any $g \in \mathrm{Aut}(Y)$ acts on both varieties as an isomorphism of abelian varieties, i.e. preserving the origin. Note however, that this action is trivial on the connected component $\mathrm{id}_Y \in \mathrm{Aut}_0(Y) \subset \mathrm{Aut}(Y)$, as any continuous group acts trivially on the discrete cohomology $H^1(Y, \mathbb{Z})$.[3] However, there is another action of $\mathrm{Aut}_0(Y)$ on the Albanese variety defined via the universality property by the commutativity of the diagram:

$$\begin{tikzcd} Y \arrow[r, "\mathrm{alb}_y"] \arrow[d, "g"'] & \mathrm{Alb}(Y) \arrow[d, dotted, "\mathrm{Alb}(g)"] \\ Y \arrow[r, "\mathrm{alb}_y"'] & \mathrm{Alb}(Y). \end{tikzcd}$$

Note that with this definition $\mathrm{Alb}(g)$ is only an isomorphism of varieties but does not necessarily fix the origin. The definition of $\mathrm{Alb}(g)$ is independent of the choice of $y \in Y$. A point $x \in Y$ is being sent to the functional given by integrating 1-forms along $\int_y^x$ and its image under $\mathrm{Alb}(g)$ is given by integration along $\int_y^{g(x)} = \int_y^x + \int_x^{g(x)}$. The second summand is independent of y.

In particular, with this definition, the action of $\mathrm{Aut}_0(Y) \subset \mathrm{Aut}(Y)$ on $\mathrm{Alb}(Y)$ is usually not trivial.[4] Using that $\mathrm{Alb}(Y)$ is an abelian variety and $\mathrm{Aut}_0(Y)$ is connected, every $\mathrm{Alb}(g)$ is in fact given by translation. This yields a (natural) morphism of group schemes

$$\mathrm{Aut}_0(Y) \longrightarrow \mathrm{Alb}(Y). \tag{2.1}$$

[3]We consider $\mathrm{Aut}_0(Y)$ as a group variety, which in general is neither compact nor abelian. Also note that $\mathrm{Alb}(Y)$ is a birational invariant while $\mathrm{Aut}_0(Y)$ is not.

[4]Note that for the same reason as before this action of $\mathrm{Aut}_0(Y)$ on the variety(!) $\mathrm{Alb}(Y)$ induces a trivial action on its Picard variety $\mathrm{Pic}^0(\mathrm{Alb}(Y)) \simeq \mathrm{Pic}^0(Y)$.

2.2. We next consider the case that Y is isomorphic to an abelian variety A, but we do not fix any such isomorphism and, in particular, we do not fix a distinguished point of Y as the origin. In this case, the universality property of the Albanese variety shows that as varieties $Y \simeq \mathrm{Alb}(Y)$, but there is no natural choice of such an isomorphism, a point $y \in Y$ needs to be fixed. In fact, the more natural thing to do is to view Y as a torsor.

Lemma 2.3. *If Y is isomorphic to an abelian variety, then there exists a canonical isomorphism of group varieties*

$$\mathrm{Aut}_0(Y) \simeq \mathrm{Alb}(Y).$$

Via this isomorphism, Y can be naturally considered as a torsor over $\mathrm{Alb}(Y)$.

Proof. Since Y is isomorphic to an abelian variety, the scheme $\mathrm{Aut}_0(Y)$ is isomorphic to Y and acts on Y as well as $\mathrm{Alb}(Y)$ free and transitively. Hence, the morphism

$$\mathrm{Aut}_0(Y) \longrightarrow \mathrm{Alb}(Y)$$

from (2.1) is injective as well as surjective and sends the identity to 0_{Alb_Y}. □

2.3. Let us recall another bit of the classical theory of abelian varieties. Any line bundle L on a smooth projective variety Y induces a natural homomorphism of algebraic groups

$$\phi_L\colon \mathrm{Aut}_0(Y) \longrightarrow \mathrm{Pic}^0(Y),\ g \longmapsto g^*L \otimes L^{-1}. \tag{2.2}$$

In the case that Y is isomorphic to an abelian variety, i.e. for any point $y \in Y$ the Albanese map $\mathrm{alb}_y\colon Y \xrightarrow{\sim} \mathrm{Alb}(Y)$ is an isomorphism, the morphism ϕ_L can be described alternatively as

$$\varphi_{L_y}\colon \mathrm{Aut}_0(Y) \simeq \mathrm{Alb}(Y) \longrightarrow \mathrm{Alb}(Y)^\vee \simeq \mathrm{Pic}^0(Y). \tag{2.3}$$

Here, L_y is the line bundle on $\mathrm{Alb}(Y)$ satisfying $\mathrm{alb}_y^*L_y \simeq L$ and $\varphi_M\colon A \longrightarrow A^\vee$ for a line bundle M on an abelian variety A is the standard morphism $x \longmapsto t_x^*M \otimes M^{-1}$ (which only depends on the class $M \in \mathrm{NS}(A)$). Note although L_y depends on the choice of the point $y \in Y$, the morphism φ_{L_y} does not, simply because it coincides with φ_L. Also, as explained earlier, the two isomorphisms involved in (2.3) are canonical.

2.4. We now turn to the relative situation. Assume $f\colon \mathcal{Y} \longrightarrow T$ is a smooth projective, connected morphism. The relative versions of Albanese and Picard provide two abelian schemes

$$g\colon \mathrm{Alb}(\mathcal{Y}/T) \longrightarrow T \ \text{ and } h\colon \mathrm{Pic}^0(\mathcal{Y}/T) \longrightarrow T. \tag{2.4}$$

By definition, an abelian scheme $\mathcal{A} \longrightarrow T$ is a smooth projective connected morphism with a group structure over T. In particular, it comes with a zero-section $\sigma\colon T \longrightarrow \mathcal{A}$ and each fibre $\mathcal{A}_t$, $t \in T$, is an abelian variety with its origin given by $\sigma(t) \in \mathcal{A}_t$. For the two abelian schemes (2.4) the fibres are $\mathrm{Alb}(\mathcal{Y}_t)$ and $\mathrm{Pic}^0(\mathcal{Y}_t)$. For the latter, σ is simply $t \longmapsto \mathcal{O}_{\mathcal{Y}_t}$.

The Hodge-theoretic description and the duality in the absolute case carry over to the relative description. For example, there exists a natural isomorphism of VHS

$$R^1 f_* \mathbb{Z} \simeq R^1 g_* \mathbb{Z}$$

and the dual abelian scheme of $\mathrm{Alb}(\mathcal{Y}/T) \longrightarrow T$ is $\mathrm{Pic}^0(\mathcal{Y}/T) \longrightarrow T$, i.e.

$$\mathrm{Pic}^0(\mathrm{Alb}(\mathcal{Y}/T)/T) \simeq \mathrm{Pic}^0(\mathcal{Y}/T) \text{ and } \mathrm{Alb}(\mathcal{Y}/T) \simeq \mathrm{Pic}^0(\mathrm{Pic}^0(\mathcal{Y}/T)/T).$$

Remark 2.4. As in Remark 2.2, the situation generalizes in a straightforward way to projective connected families $f \colon \mathcal{Y} \longrightarrow T$ with normal fibres over a smooth (reduced is enough) base. The relative Picard variety $\mathrm{Pic}^0(\mathcal{Y}/T) \longrightarrow T$ is known to exist as a smooth abelian scheme over T, see [FAG]. The Albanese $\mathrm{Alb}(\mathcal{Y}/T) \longrightarrow T$ is defined as the dual abelian scheme $\mathrm{Pic}^0(\mathrm{Pic}^0(\mathcal{Y}/T)/T)$.

For any smooth projective morphism $f \colon \mathcal{Y} \longrightarrow T$ one can also consider

$$\mathrm{Aut}_0(\mathcal{Y}/T) \longrightarrow T,$$

which in general is only a group scheme, neither abelian nor proper, cf. [Br18, Sec. 2] for standard facts and references. The natural morphisms (2.1) for all fibres glue to a morphism

$$\mathrm{Aut}_0(\mathcal{Y}/T) \longrightarrow \mathrm{Alb}(\mathcal{Y}/T)$$

of group schemes over T. Similarly, if $\mathcal{L}$ is a line bundle on $\mathcal{Y}$, then the morphisms $\phi_{\mathcal{L}_t}$ in (2.7) glue to a morphism of group schemes over T

$$\phi_{\mathcal{L}} \colon \mathrm{Aut}_0(\mathcal{Y}/T) \longrightarrow \mathrm{Pic}^0(\mathcal{Y}/T). \tag{2.5}$$

In fact, we only need the data of line bundles $\mathcal{L}_t$ on all fibres $\mathcal{Y}_t$ varying algebraically with t and not the global line bundle $\mathcal{L}$ on $\mathcal{Y}$. In other words, there exists a natural morphism $\phi \colon \mathrm{Aut}_0(\mathcal{Y}/T) \longrightarrow \mathrm{Pic}^0(\mathcal{Y}/T)$ for any section of $\mathrm{Pic}(\mathcal{Y}/T) \longrightarrow T$.

2.5. Assume now that all fibres of the smooth projective morphism $f \colon \mathcal{Y} \longrightarrow T$ are isomorphic to abelian varieties (but not assuming that any such isomorphism has been fixed). Again, the discussion in the previous sections carries over and shows that there exists a canonical isomorphism

$$P_0 \coloneqq \mathrm{Aut}_0(\mathcal{Y}/T) \simeq \mathrm{Alb}(\mathcal{Y}/T) \tag{2.6}$$

and that $\mathcal{Y} \longrightarrow T$ is naturally a torsor over $\mathrm{Aut}_0(\mathcal{Y}/T) \simeq \mathrm{Alb}(\mathcal{Y}/T)$. We denote the dual abelian scheme by

$$P_0^\vee \coloneqq \mathrm{Pic}^0(\mathcal{Y}/T) \simeq \mathrm{Pic}^0(P_0/T).$$

Thus, for any relative ample line bundle $\mathcal{L}$ on $\mathcal{Y} \longrightarrow T$ we have have a finite étale morphism

$$\phi_{\mathcal{L}} \colon P_0 \twoheadrightarrow P_0^\vee \tag{2.7}$$

of abelian schemes over T. Note that (2.5) does not necessarily admit an alternative description that globalizes the morphisms φ_{L_y} in (2.3), since typically we cannot choose points $y_t \in \mathcal{Y}_t$

in a uniform manner. As noted before, in order to define $\phi_{\mathcal{L}}$ one only needs a section of $\mathrm{Pic}(\mathcal{Y}/T) \longrightarrow T$ and not necessarily a global line bundle. Moreover, since $\varphi_M \colon A \longrightarrow A^\vee$ for an abelian variety A only depends on the class $M \in \mathrm{NS}(A)$, (2.7) is well defined for any global section of $R^2 f_* \mathbb{Z}$ that fibrewise is contained in $H^{1,1}(\mathcal{Y}_t, \mathbb{Z})$ and is locally (in T) induced by a line bundle.

2.6. Still assuming $\mathcal{Y} \longrightarrow T$ to be a smooth projective morphism with fibres isomorphic to abelian varieties, we consider the finite étale morphism $\phi_{\mathcal{L}} \colon P_0 \twoheadrightarrow P_0^\vee$. Its kernel

$$\mathcal{K} := \mathrm{Ker}(\phi_{\mathcal{L}}) \subset P_0$$

is a finite subgroup scheme over T. Furthermore, its quotient is $P_0^\vee$:

$$P_0/\mathcal{K} \simeq P_0^\vee.$$

As $\mathcal{K} \subset P_0$, it acts on P_0 but since $P_0 \simeq \mathrm{Aut}_0(\mathcal{Y}/T)$ it also acts (over T) on $\mathcal{Y}$. We denote the quotient of this action by $\mathcal{Y}^\vee := \mathcal{Y}/\mathcal{K}$. It is naturally a torsor over $P_0^\vee$:

$$\begin{tikzcd} \mathcal{Y} \arrow[r, two heads] \arrow[loop below] & \mathcal{Y}^\vee = \mathcal{Y}/\mathcal{K} \arrow[loop below] \\ P_0 \arrow[r, two heads] & P_0^\vee = P_0/\mathcal{K} \end{tikzcd} .$$

In the applications, $\mathcal{K}$ will be a constant group scheme associated with a finite group of automorphisms $K \subset \mathrm{Aut}(\mathcal{Y})$. Conversely, any automorphism of $\mathcal{Y}$ that commutes with f and is fibrewise isotopic to the identity defines a section of $P_0 \longrightarrow T$. Thus, the subgroup $\mathrm{Aut}(\mathcal{Y}, f) \subset \mathrm{Aut}(\mathcal{Y})$ of all such gives a constant subgroup scheme $\mathrm{Aut}(\mathcal{Y}, f)_T \hookrightarrow P_0$.

3. Preparations II: Elliptic K3 surfaces

This section can be read as a preparation for the subsequent parts or as an illustration of the abstract results to be presented there. The reader should feel free to skip this section at first reading. The theory of Lagrangian fibrations of K3 surfaces, so elliptic K3 surfaces, is rather explicit and can help to understand the more complicated picture in higher dimensions. However, as elliptic curves are principally polarized, at certain points the difference between the Albanese fibration and the Picard fibration is blurred and certain general features are less clear.

Let $f \colon X \longrightarrow B := \mathbb{P}^1$ be an elliptic K3 surface. As we are not assuming the existence of a section, this is often just called a genus one fibration. Denote by

$$X_0 \subset X' \subset X$$

the two open subsets consisting of all smooth fibres respectively all f-regular points. Hence, the image of the restriction $f_0 := f|_{X_0} \colon X_0 \longrightarrow B_0$ is the proper open subset $B_0 \subset B = \mathbb{P}^1$ of all points with smooth fibre and $f' \colon X' \longrightarrow B$ is still surjective, as every fibre admits at least one reduced component, cf. [Hu16, Sec. 11.1.3].

3.1. As in the general situation (2.6), two out of the three abelian schemes naturally associated with $f\colon X_0 \longrightarrow B_0$ are isomorphic:

$$P_0 \coloneqq \mathrm{Alb}(X_0/B_0) \simeq \mathrm{Aut}_0(X_0/B_0)$$

and $X_0 \longrightarrow B_0$ is naturally a torsor over $g_0\colon P_0 \longrightarrow B_0$. Furthermore, the dual abelian scheme is

$$P_0^\vee \coloneqq \mathrm{Pic}^0(X_0/B_0) \simeq \mathrm{Pic}^0(P_0/B_0).$$

To a line bundle $\mathcal{L}$ on X (on X_0 suffices) one associates a morphism $\phi_{\mathcal{L}}\colon P_0 \longrightarrow P_0^\vee$. If $\mathcal{L}$ is fibrewise ample, $\phi_{\mathcal{L}}$ is étale of degree $\deg(\mathcal{L}|_{X_t})^2$. In general, X or, equivalently, X_0 may not admit a line bundle $\mathcal{L}$ that is fibrewise of degree one. However, as explained above, $\phi_{\mathcal{L}}$ really only depends on the induced section of $R^2f_{0*}\mathbb{Z}$ over B_0, Moreover, one can in fact define $\phi_{\mathcal{L}}$ for any given section of $R^2f_{0*}\mathbb{Z}$. Since the fibres are curves, the condition to be of type $(1,1)$ is automatic.

It turns out that the local system $R^2f_{0*}\mathbb{Z}$ is trivial and that a global trivializing section restricts to a generator of $H^2(X_t,\mathbb{Z})$ for any smooth fibre X_t. Indeed, the class of a fibre $[X_t] \in H^2(X,\mathbb{Z})$ is primitive and the intersection form is unimodular. Thus, there is a distinguished isomorphism

$$\phi\colon P_0 \xrightarrow{\sim} P_0^\vee.$$

3.2. Clearly, $X_0 \longrightarrow B_0$ can be compactified to an elliptic K3 surfaces, namely to $X \longrightarrow B$ itself. What about P_0 and $P_0^\vee$? It turns out that the latter can in fact be compactified to a K3 surface. As a first step, one considers at the partial compactification provided by the relative Picard variety

$$\mathrm{Pic}^0(X/B) \longrightarrow B,$$

which is projective only over B_0. Thinking of $\mathrm{Pic}^0(X/B)$ as parametrizing sheaves on X, namely those of the form i_*M for M a line bundle of degree zero on a fibre X_t, it is natural to compactify it further by the moduli space $M_H(v)$ of sheaves with Mukai vector $v = (0,[X_t],0)$ that are stable with respect to a generic polarization. Then the general theory, see [Hu16, Sect. 11.4], shows that $\overline{P_0^\vee} \coloneqq M_H(v)$ is a K3 surface and the support map extends $\mathrm{Pic}^0(X/B) \longrightarrow B$ to an elliptic fibration

$$\overline{P_0^\vee} \longrightarrow B.$$

Moreover, it comes with a zero section given by $t \longmapsto \mathcal{O}_{X_t}$. Note that $\overline{P_0^\vee}$ does not depend on the choice of the polarization H on X, simply because two birational K3 surfaces are isomorphic.

Remark 3.1. (i) Since $P_0 \simeq P_0^\vee$, the dual K3 surface $\overline{P_0^\vee}$ can also be viewed as a compactification $\overline{P_0}$ of $P_0 \longrightarrow B_0$. Note however, that a priori we do not have a natural compactification of P_0 that would not make use of the isomorphism $P_0 \simeq P_0^\vee$, one that would only depend on the interpretation of P_0 as Alb or Aut_0. This has been attempted only recently by Saccà, in the context of higher-dimensional hyperkähler manifolds and using MMP, see Section 8.

(ii) Since $\overline{P_0} \simeq \overline{P_0^\vee} \longrightarrow B$ restricted to B_0 is isomorphic to $P_0 \simeq P_0^\vee$, the elliptic fibration $X \longrightarrow B$, restricted to $B_0 \subset B$ can be viewed as a torsor over the restriction of $\overline{P_0} \simeq \overline{P_0^\vee}$. Hence, X defines a class $[X] \in Ш(\overline{P_0}/\mathbb{P}^1)$ in the Tate–Shafarevich group, cf. [Hu16, Sect. 11.5].

(iii) In fact, every class $\alpha \in Ш(\overline{P_0}/\mathbb{P}^1)$, so a torsor $X_0(\alpha) \longrightarrow B_0$ over $P_0 \simeq P_0^\vee \longrightarrow B_0$, can be compactified to an elliptic K3 surface $X(\alpha) \longrightarrow B$, though most of these K3 surfaces $X(\alpha)$ are not projective.

3.3. There is another open set one can consider

$$X_0 \subset X_1 \subset X,$$

the union of all integral fibres, i.e. X_1 is obtained from X_0 by adding all fibres of type I_1 (rational curve with one node) and II (rational curve with one cusp). We can view $f_1 \coloneqq f|_{X_1}\colon X_1 \longrightarrow B_1 \coloneqq f(X_1) \subset B$ as a partial compactification of $X_0 \longrightarrow B_0$. In fact, the general elliptic K3 surfaces will have exactly 24 singular fibres all of type I_1, in which case $X_1 = X$.

Let us next consider the open subset $X_1' = X_1 \cap X'$ of all f_1-regular points. Hence, the fibre of $X_1' \longrightarrow B_1$ are either smooth curves of genus one, isomorphic to $\mathbb{P}^1 \backslash \{0, 1\}$ (in case I_1), or $\mathbb{P}^1 \backslash \{0\}$ (in case II).

Remark 3.2. There exists an abelian group scheme $g_1\colon P_1 \longrightarrow B_1$ acting on $X_1 \longrightarrow B_1$ and such that restriction turns $X_1' \longrightarrow B_1$ into a torsor over $P_1 \longrightarrow B_1$. The fibres of g_1 are either smooth elliptic curves, $\mathbb{G}_m$ (case I_1), or $\mathbb{G}_a$ (case II). More precisely, P_1 is constructed as a partial compactification of P_0 and there are two ways to do this:

(i) Since $P_0 \simeq P_0^\vee$, we can construct P_1 as a partial compactification of $P_0^\vee$. Namely, $P_1 \coloneqq \mathrm{Pic}^0(X_1/B_1)$. Note that Pic^0 of a rational curve with one node is indeed just $\mathbb{G}_m$ which parametrizes the gluing of the fibres of $\mathcal{O}_{\mathbb{P}^1}$ at the two points $0, 1 \in \mathbb{P}^1$. Similarly, Pic^0 of a rational curve with one cusp is the additive group $\mathbb{G}_a$. With this construction it is not easy to see that $X_1 \longrightarrow B_1$ is a torsor over $P_1 \longrightarrow B_1$.

(ii) In the second approach $X_1 \longrightarrow B_1$ is a torsor over $P_1 \longrightarrow B_1$ by construction, but the construction of P_1 itself is more involved. Consider $\mathrm{Aut}_0(X_1/B_1) \longrightarrow B_1$. Its restriction to B_0 gives back P_0. The fibre over $t \in B_1$ with X_t a singular curve of type I_1 is the subgroup of $\mathrm{Aut}(\mathbb{P}^1)$ fixing $0, 1 \in \mathbb{P}^1$, which is indeed $\mathbb{G}_m$. The automorphism group of a fibre of type II is isomorphic to $\mathbb{G}_m \rtimes \mathbb{G}_a$ and for our purposes of constructing P_1 we restrict to the action of $\mathbb{G}_a$, see Example 6.2 for details. The natural action $\mathrm{Aut}_0(X_1/B_1) \times_{B_1} X_1 \longrightarrow X_1$ restricted to P_1 has all the required properties.

(iii) This example also shows why, when one wants to construct the abelian scheme P_1 for certain singular fibres, one needs to use Aut_0 instead of Alb. Indeed, the Albanese variety is a birational invariant and the normalization of singular fibres of type I_1 or II is isomorphic to $\mathbb{P}^1$, which has trivial first cohomology. Thus, we see that the dimension of the relative automorphism scheme can increase in singular fibres and we can restrict to a certain closed subgroup of automorphisms that deform infinitessimally to nearby fibres, whereas for the Albanese variety no such construction is possible.

For K3 surfaces, this procedure can be extended to all of B. More precisely, there exists an abelian group scheme $P' \twoheadrightarrow B$ acting on $X \longrightarrow B$ and such that with respect to this action $X' \longrightarrow B$ is a torsor over P'.[5]

4. Lagrangian fibrations of abelian schemes and hyperkähler manifolds

We explain how symplectic structures are transferred between abelian schemes and torsors over them. Eventually, this is applied to the smooth part of a Lagrangian fibration $X \longrightarrow B$ of a projective hyperkähler manifold.

4.1. Let us begin with an abelian scheme $g\colon \mathcal{A} \longrightarrow T$ and assume that $\mathcal{A}$ is endowed with a symplectic structure, i.e. we are given a non-degenerate (closed) two-form $\sigma \in H^0(\mathcal{A}, \Omega^2_{\mathcal{A}})$ for which g is a Lagrangian fibration. In particular, each fibre $\mathcal{A}_t$ is a Lagrangian torus and $\dim(\mathcal{A}) = 2\dim(T)$.

Assume now that we are given another section $\tau\colon T \longrightarrow \mathcal{A}$ of $g\colon \mathcal{A} \longrightarrow T$ (different from the given zero-section). Translation by τ defines an automorphism $\tau\colon \mathcal{A} \xrightarrow{\sim} \mathcal{A}$ (over T) which induces a map τ^* on $H^0(\mathcal{A}, \Omega^2_{\mathcal{A}})$.

Lemma 4.1. *Any symplectic structure on $\mathcal{A}$ for which $g\colon \mathcal{A} \longrightarrow T$ is a Lagrangian fibration is invariant under the action of translations by sections of g.*

Proof. Consider the short exact sequence $0 \longrightarrow \mathcal{N}^*_{\mathcal{A}_t/\mathcal{A}} \longrightarrow \Omega_{\mathcal{A}}|_{\mathcal{A}_t} \longrightarrow \Omega_{\mathcal{A}_t} \longrightarrow 0$ and the induced exact sequence

$$0 \longrightarrow \mathrm{Sym}^2(\mathcal{N}^*_{\mathcal{A}_t/\mathcal{A}}) \longrightarrow \mathcal{N}^*_{\mathcal{A}_t/\mathcal{A}} \otimes \Omega_{\mathcal{A}_t} \longrightarrow \Omega^2_{\mathcal{A}}|_{\mathcal{A}_t} \longrightarrow \Omega^2_{\mathcal{A}_t} \longrightarrow 0.$$

The fact that g is a Lagrangian fibration with respect to the given symplectic structure $\sigma \in H^0(\mathcal{A}, \Omega^2_{\mathcal{A}})$ translates into the statement that the restriction $\sigma|_{\mathcal{A}_t} \in H^0(\mathcal{A}_t, \Omega^2_{\mathcal{A}}|_{\mathcal{A}_t})$ is the image of a class in $H^0(\mathcal{A}_t, \mathcal{N}^*_{\mathcal{A}_t/\mathcal{A}} \otimes \Omega_{\mathcal{A}_t})$ that induces an isomorphism $\mathcal{N}_{\mathcal{A}_t/\mathcal{A}} \simeq \Omega_{\mathcal{A}_t}$.

Since translation acts trivially on $H^0(\mathcal{A}_t, \mathcal{N}^*_{\mathcal{A}_t/\mathcal{A}} \otimes \Omega_{\mathcal{A}_t})$, the restriction $\sigma|_{\mathcal{A}_t} \in H^0(\mathcal{A}_t, \Omega^2_{\mathcal{A}}|_{\mathcal{A}_t})$ and hence σ itself remain unchanged under translation. □

Let us now consider in addition a torsor $f\colon \mathcal{Y} \longrightarrow T$ over the abelian scheme $g\colon \mathcal{A} \longrightarrow T$ or, in other words, $\mathcal{A} \simeq \mathrm{Alb}(\mathcal{Y}/T)$. The next result establishes a link between symplectic structures on the two fibrations

Proposition 4.2. *There is a natural bijection between (closed) symplectic structures on $\mathcal{A}$ and $\mathcal{Y}$ for which the two fibrations g and f are Lagrangian.*

[5] Due to the existence of multiple fibre, this is not expected to generalize to higher dimensions. The analogue of P' in higher dimension could either be (i) Pic^0 as in Nagai, see Section 5, or (ii) the relative $P_1 \subset \mathrm{Aut}_0$ by Arininkin–Fedorov, see Section 6.

Proof. Étale locally, the two fibrations $f\colon \mathcal{Y} \longrightarrow T$ and $g\colon \mathcal{A} \longrightarrow T$ are isomorphic. The glueings for the two fibrations differ by translations, which according to the lemma leave invariant the symplectic structure. For example, a given symplectic structure σ on $\mathcal{A}$, restricts to symplectic structures $\sigma|_{U_i}$ on $\mathcal{A}|_{U_i} \simeq \mathcal{Y}|_{U_i}$ for some open covering $T = \bigcup U_i$ and those then glue back to a symplectic structure on $\mathcal{Y}$. The converse is similar. □

4.2. We are interested in Lagrangian fibrations $f\colon X \longrightarrow B$ of a projective hyperkähler manifold X of dimension $2n$. According to results of Matsushita, any morphism $f\colon X \twoheadrightarrow B$ to a normal variety of dimension $0 < \dim(B) < \dim(X)$ is a Lagrangian fibration. So, all fibres are of dimension n and the smooth part of the reduction of any fibre is Lagrangian. In particular, for a smooth fibre X_t the symplectic structure induces naturally an isomorphism $\Omega_{X_t} \simeq \mathcal{N}_{X_t/X}$ and since the normal bundle is $\mathcal{N}_{X_t/X} \simeq T_tB \otimes \mathcal{O}_{X_t}$ is trivial, smooth fibres are abelian varieties. The smoothness of B or, equivalently, the flatness of f is expected and has to be assumed for most of the things that will follow. We refer to [HM21] for further information on Lagrangian fibrations and references.

As for K3 surfaces, one denotes by $X_0 \subset X$ the union of all smooth fibres which comes with a proper smooth fibration $f_0\colon X_0 \longrightarrow B_0$ over some open subset $B_0 \subset B$ with fibres isomorphic to abelian varieties.

4.3. As before, we associate two abelian schemes with the situation: The relative Albanese

$$g_0\colon P_0 = \mathrm{Alb}(X_0/B_0) \longrightarrow B_0$$

and its dual

$$P_0^\vee = \mathrm{Pic}^0(X_0/B_0) \simeq \mathrm{Pic}^0(P_0/B_0) \longrightarrow B_0.$$

Remark 4.3. As in Remark 2.4, one could consider more generally the normal part, i.e. the union of all normal fibres. The relative Picard and Albanese would still exist as abelian schemes. However, it seems that normality of a fibre is often (always?) equivalent to its smoothness. So, we do not gain anything by this generalization.

As in Section 2.6, we view $X_0 \longrightarrow B_0$ as a torsor over $P_0 \longrightarrow B_0$. Following the discussion in Section 2, we consider

$$\phi\colon P_0 \twoheadrightarrow P_0^\vee$$

associated with the generator of the image of the restriction map $H^2(X,\mathbb{Z}) \longrightarrow R^2f_{0*}\mathbb{Z}$. Its kernel $\mathcal{K} = \ker(\phi) \subset P_0 = \mathrm{Aut}(X_0/B_0)$ is a smooth finite group scheme over B_0.

4.4. The restriction of the given symplectic structure on X endows the open subset X_0 with a closed symplectic structure for which $X_0 \longrightarrow B_0$ is a Lagrangian fibration. According to the preceding discussion, this structure is passed on to the two abelian schemes $P_0 \longrightarrow B_0$ and $P_0^\vee \longrightarrow B_0$. More precisely, we draw two consequences of Proposition 4.2, see [DM96, Ma96, Na05].

Corollary 4.4 (Donagi–Markman, Markushevich)**.** *The two abelian schemes $P_0 \longrightarrow B_0$ and $P_0^\vee \longrightarrow B_0$ are both naturally (up to scaling) endowed with a closed symplectic structure for which the projections are Lagrangian fibrations*[6].

Proof. Since the polarization defines étale morphisms of abelian group schemes

$$P_0 \twoheadrightarrow P_0^\vee \twoheadrightarrow P_0,$$

the existence of a symplectic structure on one of the two abelian schemes induces a symplectic structure on the other one such that the Lagrangian properties are transferred.

By virtue of Proposition 4.2, $P_0 = \mathrm{Alb}(X_0/B_0)$ is endowed with a symplectic structure for which g is Lagrangian. This suffices to conclude. □

See also the more recent paper [BGO20], where one finds an explicit description of the symplectic structure on P_0, similar to the one by Markushevich [Ma96], and on $P_0^\vee$.

Corollary 4.5. *All torsors $h\colon Y_0 \longrightarrow B_0$ over $P_0 \longrightarrow B_0$ are naturally (up to scaling) endowed with a closed symplectic structures for which h is a Lagrangian fibration. An anologous statement holds for torsors over $P_0^\vee \longrightarrow B_0$.* □

Note that in the above proof the composition $P_0 \twoheadrightarrow P_0^\vee \twoheadrightarrow P_0$ is multiplication by the square of the degree of the polarization. Under this pull-back, a symplectic structure on P_0 is multiplied by a scalar (the fourth power of the degree).

Remark 4.6. The abelian scheme $P_0^\vee \longrightarrow B_0$ can be viewed as an open subset of a moduli space of sheaves on X, namely those of the form $i_*\mathcal{L}$, where $i\colon X_t \hookrightarrow X$ is the closed embedding of a smooth fibre and $\mathcal{L}$ is a degree zero line bundle on X_t. The tangent space of X_t and X at the point $[\mathcal{L}]$ are canonically isomorphic to $\mathrm{Ext}^1_{X_t}(\mathcal{L},\mathcal{L})$ and $\mathrm{Ext}^1_X(i_*\mathcal{L}, i_*\mathcal{L})$ respectively. It would be more natural to describe the symplectic form on $P_0^\vee$ as given by the pairing

$$\mathrm{Ext}^1_X(i_*\mathcal{L}, i_*\mathcal{L}) \times \mathrm{Ext}^1_X(i_*\mathcal{L}, i_*\mathcal{L}) \longrightarrow \mathrm{Ext}^2_X(i_*\mathcal{L}, i_*\mathcal{L}) \xrightarrow{\mathrm{tr}} H^2(X,\mathcal{O}) \simeq \mathbb{C}.$$

The Lagrangian property should then be deduced from this paring being trivial when restricted to $\mathrm{Ext}^1_{X_t}(\mathcal{L},\mathcal{L})$. Indeed, the pairing factorizes through

$$\begin{tikzcd}
H^2(X_t, \mathcal{H}om_{X_t}(\mathcal{L},\mathcal{L})) \arrow[r, "\sim"] \arrow[d] & \mathrm{Ext}^2_{X_t}(\mathcal{L},\mathcal{L}) \arrow[d] & \\
H^2(X, \mathcal{H}om_X(i_*\mathcal{L}, i_*\mathcal{L})) \arrow[r] & \mathrm{Ext}^2_X(i_*\mathcal{L}, i_*\mathcal{L}) \arrow[r, "\mathrm{tr}"] & H^2(X, \mathcal{O}_X),
\end{tikzcd}$$

but $\mathrm{tr}\colon \mathcal{H}om_X(i_*\mathcal{L}, i_*\mathcal{L}) \longrightarrow \mathcal{O}_X$ is trivial.

[6] The associated zero sections are Lagrangian as well.

5. Extended Picard after Nagai

Let as before $f\colon X \longrightarrow B$ be a Lagrangian fibration of a smooth projective hyperkähler manifold of dimension $2n$. We assume that B is smooth, which according to a result by Hwang [Hw08] is equivalent to $B \simeq \mathbb{P}^n$. Note that the smoothness of the base also implies that f is flat. It is convenient to also assume that f admits local analytic sections or, equivalently, that every fibre of f has at least one generically reduced component. If the later assumption is not met, we tacitly shrink B.

5.1. The Picard functor is the étale sheafification of the functor that associates with each $T \longrightarrow B$ the relative Picard group

$$\mathrm{Pic}(X_T/T) \coloneqq \mathrm{Pic}(X_T)/\mathrm{Pic}(T).$$

The Picard variety $\mathrm{Pic}^0(X_0/B_0)$ represents a connected component of the Picard functor for the smooth morphism $f_0\colon X_0 \longrightarrow B_0$. If f_0 admits a section, then there exists a universal line bundle on $X_0 \times_{B_0} \mathrm{Pic}^0(X_0/B_0)$. However, the Picard functor can be represented under weaker hypotheses, cf. [BLR90, Ch. 8] or [FAG, Part 5]:

• Grothendieck: Let $f_1\colon X_1 \longrightarrow B_1$ be the union of all integral fibres. Then the unit component of the associated Picard functor can be represented by a quasi-projective morphism

$$P_1 \coloneqq \mathrm{Pic}^0(X_1/B_1) \twoheadrightarrow B_1.$$

The fibres are the quasi-projective varieties $\mathrm{Pic}^0(X_t)$, which are projective for normal fibres.

• Artin: Without any assumption, the unit component of the Picard functor for the whole family $f\colon X \longrightarrow B$ is represented by a group object in the category of algebraic spaces

$$\mathrm{Pic}^0(X/B) \twoheadrightarrow B$$

which is locally of finite type, but possibly not separated. The non-separatedness is caused by the non-integral fibres.

• Mumford: Assume all fibres of $f\colon X \longrightarrow B$ are reduced (but possibly reducible). Then the unit component of its Picard functor is represented by a (possibly non-separated) scheme $\mathrm{Pic}^0(X/B) \longrightarrow B$ locally of finite type.[7]

Note that the projections $\mathrm{Pic}^0(X_1/B_1) \twoheadrightarrow B_1$ resp. $\mathrm{Pic}^0(X/B) \twoheadrightarrow B$ come with a section induced by the structure sheaf.

The fact that $\mathrm{Pic}^0(X/B)$ is typically not separated makes it unsuitable for any geometric considerations. Nagai [Na05] suggested to remedy the situation by applying a procedure originally due to Raynaud [Ra70]: Let $G \subset \mathrm{Pic}^0(X/B)$ be the closure of the zero section, which is

[7] In [AF16, Cor. 3.8] Arinkin and Fedorov complement Mumford's result by showing that the open subspace $\mathrm{Pic}^\tau(X/B) \subset \mathrm{Pic}(X/B)$ of line bundles that are numerically trivial on all fibres is separated. The assumption here is that all fibres are reduced and connected.

a group scheme over B, and define

$$P^\vee := \mathrm{Pic}^0(X/B)/G \longrightarrow B.$$

Then $P^\vee$ is a separated algebraic space.[8]

Remark 5.1. Nagai [Na05, Thm. 1.6] shows that $P^\vee \longrightarrow B$ has the Néron property asserting that every morphism $Z_0 \longrightarrow P_0^\vee$ from a smooth family $Z \longrightarrow B$ extends to a morphism $Z \longrightarrow P^\vee$.

5.2. In order for $P^\vee$ to be considered a partial compactification in the hyperkähler sense, one needs to make sure it inherits the symplectic structure.

Theorem 5.2 (Nagai)**.** *Assume that every fibre of $f\colon X \longrightarrow B$ has a reduced component or, equivalently, that f admits local sections (in the analytic or étale topology). Then the natural closed symplectic structure on $P_0^\vee$ extends to a closed symplectic structure on $P^\vee$ for which $P^\vee \twoheadrightarrow B$ is a smooth (possibly non-proper) Lagrangian fibration*[9].

Proof. The smoothness of $P^\vee \longrightarrow B$ and, therefore, of $P^\vee$ itself follows from $R^1f_*\mathcal{O}_X \simeq \Omega_B$ being locally free. The construction of the symplectic structure and the verification of the Lagrangian property rely heavily on the classification of the singular fibres of $X \longrightarrow B$ by Matsushita. $\square$

Remark 5.3. As in Remark 4.6, one could hope for a more functorial description of the symplectic structure on $P^\vee$ in terms of a pairing on $\mathrm{Ext}^1_X(i_*\mathcal{L}, i_*\mathcal{L})$.

Remark 5.4. The motivation for Nagai's work comes from results of Cho, Miyaoka, and Shepherd-Barron who attempted to prove that the base of a Lagrangian fibration $X \longrightarrow B$ with a section is $\mathbb{P}^n$. Nagai's idea now was to apply their reasoning to the Lagrangian fibration $P^\vee \longrightarrow B$. The extension of $P_0^\vee \longrightarrow B_0$ to a fibration over the whole B (and not only over B_0 or some smaller open subset) is of course essential for the idea to work. In fact, in his approach Nagai only needs to assume that $X \longrightarrow B$ admits local section over an open subset with a complement of codimension at least two.

6. Extended Albanese after Arinkin–Fedorov and Markushevich

As before, $f\colon X \longrightarrow B$ denotes a Lagrangian fibration of a projective hyperkähler manifold of dimension $2n$. In addition we assume that B is smooth, which implies that f is flat and, in

[8] It seems that Raynaud only treats the case of the base being the spectrum of a DVR, but this particular result should hold in our setting. Nagai does not addressed this issue. Also, it seems likely that $P^\vee \longrightarrow B$ is actually quasi-projective. It is possible that Raynaud's techniques actually also cover families without local sections. However, the next step in the program, the construction of a symplectic structure would fail in this broader generality.

[9] The zero section $\sigma\colon B \longrightarrow P^\vee$ is Lagrangian as well.

fact, $B \simeq \mathbb{P}^n$. By $B_1 \subset B$ we denote the open subset of all points with integral fibres and let $f_1\colon X_1 \longrightarrow B_1$ be restriction of f. Clearly, $B_0 \subset B_1$ so that we are in the following situation

$$\begin{tikzcd} X_0 \arrow[r, hook] \arrow[d, "f_0"'] \arrow[dr, phantom, "\times"] & X_1 \arrow[r, hook] \arrow[d, "f_1"] \arrow[dr, phantom, "\times"] & X \arrow[d, "f"] \\ B_0 \arrow[r, hook] & B_1 \arrow[r, hook] & B. \end{tikzcd}$$

Recall from Section 2 that $f_0\colon X_0 \longrightarrow B_0$ is naturally a torsor over the relative Albanese scheme $g_0\colon P_0 = \mathrm{Alb}(X_0/B_0) \simeq \mathrm{Aut}_0(X_0/B_0) \longrightarrow B_0$. We will use the notation $X_1' \coloneqq X_1 \cap X'$, where $X' \subset X$ is the open subset of all f-regular points.

6.1. The following is one of the main results of [AF16].

Theorem 6.1 (Arinkin–Fedorov)**.** *There exists a smooth group scheme $g_1\colon P_1 \longrightarrow B_1$ with an action $P_1 \times_{B_1} X_1 \longrightarrow X_1$ such that with this action $X_1' \longrightarrow B_1$ is a torsor over $P_1 \longrightarrow B_1$.*

Note that the projection $P_1 \longrightarrow B_1$ is necessarily surjective, but it is typically not proper over $B_1 \backslash B_0$.

We cannot give details of their proof, but here are some ideas.[10] The group scheme P_1 is constructed as a closed subgroup-scheme of $\mathrm{Aut}(X_1/B_1)$. A posteriori, it can be viewed as the closure of $\mathrm{Aut}_0(X_0/B_0)$ inside $\mathrm{Aut}(X_1/B_1)$

$$\mathrm{Aut}_0(X_0/B_0) \subset P_1 = \overline{\mathrm{Aut}_0(X_0/B_0)} \subset \mathrm{Aut}(X_1/B_1),$$

but to ensure that this results in a smooth scheme, one needs to single out those automorphisms of the singular fibre X_t that deform in all infinitesimal directions of $t \in B$.

Recall that in the absolute case, that is $Y \longrightarrow k$, the Lie algebra of $\mathrm{Aut}_0(Y)$, i.e. the tangent space at id_Y is $H^0(Y, \mathcal{T}_Y)$. In the relative setting, one considers the sheaf of Lie algebras of $\mathrm{Aut}_0(X_1/B_1)$. The key is to observe that on the level of vector fields, the automorphisms that deform in all infinitesimal directions correspond on the Lie algebra level to the Hamiltonian vector fields. In particular, the tangent space of the fibre $P_1 \longrightarrow X_1$ over t is by construction naturally isomorphic to T_t^*B. The latter observation ensures that P_1 and $P_1 \longrightarrow B_1$ are both smooth (since the flatness of $X \longrightarrow B$ implies that B is smooth).

Example 6.2. Here is the simplest example showing that $\mathrm{Aut}_0(X_1/B_1) \longrightarrow B_1$ is in general not smooth. Consider the degeneration of a smooth elliptic curve to a curve C with a cusp. Let $\rho\colon \mathbb{P}^1 \longrightarrow C$ be its normalization with coordinates $[x : y]$ such that the cusp has preimage $[1 : 0]$. Automorphisms of C can be described on the normalization as

$$[x : y] \longmapsto [ax + by, y]$$

with $a \in \mathbb{C}^*$ and $b \in \mathbb{C}$. Thus, over the point in the base corresponding to the curve C, the dimension increases by one and therefore the morphism $\mathrm{Aut}_0(X_1/B_1) \longrightarrow B_1$ cannot be flat.

[10]Thanks to D. Arinkin for an instructive email exchange.

Looking at the Lie algebra, we have e.g. for the chart $t = x/y$

$$T_{\mathrm{id}_C}\mathrm{Aut}_0(C) = \left\{(at+b)\frac{\partial}{\partial t} \mid a, b \in \mathbb{C}\right\}$$

which is of course also two-dimensional. The Hamiltonian vector fields are now exactly the abelian Lie sub-algebra of translations which correspond to setting $a = 0$. That these deform to nearby fibers can be seen easily in this example.

Remark 6.3. By construction, the restriction of $g_1\colon P_1 \longrightarrow B_1$ to the open subset $B_0 \subset B_1$ gives back $g_0\colon P_0 \longrightarrow B_0$. In this sense, $P_1 \longrightarrow B_1$ has to be viewed as a partial compactification of the relative Albanese scheme of $f_0\colon X_0 \longrightarrow B_0$:

$$\begin{tikzcd} P_0 \arrow[r, hook] \arrow[d, "g_0"'] \arrow[dr, phantom, "\times"] & P_1 \arrow[d, "g_1"] & \\ B_0 \arrow[r, hook] & B_1 \arrow[r, hook] & B. \end{tikzcd}$$

6.2. The group scheme $P_1 \longrightarrow B_1$ as a partial compactification of $P_0 \longrightarrow B_0$ is of interest to us only if it can be endowed with a symplectic structure. For this, the next result is [AF16, Prop. 7.1] is crucial.

Proposition 6.4 (Arinkin–Fedorov). *The étale morphism $\phi\colon P_0 \longrightarrow P_0^\vee$ extends naturally to an étale morphism*

$$\phi_1\colon P_1 \longrightarrow \mathrm{Pic}(X_1/B_1).$$

Proof. The morphism is extended by the same formula as in Section 2.3. The assertion follows, on the one hand, from the construction and the observation above that $T_{\mathrm{id}}g_1^{-1}(t) \simeq T_t^*B$ and, on the other hand, from $T_{\mathcal{O}_{X_t}}\mathrm{Pic}(X_t) \simeq H^1(X_t, \mathcal{O}_{X_t}) \simeq T_t^*B$, where the last isomorphism is the fibre of Matsushita's isomorphism $R^1f_*\mathcal{O}_X \simeq \Omega_B$ (which makes use of the polarization). The proof then consists of showing that with these identifications the differential of ϕ induces the identity of T_t^*B. $\square$

Since $\phi\colon P_0 \longrightarrow \mathrm{Pic}(X_0/B_0)$ takes image in $P_0^\vee$, the extension ϕ_1 composed with the projection to the maximal separated quotient defines an étale morphism

$$P_1 \longrightarrow P^\vee|_{B_1},$$

where $P^\vee \longrightarrow B$ is the restriction of the Picard scheme introduced in 5.1. As a consequence of Theorem 5.2, this immediately yields the following.

Corollary 6.5. *The smooth group scheme $g_1\colon P_1 \longrightarrow B_1$ admits a closed symplectic structure for which g_1 is a Lagrangian fibration.* $\square$

7. The dual hyperkähler variety after Kim

7.1. Assume now that there exists a subgroup $K \subset \mathrm{Aut}(X)$ of automorphisms of X that commutes with f and such that the constant group scheme K_{B_0} is isomorphic to K compatible with the action on $X_0 \twoheadrightarrow B_0$.

Under these assumptions, Kim [Ki21] introduces the notion of the dual hyperkähler fibration.

Definition 7.1. The *dual* of $f\colon X \twoheadrightarrow B$ is the fibration

$$f^\vee\colon X^\vee := X/K \twoheadrightarrow B.$$

Note that $X^\vee$ has at most finite quotient singularities and the open part

$$f^{\vee -1}(B_0) = X_0/K \subset X^\vee$$

is a torsor over $P_0^\vee$. There are good reason to call $X^\vee$ the dual variety, but be aware that $X^\vee \twoheadrightarrow B$ typically has no section (which one might expect from a dual fibration), see Remark 7.3. Note that $P_0^\vee \twoheadrightarrow B_0$ comes with a natural section, but not much is known about hyperkähler compactifications of it.

$$\begin{array}{ccccccc}
X & \twoheadrightarrow & B & \qquad & X^\vee = X/K & \twoheadrightarrow & B \\
\cup & & \cup & & \cup & & \cup \\
X_0 & \twoheadrightarrow & B_0 & & X_0^\vee = X_0/K_{B_0} & \twoheadrightarrow & B_0 \\
\circlearrowleft & & \| & & \circlearrowleft & & \| \\
P_0 & \twoheadrightarrow & B_0 & & P_0^\vee = P_0/K_{B_0} & \twoheadrightarrow & B_0
\end{array}$$

7.2. The main result of [Ki21] is the following.

Theorem 7.2 (Kim)**.** *Let $X \twoheadrightarrow B \simeq \mathbb{P}^n$ be a Lagrangian fibration of a projective hyperkähler manifold of one of the known deformation types* $\mathrm{K3}^{[n]}$, Kum_n, OG6, *or* OG10.

Then there exists a subgroup $K \subset \mathrm{Aut}(X)$ such that

$$K_{B_0} \simeq \ker(P_0 \twoheadrightarrow P_0^\vee).$$

More precisely, K is the subgroup of all automorphisms of X that act trivially on $H^2(X,\mathbb{Z})$ and commute with f:

$$K = \{\phi \in \mathrm{Aut}(X) \mid f \circ \phi = \phi \circ f \text{ and } f^* = \mathrm{id} \text{ on } H^2(X,\mathbb{Z})\}.$$

Remark 7.3. The group K can be described explicitly in all known cases:

(i) Assume first that X is deformation equivalent to a Hilbert scheme of a K3 surface. Then $\mathrm{Aut}(X) \twoheadrightarrow \mathrm{O}(H^2(X,\mathbb{Z})$ is injective [Be83]. Hence,

$$K = \{1\},\ X \simeq X^\vee, \text{ and } P_0 \simeq P_0^\vee.$$

Indeed, it is known that the the fibres of $X \longrightarrow B$ are principally polarized and that the image of the restriction map $H^2(X, \mathbb{Z}) \longrightarrow H^2(X_t, \mathbb{Z})$ to the generic fibre is spanned by the principal polarization.

(ii) For hyperkähler manifolds deformation equivalent to O'Grady's ten-dimensional example the situation is analogous. That is, the smooth fibres are again known to be principally polarized abelian varieties, the restriction map $H^2(X, \mathbb{Z}) \longrightarrow H^2(X_t, \mathbb{Z})$ is surjective onto the subgroup spanned by the principal polarization. Mongardi–Wandel [MW17] computed that again $K = \{1\}$ and, therefore, we have

$$X \simeq X^\vee, \text{ and } P_0 \simeq P_0^\vee.$$

(iii) In the other two cases, the situation is more involved, since $K \neq \{1\}$ and the fibres do not carry a principal polarization. Let X be deformation equivalent to a generalized Kummer variety of dimension $2n$. Then Kim shows, building on previous results of Wieneck [Wi16, Wi18], by studying moduli spaces of pure one-dimensional sheaves on abelian surfaces that

$$K = (\mathbb{Z}/d_1 \oplus \mathbb{Z}/d_2)^{\oplus 2}.$$

Here, $d_1 d_2 = n + 1$ and the general fibre of $f \colon X \longrightarrow B$ is a $(1, \dots 1, d_1, d_2)$-polarized abelian variety. The automorphisms of K are all induced from translations along $n+1$-torsion points on the abelian surface. The dual fibration

$$X^\vee = X/K \longrightarrow B$$

in this case is a singular primitive symplectic orbifold.

(iv) This case is similar to the previous one. The smooth fibres are $(1, 2, 2)$-polarized abelian threefolds and the group K was already computed by Mongardi–Wandel [MW17] to be

$$K = (\mathbb{Z}/2)^{\oplus 4}.$$

The dual fibration $X/K \longrightarrow B$ is again an irreducible symplectic variety.

(v) Kim constructs in the case of hyperähler manifolds deformation equivalent to Kum_n or $\mathrm{OG}(6)$ a singular symplectic compactification of the smooth $P_0^\vee$-torsor X_0/K_{B_0}. In particular, this shows that there does not exist a smooth hyperähler compactification $\bar{X} \longrightarrow B$ extending $X/K_{B_0} \longrightarrow B_0$.

Indeed, these varieties would be birational and after passing to another birational model $\tilde{X}$ of $\bar{X}$ the birational map would extend to a morphism $\tilde{X} \longrightarrow X/K$, cf. [LP16]. In particular, X/K would admit a crepant resolution, which is a contradiction.

8. Compactifying torsors à la Saccà

As before, let $f\colon X \longrightarrow B$ be a Lagrangian fibration of a projective hyperkähler manifold X. Assume that $B_1 = B$, i.e. all fibres of f are integral.[11] In this situation, the result of Arinikin–Fedorov, see Section 6, provide a commutative group scheme $P_1 \longrightarrow B_1 = B$ acting on $X \longrightarrow B = B_1$ such that the open set $X' \subset X$ of f-regular points is a torsor over it.

Theorem 8.1 (Saccà)**.** *Assume that all fibres of $X \longrightarrow B = B_1$ are integral. Then any P_1-torsor $Y_1 \longrightarrow B$ can be compactified to a smooth projective hyperkähler manifold $Y \longrightarrow B$ with a Lagrangian fibration.*

Example 8.2. For K3 surfaces, the result holds without any assumptions on the fibre, see Section 3, but in general the assumption on the integrality of all fibres is needed as shown by the following example.

Consider a K3 surface S with an ample divisor H of square $H^2 \geqslant 4$ and consider the Mukai vectors $v_1 = (0, mH, m\chi)$ and $v_2 = (0, mH, \chi')$ for $m \geqslant 3$ and χ coprime. For a generic polarization we obtain two moduli spaces of semistable sheaves with morphisms

$$f_1\colon M(v_1) \longrightarrow |mH| \text{ and } \quad f_2\colon M(v_2) \longrightarrow |mH|.$$

Over the open subset $U \subset |mH|$ of smooth curves, we have that both morphisms f_i restrict to $f_i\colon \mathrm{Pic}^{d_i}(\mathcal{C}/U) \longrightarrow U$ where $\mathcal{C} \longrightarrow U$ is the universal curve. They are both torsors over $\mathrm{Pic}^0(\mathcal{C}/U) \longrightarrow U$, which using the principal polarization is seen to be isomorphic to the restriction of P_1 to U.

Note that the variety $M(v_2)$ is a smooth hyperkähler manifold, while the results of Kaledin–Lehn–Sorger [KLS] show that the variety $M(v_1)$ does not admit a smooth resolution and therefore the restricition of f_1 to the preimage of the open set U cannot be smoothly compactified. From this argument it is not easy to see why the Lagrangian fibration $f_1\colon \mathrm{Pic}^{d_1} \longrightarrow U$ cannot be smoothly compactified when just viewing it as a smooth Lagrangian fibration over a quasi-projective base.[12]

Remark 8.3. (i) A weaker form of the above result has been used by Saccà in [Sa20] where she studied the intermediate Jacobian fibrations associated to the Fano variety of lines of a smooth cubic fourfold.

(ii) If one only assumes that the complement of $B_1 \subset B$ has codimension at least two, then Saccà constructs a $\mathbb{Q}$-factorial terminal symplectic compactification $\bar{Y} \longrightarrow B$ of P_1-torsors $Y_1 \longrightarrow B_1$. Similarly, given a finite subgroup $G \subset P_1 \subset \mathrm{Aut}_0(X_1/B_1)$, e.g. $G = \mathcal{K}$, Saccà also constructs a $\mathbb{Q}$-factorial terminal symplectic compactification of $P_1/G \longrightarrow B_1$.

[11] Since then the fibration has local sections, the base is smooth and, therefore, $B \simeq \mathbb{P}^n$. Note that integrality of the fibre is not needed for the argument, the existence of local sections is enough.

[12] Saccà uses this example to argue that the assumption on the integrality of the fibres is necessary, but for this one would also need to view f_1 over the bigger open subset of all integral fibres as a torsor over P_1.

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