Derived categories of hyper-Kähler manifolds via the extended Mukai lattice

$/\mathbb{C}$

§ Motivation

S projective K3 surface $\rightsquigarrow$ $D^b(S) := D^b(\mathrm{Coh}(S))$

lattice $\widetilde{H}(S,\mathbb{Z}) := (H^*(S,\mathbb{Z}), \tilde{b})$ $\quad \tilde{b}$ Mukai pairing

weight-two Hodge structure $\widetilde{H}^{2,0}(S,\mathbb{C}) := H^{2,0}(S)$

$\widetilde{H}^{0,2}(S,\mathbb{C}) := H^{0,2}(S)$

$\widetilde{H}^{1,1}(S,\mathbb{C}) := H^{1,1}(S) \oplus H^0(S,\mathbb{C}) \oplus H^4(S,\mathbb{C})$

Mukai, Orlov

$$\begin{tikzcd}
D^b(S) \arrow[r, "\Phi_\mathcal{E}", "\sim"'] \arrow[d, "v"'] \arrow[dr, phantom, "\circlearrowleft"] & D^b(S') \arrow[d, "v"] \\
\widetilde{H}(S,\mathbb{Z}) \arrow[r, "\Phi_\mathcal{E}^{\widetilde{H}}", "\sim"'] & \widetilde{H}(S',\mathbb{Z})
\end{tikzcd}$$

$v = ch(\_)\, td^{1/2}$

Hodge isometry

Reduces questions about $D^b(S)$ to study of $\widetilde{H}(S,\mathbb{Z})$

Goal: Find & describe similiar picture for hyper-Kähler manifolds.

§ Hyper-Kähler cohomology

X (proj) hyper-Kähler manifold (HK), i.e. compact Kähler manifold of dimension $2n$

- $\pi_1(X) = \{1\}$
- $H^0(X, \Omega_X^2) = \mathbb{C}\sigma$ nowhere degenerate

$(H^2(X,\mathbb{Z}), b)$ $\quad b$ Beauville-Bogomolov-Fujiki (BBF) form

$\lambda \in H^2(X,\mathbb{Z})$ $\quad \int_X \lambda^{2n} = c_X \frac{(2n)!}{2^n n!} b(\lambda,\lambda)^n$ $\quad c_X \in \mathbb{Q}_{>0}$ Fujiki constant

Ex:
- K3 surfaces $\quad c_X = 1 \quad H^2(X,\mathbb{Z}) \cong E_8(-1)^{\oplus 2} \oplus U^{\oplus 3}$
- $K3^{[n]}$-type $\quad c_X = 1 \quad H^2(X,\mathbb{Z}) \cong E_8(-1)^{\oplus 2} \oplus U^{\oplus 3} \oplus \mathbb{Z}(2-2n)$

$$SH(X,\mathbb{Q}) := \mathrm{Im}\left(\mathrm{Sym}^{\bullet} H^2(X,\mathbb{Q}) \longrightarrow H^{\bullet}(X,\mathbb{Q})\right) \subseteq H^{\bullet}(X,\mathbb{Q})$$

Verbitsky component

LLV algebra $\quad \mathfrak{g}(X) = \langle (e_\lambda, h, f_\lambda) \ \mathfrak{sl}_2\text{-triples} \rangle \subseteq \mathrm{End}(H^{\bullet}(X,\mathbb{Q}))$

$e_\lambda = \lambda \cup \quad \lambda \in H^2(X,\mathbb{Q})$ with Hard Lefschetz property

$$\begin{tikzcd}
H^{2n-4} \arrow[r, "e_\lambda"] \arrow[rrrr, bend right, "\cong"'] & H^{2n-2} \arrow[r, "e_\lambda"] \arrow[rr, bend right, "\cong"'] & H^{2n} \arrow[r, "e_\lambda"] & H^{2n+2} \arrow[r, "e_\lambda"] & H^{2n+4}
\end{tikzcd}$$

$$h|_{H^{2n-i}} = (2n-i)\,\mathrm{id}$$

§ Derived categories & extended Mukai lattice

Thm (Taelman) $\quad X, Y$ HK and assume $\Phi: D^b(X) \cong D^b(Y)$

$\Rightarrow \exists\, \Phi^{\mathfrak{g}}: \mathfrak{g}(X) \cong \mathfrak{g}(Y)$ such that

$\Phi^H: H^{\bullet}(X,\mathbb{Q}) \cong H^{\bullet}(Y,\mathbb{Q})$ is equivariant w.r.t. $\Phi^{\mathfrak{g}}$.

Cor: $\quad \Phi^H: H^{\bullet}(X,\mathbb{Q}) \cong H^{\bullet}(Y,\mathbb{Q})$ restricts to a Hodge isometry (w.r.t. Mukai pairings)

$\Phi^{SH}: SH(X,\mathbb{Q}) \cong SH(Y,\mathbb{Q})$

Thm (Looijenga-Lunts, Verbitsky) $\quad \mathfrak{g}(X) \cong \mathfrak{so}(\tilde{H}(X,\mathbb{Q}))$

Def: $(\tilde{H}(X,\mathbb{Q}), \tilde{b})$ extended Mukai lattice

$$\tilde{H}(X,\mathbb{Q}) := \mathbb{Q}\alpha \oplus H^2(X,\mathbb{Q}) \oplus \mathbb{Q}\beta \qquad \dim = b_2(X) + 2$$

deg -2 deg 0 deg 2

$\tilde{b}|_{H^2(X,\mathbb{Q})} = b$ $\qquad H^2(X,\mathbb{Q}) \perp \alpha, \beta$ $\qquad \tilde{b}(\alpha,\alpha) = \tilde{b}(\beta,\beta) = 0, \tilde{b}(\alpha,\beta) = -1$

$\tilde{H}^{2,0}(X,\mathbb{C}) = H^{2,0}(X)$ $\qquad \tilde{H}^{0,2}(X,\mathbb{C}) = H^{0,2}(X)$ $\qquad \tilde{H}^{1,1}(X,\mathbb{C}) = H^{1,1}(X) \oplus \mathbb{C}\alpha \oplus \mathbb{C}\beta$

$\mathfrak{g}(X) \circlearrowright \tilde{H}(X,\mathbb{Q})$ $\qquad e_\lambda(\alpha) = \lambda \qquad e_\lambda(\mu) = b(\lambda,\mu)\beta \qquad e_\lambda(\beta) = 0 \qquad \lambda,\mu \in H^2(X,\mathbb{Q})$

$\qquad h(\alpha) = -2\alpha \qquad h(\mu) = 0 \qquad h(\beta) = 2\beta$

$$\mathfrak{g}(X) \circlearrowright SH(X,\mathbb{Q}) \overset{\psi}{\hookrightarrow} \operatorname{Sym}^n \tilde{H}(X,\mathbb{Q}) \circlearrowleft \mathfrak{g}(X)$$
$$1 \longmapsto \frac{\alpha^n}{n!}$$

inclusion of $\mathfrak{g}(X)$-modules

$(SH(X,\mathbb{Q}), b_{SH})$ $\qquad b_{SH}(\lambda_1 \cdots \lambda_m, \mu_1 \cdots \mu_{2n-m}) := (-1)^m \int_X \lambda_1 \cdots \lambda_m \mu_1 \cdots \mu_{2n-m}$ Mukai pairing

$(\operatorname{Sym}^n \tilde{H}(X,\mathbb{Q}), b_{[n]})$ $\qquad b_{[n]}(v_1 \cdots v_n, w_1 \cdots w_n) := (-1)^n c_X \sum_{\sigma \in \mathfrak{S}_n} \prod_{i=1}^n \tilde{b}(v_i, w_{\sigma(i)})$

ψ respects
- $\mathfrak{g}(X)$-module structure
- quadratic forms
- Hodge structures
- gradings

X, Y defo. eq. HK & assume n or $b_2(X)$ odd.
Consider $\Phi : D^b(X) \cong D^b(Y)$. Then
$\exists!\, \Phi^{\tilde{H}} \in O(\tilde{H}(X,\mathbb{Q}))$ Hodge isometry such that

$$\begin{tikzcd}
SH(X,\mathbb{Q}) \arrow[r, "\det(\Phi^{\tilde{H}})^{n+1}\,\Phi^{SH}"] \arrow[d, "\psi"'] \arrow[dr, phantom, "\circlearrowright"] & SH(Y,\mathbb{Q}) \arrow[d, "\psi"] \\
\operatorname{Sym}^n \tilde{H}(X,\mathbb{Q}) \arrow[r, "\operatorname{Sym}^n \Phi^{\tilde{H}}"] & \operatorname{Sym}^n \tilde{H}(Y,\mathbb{Q})
\end{tikzcd}$$

$$\begin{tikzcd}
 & O(H^\bullet(X,\mathbb{Q})) \arrow[d, dotted, "\mathrm{res}"] \\
\mathrm{Aut}(D^b(X)) \arrow[ur] \arrow[r] \arrow[dr] & O(SH(X,\mathbb{Q})) \\
 & O(\tilde{H}(X,\mathbb{Q})) \arrow[u, hook]
\end{tikzcd}$$

§ Extended Mukai vector

<u>Def:</u> $q_{2i} \in SH^{4i}(X,\mathbb{Q})$ $\qquad \int_X q_{2i}\lambda^{2n-2i} = c_X \dfrac{(2n-2i)!}{2^{n-i}(n-i)!} b(\lambda,\lambda)^{n-i} \quad \forall \lambda \in H^2(X,\mathbb{Q})$

<u>Ex:</u> $q_0 = 1$, $\qquad q_{2n} = c_X[pt]$, $\qquad \int_X q_{2n-2}\lambda\mu = c_X b(\lambda,\mu) \quad \lambda,\mu \in H^2(X,\mathbb{Q})$

$$SH(X,\mathbb{Q}) \underset{\psi}{\overset{T}{\rightleftarrows}} \mathrm{Sym}^n \tilde{H}(X,\mathbb{Q}) \qquad \text{orthogonal split}$$

<u>Lem:</u> $T\left(\dfrac{\alpha^{n-i}\beta^i}{(n-i)!}\right) = q_{2i}$

$$H^\bullet(X,\mathbb{Q}) \xrightarrow{\overline{T}} SH(X,\mathbb{Q}), \qquad H^\bullet(X,\mathbb{Q}) = SH(X,\mathbb{Q}) \oplus SH(X,\mathbb{Q})^\perp$$

<u>Prop</u> (B) $\exists\, r_X \in \mathbb{Q}_{>0}$ such that

$$\underset{H^\bullet(X,\mathbb{Q})}{\underset{\cap}{\mathrm{td}^{1/2}}} \qquad\qquad \underset{\mathrm{Sym}^n\tilde{H}(X,\mathbb{Q})}{\underset{\cap}{\frac{(\alpha+r_X\beta)^n}{n!}}}$$

<u>Def:</u> $r_X = \dfrac{C(c_2(X))}{c_X} \cdot \dfrac{2n-1}{24} \cdot \dfrac{2^n n!}{(2n)!}$ $\qquad C(c_2(X))\, b(\lambda,\lambda)^{n-1} = \int_X c_2(X)\lambda^{2n-2}$

$$r_X = \begin{cases} \dfrac{n+3}{4} & K3^{[n]},\ OG^{10} \\ \dfrac{n+1}{4} & \mathrm{Kum}^n,\ OG^6 \end{cases}$$

<u>Def:</u> L line bundle. Then its extended Mukai vector

$$\tilde{v}(L) := \alpha + c_1(L) + \left(r_X + \frac{b(c_1(L),c_1(L))}{2}\right)\beta \in \tilde{H}(X,\mathbb{Q})$$

$$\overline{v(L)} = T\left(\frac{\tilde{v}(L)^n}{n!}\right)$$

$L\otimes\_ \in \mathrm{Aut}\,D^b(X)$ with $\lambda = c_1(L)$ induces $B_\lambda \in O(\tilde{H}(X,\mathbb{Q}))$

$$r\alpha+\mu+s\beta \longmapsto r\alpha+\mu+r\lambda+\left[s+b(\lambda,\mu)+r\frac{b(\lambda,\lambda)}{2}\right]\beta.$$

Rem: In general: $\mathcal{E}\in D^b(X)$ admits extended Mukai vector, if $\exists\, \tilde{v}(\mathcal{E})\in\tilde{H}(X,\mathbb{Q})$

$$\overline{v(\mathcal{E})} = \frac{1}{c}T(\tilde{v}(\mathcal{E})^n) \qquad c\in\mathbb{Q}$$

Ex: • $k(x)$ $\quad \tilde{v}(k(x)) = \beta$, $c = c_X$ i.e. $v(k(x)) = T\left(\frac{\beta^n}{c_X}\right)$

• $\mathbb{P}^1\cong C \hookrightarrow S \rightsquigarrow \mathbb{P}^n\cong C^{[n]} \hookrightarrow S^{[n]}$ $\quad \overline{v(\mathcal{O}_{\mathbb{P}^n})} = T\left(\frac{([C]+\delta/2+\frac{n+1}{2}\beta)^n}{n!}\right)$

Thm (B) X, Y defo. eq. HK $\quad \Phi_{\mathcal{E}}: D^b(X)\cong D^b(Y)$

• $\mathrm{rk}(\mathcal{E}) = \frac{r^n n!}{c_X}$ $\quad r\in\mathbb{Q}$ $\quad (c_X\in\mathbb{Z} \Rightarrow r\in\mathbb{Z})$

• $\mathrm{rk}(\mathcal{E}) = 0 \Rightarrow$ a) $\mathrm{supp}(\mathcal{E})$ covers X resp. Y with Lagrangians

or b) $H^2(X,\mathbb{Z})\cong H^2(Y,\mathbb{Z})$ Hodge isometry

§ Derived monodromies for $K3^{[n]}$-type

X $\quad K3^{[n]}$-type, $n>1$

Hilbert scheme $S^{[n]}$ $\quad H^2(S^{[n]},\mathbb{Z})\cong H^2(S,\mathbb{Z})\oplus\mathbb{Z}\delta$ $\quad 2\delta = [E]$ exceptional divisor

For general X fix $\quad H^2(X,\mathbb{Z})\cong H^2(S^{[n]},\mathbb{Z})$

$$\delta \longleftarrow\!\shortmid\ \delta$$

Def: $\Lambda_X := B_{-\frac{\delta}{2}}(\tilde{H}(X,\mathbb{Z})) \subseteq \tilde{H}(X,\mathbb{Q})$

$$\tilde{\alpha} := B_{-\frac{\delta}{2}}(\alpha) = \alpha - \frac{\delta}{2} + \frac{1-n}{4}\beta \qquad \tilde{\delta} := B_{-\frac{\delta}{2}}(\delta) = \delta + (n-1)\beta$$

$$\Lambda_X = \mathbb{Z}\tilde{\alpha} \oplus H^2(X,\mathbb{Z}) \oplus \mathbb{Z}\beta \subseteq \tilde{H}(X,\mathbb{Q})$$

Thm (B) i) $Aut(D^b(X)) \xrightarrow{\rho^{\tilde{H}}} Aut(\Lambda_X)$ group of Hodge isometries

ii) X, Y $K3^{[n]}$-type $\quad \Phi: D^b(X) \cong D^b(Y)$ induces

$\Phi^{\tilde{H}}: \Lambda_X \cong \Lambda_Y$ Hodge isometry

$\quad\quad \cup\!\mid \quad\quad \cup\!\mid$

$T(X) \cong T(Y)$ transcendental lattices

Sketch of proof: 1) Consider all equivalences:

$$DMon(X) := \langle \gamma' \circ \Phi^{\tilde{H}} \circ \gamma \rangle \subseteq O(\tilde{H}(X,\mathbb{Q}))$$ derived monodromy group

$\gamma: \tilde{H}(X,\mathbb{Q}) \cong \tilde{H}(X',\mathbb{Q})$ parallel transport isometry

$\Phi: D^b(X') \cong D^b(X'') \rightsquigarrow \Phi^{\tilde{H}}: \tilde{H}(X',\mathbb{Q}) \cong \tilde{H}(X'',\mathbb{Q})$

$\gamma': \tilde{H}(X'',\mathbb{Q}) \cong \tilde{H}(X,\mathbb{Q})$ parallel transport isometry

2) Lower bound: Study known equivalences

Bridgeland-King-Reid, Krug: $\quad D^b(S^{[n]}) \cong D^b(S^n)_{\mathfrak{S}_n}$

a) $\sigma_n \in Aut(D^b(S^n)_{\mathfrak{S}_n})$ sign representation

Prop $\quad \sigma_n^{\tilde{H}} = s_{\tilde{\delta}}$ reflection along $\tilde{\delta} \in \Lambda_{S^{[n]}}$

b) Ploog: $\Phi_{[n]}: Aut(D^b(S)) \longrightarrow Aut(D^b(S^{[n]}))$

$\quad\quad \Phi_{\mathcal{E}} \longmapsto \Phi_{\mathcal{E}^{\boxtimes n}}$

Prop $\quad \Phi_{[n]}(ST_{\mathcal{O}_S})^{\tilde{H}} = s_{(\tilde{\alpha}+\beta)}$ reflection along $\tilde{\alpha}+\beta \in \Lambda_X$

Recall: $ST_{\mathcal{O}_S}^{\tilde{H}} = s_{\alpha+\beta}$

$\rightsquigarrow \hat{O}^+(\Lambda_X) \subset \mathrm{DMon}(X)$

orientation-preserving isometries acting with $\pm id$ on discriminant

3) Upper bound:

Thm(B) $\hat{O}^+(\Lambda_X) \subset \mathrm{DMon}(X) \subseteq O(\Lambda_X)$

Proof uses:
- Lattice theory
- Further restrictions from geometry via the extended Mukai vector $\square$

Cor
- Number of $K3^{[n]}$-FM partners is finite
- $D^b(X) \cong D^b(M_\sigma^S(v)) \Rightarrow X \cong M_{\sigma'}^S(v')$
- $D^b(S^{[n]}) \cong D^b(S'^{[n]}) \Leftrightarrow D^b(S) \cong D^b(S')$
- $\hat{\mathrm{Aut}}^+(\Lambda_{S^{[n]}}) \subseteq \mathrm{Im}(\rho^{\tilde{H}}) \subseteq \mathrm{Aut}(\Lambda_{S^{[n]}})$ for S with $U \subseteq NS(S)$