Vortrag
Dienstag, 18. Mai 2021 09:05

Derived categories of hyper-Kähler manifolds

§1 Motivation & Recollection:

Goal: Study $D^b(X) := D^b(\mathrm{Coh}(X))$ & $\mathrm{Aut}(D^b(X))$ for X HK.

Interest: Produces naturally cycles:

Thm (Orlov) $\Phi : D^b(X) \cong D^b(Y)$, X, Y smooth projective $\Rightarrow$ $\Phi = FM_{\mathcal{E}}$ for $\mathcal{E} \in D^b(X \times Y)$, i.e.

$$FM_{\mathcal{E}} = p_{Y*}(\mathcal{E} \otimes \_\,) \circ (p_X^*(\,))$$

$\rightsquigarrow$
$$\begin{tikzcd} D^b(X) \arrow[r, "FM_{\mathcal{E}}"] \arrow[d, "v"'] \arrow[dr, phantom, "\circlearrowright"] & D^b(Y) \arrow[d, "v"] \\ H^\cdot(X, \mathbb{Q}) \arrow[r, "FM^H_{v(\mathcal{E})}"] & H^\cdot(Y, \mathbb{Q}) \end{tikzcd}$$

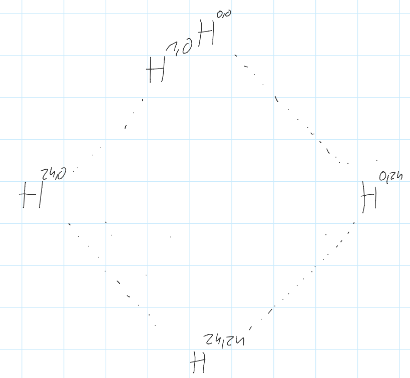
v Mukai vector

$v(\mathcal{E}) \in H^\cdot(X \times Y, \mathbb{Q})$

cohomological Fourier–Mukai transform

Properties: 1) Preserves columns of Hodge diamond

$v(\mathcal{E}) \in \bigoplus_p H^{p,p}(X \times Y, \mathbb{Q})$



2) Isometry w.r.t. Mukai pairing: signed intersection pairing

$$\rightsquigarrow \rho^H : \mathrm{Aut}\, D^b(X) \longrightarrow O(\tilde{H}(X, \mathbb{Q}))$$

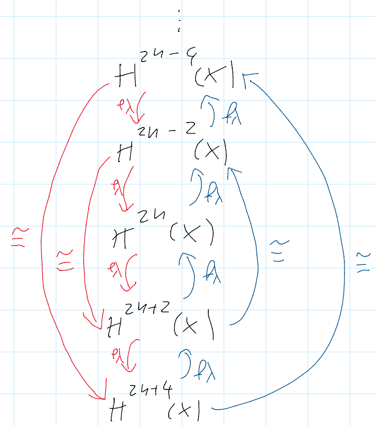
Rem: Neither cohomological grading nor ring structure is preserved!

§2 LLV Lie algebra

X HK $\dim X = 2n$

$\lambda \in H^2(X, \mathbb{Q}) \rightsquigarrow e_\lambda := \lambda \cup \_ \in \mathrm{End}(H^\cdot(X, \mathbb{Q}))$ f_λ Dual Lefschetz operator

λ has HL-property, if



$\vdots$

$h \in \mathrm{End}(H^*(X,\mathbb{Q}))$ grading operator

$$h|_{H^i} = (i-2n)\cdot \mathrm{id}$$

$\rightsquigarrow$ $(e_\lambda, h, f_\lambda)$ $\mathfrak{sl}_2$-triple

$\rightsquigarrow$ $\mathfrak{g}(X) = \langle (e_\lambda, h, f_\lambda) \mid \lambda \in H^2(X,\mathbb{Q})$ Hard Lefschetz $\rangle \subseteq \mathrm{End}\, H^*(X,\mathbb{Q})$

LLV Lie algebra

<u>Goal:</u> Relate $D^b(X)$ and $\mathfrak{g}(X)$, s.t. $\mathrm{Aut}(D^b(X)) \curvearrowright \mathfrak{g}(X)$

§3 Polyvector fields

<u>Def:</u> $HT^*(X) := \bigoplus_{p+q=*} H^q(X, \Lambda^p T_X)$ ring of polyvector fields

graded $\mathbb{C}$-algebra

X HK $\sigma \in H^0(X, \Omega_X^2)$ symplectic form $\rightsquigarrow \sigma^p : \Lambda^p T_X \cong \Lambda^p \Omega_X$

$$\rightsquigarrow \quad HT^*(X) = \bigoplus_{p,q} H^q(X, \Lambda^p T_X) \cong \bigoplus_{p,q} H^q(X, \Lambda^p \Omega_X) \underset{(*)}{=} H^*(X,\mathbb{C})$$

$\mathbb{C}$-algebra isomorphism

$(*) \rightsquigarrow$ $HT^*(X) \curvearrowright H^*(X,\mathbb{C})$

$v \in H^q(X, \Lambda^p T_X)$ $\quad x \in H^{q'}(X, \Lambda^{p'} \Omega_X)$

$\rightsquigarrow$ $v \lrcorner x \in H^{q+q'}(X, \Lambda^{p'-p} \Omega_X)$

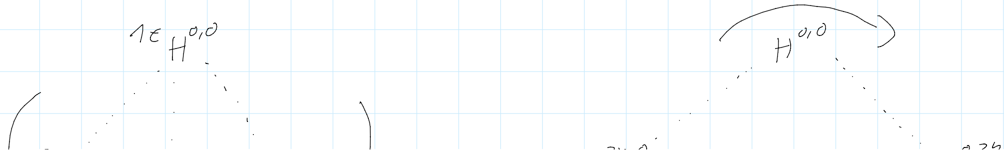
<u>Lemma:</u> X HK. $H^*(X,\mathbb{C})$ is a free $HT^*(X)$ module of rank 1 generated by $\sigma^n \in H^0(X, \Omega_X^{2n})$:

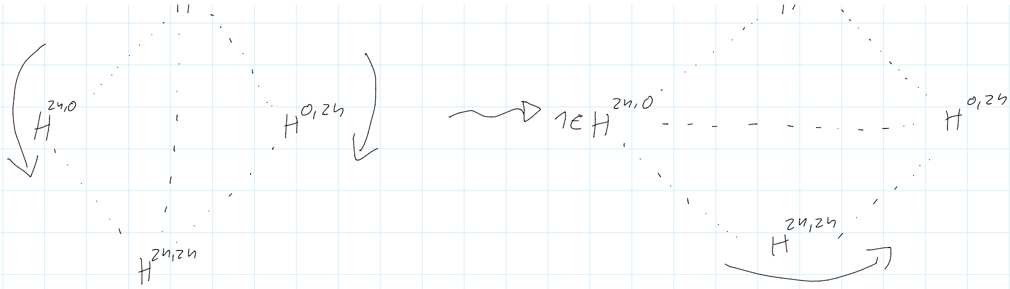
$$HT^*(X) \lrcorner \sigma^n \cong H^*(X,\mathbb{C})$$

(! not graded!)

$HT^0(X)$	$HT^1(X)$	$HT^2(X)$	$\bigoplus_{p-q=2n} H^q(X,\Omega_X^p)$	$\bigoplus_{p-q=2n-1} H^q(X,\Omega_X^p)$	$\bigoplus_{p-q=2n-2} H^q(X,\Omega_X^p)$
		$H^0(X,\Lambda^2 T_X)$			$H^0(X,\Omega_X^{2n-2})$
	$H^0(X,\Lambda^1 T_X)$			$H^0(X,\Omega_X^{2n-1})$	
$H^0(X,\Lambda^0 T_X)$		$H^1(X,T_X^1)$	$H^0(X,\Omega_X^{2n})$		$H^1(X,\Omega_X^{2n-1})$
	$H^1(X,\Lambda^0 T_X)$			$H^1(X,\Omega_X^{2n})$	
		$H^2(X,\mathcal{O}_X)$			$H^2(X,\Omega_X^{2n})$

From this viewpoint, the ring structure on $H^*(X,\mathbb{C})$ is rotated by 90 degrees:





de Rham   Polyvector

Important fact: $HT^*(X)$ + algebra action on $H^*(X,\mathbb{C})$ is a derived invariant:

For $\Phi : D^b(X) \cong D^b(Y)$ derived equivalence

$\rightsquigarrow \exists$ natural $\Phi^{HT} : HT^*(X) \cong HT^*(Y)$ graded $\mathbb{C}$-algebra isomorphism which is equivariant w.r.t. Φ^H:

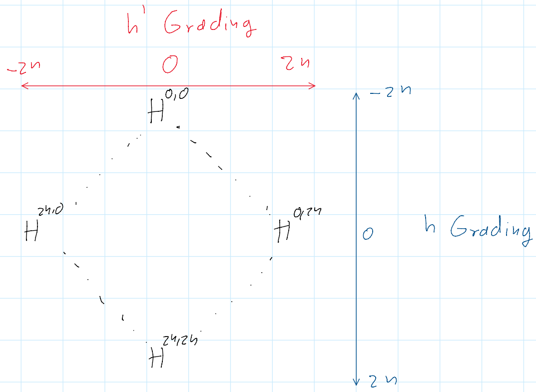
$v \in HT^*(X)$ $\quad x \in H^*(X,\mathbb{C})$: $\quad \Phi^H(v \lrcorner x) = \Phi^{HT}(v) \lrcorner \Phi^H(x) \in H^*(Y,\mathbb{C})$

§4 Reinventing $g(X)$

Define $h_p, h_q \in \mathrm{End}(H^*(X,\mathbb{C}))$ $\quad h_p|_{H^{p,q}} = (p-n)\,\mathrm{id}$ $\quad h_q|_{H^{p,q}} = (q-n)\,\mathrm{id}$

Then $h = h_p + h_q$

Def: $h' = h_q - h_p$



$\mu \in HT^2(X) \rightsquigarrow e_\mu \in \mathrm{End}(H^*(X,\mathbb{C}))$ degree 2 w.r.t. h' grading

μ has Hard-Lefschetz property (w.r.t. h') if

$$\begin{tikzcd} \bigoplus\limits_{p-q=2} H^{q,p}(X) \arrow[rr, bend right, "e_\mu"'] \arrow[rrrr, bend right=40, "\cong"'] & \bigoplus\limits_{p-q=1} H^{q,p}(X) \arrow[rr, bend right, "e_\mu"', "\cong"] & \bigoplus\limits_{p-q=0} H^{q,p}(X) \arrow[rr, bend right, "e_\mu"'] & \bigoplus\limits_{p-q=-1} H^{q,p}(X) & \bigoplus\limits_{p-q=-2} H^{q,p}(X) \end{tikzcd}$$

Jacobson Morozov Theorem: μ Hard Lefschetz $\Leftrightarrow$ $\exists f_\mu \in \mathrm{End}(H^*(X,\mathbb{C}))$ s.t. (e_μ, h', f_μ) $\mathfrak{sl}_2$-triple

$\rightsquigarrow \quad \mathfrak{g}'(X) := \langle (e_\mu, h', f_\mu) \mid \mu \in HT^2(X) \text{ Hard Lefschetz} \rangle \subseteq \operatorname{End}(H^\cdot(X,\mathbb{Q}))$

Rem: i) $\mathfrak{g}'(X) \cong \mathfrak{g}(X) \otimes_{\mathbb{Q}} \mathbb{C}$ $\quad$ ($H^\cdot(X, \Lambda^\cdot \Omega_X) \underset{\sigma}{\cong} H^\cdot(X, \Lambda^\cdot T_X)$ $\mathbb{C}$-algebra isomorphism)

ii) $\Phi: D^b(X) \cong D^b(Y)$ induces $\Phi^{\mathfrak{g}}: \mathfrak{g}'(X) \cong \mathfrak{g}'(Y)$ s.t.

$\Phi^H: H^\cdot(X,\mathbb{C}) \cong H^\cdot(Y,\mathbb{C})$ is equivariant w.r.t. $\Phi^{\mathfrak{g}}$, i.e. $\quad j \in \mathfrak{g}'(X) \quad x \in H^\cdot(X,\mathbb{C})$

$$\Phi^H(j.x) = \Phi^{\mathfrak{g}}(j).\Phi^H(x) \in H^\cdot(Y,\mathbb{C})$$

Pf: $\Phi^{HT}|_{HT^2}: HT^2(X) \cong HT^2(Y) \rightsquigarrow e_\mu \mapsto e_{\Phi^{HT}(\mu)} = \Phi^H \circ e_\mu \circ (\Phi^H)^{-1}$

Φ^H respects Hodge diamond columns: $\quad h'_Y \circ \Phi^H = \Phi^H \circ h'_X$ $\square$

Thm (Verbitsky '95, Taelman '19)

$\mathfrak{g}'(X) = \mathfrak{g}(X) \otimes_{\mathbb{Q}} \mathbb{C}$

Cor: $\Phi: D^b(X) \cong D^b(Y) \rightsquigarrow \Phi^{\mathfrak{g}}: \mathfrak{g}(X) \cong \mathfrak{g}(Y)$ $\quad (/\mathbb{Q})$

$SH(X,\mathbb{Q}) = \operatorname{Im}(\operatorname{Sym}^\cdot H^2(X,\mathbb{Q}) \to H^\cdot(X,\mathbb{Q})) \subseteq H^\cdot(X,\mathbb{Q})$ Verbitsky component

Cor: $\Phi^H: H^\cdot(X,\mathbb{Q}) \cong H^\cdot(Y,\mathbb{Q})$ restricts to $\Phi^{SH}: SH(X,\mathbb{Q}) \cong SH(Y,\mathbb{Q})$

§5 Extended Mukai lattice

Goal: Use $\mathfrak{g}(X)$ & $SH(X,\mathbb{Q})$ to refine study of $D^b(X)$ & $\operatorname{Aut} D^b(X)$.

- Dimension of $SH(X,\mathbb{Q})$ is too large for the information it encodes
- $\Phi: D^b(X) \cong D^b(Y)$ induces $\Phi^{\mathfrak{g}}: \mathfrak{g}(X) \cong \mathfrak{g}(Y)$ via $\Phi^{HT^2}: HT^2(X) \cong HT^2(Y)$

Problems: a) Integral / rational structure of $HT^2(X)$.
b) How to determine Φ^{HT^2} ?

Recall: Decompose $\mathfrak{g}(X)$:

$$\begin{array}{l} \mathfrak{g}(X)_{-2} \\ \oplus \\ \mathfrak{g}(X)_0 = \overline{\mathfrak{g}(X)} \oplus \mathbb{Q}h \cong \mathfrak{so}(H^2(X,\mathbb{Q})) \oplus \mathbb{Q}h \\ \oplus \\ \mathfrak{g}(X)_{+2} \end{array}$$

$$\mathfrak{g}(X) \cong \mathfrak{so}(\tilde{H}(X,\mathbb{Q}))$$

Def: $(\tilde{H}(X,\mathbb{Q}), \tilde{b})$ extended Mukai lattice

$$\tilde{H}(X,\mathbb{Q}) := \underset{\deg -2}{\mathbb{Q}\alpha} \oplus \underset{\deg 0}{H^2(X,\mathbb{Q})} \oplus \underset{\deg 2}{\mathbb{Q}\beta} \qquad \dim = b_2(X)+2$$

$\tilde b|_{H^2(X,\mathbb{Q})} = b$ BBF-form, $\quad H^2(X,\mathbb{Q}) \perp \alpha,\beta \quad \tilde b(\alpha,\alpha)=\tilde b(\beta,\beta)=0, \tilde b(\alpha,\beta)=-1$

$$\tilde H^{2,0}(X,\mathbb{C}) = H^{2,0}(X) \qquad \tilde H^{0,2}(X,\mathbb{C}) = H^{0,2}(X) \qquad \tilde H^{1,1}(X,\mathbb{C}) = H^{1,1}(X)\oplus\mathbb{C}\alpha\oplus\mathbb{C}\beta$$

$$\mathfrak{g}(X) \curvearrowright \tilde H(X,\mathbb{Q}) \qquad \begin{array}{lll} e_\lambda(\alpha)=\lambda & e_\lambda(\mu)=b(\lambda,\mu)\beta & e_\lambda(\beta)=0 \quad \lambda,\mu\in H^2(X,\mathbb{Q}) \\ h(\alpha)=-2\alpha & h(\mu)=0 & h(\beta)=2\beta \end{array}$$

$$\mathfrak{g}(X) \circlearrowright SH(X,\mathbb{Q}) \xhookrightarrow{\Psi} \mathrm{Sym}^n \tilde H(X,\mathbb{Q}) \circlearrowleft \mathfrak{g}(X) \qquad 1 \longmapsto \frac{\alpha^n}{n!}$$

inclusion of $\mathfrak{g}(X)$-modules

$$\rightsquigarrow \quad 0 \to SH(X,\mathbb{Q}) \xrightarrow{\Psi} \mathrm{Sym}^n \tilde H(X,\mathbb{Q}) \xrightarrow{\Delta} \mathrm{Sym}^{n-2} \tilde H(X,\mathbb{Q}) \to 0$$

$$\Delta(v_1\cdots v_n) = \sum_{i<j} \tilde b(v_i,v_j)\, v_1\cdots\hat v_i\cdots\hat v_j\cdots v_n$$

$(SH(X,\mathbb{Q}), b_{SH})$ $\quad b_{SH}(\lambda_1\cdots\lambda_m, \mu_1\cdots\mu_{2n-m}) := (-1)^m \int_X \lambda_1\cdots\lambda_m\mu_1\cdots\mu_{2n-m}$ Mukai pairing

$(\mathrm{Sym}^n\tilde H(X,\mathbb{Q}), b_{[n]})$ $\quad b_{[n]}(v_1\cdots v_n, w_1\cdots w_n) := (-1)^n c_X \sum_{\sigma\in\mathfrak{S}_n}\prod_{i=1}^n \tilde b(v_i, w_{\sigma(i)})$ $\quad$ (c_X Fujiki constant)

Ψ respects
- $\mathfrak{g}(X)$-module structure
- quadratic forms
- Hodge structures
- gradings

We have $\mathrm{Aut}(D^b(X)) \longrightarrow \mathrm{Aut}(SH(X,\mathbb{Q}), b_{SH}, \mathfrak{g}(X))$

isometry w.r.t. b_{SH}

isomorphism normalizes $\mathfrak{g}(X)$, i.e. $\Phi^H \mathfrak{g}(X) \Phi^{H^{-1}} = \mathfrak{g}(X) \subseteq \mathrm{End}(SH(X,\mathbb{Q}))$

Prop: $\mathrm{Aut}(SH(X,\mathbb{Q}), b_{SH}, \mathfrak{g}(X)) \cong O(\tilde H(X,\mathbb{Q}))$ $\quad$ if n odd or $b_2(X)$ odd.

X,Y deformation equivalent:

$$\begin{tikzcd} SH(X,\mathbb{Q}) \arrow[r, "\det(\Phi^{\tilde H})^{n+1}\,\Phi^{SH}"] \arrow[d, "\Psi"'] \arrow[dr, phantom, "\circlearrowleft"] & SH(Y,\mathbb{Q}) \arrow[d, "\Psi"] \\ \mathrm{Sym}^n\tilde H(X,\mathbb{Q}) \arrow[r, "\mathrm{Sym}^n(\Phi^{\tilde H})"] & \mathrm{Sym}^n\tilde H(Y,\mathbb{Q}) \end{tikzcd}$$

$\mathrm{Aut}\, D^b(X) \xrightarrow{\rho^{\tilde H}} \mathrm{Aut}(\tilde H(X,\mathbb{Q}))$ $\quad$ Hodge isometries

In general:

Thm: X,Y HK s.t. $D^b(X)\cong D^b(Y)$ $\rightsquigarrow$ $\exists\, \Phi^{\tilde H}: \tilde H(X,\mathbb{Q}) \to \tilde H(Y,\mathbb{Q})$ Hodge similitude
$\lambda\in\mathbb{Q}$

$$SH(X,\mathbb{Q}) \xrightarrow{\Phi^{SH}} SH(Y,\mathbb{Q}) \qquad \circlearrowleft$$

$$\begin{tikzcd}
SH^*(X,\mathbb{Q}) \arrow[r] \arrow[d] & SH^*(Y,\mathbb{Q}) \arrow[d] \\
\mathrm{Sym}^n \tilde{H}(X,\mathbb{Q}) \arrow[r, "\lambda\, \mathrm{Sym}^n(\Phi^{\tilde{H}})"] \arrow[ur, phantom, "\circlearrowleft"] & \mathrm{Sym}^n \tilde{H}(X,\mathbb{Q})
\end{tikzcd}$$

Moreover, $\exists\ H^*(X,\mathbb{Q}) \cong H^*(Y;\mathbb{Q})$ isomorphism of $\mathbb{Q}$-Hodge structures.

<u>Idea Pf:</u> (V,q) quadratic space $\rightsquigarrow S_{[n]}V = \mathrm{Ker}(\mathrm{Sym}^n V \to \mathrm{Sym}^{n-2} V)$

$$N(V) = \{ g \in GL(S_{[n]} V) \mid g\, so(V)\, g^{-1} = so(V) \}$$

<u>Claim:</u>
$$1 \to \{\pm 1\} \longrightarrow \mathbb{Q}^* \times O(V) \longrightarrow N(V) \to 1$$
$$e \longmapsto (e^n, e)$$
$$(\lambda, \varphi) \longmapsto \lambda\, \mathrm{Sym}^n(\varphi)$$

<u>Hard part:</u> Surjectivity! Use
$$1 \to \mathbb{Q}^* \to N(V) \longrightarrow \mathrm{Aut}(so(V))$$
$$1 \to SO(V) \longrightarrow \mathrm{Aut}(so(V)) \to \mathrm{Out}(so(V)) \to 1$$

<u>General case:</u> If (V_1,q_1), (V_2,q_2) quadratic spases s.t.

$S_{[n]}V_1 \underset{\substack{\uparrow \\ \text{isometry}}}{\cong} S_{[n]}V_2$ then $\exists$ isometry $(V_1,q_1) \cong (V_2, \lambda q_2)$, $\lambda \in \mathbb{Q}$.

<u>Hodge structures:</u>

<u>Thm</u> (Soldaten|cov) X, Y HK. Assume $\varphi: H^2(X,\mathbb{Q}) \cong H^2(Y,\mathbb{Q})$ Hodge isometry s.t. $\varphi = \psi|_{H^2(X,\mathbb{Q})}$, $\psi \in \mathrm{Aut}(H^*(M,\mathbb{Q}))$ algebra automorphism (view $X = (M,I)$, $Y = (M,I')$)

Then $\psi|_{H^i}: H^i(X,\mathbb{Q}) \cong H^i(Y,\mathbb{Q})$ isomorphism of $\mathbb{Q}$-HS.

<u>Pf:</u> $ad(\psi): g(X) \cong g(Y)$
$$h_X \longmapsto h_Y \quad \text{respects cohomological grading}$$
$$h'_X \longmapsto h'_Y \quad \text{respects Hodge structure on } H^2$$
$$h_p \longmapsto h_p$$
$$h_q \longmapsto h_q$$
$\Rightarrow \psi$ sends $H^{p,q}(X)$ to $H^{p,q}(Y)$. $\square$

$\Phi: D^b(X) \cong D^b(Y) \rightsquigarrow \Phi^g:$
$$\begin{array}{ccc} g(X) & \cong & g(Y) \\ \text{\rotatebox{90}{$\cong$}} & & \text{\rotatebox{90}{$\cong$}} \\ ad(\phi): so(\tilde{H}(X,\mathbb{Q})) & \longrightarrow & so(\tilde{H}(Y,\mathbb{Q})) \end{array}$$
$\phi: \tilde{H}(X,\mathbb{Q}) \cong \tilde{H}(Y,\mathbb{Q})$ Hodge similitude

With cancelation: $\exists\ \psi \in SO(\tilde{H}(Y,\mathbb{Q}))$ s.t. $\psi\circ\phi$ <u>graded</u> Hodge similitude, i.e.
$$ad(\psi\circ\phi)(h_X) = h_Y$$
$$ad(\psi\circ\phi)(h'_X) = h'_Y$$

LLV representation: $SO(\tilde{H}(X,\mathbb{Q})) \longrightarrow GL(H^{even}(X,\mathbb{Q}))$

LLV representation: $SO(H(X,\mathbb{Q})) \longrightarrow GL(H^\bullet(X,\mathbb{Q}))$

Need to lift $\psi \in SO(\tilde{H}(X,\mathbb{Q}))$ to $GSpin(\tilde{H}(X,\mathbb{Q}))$

$$1 \to \mathbb{Q}^* \to GSpin(\tilde{H}(X,\mathbb{Q})) \to SO(\tilde{H}(X,\mathbb{Q})) \to 1$$

$$\begin{array}{ccc} \exists\ GSpin(\tilde{H}(X,\mathbb{Q})) & \longrightarrow & GL(H^\bullet(X,\mathbb{Q})) \\ \cup\!| & \nearrow_{LLV} & \\ Spin(\tilde{H}(X,\mathbb{Q})) & & \end{array}$$

Upshot: $\exists\, \eta \in GL(H^\bullet(X,\mathbb{Q})) : \eta \circ \bar{\Phi}^H : H^\bullet(X,\mathbb{Q}) \xrightarrow{\sim} H^\bullet(Y,\mathbb{Q})$

$ad(\eta\circ\bar{\Phi}^H)(h_X) = h_Y$

$ad(\eta\circ\bar{\Phi}^H)(h'_X) = h'_Y$ $\square$

Thm (Verbitsky, Taelman)

$\mathfrak{g}'(X) = \mathfrak{g}(X) \otimes_{\mathbb{Q}} \mathbb{C}$

Pf:

Show inclusion $\mathfrak{g}'(X) = \mathfrak{g}(X)\otimes_{\mathbb{Q}} \mathbb{C}$

1) e_σ $\sigma \in H^0(X,\Omega^2_X)$ $e_{\check\sigma}$ $\check\sigma \in H^0(X,\Lambda^2 T_X)$

$\rightsquigarrow (e_\sigma, h_p, e_{\check\sigma})$ $\mathfrak{sl}_2$-triple

2) Analogously: $\bar\sigma \in H^2(X,\mathcal{O}_X)$ $e_{\bar\sigma}$ $\rightsquigarrow \exists\, e_{\check{\bar\sigma}} \in End(H^\bullet(X,\mathbb{Q}))$ s.t.

$(e_{\bar\sigma}, h_q, e_{\check{\bar\sigma}})$ $\mathfrak{sl}_2$-triple.

3) $\mathfrak{sl}_2$-triples from 1) & 2 commute

$\Rightarrow (e_\sigma + e_{\bar\sigma}, h, e_{\check\sigma} + e_{\check{\bar\sigma}})$ and $(e_\sigma - e_{\bar\sigma}, h, e_{\check\sigma} - e_{\check{\bar\sigma}})$ $\mathfrak{sl}_2$-triples

$\Rightarrow e_{\check\sigma} \in \mathfrak{g}(X)\otimes_{\mathbb{Q}}\mathbb{C} \overset{1)}{\Rightarrow} h_p, h_q \in \mathfrak{g}(X)\otimes_{\mathbb{Q}}\mathbb{C} \Rightarrow h' \in \mathfrak{g}(X)\otimes_{\mathbb{Q}}\mathbb{C}$

4) $\mu \in H^1(X,\Omega^1_X) \rightsquigarrow [e_{\check\sigma}, e_\mu] = e_\eta$ $\quad \eta = \mu \lrcorner \check\sigma \in H^1(X,T_X)$ $\square$

§7 Geometry of the extended Mukai lattice

$\psi : SH(X,\mathbb{Q}) \hookrightarrow Sym^n \tilde{H}(X,\mathbb{Q})$

T orthogonal split

Lem: $T\left(\frac{\alpha^{n-i}\beta^i}{(n-i)!}\right) = q_{2i}$

$\int_X q_{2i}\lambda^{2n-2i} = c_X \frac{(2n-2i)!}{2^{n-i}(n-i)!} b(\lambda,\lambda)^{n-i} \quad \forall \lambda \in H^2(X,\mathbb{Q})$

Ex: • $K3^{[2]}$: $H^\bullet(X,\mathbb{Q}) = SH^\bullet(X,\mathbb{Q})$:

$\frac{\alpha^2}{2} \longleftrightarrow [1]$

$\alpha\cdot\beta \longleftrightarrow q_2$ BBF form

$\beta^2 \longleftrightarrow [pt]$

$\lambda\cdot\alpha \longleftrightarrow \lambda \in H^2(X,\mathbb{Q})$

$\lambda\cdot\beta \longleftrightarrow q_2\lambda \in H^6(X,\mathbb{Q})$

• $K3^{[3]}$: $H^\bullet(X,\mathbb{Q}) = SH(X,\mathbb{Q}) \oplus \Lambda^2 \tilde{H}(X,\mathbb{Q})$

Monodromy invariant classes: $SH(X,\mathbb{Q})$: $\alpha^3,\ \alpha^2\beta,\ \alpha\beta^2,\ \beta^3$

$\Lambda^2\tilde{H}(X,\mathbb{Q})$: $\alpha\wedge\beta$

$$\begin{array}{cccc} & & \mathbb{C}\alpha & \\ H^2(X,\mathbb{C}) & \mathbb{C}\sigma & H^{1,1} & \mathbb{C}\bar\sigma \\ & & \mathbb{C}\beta & \end{array} \rightsquigarrow \begin{array}{ccc} & HT^2(X) & \\ & \mathbb{C}\check\sigma & \\ \mathbb{C}\sigma & H^1(X,T_X) & \mathbb{C}\varepsilon \\ & \mathbb{C}\bar\sigma & \end{array}$$

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polyvector fields