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Winter term 2015/16

# Exercises, Algebraic Geometry I – Week 10

**Exercise 54.** (2 points) *Restriction of sections on affine schemes.*
Let $X$ be the affine scheme $\mathrm{Spec}(A)$ and $\mathcal{F}$ a quasi-coherent sheaf on $X$. Prove the following assertions:

i) Assume $0 = s|_{D(a)} \in \Gamma(D(a), \mathcal{F})$ for some $s \in \Gamma(X, \mathcal{F})$ and $a \in A$. Then there exists $n > 0$ such that $a^n \cdot s = 0$ in $\Gamma(X, \mathcal{F})$.

ii) Let $t \in \Gamma(D(a), \mathcal{F})$. Then there exists $n > 0$ and $s \in \Gamma(X, \mathcal{F})$ such that $s|_{D(a)} = a^n \cdot t$ in $\Gamma(D(a), \mathcal{F})$.

**Exercise 55.** (3 points) *Tensor products of quasi-coherent sheaves.*
Let $A$ be a ring and $M, N$ be two $A$-modules. Show that $(M \otimes_A N)^\sim \cong \tilde{M} \otimes_{\mathcal{O}_X} \tilde{N}$ on $\mathrm{Spec}(A)$. Is there an analogous version in the graded case, i.e. for sheaves on $\mathrm{Proj}(B)$, assuming that $B$ is generated by $B_1$ as a $B_0$-algebra?

**Exercise 56.** (2 points) *Direct images of (quasi)-coherent sheaves.*
Find examples of a scheme morphism $f\colon X \to Y$ and a coherent sheaf $\mathcal{F}$ such that $f_*\mathcal{F}$ is not coherent. Is $f_*$ or $f^*$ compatible with tensor products?

**Exercise 57.** (4 points). *Veronese embedding.*
Let $B = \bigoplus_{i \geq 0} B_i$ be a graded ring generated by $B_1$ as a $B_0$-algebra. Then $B^{(d)}$ denotes the graded ring defined by $B^{(d)} := \bigoplus_{i \geq 0} B_i^{(d)}$ with $B_i^{(d)} := B_{di}$.

i) Show that there exists an isomorphism $\varphi\colon \mathrm{Proj}(B) \xrightarrow{\sim} \mathrm{Proj}(B^{(d)})$ with $\varphi^*\mathcal{O}(1) \cong \mathcal{O}(d)$.

ii) Consider the case $B = k[x_0, \ldots, x_n]$. Show that the surjection $k[y_0, \ldots, y_N] \twoheadrightarrow B^{(d)}$ mapping $y_i$ to the $i$-th monomial of degree $d$ in the variables $x_i$ defines a closed embedding
$$\mathbb{P}^n_k \cong \mathrm{Proj}(B) \cong \mathrm{Proj}(B^{(d)}) \hookrightarrow \mathbb{P}^N_k.$$
What is $N$? Show that for $k$ algebraically closed the morphism on the closed points is given by $[\lambda_0 : \ldots : \lambda_n] \mapsto [\lambda_0^d : \ldots : \lambda^I : \ldots : \lambda_n^d]$ with $\lambda^I$ running through all monomials of degree $d$.

The last exercise is not necessary for the understanding of the lectures at this point.

**Exercise 58.** (8 extra points) *Relative* **Spec**.
Work through Exercise II.5.17 in Hartshorne's book.

**Please turn over.**

Due Monday 11 January, 2016. Before the lecture.

# Reflex test[1]

1. Let $A = k[[t]]$ be the ring of formal power series over a field. Describe the topological space $\mathrm{Spec}(A)$.
2. Let $\mathcal{F}$ be a sheaf of sets on a topological space. What is $\mathcal{F}(\emptyset)$?
3. Let $B = \bigoplus_{i\geq 0} B_i$ be a graded algebra over a field $k = B_0$. If $B$ is finite-dimensional as a $k$-vector space, what is $\mathrm{Proj}(B)$?
4. For a graded ring $B$, is the nilradical a homogeneous ideal?
5. Is the scheme $\mathrm{Spec}(\bar{\mathbb{Q}} \otimes_{\mathbb{Q}} \bar{\mathbb{Q}})$ connected?
6. Let $X = \mathrm{Spec}(\mathbb{Z})$ and $p \in \mathbb{Z}$ be a prime number. What is the stalk $\mathcal{O}_{X,(p)}$ of $\mathcal{O}_X$ at $(p) \in X$? What is the stalk $\mathcal{O}_{X,(0)}$ at the generic point?
7. Let $X = \mathrm{Spec}(\mathbb{Z}[x])$. Describe the fibre product $X \times_{\mathrm{Spec}\,\mathbb{Z}} \mathrm{Spec}\,\mathbb{F}_p$.
8. Let $X = \mathrm{Spec}(k[x])$, $Y = \mathrm{Spec}(k[y])$ and $f : X \to Y$ be induced by the map $y \mapsto x^2$. Describe the fibre of $f$ over the closed point $(y)$.
9. Let $X = \mathrm{Spec}(k[x,y]/(xy))$ and $o \in X$ be the point corresponding to the maximal ideal $\mathfrak{m} = (x,y)$. What is the dimension of the tangent space $T_oX$ of $X$ at $o$?
10. Let $X = \mathrm{Spec}(k[x])$. Consider the two projections $p_1, p_2 \colon X \times X \to X$. Let $a \in X \times X$ be a point, such that $p_1(a) = p_2(a)$. Is it true that $a$ lies in the image of the diagonal morphism $\Delta\colon X \to X \times X$?
11. Let $X = \mathrm{Spec}(k[x,y]/(xy^2))$. Describe the reduction $X_{red}$. Describe the normalization of the reduction $\widetilde{X_{red}}$.
12. Let $k$ be a field, $X = \mathrm{Spec}(k[x])$ and $f : X \to \mathrm{Spec}(k)$ be the natural morphism. Describe $f_*\mathcal{O}_X$. Same question for $X = \mathrm{Proj}(k[x,y])$.
13. Consider the ideal $I = (x) \subset k[x]$ and the corresponding ideal sheaf $\widetilde{I}$ on $\mathrm{Spec}(k[x])$. Is $\widetilde{I}$ locally free? Same question for $I = (x,y) \subset k[x,y]$.
14. If for a graded algebra $B = \bigoplus_{i\geq 0} B_i$ over a field $k = B_0$ we have $\mathrm{Proj}(B) = \emptyset$, is it true that $B$ is a finite-dimensional $k$-vector space?
15. Let $X$ be the complement of a closed point in $\mathrm{Spec}(\mathbb{C}[x,y])$. Is $X$ affine?
16. Consider the ideal $\mathfrak{a} = (x,y) \subset \mathbb{C}[x,y]$ and the corresponding ideal sheaf $\tilde{\mathfrak{a}}$ on $X = \mathrm{Spec}(\mathbb{C}[x,y])$. What is the dimension $\dim(\widetilde{\mathfrak{a}}_p \otimes_{\mathcal{O}_{X,p}} \mathbb{C})$ for a closed point $p \in X$?
17. Let $G$ be an abelian group. What is the difference between the constant presheaf $\mathbf{G}$ and the constant sheaf $\underline{G}$ on the scheme $\mathrm{Spec}(k[x,y]/(x, y^2 - 1))$?
18. Find an example of a quasi-finite morphism of schemes that is not finite.
19. What is an example of a surjection of sheaves $\mathcal{F} \twoheadrightarrow \mathcal{G}$ for which $\Gamma(X, \mathcal{F}) \to \Gamma(X, \mathcal{G})$ is not surjective.
20. For which of the following constructions for sheaves the natural pre-sheaf needs to be sheafified: $\mathrm{Ker}(\varphi)$, $f_*\mathcal{F}$, $\mathrm{Coker}(\varphi)$, $\mathrm{Im}(\varphi)$, $\mathcal{F} \otimes \mathcal{G}$.
21. For a graded ring $B$ is the natural inclusion $\mathrm{Proj}(B) \hookrightarrow \mathrm{Spec}(B)$ a morphism of schemes?
22. Is $\mathcal{O}_X$ a flasque sheaf?
23. Is the projection $\mathbb{A}^2_k \to \mathbb{A}^1_k$ on the $x$-axis closed.
24. Let $X$ be a scheme of finite type over $k$. Are the local rings $\mathcal{O}_{X,x}$ finite type $k$-algebras?
25. What does adjunction say for $f_*$ and $f^*$?
26. Describe a sheaf on $\mathbb{A}^1_k$ that is not quasi-coherent.
27. Consider the inclusion $U \subset X$ of an open subscheme as a morphism of schemes and decide whether it is (in general) separated, proper, finite type, quasi-finite, finite, quasi-compact with irreducible, integral, connected, reduced, fibres. The same for a closed immersion $Z \subset X$.

# Happy holidays!

[1] These quick exercises are meant to test your reflexes. These are the things you should know instantaneously and without much thinking. If you do not get them right the first time, try again later. Do not hand in solutions for them.