Prof. Dr. Daniel Huybrechts
Dr. Andrey Soldatenkov

Winter term 2015/16

# Exercises, Algebraic Geometry I – Week 9

**Exercise 48.** (2 points) *Homogeneous prime ideals.*
Let $B$ be a graded ring and $a \in B_+$ a homogeneous element of positive degree. Show that with $\mathfrak{q} \in \mathrm{Spec}(B_{(a)})$ also $\sqrt{\mathfrak{q}B_a} \in \mathrm{Spec}(B_a)$.

**Exercise 49.** (3 points) *Ideal sheaf of the diagonal.*
Let $X$ be a separated $k$-scheme and $\mathcal{I} := \mathcal{I}_\Delta$ be the ideal sheaf of its diagonal $\Delta \subset X \times_k X$. Show that there is an exact sequence

$$0 \to \mathcal{I}/\mathcal{I}^2 \to \mathcal{O}_{X\times_k X}/\mathcal{I}^2 \to \mathcal{O}_\Delta \to 0.$$

Prove that $\mathcal{I}/\mathcal{I}^2$ can be naturally viewed as the direct image under the diagonal morphism $\Delta\colon X \to X \times_k X$ of a sheaf, called the *cotangent sheaf* $\Omega_{X/k}$, on $X$. Prove that $\Omega_{X/k}$ is (locally) free for $X = \mathbb{A}^n_k$.

**Exercise 50.** (4 points) *Products of* Proj.
Let $B = \bigoplus_{d\geq 0} B_d$ and $C = \bigoplus_{d\geq 0} C_d$ be two graded rings with $A := B_0 \cong C_0$. Assume that $B$ and $C$ are generated by finite number of elements of degree one as $A$-algebras. Consider $B \times_A C := \bigoplus_{d\geq 0} B_d \otimes_A C_d$ and the schemes $X := \mathrm{Proj}(B)$ and $Y := \mathrm{Proj}(C)$.

i) Show that $X \times_{\mathrm{Spec}(A)} Y \cong \mathrm{Proj}(B \times_A C)$ and that

ii) under this isomorphism $\mathcal{O}(1)$ on $\mathrm{Proj}(B \times_A C)$ is isomorphic to $p_1^*\mathcal{O}_X(1) \otimes p_2^*\mathcal{O}_Y(1)$, where $p_1$ and $p_2$ are the two projections from $X \times_{\mathrm{Spec}(A)} Y$.

**Exercise 51.** (4 points) *Examples of* Proj.
Describe the following schemes:
$\mathrm{Proj}(\mathbb{Z}[X])$; $\mathbb{P}^1_k$ for $k = \bar{k}$; $\mathbb{P}^2_k \setminus D_+(x^2 + y^2 - z^2)$ (you may assume that $\mathrm{char}(k) \neq 2$); $\mathrm{Proj}(k[x,y]/(x^2, y^2))$.

**Exercise 52.** (2 points) *Noetherian graded rings.*
Show that a graded ring $B = \bigoplus_{d\geq 0} B_d$ is Noetherian if and only if $B_0$ is Noetherian and $B_+$ is a finitely generated ideal.

**Exercise 53.** (3 points) *The irrelevant ideal.*
Let $B = \bigoplus_{d\geq 0} B_d$ be a graded ring and $\mathfrak{a} \subset B$ a homogeneous ideal. Show that the following conditions are equivalent.

i) $V_+(\mathfrak{a}) = \emptyset$.

ii) $B_+ \subset \sqrt{\mathfrak{a}}$.

iii) If $\mathfrak{a} = (a_i)$ with $a_i$ homogeneous, then $\bigcup D_+(a_i) = \mathrm{Proj}(B)$.

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Due Monday 21 December, 2015. Before the lecture.