Prof. Dr. Daniel Huybrechts
Dr. Andrey Soldatenkov Winter term 2015/16

Exercises, Algebraic Geometry I – Week 8

Exercise 42. (2 points) *Adjoint functors f^* and f_*.*
Consider a morphism of ringed spaces $f\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$. Show that f^* is left adjoint to f_*, i.e. for all $\mathcal{F} \in \mathrm{Mod}(X, \mathcal{O}_X)$ and $\mathcal{G} \in \mathrm{Mod}(Y, \mathcal{O}_Y)$ there exists an isomorphism (functorial in $\mathcal{F}$ and $\mathcal{G}$):

$$\mathrm{Hom}_{\mathcal{O}_X}(f^*\mathcal{G}, \mathcal{F}) \cong \mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{G}, f_*\mathcal{F}).$$

Exercise 43. (3 points) *$M \mapsto \tilde{M}$ and adjunction.*
Let X be an affine scheme $\mathrm{Spec}(A)$ and consider an A-module M and a sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules. Show that (A-mod) $\to \mathrm{Mod}(X, \mathcal{O}_X)$, $M \mapsto \tilde{M}$ is left adjoint to $\mathrm{Mod}(X, \mathcal{O}_X) \to$ (A-mod), $\mathcal{F} \mapsto \Gamma(X, \mathcal{F})$, i.e. that there exist functorial (in M and $\mathcal{F}$) isomorphisms

$$\mathrm{Hom}_A(M, \Gamma(X, \mathcal{F})) \cong \mathrm{Hom}_{\mathcal{O}_X}(\tilde{M}, \mathcal{F}).$$

Exercise 44. (3 points) *f^* and $\otimes$ are only left exact.*
Let $(X, \mathcal{O}_X)$ be a ringed space and consider $\mathcal{F}, \mathcal{G} \in \mathrm{Mod}(X, \mathcal{O}_X)$. Show that for all $x \in X$ there exists a natural isomorphism

$$(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G})_x \cong \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{G}_x.$$

Prove that $\mathcal{F} \otimes_{\mathcal{O}_X} (\ )\colon \mathrm{Mod}(X, \mathcal{O}_X) \to \mathrm{Mod}(X, \mathcal{O}_X)$ and $f^*\colon \mathrm{Mod}(Y, \mathcal{O}_Y) \to \mathrm{Mod}(X, \mathcal{O}_X)$ for a morphism of ringed spaces $f\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ are both right exact functors. Describe examples showing that in general they are not left exact.

Exercise 45. (3 points) *Projection formula.*
Consider a morphism of ringed spaces $f\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ and $\mathcal{F} \in \mathrm{Mod}(X, \mathcal{O}_X)$ and $\mathcal{G} \in \mathrm{Mod}(Y, \mathcal{O}_Y)$. Suppose $\mathcal{G}$ is locally free of finite rank. Show that there exists a natural isomorphism

$$f_*(\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{G}) \cong f_*\mathcal{F} \otimes_{\mathcal{O}_Y} \mathcal{G}.$$

Exercise 46. (5 points) *Fibre dimension.*
Let X be a Noetherian scheme and let $\mathcal{F}$ be a coherent sheaf on X. We will consider the function

$$\varphi(x) := \dim_{k(x)} \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} k(x),$$

where $k(x) = \mathcal{O}_{X,x}/\mathfrak{m}_x$ is the residue field of the point $x \in X$. Use Nakayama lemma to prove the following statements.

i) The function φ is upper semi-continuous, i.e. for any $n \in \mathbb{Z}$ the set $\{x \in X \mid \varphi(x) \geq n\}$ is closed.

ii) If $\mathcal{F}$ is locally free and X is connected, then φ is a constant function.

iii) Conversely, if X is reduced and φ is constant, then $\mathcal{F}$ is locally free.

Exercise 47. (2 points) *Proper affine varieties.*
Let A be a finite type k-algebra. Assume that the morphism $f : \mathrm{Spec}(A) \to \mathrm{Spec}(k)$ is proper. Prove that f is finite (equivalently, that $\dim_k(A) < \infty$).

Due Monday 14 December, 2015. Before(!!) the lecture.