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Winter term 2015/16

Exercises, Algebraic Geometry I – Week 7

Exercise 37. (4 points) *Properties of morphisms.*
Verify the following assertions.

i) Show that a morphism $f\colon X \to Y$ of schemes which is surjective, of finite type, and quasi-finite, need not be finite.

ii) Show that 'quasi-finite' and 'injective' are not preserved under base change.

iii) Show that 'being an open/closed immersion' are preserved under base change.

iv) Show that 'having reduced/integral/connected fibres' is not preserved under base change.

Exercise 38. (6 points) *Proper and separated morphisms.*
Decide which of the following morphisms are separated and which are proper.

i) $\mathbb{A}^n_k \to \mathrm{Spec}(k)$; ii) $\mathrm{Spec}(\mathbb{Q}) \to \mathrm{Spec}(\mathbb{Z})$, iii) $V(xy-1) \subset \mathbb{A}^2_k \to \mathbb{A}^1_k$.

iv) Let X, Y be the schemes obtained by glueing two copies of $\mathbb{A}^1$ over the the open set $D(t)$ via $k[t, t^{-1}] \to k[t, t^{-1}]$, $t \mapsto t$, resp. $t \mapsto t^{-1}$. Consider the natural morphisms $X, Y \to \mathrm{Spec}(k)$ and $\mathbb{A}^1_k \to X, Y$.

Exercise 39. (2 points) *Surjective morphisms and base change.*
Let $X \to S$ be a surjective morphism of schemes. Given another S-scheme $Y \to S$, is it true that $X \times_S Y \to Y$ is also surjective? Prove this or give a counterexample.

Exercise 40. (2 points) *Intersections of affine open subschemes.*
Let $U, V \subset X$ be two open subschemes. Show that the intersection $U \cap V$ need not be affine. Prove, however, that this is true for separated schemes X (i.e. for which $X \to \mathrm{Spec}(\mathbb{Z})$ is separated). For the latter you first need to show that $\Delta \cap (U \times_{\mathbb{Z}} V) \cong U \cap V$, where $\Delta \subset X \times_{\mathbb{Z}} X$ is the diagonal.

Exercise 41. (4 points) *The image of a proper scheme is proper.*
Let $f\colon X \to Y$ be a morphism of S-schemes. Suppose that $Y \to S$ is separated.

i) Show that the graph $\Gamma_f\colon X \to X \times_S Y$ is a closed immersion.

ii) Let $Z \subset X$ be a closed subscheme that is proper over S. Show that $f(Z) \subset Y$ is closed.

Due Monday 7 December, 2015. Before(!!) the lecture.