Prof. Dr. Daniel Huybrechts
Dr. Andrey Soldatenkov Winter term 2015/16

Exercises, Algebraic Geometry I – Week 6

Exercise 31. (4 points) *Integral and irreducible fibres.*
Find examples for the following phenomena:

i) Show that there exist morphisms $X \to Y$ with Y integral and such that all fibres X_y are irreducible without X being irreducible.

ii) Show that there exist morphisms $X \to \mathrm{Spec}(\mathbb{C}[x])$ with X integral, the generic fibre X_η non-empty and integral but no closed fibre integral.

iii) Show that there exist morphisms $X \to \mathrm{Spec}(\mathbb{Q}[x])$ with X integral and infinitely many irreducible and infinitely many reducible closed fibres. What happens for the geometric closed fibres in your example?

Exercise 32. (2 points) *Fibres.*
Consider the subscheme $X \subset \mathbb{A}^2_{\mathbb{Z}} := \mathrm{Spec}(\mathbb{Z}[x, y])$ given by $xy^2 - m$, for some $m \in \mathbb{Z}$. Study the fibres of $X \to \mathrm{Spec}(\mathbb{Z})$. Which ones are irreducible?

Exercise 33. (6 points) *Affine schemes under base field extensions.*
Let k be an algebraically closed field, $k_0 \subset k$ a subfield, and X_0 a scheme over k_0. In this exercise we study the relation between X_0 and $X := X_0 \times_{k_0} k$ in examples.

i) Let $k = \mathbb{C}$ and $k_0 = \mathbb{R}$. Consider $X_0 = \mathrm{Spec}(k_0[x, y]/(y^2 - x(x^2 - 1)))$. Which residue fields are possible for points in X_0 and how does the set $X_0(\mathbb{R})$ of $\mathbb{R}$-rational points look like? The Galois group $G := \mathrm{Gal}(k/k_0) \cong \mathbb{Z}/2\mathbb{Z}$ acts on X and $X(\mathbb{C})$. Study the fibres of $X \to X_0$ in terms of G-orbits.

ii) Let again $k = \mathbb{C}$ and $k_0 = \mathbb{R}$. Let $X_0 = \mathrm{Spec}(k_0[x, y]/(x^2 + y^2))$. Check that X_0 is irreducible while X is not. Describe X geometrically.

iii) Let k_0 be an imperfect field of characteristic $p > 0$ and let $a \in k_0 \setminus k_0^p$. Let ℓ be a non-trivial homogeneous linear polynomial in x, y and $X_0 = \mathrm{Spec}(k_0[x, y]/(\ell^p - a))$. Prove that X_0 is reduced. Prove further that $X \cong \mathrm{Spec}(k[x, y]/((\ell - a^{1/p})^p))$ and thus not reduced.

Exercise 34. (4 points) *Points under base change.*
Consider the natural morphism $\mathbb{A}^2_{\bar{\mathbb{Q}}} \to \mathbb{A}^2_{\mathbb{Q}}$ and determine the images of the following points: i) $(x - \sqrt{2}, y - \sqrt{2})$; ii) $(x - \sqrt{2}, y - \sqrt{3})$; iii) $(\sqrt{2}x - \sqrt{3}y)$.

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Due Monday 30 November, 2015. Before(!!) the lecture.

Exercise 35. (3 points) *Morphisms into separated schemes.*
Consider schemes X and Y over a base scheme S. Assume that X is reduced (or even stronger integral) and that $Y \to S$ is separated. Show that two morphisms $f, g : X \to Y$ over S that coincide on a dense open subset $U \subset X$ are actually equal. (*Hint*: Consider the composition of the graph $X \to X \times_S Y$ of f with $(g, \mathrm{id}) : X \times_S Y \to Y \times_S Y$) Give counterexamples if one of the hypotheses is dropped.

The last exercise is not necessary for the understanding of the lectures at this point.

Exercise 36. (6 extra points) *Functors of points.*
Consider the functor of points $h_X \colon (\mathit{Sch}/S)^{\mathrm{op}} \to (\mathit{Sets})$ that for a fixed S-scheme $X \to S$ maps any S-scheme $Y \to S$ to the set of morphisms $\mathrm{Mor}_S(Y, X)$ of S-schemes $Y \to X$.

i) Define the notion of fibre product functor $h_X \times h_Y \colon (\mathit{Sch}/S)^{\mathrm{op}} \to (\mathit{Sets})$ and show that this functor is isomorphic to $h_{X \times_S Y}$, i.e.

$$h_{X \times_S Y} = h_X \times h_Y.$$

Compare this to the fact that the underlying topological space $|X \times_S Y|$ of $X \times_S Y$ is usually different from $|X| \times_{|S|} |Y|$ (example!).

ii) For $S = \mathrm{Spec}(k)$ and a field extension K/k, show that $h_X(\mathrm{Spec}(K)) = X(K)$.

iii) Find examples of morphisms $X \to Y$ (of S-schemes) which are not determined by their underlying continuous maps. Observe, however, that $X \to Y$ is determined by $h_X \to h_Y$.