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Exercises, Algebraic Geometry I – Week 5

Exercise 25. (2 points) *Schemes are T_0-spaces.*
Let X be a scheme. Prove the following assertions.

i) If X is irreducible and consists of at least two points, then X is not Hausdorff.

ii) Show that X is a T_0-space, i.e. for any two distinct points $x, y \in X$ there exists an open subset $U \subset X$ containing exactly one of the two points.

Exercise 26. (3 points) *Morphisms to affine schemes.*
Let $(f, f^\sharp) : X \to \mathrm{Spec}(A)$ be a morphism of schemes. Taking global sections of $f^\sharp\colon \mathcal{O}_{\mathrm{Spec}(A)} \to f_*\mathcal{O}_X$ yields a homomorphism of rings $A \to \Gamma(X, \mathcal{O}_X)$. Show that this defines a bijection

$$\mathrm{Hom}_{(Sch)}(X, \mathrm{Spec}(A)) \to \mathrm{Hom}_{(Rings)}(A, \Gamma(X, \mathcal{O}_X)).$$

Exercise 27. (4 points) *Normalization.*
A scheme X is *normal* if all its local rings $\mathcal{O}_{X,x}$ are integrally closed domains.

Let X be an arbitrary integral scheme and $\eta \in X$ its generic point. For any open $\mathrm{Spec}(A) \subset X$ consider the integral closure $A \subset \tilde{A} \subset Q(A) = k(\eta) = K(X)$ and the associated affine scheme $\widetilde{U} := \mathrm{Spec}(\widetilde{A})$.

i) Show that the schemes $\widetilde{U}$ can be glued to a scheme $\widetilde{X}$ (the *normalization* of X) that comes with a natural morphism $\nu : \widetilde{X} \to X$ extending $\widetilde{U} \to U$. The normalization $\widetilde{X}$ is normal.

ii) Prove the following universal property: Every dominant morphism $Z \to X$ from a normal integral scheme Z factors uniquely through $\nu : \widetilde{X} \to X$.

Exercise 28. (4 points) *Reduced schemes and reduction of schemes.*
Let X be a scheme. Prove the following assertions:

i) Show that X is reduced if and only if all local rings $\mathcal{O}_{X,x}$ are reduced.

ii) Construct the *reduction* X_{red} of X. Its topological space is the same as that of X and its structure sheaf is given by $\mathcal{O}_{X_{\mathrm{red}}} = \mathcal{O}_X/\mathfrak{N}$, where $\mathfrak{N}$ is the subsheaf of nilpotent elements in $\mathcal{O}_X$. More precisely, if we denote by $N(A)$ the nilradical of a ring A, then $\mathfrak{N}(U) = \{s \in \mathcal{O}_X(U)\colon s_x \in N(\mathcal{O}_{X,x}), \forall x \in U\}$ for any open subset $U \subset X$. Show that this defines a scheme and a natural morphism of schemes $X_{\mathrm{red}} \to X$ which is a homeomorphism of topological spaces.

iii) Show that X_{red} has the following universal property: If $Y \to X$ is a morphism of schemes with Y reduced, then it factors uniquely over $X_{\mathrm{red}} \to X$.

Due Monday 23 November, 2015. Before(!!) the lecture.

Exercise 29. (3 points) *Distinguished open sets.*
Recall that the open sets $D(f) \subset \mathrm{Spec}(A)$ of all prime ideals not containing $f \in A$ form a basis of the topology. Define similar sets X_f for any scheme X and any $f \in \Gamma(X, \mathcal{O}_X)$. More precisely, let $X_f \subset X$ be the set of points $x \in X$ such that the stalk $f_x \in \mathcal{O}_{X,x}$ is not contained in the maximal ideal $\mathfrak{m}_x \subset \mathcal{O}_{X,x}$ (or, equivalently, that the image of f in the residue field $k(x)$ is non-trivial). Prove that X_f is an open subset.
(*Warning*: For general schemes X the ring $\Gamma(X, \mathcal{O}_X)$ is too small for the sets X_f to form a basis of the topology.)

The last exercise is not necessary for the understanding of the lectures at this point.

Exercise 30. (6 extra points) *Injective resolutions of abelian groups and modules.*
Let A be a ring and I be an A-module.

i) Show that I is injective if for any ideal $\mathfrak{a} \subset A$ the induced map

$$\mathrm{Hom}_A(A, I) \to \mathrm{Hom}_A(\mathfrak{a}, I)$$

is surjective.

ii) Show that any divisible abelian group G (i.e. $g \mapsto ng$ is surjective for all $n > 0$) is an injective object in (Ab). In particular, $\mathbb{Q}$ and $\mathbb{Q}/\mathbb{Z}$ are injective.

iii) Show that $I(G) = \prod_{J(G)} \mathbb{Q}/\mathbb{Z}$ is a divisible (and hence injective) group. Here, the index set $J(G)$ is the set $\mathrm{Hom}_{(Ab)}(G, \mathbb{Q}/\mathbb{Z})$.

iv) Show that the natural map $G \to I(G)$, $g \mapsto (f(g))_f$ is injective. (Pick for $g \in G$ a non-trivial homomorphism $\langle g \rangle \to \mathbb{Q}/\mathbb{Z}$ and use the injectivity of $\mathbb{Q}/\mathbb{Z}$ to extend it to a homomorphism $G \to \mathbb{Q}/\mathbb{Z}$.) Conclude that the category (Ab) of abelian groups has enough injectives.

v) Check that the same argument shows that the category of A-modules has enough injectives for any ring.