Prof. Dr. Daniel Huybrechts
Dr. Andrey Soldatenkov

Winter term 2015/16

Exercises, Algebraic Geometry I – Week 4

Exercise 18. (2 points) *Universality of δ-functors.*
Let $(H^i)\colon \mathcal{A} \to \mathcal{B}$ be a δ-functor. Show that if H^1 is erasable and $(\tilde{H}^i)\colon \mathcal{A} \to \mathcal{B}$ is another δ-functor with a natural transformation $H^0 \to \tilde{H}^0$, then for any object $A \in \mathcal{A}$ there exists a natural map $H^1(A) \to \tilde{H}^1(A)$. (This is the key step towards the proof of Grothendieck's theorem that δ-functors with erasable H^i, $i > 0$, are universal.)

Exercise 19. (2 points) *Flasque resolutions.*
Let $\mathcal{F}$ be a sheaf on X and consider $\mathcal{F}_0\colon U \mapsto \{s\colon U \to \coprod \mathcal{F}_x \mid s(x) \in \mathcal{F}_x\}$. Show that $\mathcal{F}_0$ is a flasque sheaf and deduce from this that every sheaf $\mathcal{F}$ admits a flasque resolution $0 \to \mathcal{F} \to \mathcal{F}_0 \to \mathcal{F}_1 \to \ldots$.

Exercise 20. (4 points) *Cohomology of the circle.*
Let S^1 be the circle with the usual topology. Let $\mathcal{C}$ be the sheaf of continuous real functions on S^1. Prove that

$$H^1(S^1, \underline{\mathbb{Z}}) = \mathbb{Z} \text{ and } H^1(S^1, \mathcal{C}) = 0.$$

Exercise 21. (2 points) *Morphism of locally ringed spaces.*
Let A be a local integral domain which is not a field. The natural inclusion $A \hookrightarrow Q(A)$ is not local but $(\mathrm{Spec}(Q(A)), \mathcal{O}) \to (\mathrm{Spec}(A), \mathcal{O})$ is a morphism of locally ringed spaces. Is this a contradiction?

Exercise 22. (2 points) *Direct image under point inclusion.*
Let $x \in X$ be an arbitrary point of a topological space. Is the direct image $i_{x*}\colon \mathrm{Sh}(\{x\}) \to \mathrm{Sh}(X)$ associated with the inclusion $i_x\colon \{x\} \to X$ exact?

Exercise 23. (3 points) *Rational points.*
Let $(X, \mathcal{O}_X)$ be an affine scheme and let $x \in X$ with residue field $k(x) \coloneqq \mathcal{O}_{X,x}/\mathfrak{m}_x$.

i) Show that for a field K to give a morphism of affine schemes $(\mathrm{Spec}(K), \mathcal{O}) \to (X, \mathcal{O}_X)$ with image x is equivalent to give a field inclusion $k(x) \hookrightarrow K$.

ii) If $(X, \mathcal{O}_X)$ is an affine k-scheme for some field k, i.e. a morphism of schemes

$$(X, \mathcal{O}_X) \to (\mathrm{Spec}(k), \mathcal{O})$$

is fixed, show that every residue field $k(x)$ is naturally a field extension $k \subset k(x)$. A point $x \in X$ is *rational* if this extension is bijective, i.e. $k = k(x)$. The *set of rational points* is denoted by $X(k)$. Show using i) that this set can also be described as the set of k-morphisms $(\mathrm{Spec}(k), \mathcal{O}) \to (X, \mathcal{O}_X)$, i.e. morphisms such that the composition

$$(\mathrm{Spec}(k), \mathcal{O}) \to (X, \mathcal{O}_X) \to (\mathrm{Spec}(k), \mathcal{O})$$

is the identity.

Please turn over.

Due Monday 16 November, 2015. Before(!!) the lecture.

Exercise 24. (3 points) *Zariski tangent space.*
Let $(X, \mathcal{O}_X)$ be an affine scheme. For a point $x \in X$ the quotient $\mathfrak{m}_x/\mathfrak{m}_x^2$ is considered as a vector space over the residue field $k(x) = \mathcal{O}_{X,x}/\mathfrak{m}_x$. The Zariski tangent space T_x of X at $x \in X$ is defined as the dual of this vector space, i.e.

$$T_x = (\mathfrak{m}_x/\mathfrak{m}_x^2)^* = \mathrm{Hom}_{k(x)}(\mathfrak{m}_x/\mathfrak{m}_x^2, k(x)).$$

Assume $(X, \mathcal{O}_X)$ is an affine k-scheme (see previous exercise) and denote the *ring of dual numbers* $k[T]/(T^2)$ by $k[\varepsilon]$.

Show that to give a morphism $(\mathrm{Spec}(k[\varepsilon]), \mathcal{O}_{\mathrm{Spec}(k[\varepsilon])}) \to (X, \mathcal{O}_X)$ that commutes with the morphisms to $(\mathrm{Spec}(k), \mathcal{O})$ is equivalent to give a rational point $x \in X$ (see previous exercise) and an element $v \in T_x$.