

## Exercises, Algebraic Geometry I – Week 2

Do not be seduced by the lotus-eaters into infatuation with untethered abstraction.  
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**Exercise 6.** (4 points) *Sheafification*

Describe examples of presheaves (of abelian groups)  $\mathcal{F}$  for which the sheafification  $\mathcal{F} \rightarrow \mathcal{F}^+$  is not injective resp. not surjective. Find an example with  $\mathcal{F} \neq 0$  but  $\mathcal{F}^+ = 0$ .

**Exercise 7.** (3 points) *Support of a section.*

Let  $\mathcal{F}$  be a sheaf on a topological space  $X$  and let  $s, t \in \mathcal{F}(U)$  be two sections over an open set  $U \subset X$ . Show that the set of points  $x \in U$  with  $s_x = t_x$  in  $\mathcal{F}_x$  is an open subset of  $U$ .

If  $\mathcal{F}$  is a sheaf of abelian groups, one defines the *support*  $\text{supp}(s)$  of a section  $s \in \mathcal{F}(U)$  as the set of points  $x \in U$  such that  $0 \neq s_x \in \mathcal{F}_x$ . Show that  $\text{supp}(s)$  is a closed subset of  $U$ .

**Exercise 8.** (4 points) *'Espace étale' of a presheaf.*

Let  $\mathcal{F}$  be a presheaf on a topological space  $X$ . Define

$$|\mathcal{F}| := \bigsqcup_{x \in X} \mathcal{F}_x,$$

which comes with a natural projection  $\pi : |\mathcal{F}| \rightarrow X$ ,  $(s \in \mathcal{F}_x) \mapsto x$ . Then any  $s \in \mathcal{F}(U)$  defines a section of  $\pi$  over  $U$  by  $x \mapsto s_x$ . One endows  $|\mathcal{F}|$  with the strongest topology such that all  $s \in \mathcal{F}(U)$  define continuous sections  $x \mapsto s_x$ . Show that the sheafification  $\mathcal{F}^+$  can be described as the sheaf of continuous sections of  $|\mathcal{F}| \rightarrow X$ .

**Exercise 9.** (3 points) *Subsheaf with support.*

Let  $Z \subset X$  be a closed subset. For any sheaf  $\mathcal{F}$  of abelian groups on  $X$  one defines for any open  $U \subset X$  the subgroup  $\Gamma_{Z \cap U}(U, \mathcal{F})$  of all sections  $s \in \Gamma(U, \mathcal{F})$  with  $\text{supp}(s) \subset Z \cap U$ . Show that this defines a sheaf (denoted  $\mathcal{H}_Z^0(\mathcal{F})$ ).

(A sheaf  $\mathcal{F}$  is said to be *supported on*  $Z$  if  $\mathcal{H}_Z^0(\mathcal{F}) = \mathcal{F}$ .)

**Exercise 10.** (4 points) *Exponential map.*

Consider  $X = \mathbb{C} \setminus \{0\}$  with its usual topology and let  $\mathcal{O}_X$  be the sheaf of holomorphic functions, i.e.  $\mathcal{O}_X(U) = \{f : U \rightarrow \mathbb{C} \mid \text{holomorphic}\}$ . Similarly, let  $\mathcal{O}_X^*$  be the sheaf of holomorphic functions without zeroes. (Throughout, you may work with differentiable function instead of holomorphic ones if you prefer.)

Show that the exponential map defines a morphism of sheaves (of abelian groups)

$$\exp : \mathcal{O}_X \rightarrow \mathcal{O}_X^*, f \in \mathcal{O}_X(U) \mapsto \exp(f) \in \mathcal{O}_X^*(U).$$

Find a basis of the topology such that  $\exp_U : \mathcal{O}_X(U) \rightarrow \mathcal{O}_X^*(U)$  is surjective for all  $U$  in this basis. Note that  $\mathcal{O}_X(X) \rightarrow \mathcal{O}_X^*(X)$  is not surjective. Describe the kernel of  $\exp_U$ .

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The last exercise is not necessary for the understanding of the lectures at this point.

**Exercise 11.** (4 extra points) *Grothendieck topology.*

The notion of a (pre)sheaf on a topological space can be formalized as follows.

A Grothendieck topology  $(\mathcal{C}, \text{Cov}_{\mathcal{C}})$  consists of a category  $\mathcal{C}$  with a set  $\text{Cov}_{\mathcal{C}}$  of collections  $\{\pi_i : U_i \rightarrow U\}_i$  of morphisms in  $\mathcal{C}$  (called *coverings* of  $U$ ) subject to the following conditions:

- (1) Any isomorphism  $\varphi : U \xrightarrow{\sim} U$  defines a covering  $\{U \rightarrow U\} \in \text{Cov}_{\mathcal{C}}$ .
- (2) Suppose we are given  $\{\pi_i : U_i \rightarrow U\}_i \in \text{Cov}_{\mathcal{C}}$  and for each  $i$  a covering  $\{\pi_{ij} : U_{ij} \rightarrow U_i\}_j \in \text{Cov}_{\mathcal{C}}$ . Then  $\{\pi_i \circ \pi_{ij} : U_{ij} \rightarrow U\}_{ij} \in \text{Cov}_{\mathcal{C}}$  is a covering.
- (3) If  $\{\pi_i : U_i \rightarrow U\}_i$  is a covering and  $V \rightarrow U$  is a morphism in  $\mathcal{C}$ , then  $\{\tilde{\pi}_i : U_i \times_U V \rightarrow V\}_i$  is a covering.

(In particular, one assumes that the fibre products in (2) and (3) exist. Recall the abstract notion of a fibre product.)

(i) Show that for a topological space  $X$  the category of open sets  $\text{Ouv}_X$  comes with a natural Grothendieck topology given by the usual open coverings  $U = \bigcup U_i$ . Show that the notions presheaf, sheaf, stalk, morphism of (pre)sheaves, etc., can be phrased entirely in terms of this Grothendieck topology.

Here is another example of a Grothendieck topology: For a finite group  $G$  consider the category  $G\text{-Sets}$  of sets  $S$  with a left  $G$ -action  $G \times S \rightarrow S$ . Morphisms in this category are maps that commute with the  $G$ -action.

(ii) Show that the collections of  $\{S_i \rightarrow S\}_i$  with  $\bigcup S_i \rightarrow S$  surjective define a Grothendieck topology on  $G\text{-Sets}$ .

The group  $G$  itself comes with a natural left  $G$ -action (by multiplication). The corresponding object is denoted  $\langle G \rangle \in G\text{-Sets}$ .

(iii) Show that any sheaf  $\mathcal{F}$  on  $G\text{-Sets}$  yields a set  $\mathcal{F}(\langle G \rangle)$  that is endowed with a natural left  $G$ -action. (In fact,  $\mathcal{F}$  is determined by this set, as  $\mathcal{F}(S) = \text{Hom}_G(S, \mathcal{F}(\langle G \rangle))$ .)

The final object in  $G\text{-Sets}$  consists of a set  $\{*\}$  of one element.

(iv) One can show that for a sheaf  $\mathcal{F}$  the space of sections  $\mathcal{F}(\{*\})$  is the fixed point set  $\mathcal{F}(\langle G \rangle)^G$ .