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Winter term 2015/16

Exercises, Algebraic Geometry I – Week 13

Exercise 70. (2 points) *Codimension.*
Let $Y \subset X$ be an integral subscheme and $\eta_Y \in Y$ be its generic point. Show that

$$\dim \mathcal{O}_{X,\eta_Y} = \operatorname{codim}(Y).$$

Exercise 71. (4 points) *Ample invertible sheaves.*
Let X be a noetherian scheme.

i) Show that if $\mathcal{L}$ and $\mathcal{M}$ are two invertible sheaves on X such that $\mathcal{L}$ is ample, then $\mathcal{L}^n \otimes \mathcal{M}$ is ample for $n \gg 0$. Conclude that any invertible sheaf $\mathcal{M}$ is isomorphic to some $\mathcal{L}_1 \otimes \mathcal{L}_2^*$ with $\mathcal{L}_1$ and $\mathcal{L}_2$ ample if there exists an ample invertible sheaf at all.

ii) Is the tensor product $\mathcal{L}_1 \otimes \mathcal{L}_2$ of two ample (resp. very ample) invertible sheaves again ample (resp. very ample)?

Exercise 72. (4 points) *Ample invertible sheaves on the quadric.*
Consider the quadric $Q = \mathbb{P}^1_k \times \mathbb{P}^1_k$ and use that $\operatorname{Pic}(Q) \cong \mathbb{Z} \oplus \mathbb{Z}$, i.e. every invertible sheaf on Q is isomorphic to a unique $\mathcal{O}(a,b) := p_1^*\mathcal{O}(a) \otimes p_2^*\mathcal{O}(b)$.

i) Determine all ample invertible sheaves on Q. Are they all very ample?

ii) Compute the cohomology groups $H^1(Q, \mathcal{O}(a,b))$.

Exercise 73. (4 points). *Trivial and torsion invertible sheaves.*
Let X be an integral projective scheme over an algebraically closed field k.

i) Assume $H^0(X,\mathcal{L}) \neq 0$ and $H^0(X,\mathcal{L}^*) \neq 0$ for some invertible sheaf $\mathcal{L}$. Show that then $\mathcal{L} \cong \mathcal{O}_X$.

ii) Let $\mathcal{L} \in \operatorname{Pic}(X)$ be of order n. Show $H^0(X,\mathcal{L}^m) = k$ for $n|m$ and $= 0$ otherwise.

Exercise 74. (6 points) *Base locus.*
Let X be a projective integral scheme over $k = \bar{k}$ and $\mathcal{L}$ an invertible sheaf on X. Let V be a subspace in $H^0(X,\mathcal{L})$. A point $x \in X$ is a base point of the linear system $\mathbb{P}(V) \subset |\mathcal{L}|$ if $s_x \in \mathfrak{m}_x\mathcal{L}$ for all $s \in V$. Thus, $\mathcal{L}$ is globally generated if and only if $|\mathcal{L}|$ has no base points.

i) Prove that the base locus $\operatorname{Bs} \subset X$, i.e. the set of all base points, is closed.

ii) Assume in addition that X is locally factorial. Show that for any $\mathcal{L}$ there exists an effective divisor D such that the base locus of the complete linear system given by $\mathcal{L}(-D) := \mathcal{L} \otimes \mathcal{O}(-D)$ is of codimension ≥ 2.

Due Monday 1 February, 2016.
This is the last exercise sheet for which you can get points.
Exams:
23.02.2016, 13.00 - 15.00, Großer Hörsaal, Wegelerstr. 10;
31.03.2016, 9.00 - 11.00, Kleiner Hörsaal, Wegelerstr. 10.