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Winter term 2015/16

Exercises, Algebraic Geometry I – Week 12

Exercise 64. (6 points) *Extension of coherent sheaves. Part I*
Let X be a Noetherian affine scheme.

i) Prove that any quasi-coherent sheaf on X is the union of its coherent subsheaves. We say that $\mathcal{F}$ is the union of subsheaves $\mathcal{F}_\alpha$ if for any open subset $U \subset X$ the group $\mathcal{F}(U)$ is the union of $\mathcal{F}_\alpha(U)$.

ii) Let $U \subset X$ be an open subset and $\mathcal{F}$ a coherent sheaf on U. Prove that there exists a coherent sheaf $\mathcal{F}'$ on X, such that $\mathcal{F}'|_U = \mathcal{F}$.

iii) Let X, U, $\mathcal{F}$ be as in ii), and suppose that $\mathcal{G}$ is a quasi-soherent sheaf on X such that $\mathcal{F} = \mathcal{G}|_U$. Prove that one can find a coherent subsheaf $\mathcal{F}' \subset \mathcal{G}$ such that $\mathcal{F}'|_U = \mathcal{F}$.

Exercise 65. (4 points) *Extension of coherent sheaves. Part II*
Let X be a Noetherian scheme, $U \subset X$ an open subset and $\mathcal{F}$ a coherent sheaf on U.

i) Prove that there exists a coherent sheaf $\mathcal{F}'$ on X such that $\mathcal{F}'|_U = \mathcal{F}$.

ii) Prove that any quasi-coherent sheaf on X is the union of its coherent subsheaves.

Exercise 66. (3 points) *Invertible sheaves.*
Let $\varphi : \mathcal{L} \to \mathcal{M}$ be a homomorphism of invertible sheaves on a scheme X.

i) Show that φ is an isomorphism if φ is surjective.

ii) Give an example where φ is injective but not an isomorphism.

Exercise 67. (3 points). *Morphisms from $\mathbb{P}^n_k$.*
Let $\varphi : \mathbb{P}^n_k \to X$ be a morphism between projective k-schemes. Suppose $Y \subset \mathbb{P}^n_k$ is an irreducible closed subscheme such that $\varphi(Y)$ is zero-dimensional. Show that then $\dim(Y) = 0$ or $\varphi(\mathbb{P}^n_k)$ consists of one point.

Exercise 68. (4 points) *Linear systems.*
Consider $s_0 := x_0^2, s_1 := x_1^2, s_2 := x_0x_1, s_3 := x_0x_2, s_4 := x_1x_2, s_5 := x_2^2 \in H^0(\mathbb{P}^2_k, \mathcal{O}(2))$.

i) Determine the maximal open set $U \subset \mathbb{P}^2_k$ on which the map to $\mathbb{P}^4_k$ determined by s_0, s_1, s_2, s_3, s_4 is regular.

ii) Consider the map to $\mathbb{P}^4_k$ given by s_0, s_1, s_2, s_3, s_5. Prove that it is well-defined on $\mathbb{P}^2_k$. Is it a closed immersion?

Exercise 69. (2 points) *Divisors on $\mathbb{P}^n_k$.*
Use the isomorphism $\mathrm{Cl}(X) \cong \mathrm{Pic}(X)$ (for X a factorial, Noetherian, separated, integral scheme) and the fact that $\mathrm{Cl}(\mathbb{P}^n_k) \cong \mathbb{Z}$ generated by a hyperplane $H \cong \mathbb{P}^{n-1}_k \subset \mathbb{P}^n_k$ to show that $\mathrm{Pic}(\mathbb{P}^n_k) \cong \mathbb{Z}$ generated by $\mathcal{O}(1)$. In particular, show that under the isomorphism H is mapped to $\mathcal{O}(1)$ (and not e.g. to $\mathcal{O}(-1)$).

Due Monday 25 January, 2016. Before the lecture.