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Winter term 2015/16

Exercises, Algebraic Geometry I – Week 11

Exercise 59. (2 points) *Segre embedding.*
Let S and T be graded rings with $S_0 = T_0 = A$. Let S be generated by $x_0, \dots, x_r \in S_1$ and T by $y_0, \dots, y_s \in T_1$. This gives projective embeddings $\operatorname{Proj} S \to \mathbb{P}^r_A$ and $\operatorname{Proj} T \to \mathbb{P}^s_A$. Recall from Exercise 50 the definition of Cartesian product $S \times_A T$ and prove that $S \times_A T$ is generated by $x_i \otimes y_j$, which gives an embedding $\operatorname{Proj}(S \times_A T) \to \mathbb{P}^{rs+r+s}_A$.

Exercise 60. (4 points) *Tensor products of ample line bundles.*
Work through Exercise II.5.12 in Hartshorne's book.

Exercise 61. (4 points) *Global sections of the structure sheaf.*
Let X be a projective scheme over a field k. Then $H^0(X, \mathcal{O}_X)$ is a finite-dimensional vector space by Serre's theorem. Show that $H^0(X, \mathcal{O}_X) \cong k$ if X is reduced and connected and k is algebraically closed.

Exercise 62. (3 points) *Euler characteristic.*
Let X be a projective scheme over a field k. Recall that by Serre's theorem for every $\mathcal{F} \in \operatorname{Coh}(X)$ the k-vector spaces $H^i(X, \mathcal{F})$ are finite-dimensional. Define the Euler characteristic of $\mathcal{F}$ as

$$\chi(X, \mathcal{F}) := \sum_{i=0}^{n} (-1)^i \dim_k H^i(X, \mathcal{F}).$$

Show that for a short exact sequence of coherent sheaves $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ one has

$$\chi(X, \mathcal{G}) = \chi(X, \mathcal{F}) + \chi(X, \mathcal{H}).$$

Exercise 63. (3 points) *Arithmetic genus.*
The arithmetic genus of a projective scheme X of dimension n over a field k is defined as

$$p_a(X) := (-1)^n (\chi(X, \mathcal{O}_X) - 1),$$

So if X is an integral curve, i.e. $n = 1$, and $k = \bar{k}$, then $p_a(X) = \dim_k H^1(X, \mathcal{O}_X)$. Show that for $X \subset \mathbb{P}^2_k$ given by a polynomial of degree d, $p_a(X) = (d-1)(d-2)/2$.

Due Monday 18 January, 2016. Before the lecture.