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Winter term 2015/16

Exercises, Algebraic Geometry I – Week 1

Exercise 1. (4 points)
Let $\mathcal{C}$ be a category, for example the category of sets $(Sets)$ or abelian groups (Ab), and let $\varphi_1, \varphi_2 \colon M \rightrightarrows N$ be two morphisms in $\mathcal{C}$. A morphism $\varphi \colon K \to M$ with $\varphi_1 \circ \varphi = \varphi_2 \circ \varphi$ is called an *equalizer* of (φ_1, φ_2) if for all $\psi \colon P \to M$ in $\mathcal{C}$ with $\varphi_1 \circ \psi = \varphi_2 \circ \psi$ there exists a unique morphism $\tilde{\psi} \colon P \to K$ with $\varphi \circ \tilde{\psi} = \psi$. If an equalizer exists it is unique up to unique isomorphism.

i) Show that in $\mathcal{C} = (Sets)$ the subset $K := \{x \mid \varphi_1(x) = \varphi_2(x)\} \subset M$ is an equalizer.

ii) Show that in $\mathcal{C} = (Ab)$ the kernel $\mathrm{Ker}(\varphi_1 - \varphi_2)$ is an equalizer.

Exercise 2. (4 points)
Let X be a topological space. For any non-empty open subset $U \subset X$ let $\mathcal{C}(U) := \{f \colon U \to \mathbb{R} \text{ continuous}\}$ and for an inclusion of open subsets $V \subset U$ let $\rho_{UV} \colon \mathcal{C}(U) \to \mathcal{C}(V)$ be the restriction map $\rho_{UV}(f) := f|_V$. For the empty set define $\mathcal{C}(\emptyset) := \{*\}$, where $\{*\}$ is the one-element set. The restriction map $\rho_{U\emptyset} \colon \mathcal{C}(U) \to \mathcal{C}(\emptyset)$ is the unique map to the one-element set.

i) Show that $\rho_{VW} \circ \rho_{UV} = \rho_{UW} \colon \mathcal{C}(U) \to \mathcal{C}(W)$ for open subsets $W \subset V \subset U$.

ii) Let $U = \bigcup V_i$ be an open cover, set $V_{ij} := V_i \cap V_j$, and $\varphi_1, \varphi_2 \colon \prod \mathcal{C}(V_i) \to \prod \mathcal{C}(V_{ij})$ be the two maps $(f_i)_i \mapsto (f_i|_{V_{ij}})_{ij}$ and $(f_i)_i \mapsto (f_j|_{V_{ij}})_{ij}$. Show that $\mathcal{C}(U) \to \prod \mathcal{C}(V_i)$, $f \mapsto (f|_{V_i})_i$ is an equalizer for (φ_1, φ_2).

Exercise 3. (4 points)
Consider a topological space X and fix an abelian group M_x for all $x \in X$. Define for any open set $U \subset X$ the abelian group $\mathcal{F}(U) := \prod_{x \in U} M_x$. Imitate the previous exercise, define natural restriction maps $\rho_{UV} \colon \mathcal{F}(U) \to \mathcal{F}(V)$, and prove i) and ii) as above.

Exercise 4. (4 points)
As a special case of Exercise 3 study $X = \mathrm{Spec}(\mathbb{Z})$ with $M_{(p)} := \mathbb{Z}_{(p)}$, for $p \in \mathbb{Z}$ prime, and $M_{(0)} := \mathbb{Q}$. Recall that the open sets are of the form $U = \mathrm{Spec}(\mathbb{Z}[1/n])$, with $n \in \mathbb{Z}$.

Modify the above construction and study $\mathcal{O}(U) := \mathbb{Z}[1/n]$ viewed as a subgroup of $\prod \mathbb{Z}_{(p)}$. Verify that the natural restriction maps satisfy i) and ii).

Exercise 5. (4 points)
Let X be a topological space and G an abelian group. For an open subset $U \subset X$ let $\mathbf{G}(U)$ be the group of constant maps $f \colon U \to G$ and define again $\rho_{UV} \colon \mathbf{G}(U) \to \mathbf{G}(V)$ as the natural restriction maps. Are i) and ii) in Exercise 2 still valid? What happens if instead one considers the groups $\underline{G}(U)$ of locally constant functions (i.e. continuous functions $U \to G$, where G is endowed with the discrete topology)?

Please turn over.

Due Monday 26 October, 2015. Before(!!) the lecture.

Exercises will be handed out each Monday or can be downloaded from:
`http://www.math.uni-bonn.de/people/aosoldat/alggeom_V4A1_WS15.htmpl`
Solutions have to be handed in the following Monday before(!) the lecture.
It is necessary to collect half of the points to be admitted to the exam.

Tutorial groups:
1st group: Wednesdays 8-10, room 1.008, tutor Kai Behrens
2nd group: Wednesdays 14-16, room 0.006, tutor Paul Görlach
3rd group: Fridays 10-12, room 0.007, tutor Isabell Große-Brauckmann