Solutions of the exam problems (V3A1, Algebra I)

Exercise A. 1) Let $x \in \mathfrak{N}(A)$, then $x^n = 0$ for some n. For any $s \in S$ we have $(x/s)^n = x^n/s^n = 0$, so $x/s \in \mathfrak{N}(S^{-1}A)$. Conversely, let $x/s \in \mathfrak{N}(S^{-1}A)$. Then for some n we have $x^n/s^n = 0$, which implies that $tx^n = 0$ for some $t \in S$. It follows that $tx \in \mathfrak{N}(A)$, hence $x/s \in S^{-1}(\mathfrak{N}(A))$.

2) If $\mathfrak{N}(A) = 0$, then using part 1) for any prime ideal $\mathfrak{p} \subset A$ we have $\mathfrak{N}(A_\mathfrak{p}) = (\mathfrak{N}(A))_\mathfrak{p} = 0$. This proves that i) implies ii). Since every maximal ideal is prime, ii) implies iii). Let us prove that iii) implies i). Assume that A is not reduced. Then the nilradical of A is a non-trivial A-module. The support of $\mathfrak{N}(A)$ is non-empty, so it contains a maximal ideal $\mathfrak{m}$. Hence $\mathfrak{N}(A_\mathfrak{m}) = \mathfrak{N}(A)_\mathfrak{m} \neq 0$ and $A_\mathfrak{m}$ is non-reduced.

Exercise B. Let $\mathfrak{a} = \ker(\varphi)$, so that $B = A/\mathfrak{a}$. We identify prime ideals of B with their preimages in A, which are prime ideals of A containing $\mathfrak{a}$.

Let us prove that $\mathrm{Ass}_A(M) = \mathrm{Ass}_B(M)$. Note that for any element $m \in M$ we have $\mathrm{Ann}_A(m) = \varphi^{-1}(\mathrm{Ann}_B(m))$. Indeed, if $x \in \mathrm{Ann}_A(M)$ then $\varphi(x)m = 0$, so $x \in \varphi^{-1}(\mathrm{Ann}_B(m))$; if $y \in \mathrm{Ann}_B(m)$ and $\varphi(x) = y$, then $\varphi(x)m = 0$ so $x \in \mathrm{Ann}_A(m)$.

If $\mathfrak{p} \in \mathrm{Ass}_A(M)$ then there exists an element $m \in M$ such that $\mathfrak{p} = \mathrm{Ann}_A(m) = \varphi^{-1}(\mathrm{Ann}_B(m))$, so $\mathfrak{p}$ is identified with $\mathrm{Ann}_B(m)$. Conversely, for $\mathfrak{q} \in \mathrm{Ass}_B(M)$ we have $\mathfrak{q} = \mathrm{Ann}_B(m)$ for some $m \in M$, so the preimage $\mathfrak{p} = \varphi^{-1}(\mathfrak{q}) = \mathrm{Ann}_A(m)$ and $\mathfrak{p} \in \mathrm{Ass}_A(M)$.

Let us prove that $\mathrm{Supp}_A(M) = \mathrm{Supp}_B(M)$. The support of M is the set of those prime ideals $\mathfrak{p}$ for which $M_\mathfrak{p} \neq 0$. Note that for a prime ideal $\mathfrak{q} \subset B$ and $\mathfrak{p} = \varphi^{-1}(\mathfrak{q})$ we have $B_\mathfrak{q} = B \otimes_A A_\mathfrak{p}$. It follows that $M_\mathfrak{q} = M \otimes_B B_\mathfrak{q} = M \otimes_B B \otimes_A A_\mathfrak{p} = M \otimes_A A_\mathfrak{p} = M_\mathfrak{p}$. From this we see that $\mathrm{Supp}_B(M) \subset \mathrm{Supp}_A(M)$. Conversely, if $\mathfrak{p} \in \mathrm{Supp}_A(M)$ then $\mathfrak{a} \subset \mathfrak{p}$, because otherwise $A\backslash\mathfrak{p}$ contains elements that annihilate M and $M_\mathfrak{p} = 0$. So we have $\mathfrak{p} = \varphi^{-1}(\mathfrak{q})$ for some prime ideal $\mathfrak{q}$, which is in $\mathrm{Supp}_B(M)$ by the equality $M_\mathfrak{q} = M_\mathfrak{p}$ proven above.

Exercise C. i) The ring $\mathbb{Z}$ is a PID, so it is of dimension one. The chain $(0) \subset (2)$ of prime ideals is of length one.

ii) The ring $A = k[X,Y]/(X^2 - Y^3)$ is an integral domain because $X^2 - Y^3$ is an irreducible polynomial. The normalization of A is $\bar{A} = k[T]$, where $X = T^3$ and $Y = T^2$. Since $k[T]$ is a PID, we have $\dim(\bar{A}) = 1$. The extension $A \to \bar{A}$ is integral, so $\dim(A) = \dim(\bar{A}) = 1$. The chain $(0) \subset (X-1, Y-1)$ is of length one.

Another proof. By Krull's principal ideal theorem, for any minimal prime ideal $\mathfrak{p} \subset k[X,Y]$ *containing* $X^2 - Y^3$ *we have* $\mathrm{ht}(\mathfrak{p}) = 1$, *because* $X^2 - Y^3$ *is neither a unit nor a zero-divisor. Then* $\dim(A) \leq \dim(k[X,Y]) - 1 = 1$. *But the chain* $(0) \subset (X-1, Y-1)$ *is of length one, so* $\dim(A) = 1$.

iii) The tensor product $k[X] \otimes_k k[X]$ is isomorphic to $k[X,Y]$ which is of dimension two. The chain $(0) \subset (X) \subset (X,Y)$ is of length two.

iv) The ring $A = \prod_{i=1}^n k_i$ consists of n-tuples $x = (x_1, \ldots, x_n)$ with $x_i \in k_i$. Denote by e_i the image of the identity in k_i under the natural embedding $k_i \hookrightarrow A$. If $I \subset A$ is a non-zero ideal, pick a non-zero element $x \in I$. If $x_i \neq 0$, then $xe_i = x_i e_i \in I$, and I contains the image of k_i because x_i is invertible in k_i. It follows that any ideal is of the form $I = \prod_{i \in S} k_i$ where $S \subset \{1, \ldots, n\}$ is a subset. The quotient is $A/I = \prod_{i \in \bar{S}} k_i$ for $\bar{S} = \{1, \ldots, n\}\backslash S$, and the ideal I is prime if and only if $\bar{S}$ consists of one element. Hence there are no inclusions between prime ideals and $\dim(A) = 0$.

Exercise D. The degree of polynomials is additive: $\deg(FG) = \deg(F) + \deg(G)$. It follows that the map ν is well-defined: if $F/G = F'/G'$ then $FG' = F'G$ and $\deg(F) + \deg(G') = \deg(F') + \deg(G)$; and ν is a group homomorphism: $\nu(FF'/GG') = \nu(F/G) + \nu(F'/G')$; the surjectivity of ν is obvious. Since $F/G + F'/G' = (FG' + F'G)/GG'$ and since we have the inequality $\deg(FG' + F'G) \leq \max(\deg(FG'), \deg(F'G))$, we get $\nu(F/G + F'/G') = \deg(GG') - \deg(FG' + F'G) \geq \min(\nu(F/G), \nu(F'/G'))$.

Let A be the corresponding valuation ring. Then $F/G \in A$ if and only if $\deg(G) \geq \deg(F)$. Let $F = f_0 + f_1X + \cdots + f_nX^n$ and $G = g_0 + g_1X + \cdots + g_mX^m$ with $f_n \neq 0$, $g_m \neq 0$. Then $F/G = (1/X)^{m-n}(f_0/X^n + \cdots + f_n)/(g_0/X^m + \cdots + g_m)$. This element is in A if and only if $m \geq n$, that is if $F/G \in k[1/X]_{(1/X)}$ (localization at the prime ideal generated by $1/X$). The maximal ideal of the latter ring is generated by $1/X$, which is the uniformizing parameter.

Exercise E. i) We have $A/(x) = k[y,z]/(z^2)$. Any element of this ring is of the form $p(y)+zq(y)$. Suppose this element is a zero-divisor: $0 = (p(y)+zq(y))(r(y)+zs(y)) = p(y)r(y)+z(p(y)s(y)+q(y)r(y))$. This can happen if and only if $p = r = 0$. But all elements of the form $zq(y)$ are nilpotent. So all zero-divisors of $A/(x)$ are nilpotent, and (x) is primary. Everything is symmetric in x and y, so (y) is also primary. The ring $A/(z) = k[x,y]$ is integral, so (z) is prime.

ii) We have a primary decomposition $(0) = (x) \cap (y) \cap (z)$ in A. Indeed, in $k[x,y,z]$ we have $(xyz) = (x) \cap (y) \cap (z)$, because every polynomial in $k[x,y,z]$ which is divisible by x, y and z is divisible by xyz. The ideal (x) is (x,z)-primary, because (x,z) is prime and it is clearly contained in the radical of (x). Analogously, (y) is (y,z)-primary, so all the associated primes in the decomposition are distinct and the decomposition is minimal. We have $(z) \subset (x,z)$ and $(z) \subset (y,z)$, so (z) is isolated and (x,z), (y,z) are embedded.

Exercise F. i) Let $A \subset B$ be an integral extension of rings. Then the map $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is surjective. More precisely, if $\mathfrak{p} \subset \mathfrak{p}' \subset A$ are prime ideals and $\mathfrak{q} \subset B$ is a prime ideal such that $\mathfrak{q} \cap A = \mathfrak{p}$, then there exists a prime ideal $\mathfrak{q}' \subset B$ with $\mathfrak{q} \subset \mathfrak{q}'$ and $\mathfrak{q}' \cap A = \mathfrak{p}'$.

ii) Consider the prime ideals $(0) \subset (5) \subset \mathbb{Z}$ and $(0) \subset \mathbb{Z}[1/5]$. If the extension $\mathbb{Z} \subset \mathbb{Z}[1/5]$ had the going-up property, then there would exist a prime ideal $\mathfrak{q} \subset \mathbb{Z}[1/5]$, such that $\mathfrak{q} \cap \mathbb{Z} = (5)$, in particular $5 \in \mathfrak{q}$. But 5 is invertible in $\mathbb{Z}[1/5]$, so this is impossible.

iii) Denote by A the ring $k[x,y,z]/(zy-x)$. Consider the prime ideals $(x-1) \subset (x-1,y) \subset k[x,y]$ and $(x-1) \subset A$. If the ring extension had going-up property, we would find a prime ideal in A lying over $(x-1,y)$ and containing $(x-1)$. Taking the quotient of both rings by $(x-1)$ we would obtain a prime ideal $\mathfrak{q} \subset k[y,z]/(yz-1)$ lying over $(y) \subset k[y]$. In particular $\mathfrak{q}$ would contain y which is invertible in $k[y,z]/(yz-1)$, so this is impossible.

Another proof: one can rewrite the given map as follows: $k[x,y] \to k[y,z] \simeq A$, $x \mapsto yz$, $y \mapsto y$. From this description it is clear that the image of the corresponding map of spectra does not contain points of the form $y = 0$, $x \neq 0$. So the map of spectra is not surjective and the extension is not integral.

Exercise G. i) If A is a field then an A-module is a vector space, so it has a basis and hence it is free. Conversely, if any A-module is free, then for any ideal $I \subsetneq A$ the module A/I is free. It follows that there is an injective morphism $f\colon A \to A/I$, but in this case $I \subset \ker(f)$, so $I = 0$. We see that any ideal in A is trivial, so A is a field.

ii) The first part is the same as in i) because any free module is flat. For the converse pick a non-zero element $x \in A$. Consider a morphism of A-modules $f\colon A \to A$, $f(y) = xy$. This morphism is injective because A has no zero-divisors. By assumption, the module $A/(x)$ is flat, so after we tensor f with $A/(x)$ the resulting map has to be injective. But this map $A/(x) \to A/(x)$ is multiplication by x, hence it is a zero map. It follows that $A/(x) = 0$, and the element x is invertible.