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SS 2015

Exam: Commutative Algebra (V3A1, Algebra I)

Solutions can be written in English or German

Exercise A. $(2+3$ points)
1) Let A be a ring and $S \subset A$ a multiplicative subset. Prove the following equality for the nilradical: $\mathfrak{N}(S^{-1}A) = S^{-1}(\mathfrak{N}(A))$.
2) A ring is called reduced if its nilradical is trivial. Prove that the following assertions are equivalent:

i) A is reduced.

ii) $A_{\mathfrak{p}}$ is reduced for all prime ideals $\mathfrak{p} \subset A$.

iii) $A_{\mathfrak{m}}$ is reduced for all maximal ideal $\mathfrak{m} \subset A$.

Exercise B. (4 points)
Let $\varphi\colon A \twoheadrightarrow B$ be a surjective ring homomorphism. Consider any B-module M simultaneously as an A-module and identify $\mathrm{Spec}(B)$ with the closed subset $V(\mathrm{Ker}(\varphi)) \subset \mathrm{Spec}(A)$. Prove that under this identification the equalities $\mathrm{Ass}_B(M) = \mathrm{Ass}_A(M)$ and $\mathrm{Supp}_B(M) = \mathrm{Supp}_A(M)$ hold.

Exercise C. $(1+2+1+3$ points)
Compute the dimensions of the following rings and provide a chain of prime ideals of maximal length in each case:
i) $\mathbb{Z}$; ii) $k[X,Y]/(X^2-Y^3)$ for a field k;
iii) $k[X] \otimes_k k[X]$ for a field k; iv) $\prod_{i=1}^n k_i$ for fields k_i.

Exercise D. (4 points)
Let k be a field and $\nu\colon k(x)^* \to \mathbb{Z}$, $F(x)/G(x) \mapsto \deg(G) - \deg(F)$. Show that ν is a discrete valuation. Determine its valuation ring and a uniformizing parameter.

Exercise E. $(3+3$ points) Consider the ring $A = k[x,y,z]/(xyz, z^2)$, with k a field.

i) Show that (x), (y) are primary ideals and that (z) is a prime ideal.

ii) Determine a minimal primary decomposition of (0) and decide which of the associated prime ideals are isolated and which ones are embedded.

Exercise F. $(2+1+2$ points)

i) State the 'going-up' theorem for ring extensions $A \subset B$.

ii) Show, by describing a counterexample, that the going-up property does not hold for the ring extension $\mathbb{Z} \subset \mathbb{Z}[1/5]$.

iii) Explain why $k[x,y] \hookrightarrow k[x,y,z]/(zy-x)$ (k a field) cannot be integral.

Exercise G. $(2+2$ points)

i) Prove that a ring A is a field if and only if every A-module is free.

ii) Prove that an integral domain A is a field if and only if every A-module is flat.

Klausureinsicht (review of corrected exam): Thursday July 30, 14.15 – 15.45. Seminar rooms 0.007 and 0.008.