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Summer term 2015

Exercises, Algebra I (Commutative Algebra) – Week 9

Exercise 48. (4 points)
Let $f_1, \ldots, f_k \in \mathbb{Z}[x_1, \ldots, x_n]$ be polynomials without any common zero $(a_1, \ldots, a_n) \in \mathbb{C}^n$. Show that there exist $g_1, \ldots, g_k \in \mathbb{Z}[x_1, \ldots, x_n]$ with $0 \neq g_1 f_1 + \ldots + g_k f_k \in \mathbb{Z}$. Is this still true if $\mathbb{C}^n$ is replaced by $\mathbb{R}^n$?

Exercise 49. (4 points)
Show that any field that is finitely generated as a $\mathbb{Z}$-algebra is in fact a finite field.

Exercise 50. (2 points) (Noether normalization for infinite fields)
Show that for any closed set $V(\mathfrak{a}) \subset \mathbb{A}^n_k$ with $|k| = \infty$ there exists a linear projection $\mathbb{A}^n_k \twoheadrightarrow \mathbb{A}^m_k$, i.e. given by $k[y_1, \ldots, y_m] \hookrightarrow k[x_1, \ldots, x_n]$, $y_i \mapsto f_i(x_1, \ldots, x_n)$ with f_i linear, such that the induced map $V(\mathfrak{a}) \to \mathbb{A}^m_k$ is finite, closed, and surjective.

Exercise 51. (4 points)
Show that $\mathbb{Z}$ is a Jacobson ring and that the ring of formal power series $k[[x]]$ over a field k is not.

Exercise 52. (3 points)
Let $V(\mathfrak{a}) \subset \mathbb{A}^3_k$ with $\mathfrak{a} = (y - z^2, xz - y^2) \subset k[x, y, z]$. Determine explicitly a linear projection $V(\mathfrak{a}) \to \mathbb{A}^1_k$ which is finite, closed and surjective.

Due Monday Jun 15.