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Summer term 2015

Exercises, Algebra I (Commutative Algebra) – Week 7

Exercise 36. (2 points)
Describe the normalization of the cusp. More precisely, show that the normalization of $A = k[x_1, x_2]/(x_2^2 - x_1^3)$ is isomorphic to $k[x]$ and describe (for $k = \bar{k}$) the induced map $\mathbb{A}^1_k \to \operatorname{Spec}(A) \subset \mathbb{A}^2_k$.

Exercise 37. (4 points)
Consider $\mathbb{Q}(\sqrt{n})$ for a square free $n \in \mathbb{N}$ and let

$$\alpha \coloneqq \begin{cases} (1+\sqrt{n})/2 & \text{if } n \equiv 1\,(4) \\ \sqrt{n} & \text{if } n \equiv 2 \,\text{or}\, n \equiv 3\,(4). \end{cases} \tag{1}$$

Show that the integral closure of $\mathbb{Z}$ in $\mathbb{Q}(\sqrt{n})$ is $\mathbb{Z}[\alpha]$.

Exercise 38. (4 points)
Let $A \coloneqq k[x_1, x_2, y]/(x_2^2 - x_1^2(x_1 + 1))$ and $\operatorname{Spec}(A) \hookrightarrow \mathbb{A}^3_k$ the natural inclusion induced by the projection $k[x_1, x_2, y] \twoheadrightarrow A$. Consider the map

$$f \colon \mathbb{A}^2_k \to \operatorname{Spec}(A),$$

induced by the ring homomorphism $A \to k[x, y]$, $x_1 \mapsto x^2 - 1$, $x_2 \mapsto x(x^2 - 1)$, $y \mapsto y$. Show that f does not have the 'going-down' property. Hint: Consider the prime ideals $(x - 1, y)$ and $(y - (x + 1))$. Try to draw a picture!

Exercise 39. (4 points)
An ideal $\mathfrak{q} \subset A$ is called *primary* if $a \cdot b \in \mathfrak{q}$ with $a \notin \mathfrak{q}$ implies that $b^n \in \mathfrak{q}$ for some $n > 0$.

i) Show that $\mathfrak{q}$ is primary if and only if all zero divisors in $A/\mathfrak{q}$ are nilpotent.

ii) Show that for a primary ideal $\mathfrak{q}$ its radical $\mathfrak{p} \coloneqq \sqrt{\mathfrak{q}}$ is a prime ideal and that it is the smallest prime ideal containing $\mathfrak{q}$. (Then $\mathfrak{q}$ is called $\mathfrak{p}$-*primary*.)

Please turn over

Due Monday Jun 1.

Exercise 40. (6 points)

i) Show that primary ideals in $\mathbb{Z}$ are of the form (p^n) with p a prime number or (0).

ii) Show that $\mathfrak{q} = (x, y^2) \subset k[x, y]$ is a (x, y)-primary ideal which is not of the form $\mathfrak{p}^n$ for any prime ideal $\mathfrak{p}$.

iii) Show that $\mathfrak{p} = (\bar{x}, \bar{z}) \subset k[x, y, z]/(xy - z^2)$ is a prime ideal, but that $\mathfrak{p}^2$ is not primary ($\bar{x}$ and $\bar{z}$ denote the images of x and z in the quotient ring).

Exercise 41. (6 points)
Let M be an A-module. A prime ideal $\mathfrak{p} \subset A$ is *associated* to M if there exists an $m \in M$ such that $\mathfrak{p} = \mathrm{Ann}(m) := \{a \in A \mid am = 0\}$. The set of associated prime ideals is denoted $\mathrm{Ass}(M) \subset \mathrm{Spec}(A)$.

i) Show that $\mathfrak{p}$ is associated to M if and only if there exists an injective A-module homomorphism $A/\mathfrak{p} \hookrightarrow M$.

ii) Let $\mathfrak{p} = \mathrm{Ann}(m) \in \mathrm{Ass}(M)$ and $0 \neq n \in A \cdot m$. Show that $\mathrm{Ann}(n) = \mathfrak{p}$.

iii) Show that $\mathrm{Ass}(A/\mathfrak{p}) = \{\mathfrak{p}\}$.