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Summer term 2015

Exercises, Algebra I (Commutative Algebra) – Week 6

Exercise 30. (3 points)
Describe $\mathrm{Spec}(A)$ for $A = k[X]/(X^4 - 1)$, with $k = \mathbb{R}, \mathbb{C}$, and $A = \mathbb{Z}_{30}$.

Exercise 31. (3 points)
Let A be a Noetherian ring and B a finite type A-algebra. Suppose $G = \{g_i\}$ is a finite group of A-algebra homomorphisms $g_i \colon B \to B$. Show that $B^G := \{b \in B \mid g_i(b) = b \text{ for all } i\}$ is a finite type A-algebra.

Exercise 32. (6 points)
A ring homomorphism $f \colon A \to B$ is said to have the *going-up property* if for any $\mathfrak{q} \in \mathrm{Spec}(B)$ and any $\mathfrak{p}' \in \mathrm{Spec}(A)$ containing $\mathfrak{p} := \mathfrak{q}^c$ there exists a prime ideal $\mathfrak{q}' \in \mathrm{Spec}(B)$ containing $\mathfrak{q}$ and such that $\mathfrak{p}' = (\mathfrak{q}')^c$. Prove that f has the going-up property if and only if the induced map $\varphi \colon \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is closed (i.e. the image of every closed set is again closed).

For the 'only if direction' prove first that φ is closed if its image is closed under specialization, i.e. if $\mathfrak{p}' \in \mathrm{Spec}(A)$ contains $\mathfrak{p} := \mathfrak{q}^c$, then $\mathfrak{p}'$ is contained in the image.

Exercise 33. (4 points)
Decide whether the following rings are normal i) $k[x, y]/(x^5 - y^2)$; ii) $k[x, y]/(x^2 - y^2)$; iii) $k[x, y]/(y^2 - x^3 - x^n)$ for $n \geq 3$; iv) $\mathbb{Z}[1/n]$.

Exercise 34. (3 points)
A topological space X is *Noetherian* if every ascending chain of open subsets $U_1 \subset U_2 \subset \ldots$ becomes stationary (i.e. $\bigcup U_i = U_n$ for $n \gg 0$) or, equivalently, if every descending chain of closed sets $V_1 \supset V_2 \supset \ldots$ becomes stationary (i.e. $\bigcap V_i = V_n$ for $n \gg 0$).

i) Show that $\mathrm{Spec}(A)$ of a Noetherian ring is a Noetherian topological space and find a counter-example for the converse.

ii) Show that for a finite type A-algebra B the fibres of $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ are Noetherian topological spaces.

Exercise 35. (3 points)
Suppose $A \to B$ is integral. For a maximal ideal $\mathfrak{n} \subset B$ let $\mathfrak{m} := \mathfrak{n}^c \subset A$ (which is again maximal, as will be shown in class). Is then the induced ring homomorphism $A_{\mathfrak{m}} \to B_{\mathfrak{n}}$ always integral? (*Hint*: Consider $k[X^2 - 1] \subset k[X]$.)

Due Monday May 18.