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Summer term 2015

Exercises, Algebra I (Commutative Algebra) – Week 12

Exercise 63. (6 points) Let $A = k[x, y, z]$ and $\mathfrak{a} = (xy, x - yz)$.

i) Show that $V(\mathfrak{a}) = (V(x) \cup V(y)) \cap V(x - yz) = V(x, y) \cup V(x, z)$.

ii) Prove that $\mathrm{Ass}(A/\mathfrak{a}) = \{(x, y), (x, z)\}$.

iii) Find a minimal primary decomposition of $\mathfrak{a}$.

Exercise 64. (4 points) Describe examples (different from the ones in class): i) of an ideal with embedded prime ideals; ii) of an ideal with two distinct minimal primary decompositions.

Exercise 65. (6 points)
Let A be a Noetherian ring and M be a finite A-module. Show that there exists a finite chain of submodules

$$0 = M_0 \subset M_1 \subset \ldots \subset M_n = M,$$

such that $M_i/M_{i-1} \cong A/\mathfrak{p}_i$ for some prime ideal $\mathfrak{p}_i$ (and so, in particular, $\mathrm{Ass}(M_i/M_{i-1}) = \{\mathfrak{p}_i\}$). Show that then $\mathrm{Ass}(M) \subset \{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$ and find an example for which the inclusion is strict.

Exercise 66. (3 points)
Let A be a k-algebra and d a derivation of A, that is a k-linear map $d\colon A \to A$ which satisfies $d(xy) = x\ dy + y\ dx$. Assume that A is reduced (i.e. nilradical is trivial). Let $\mathfrak{p} \in \mathrm{Ass}(A)$. Prove that $d(\mathfrak{p}) \subset \mathfrak{p}$.

Exercise 67. (6 points)
Let A be a Noetherian ring and M a finite A-module. Consider the ring $A[[t]] = \{\sum_{i\geq 0} a_i t^i | a_i \in A\}$ of formal power series with coefficients in A and the corresponding module $M[[t]]$. Prove that the associated prime ideals of $M[[t]]$ (considered as an $A[[t]]$-module) are prime ideals of the form $\mathfrak{p}[[t]]$ for all $\mathfrak{p} \in \mathrm{Ass}(M)$.

Please turn over

Due Monday Jul 6. Admittance to the examen will be announced on Thursday July 9

Test[1]

1. Is the nilradical $\mathfrak{N}(A)$ of a ring always contained in its Jacobson radical $\mathfrak{R}(A)$?
2. Do you know rings with exactly one, two, resp. three prime ideals?
3. Is every open set in $\mathrm{Spec}(A)$ of the form $\mathrm{Spec}(S^{-1}A)$?
4. Is the localization $S^{-1}A$ of a Noetherian ring A again Noetherian?
5. Is the localization $S^{-1}A$ of a finite type k-algebra A again a finite type k-algebra?
6. Do you know an example of two non-trivial A-modules with $M \otimes_A N = 0$?
7. Suppose $\mathfrak{p}_1 \subset \mathfrak{p}_2$ are two prime ideals. What is the relation between $A_{\mathfrak{p}_1}$ and $A_{\mathfrak{p}_2}$?
8. When is a prime ideal called an associated ideal of an ideal $\mathfrak{a}$ and when is it an isolated associated ideal?
9. Is $\mathrm{MaxSpec}(A) \subset \mathrm{Spec}(A)$ always closed? What about DVR or Dedekind rings?
10. What is the relation between the localizations $\mathbb{Z}_p$ and $\mathbb{Z}_{(p)}$?
11. Is $\mathbb{Q}$ a normal ring?
12. Describe $\mathrm{Spec}(k[x])$ and $\mathrm{Spec}(\mathbb{Z}_p)$.
13. Let $k \subset A$ be an integral ring extension with k a field and A an integral domain. How big is $\mathrm{Spec}(A)$
14. Could $V(\mathfrak{a})$ consist of just one point without $\mathfrak{a}$ being a maximal ideal?
15. Is $(p) \in \mathrm{MaxSpec}(\mathbb{Z}[X])$ for a prime number p?
16. Let $\mathbb{F}_q$ be a finie field. Is $\mathrm{MaxSpec}(\mathbb{F}_q[X])$ finite?
17. Is every $\sqrt{\mathfrak{q}_i}$ of a primary decomposition $\mathfrak{a} = \bigcap \mathfrak{q}_i$ an associated ideal of $\mathfrak{a}$?
18. When exactly is $D(a) = \mathrm{Spec}(A)$ and when $D(a) = \emptyset$?
19. What is the relation between the existence of a prime factor decomposition in a Dedekind ring A and its class group $\mathrm{Cl}(A)$?
20. Is any factorial ring normal?
21. For a prime number $p \in \mathbb{Z}$, consider the two DVR $\mathbb{Z}_{(p)}$ and $\mathbb{F}_p[x]_{(x)}$. Are they isomorphic?
22. Suppose K is a number field. What is $\mathcal{O}_K$ and when is $\mathcal{O}_K = \mathbb{Z}$?
23. Why is the preimage of a prime ideal under a ring homomorphism never the unit ideal?
24. State the assertions of Noether normalization and of the Hilbert Nullstellensatz.
25. Over a local ring, could there be a flat module that is not free?
26. Is $\mathrm{Spec}(k[x]_x)$ homeomorphic to a closed subset of $\mathbb{A}^2_k$?

[1]Do not hand in solutions for the following quick questions. They merely serve as as self-evaluation.