Prof. Dr. Daniel Huybrechts
Dr. Andrey Soldatenkov Summer term 2015

Exercises, Algebra I (Commutative Algebra) – Week 10

Exercise 53. (2 points) Write the normalization of $A = k[x,y]/(y^2 - x^2(x+1))$ as the intersection of discrete valuation rings.

Exercise 54. (4 points) An *absolute value* on an integral domain A is a map $|\cdot|\colon A \to \mathbb{R}$ such that for all $a, b \in A$:
1) $|a| \geq 0$; 2) $|a| = 0$ if and only if $a = 0$; 3) $|ab| = |a| \cdot |b|$; and 4) $|a+b| \leq |a| + |b|$.

i) Check that any absolute value on A extends to an absolute value on its fraction field $Q(A)$ (define $|a/b| = |a|/|b|$).

ii) Suppose 4) is replaced by the stronger requirement $|a+b| \leq \max\{|a|, |b|\}$. Show that then for any $\alpha > 1$ the map $\nu\colon Q(A)^* \to \mathbb{R}$, $x \mapsto -\log_\alpha |x|$ is a valuation.

iii) What goes wrong for $\mathbb{C}^* \to \mathbb{R}$, $x \mapsto -\log_\alpha |x|$?

iiii) Determine an absolute value describing $\mathbb{Z}_{(p)}$ for a prime p.

Exercise 55. (2 points)
Consider a field extension $K \subset L$, B a valuation ring with quotient field L, and let $A := K \cap B$. Prove the following statements:

i) A is a valuation ring with quotient field K.

ii) If L/K is algebraic and B is not a field, then also A is not a field.

Exercise 56. (6 points) (Associated prime ideals)
Let A be a ring and M an A-module. Recall that a prime ideal $\mathfrak{p} \subset A$ is associated with M if there exists an $m \in M$ with $\mathrm{Ann}(m) = \mathfrak{p}$. The set of associated prime ideals is denoted $\mathrm{Ass}(M)$.

i) Let N be a submodule of M. Prove that $\mathrm{Ass}(N) \subset \mathrm{Ass}(M) \subset \mathrm{Ass}(N) \cup \mathrm{Ass}(M/N)$.

ii) Show that for all $\mathfrak{p} \in \mathrm{Ass}(M)$ one has $M_\mathfrak{p} \neq 0$, i.e. $\mathfrak{p} \in \mathrm{Supp}(M)$.

iiI) Assuming A to be Noetherian, prove that the natural map $M \to \prod_{\mathfrak{p} \in \mathrm{Ass}(M)} M_\mathfrak{p}$ is injective.

Exercise 57. (6 points) (Primary ideals)
Recall that an ideal $\mathfrak{q} \subset A$ is called *primary* if $ab \in \mathfrak{q}$ and $a \notin \mathfrak{q}$ implies $b^n \in \mathfrak{q}$ for some $n > 0$.

i) Let $\mathfrak{q}$ be a primary ideal. Show that $V(\mathfrak{q})$ is irreducible.

ii) Assume that A is Noetherian. Prove that an ideal $\mathfrak{q}$ is $\mathfrak{p}$-primary (that is, $\mathfrak{q}$ is primary and its radical is a prime ideal $\mathfrak{p}$) if and only if $\mathrm{Ass}(A/\mathfrak{q}) = \{\mathfrak{p}\}$.

Due Monday Jun 22.