

Holonomy of the Obata connection on $SU(3)$

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Abstract

A hypercomplex structure on a smooth manifold is a triple of integrable almost complex structures satisfying quaternionic relations. The Obata connection is the unique torsion-free connection that preserves each of the complex structures. The holonomy group of the Obata connection is contained in $GL(n, \mathbb{H})$. There is a well-known construction of hypercomplex structures on Lie groups due to Joyce. In this paper we show that the holonomy of the Obata connection on $SU(3)$ coincides with $GL(2, \mathbb{H})$. In this version of the paper we present an alternative proof of the main theorem, avoiding the classification of irreducible holonomy groups.

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1 Introduction

Consider a smooth manifold equipped with a triple of almost complex structures satisfying quaternionic relations. The manifold is called hypercomplex if these almost complex structures are integrable. Hypercomplex manifolds were defined by Boyer [Bo] and they were much studied since then. There exist many examples of such manifolds including hyperkähler manifolds, nilmanifolds, Lie groups with hypercomplex structures and others. Boyer also classified compact hypercomplex manifolds of real dimension four. Homogeneous hypercomplex structures on Lie groups appeared in the context of string theory (see [SSTV]) and then in the work of Joyce [J].

Each hypercomplex manifold is endowed with a torsion-free connection preserving all the complex structures which is called the Obata connection. The holonomy group of this connection is an important characteristic of the hypercomplex structure. Since the Obata connection preserves the quaternionic structure, its holonomy is contained in $GL(n, \mathbb{H})$ which is one of the groups in the list of possible irreducible holonomies.

The classification of irreducible holonomy groups of torsion-free connections has a long history. For locally symmetric connections the problem essentially reduces to the classification of symmetric spaces which was known since Élie Cartan (see e.g. [Bes]). For connections that are not locally symmetric a major breakthrough was made in 1955 by Berger. He obtained a list of irreducible metric holonomies (i.e. holonomies of the connections that preserve some non-degenerate symmetric bilinear form) and a part of the list of non-metric ones. The classification was completed in 1999 by Merkulov and Schwachhöfer [MS] thus providing a full list of all possible irreducible holonomy groups.

The subgroups of $GL(n, \mathbb{H})$ which appear in the list of irreducible holonomies are $Sp(n)$ and $SL(n, \mathbb{H})$. For both of these subgroups, there exist examples of manifold with holonomy contained in it. These are hyperkähler manifolds for $Sp(n)$ and, for example, nilmanifolds for $SL(n, \mathbb{H})$ (see e.g. [BDV]). The group $GL(n, \mathbb{H})$ appears as a possible local holonomy group (see [MS]), but it was apparently unknown if it could occur as a holonomy of a compact hypercomplex manifold. The purpose of the present paper is to prove that the holonomy of the Obata connection on $SU(3)$ is $GL(2, \mathbb{H})$, thus providing the first compact example.

In Section 2 we recall the definition of the hypercomplex structure and obtain some useful properties of the Obata connection. In Section 3 we review

the construction of the hypercomplex structures on Lie groups. In Section 4 we study the Obata connection on $SU(3)$ and prove the main theorem.

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2 Hypercomplex manifolds and the Obata connection

In this section we recall the definition of a hypercomplex manifold and establish some useful properties of the Obata connection.

2.1 Hypercomplex structures

Let M be a smooth manifold. Recall that an *almost complex structure* on M is an endomorphism $I: TM \rightarrow TM$ satisfying $I^2 = -Id$. The *Nijenhuis tensor* for I is given by

$$N_I(X, Y) = [X, Y] + I[IX, Y] + I[X, IY] - [IX, IY]. \quad (2.1)$$

If the Nijenhuis tensor vanishes, the almost complex structure is called *integrable*. It is a well-known result of Newlander and Nirenberg that every integrable almost complex structure arises from a complex analytic structure on M .

Definition 2.1. *A hypercomplex structure on a smooth manifold M is a triple of integrable almost complex structures I, J, K satisfying*

$$IJ = -JI = K.$$

Note that a hypercomplex structure induces a natural action of the quaternion algebra \mathbb{H} on the tangent bundle of M . Thus, every hypercomplex manifold is equipped with a two-dimensional sphere of complex structures corresponding to imaginary quaternions of unit length.

2.2 The Obata connection

Let (M, I, J, K) be a hypercomplex manifold. It has been shown by Obata [Ob] that M admits a unique torsion-free connection ∇ that preserves the hypercomplex structure, i.e.

$$\nabla I = \nabla J = \nabla K = 0.$$

This connection is called *the Obata connection*.

Consider the decomposition of the complexified tangent bundle of M with respect to I :

$$T_{\mathbb{C}}M = TM \otimes_{\mathbb{R}} \mathbb{C} = T_I^{1,0}M \oplus T_I^{0,1}M,$$

where $T_I^{1,0}M = \{X \in T_{\mathbb{C}}M : IX = \sqrt{-1}X\}$, $T_I^{0,1}M = \{X \in T_{\mathbb{C}}M : IX = -\sqrt{-1}X\}$.

Since the complex structure I anticommutes with J , the latter interchanges the eigenspaces of I :

$$J: T_I^{1,0}M \rightarrow T_I^{0,1}M, \quad J: T_I^{0,1}M \rightarrow T_I^{1,0}M.$$

Recall that the bundle $T_I^{1,0}M$ can be endowed with a holomorphic structure given by an operator

$$\bar{\partial}: \Gamma(T_I^{1,0}M) \rightarrow \Omega_I^{0,1}M \otimes \Gamma(T_I^{1,0}M),$$

where $\Omega_I^{0,1}M$ is a space of $(0, 1)$ -forms with respect to I . Similarly, $T_I^{0,1}M$ can be endowed with an antiholomorphic structure

$$\partial: \Gamma(T_I^{0,1}M) \rightarrow \Omega_I^{1,0}M \otimes \Gamma(T_I^{0,1}M).$$

We will identify the complex bundle (TM, I) with $T_I^{1,0}M$ via the isomorphism

$$X \mapsto \frac{1}{2}(X - \sqrt{-1}IX). \quad (2.2)$$

Since ∇ preserves I , this isomorphism enables us to view the Obata connection as a connection on $T_I^{1,0}M$. Considered from this perspective, ∇ admits an especially simple description.

Proposition 2.2. *Let (M, I, J, K) be a hypercomplex manifold, $\dim_{\mathbb{R}} M = 4n$.*

1. The Obata connection $\nabla: \Gamma(T_I^{1,0}M) \rightarrow \Omega_{\mathbb{C}}M \otimes \Gamma(T_I^{1,0}M)$ is given by

$$\nabla = \bar{\partial} - J \partial J. \quad (2.3)$$

2. The curvature of the Obata connection is an $SU(2)$ -invariant 2-form with coefficients in $\text{End}_{\mathbb{H}}(TM)$, where $SU(2)$ is identified with the group of unit quaternions:

$$R(IX, IY)Z = R(JX, JY)Z = R(KX, KY)Z = R(X, Y)Z.$$

Proof. It is clear that the formula (2.3) defines a complex connection on $T_I^{1,0}M$. Note that under identification (2.2) the endomorphism J of the real tangent bundle maps to complex-antilinear operator $A: T_I^{1,0}M \rightarrow T_I^{1,0}M$, $AX = J\bar{X}$. A short calculation shows that ∇ preserves A :

$$\begin{aligned} (\nabla A)X &= \nabla(AX) - A\nabla X \\ &= \bar{\partial}J\bar{X} - J\partial J^2\bar{X} - J(\bar{\partial}X) + J(J\bar{\partial}J\bar{X}) \\ &= \bar{\partial}J\bar{X} + J\bar{\partial}X - J\partial\bar{X} - \bar{\partial}(J\bar{X}) = 0. \end{aligned}$$

This proves that ∇ preserves the hypercomplex structure.

It remains to check that the corresponding connection on TM is torsion-free. Let $e_i = \frac{1}{2}(\xi_i - \sqrt{-1}I\xi_i)$, $i = 1, \dots, 2n$ be a local holomorphic basis of $T_I^{1,0}M$, where ξ_i are pairwise commuting real vector fields. We have to show that $\nabla_{\xi_i}\xi_j = \nabla_{\xi_j}\xi_i$ and $\nabla_{I\xi_i}\xi_j = \nabla_{\xi_j}I\xi_i$ for all $i, j = 1, \dots, 2n$. Note that in view of the isomorphism (2.2)

$$\nabla_{\xi_i}\xi_j = \nabla_{\xi_i}e_j = \nabla_{e_i}^{1,0}e_j,$$

because $\nabla^{0,1}e_j = \bar{\partial}e_j = 0$. So it suffices to show that $\nabla_{e_i}^{1,0}e_j = \nabla_{e_j}^{1,0}e_i$, which is equivalent to $\partial_{e_i}Je_j = \partial_{e_j}Je_i$ according to (2.3).

Consider the vector fields

$$f_i = e_i - \sqrt{-1}Je_i \in T_J^{1,0}M.$$

Since the almost complex structure J is integrable we have $[f_i, f_j] \in T_J^{1,0}M$. We claim that $[f_i, f_j]$ is also contained in $T_I^{0,1}M$. Indeed,

$$[f_i, f_j] = -[Je_i, Je_j] - \sqrt{-1}([Je_i, e_j] + [e_i, Je_j]).$$

But since $Je_i \in T_I^{0,1}M$ we have $[Je_i, Je_j] \in T_I^{0,1}M$; moreover, because e_i are holomorphic $[Je_i, e_j] = -\partial_{e_j}Je_i \in T_I^{0,1}M$ and $[e_i, Je_j] = \partial_{e_i}Je_j \in T_I^{0,1}M$. So we have proved that $[f_i, f_j] \in T_I^{0,1}M \cap T_J^{1,0}M$. But the operators I and J anticommute and the intersection of their eigenspaces is trivial. We conclude that $[e_i - \sqrt{-1}Je_i, e_j - \sqrt{-1}Je_j] = 0$. Analogously, $[e_i + \sqrt{-1}Je_i, e_j + \sqrt{-1}Je_j] = 0$, and it follows from these two equalities that $\partial_{e_i}Je_j - \partial_{e_j}Je_i = [e_i, Je_j] - [e_j, Je_i] = 0$. This completes the proof of the first part.

To prove the second part, note that according to (2.3) $\nabla^{1,0} = -J\partial J$, $\nabla^{0,1} = \bar{\partial}$ and since $\bar{\partial}^2 = 0$, $\partial^2 = 0$, $J^2 = -Id$ we have $(\nabla^{0,1})^2 = 0$, $(\nabla^{1,0})^2 = 0$. The standard argument (which works for the Chern connection, for example) shows that the curvature R is contained in $\Lambda_I^{1,1}M \otimes \text{End}(T_I^{1,0}M)$, and this implies $R(IX, IY)Z = R(X, Y)Z$. Next, note that the complex structure I has been chosen arbitrarily from the whole 2-dimensional sphere of complex structures on M , thus if we replace I with J and K the analogous reasoning shows that $R(JX, JY)Z = R(KX, KY)Z = R(X, Y)Z$. Finally, for every X and Y the endomorphism $R(X, Y)$ is \mathbb{H} -linear since the Obata connection preserves the hypercomplex structure. \square

In order to study the hypercomplex structures on Lie groups, it will be convenient to express the Obata connection in terms of the commutator of real vector fields. We are going to use the following well-known formula for the $\bar{\partial}$ -operator (see [Ga], where this operator appears in a similar fashion).

Proposition 2.3. *Let M be a smooth manifold and I a complex structure on it. Considering (TM, I) as a holomorphic bundle (using the isomorphism (2.2)), we can write the corresponding $\bar{\partial}$ -operator as*

$$\bar{\partial}_X Y = \frac{1}{2}([X, Y] + I[IX, Y]). \quad (2.4)$$

Proof. It is clear that (2.4) is \mathbb{R} -linear in both X and Y . Moreover, since the Nijenhuis tensor of I (2.1) vanishes we see that $\bar{\partial}_X(IY) = I\bar{\partial}_X Y$, i.e (2.4) is \mathbb{C} -linear in Y . Next, observe that it satisfies the Leibniz rule:

$$\begin{aligned} \bar{\partial}_X(fY) &= \frac{1}{2}([X, fY] + I[IX, fY]) \\ &= \frac{1}{2}(f([X, Y] + I[IX, Y]) + (\mathcal{L}_X f)Y + (\mathcal{L}_{IX} f)IY) \\ &= f\bar{\partial}_X Y + \frac{1}{2}(\mathcal{L}_X f + \sqrt{-1}\mathcal{L}_{IX} f)Y = f\bar{\partial}_X Y + (\bar{\partial}_X f)Y, \end{aligned}$$

and that it is $C^\infty(M)$ -linear in X :

$$\bar{\partial}_{fX}Y = \frac{1}{2}([fX, Y] + I[fIX, Y]) = f\bar{\partial}_X Y - \frac{1}{2}(\mathcal{L}_Y f)(X + I^2X) = f\bar{\partial}_X Y.$$

Next we have to show that (2.4) vanishes when Y is holomorphic. But it is known that Y is a holomorphic section of (TM, I) if and only if $\mathcal{L}_Y I = 0$. Now, $(\mathcal{L}_Y I)(IX) = [Y, I^2X] - I[Y, IX] = 2\bar{\partial}_X Y = 0$.

Since the properties that we have checked above uniquely determine the $\bar{\partial}$ -operator of a holomorphic vector bundle, this completes the proof. \square

It follows from Propositions 2.2 and 2.3 that the Obata connection on a hypercomplex manifold (M, I, J, K) can be written in the following form:

$$\nabla_X Y = \frac{1}{2}([X, Y] + I[IX, Y] - J[X, JY] + K[IX, JY]). \quad (2.5)$$

3 Hypercomplex structures on Lie groups

In this section we review the construction of homogeneous hypercomplex structures on compact Lie groups following Joyce [J]. Let G be a compact semisimple Lie group and \mathfrak{g} its Lie algebra. Let $\mathfrak{t} \subset \mathfrak{g}$ be a maximal torus.

The first step in constructing the hypercomplex structure is to obtain the following decomposition of \mathfrak{g} (cf. [J], Lemma 4.1):

$$\mathfrak{g} = \mathfrak{b} \oplus \bigoplus_{k=1}^n \mathfrak{d}_k \oplus \bigoplus_{k=1}^n \mathfrak{f}_k,$$

where \mathfrak{b} is an abelian subalgebra, \mathfrak{d}_k are subalgebras isomorphic to $\mathfrak{su}(2)$ and \mathfrak{f}_k are subspaces with the following properties:

1. $[\mathfrak{d}_k, \mathfrak{b}] = 0$ and $\mathfrak{t} \subset \mathfrak{b} \oplus \bigoplus_{k=1}^n \mathfrak{d}_k$;
2. $[\mathfrak{d}_k, \mathfrak{f}_j] = 0$ for $j > k$;
3. $[\mathfrak{d}_k, \mathfrak{f}_k] \subset \mathfrak{f}_k$ and this Lie bracket action of \mathfrak{d}_k on \mathfrak{f}_k is isomorphic to the direct sum of some number of copies of $\mathfrak{su}(2)$ -action on \mathbb{C}^2 by matrix multiplication from the left.

Note that $\mathfrak{d}_k \oplus \mathfrak{u}(1) \simeq \mathfrak{su}(2) \oplus \mathfrak{u}(1)$ can be identified with the quaternion algebra \mathbb{H} . Since the subalgebra \mathfrak{b} is isomorphic to a direct sum of $\mathfrak{u}(1)$'s we can (after possibly adding some extra copies of $\mathfrak{u}(1)$, i.e. multiplying G by some number of S^1) identify $\mathfrak{b} \oplus \bigoplus_{k=1}^n \mathfrak{d}_k$ with \mathbb{H}^m for some m . Denote by $\mathcal{I}_k, \mathcal{J}_k, \mathcal{K}_k$ the elements of \mathfrak{d}_k corresponding to the standard imaginary quaternions under the identification $\mathfrak{d}_k \oplus \mathfrak{u}(1) \simeq \mathbb{H}$. We define a triple of complex structures $I, J, K \in \text{End}(\mathfrak{g})$ as follows: the action of I, J, K on $\mathfrak{b} \oplus \bigoplus_{k=1}^n \mathfrak{d}_k \simeq \mathbb{H}^m$ is multiplication by the corresponding imaginary quaternion from the left and the action on \mathfrak{f}_k is given by

$$IX = [\mathcal{I}_k, X], \quad JX = [\mathcal{J}_k, X], \quad KX = [\mathcal{K}_k, X]$$

for $X \in \mathfrak{f}_k$. The endomorphisms I, J, K define three left-invariant almost-complex structures on G . One can check ([J], Lemma 4.3) that they are integrable and satisfy the quaternionic relations thus giving a hypercomplex structure on G .

We are interested in the case when $G = SU(3)$. The Lie algebra \mathfrak{g} is the algebra of 3×3 skew-Hermitian trace-free matrices. Such a matrix can be represented in the form

$$\begin{pmatrix} D & f \\ -\bar{f}^t & b \end{pmatrix} \tag{3.1}$$

where $D \in \mathfrak{u}(2)$, $f \in \mathbb{C}^2$ is a column-vector and $b \in \mathbb{C}$ with $\text{tr}(D) + b = 0$. The decomposition of \mathfrak{g} described above takes form $\mathfrak{g} = \mathfrak{b} \oplus \mathfrak{d} \oplus \mathfrak{f}$ where \mathfrak{d} consists of matrices with zero f and b , \mathfrak{f} — of matrices with zero D and b and \mathfrak{b} consists of diagonal matrices commuting with \mathfrak{d} . Note that the adjoint action of \mathfrak{b} preserves \mathfrak{f} and $[\mathfrak{f}, \mathfrak{f}] \subset \mathfrak{b} \oplus \mathfrak{d}$ thus we obtain $\mathbb{Z}/2\mathbb{Z}$ -grading: $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ with $\mathfrak{g}_0 = \mathfrak{b} \oplus \mathfrak{d}$ and $\mathfrak{g}_1 = \mathfrak{f}$.

We can also mention that it is possible to choose the identification $\mathfrak{b} \oplus \mathfrak{d} \simeq \mathbb{H}$ and thus the corresponding hypercomplex structure in such a way that the Killing form will be quaternionic Hermitian. This turns G into an HKT-manifold [GP].

4 Holonomy of the Obata connection

4.1 The Euler vector field

Consider the Lie group $G = SU(3)$ with the hypercomplex structure described above. The Lie algebra of G is $\mathbb{Z}/2\mathbb{Z}$ -graded: $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$, where

$\mathfrak{g}_0 \simeq \mathfrak{su}(2) \oplus \mathfrak{u}(1)$ will be identified with the algebra of quaternions \mathbb{H} , and \mathfrak{g}_1 is a \mathfrak{g}_0 -module with the action of \mathbb{H} obtained from the adjoint action of \mathfrak{g}_0 as described in the previous section.

We will identify the elements of \mathfrak{g} and left-invariant vector fields on G . Denote by \mathcal{E} the element of \mathfrak{g}_0 (and the vector field) corresponding to $-1 \in \mathbb{H}$ under the isomorphism $\mathfrak{g}_0 \simeq \mathbb{H}$. We will call \mathcal{E} the Euler vector field. Choose also some non-zero element $W \in \mathfrak{g}_1$. Then $\langle \mathcal{E}, W \rangle$ form an \mathbb{H} -basis in \mathfrak{g} . Recall that the action of \mathbb{H} on \mathfrak{g}_1 is given by

$$IW = [W, I\mathcal{E}], \quad JW = [W, J\mathcal{E}], \quad KW = [W, K\mathcal{E}].$$

Remark 4.1. Note that the subgroup G_0 corresponding to \mathfrak{g}_0 is isomorphic to $SU(2) \times U(1)$ and it is a hypercomplex submanifold of G . If we identify \mathfrak{g}_0 with the quaternion algebra \mathbb{H} then the hypercomplex structure is given by left quaternionic multiplication. It follows from (2.5) that the Obata connection in this case is given by

$$\nabla_X Y = -Y \cdot X$$

for any $X, Y \in \mathfrak{g}_0$, where \cdot is multiplication in $\mathbb{H} \simeq \mathfrak{g}_0$. It is easy to check that the Obata connection on G_0 is flat. The group $G_0 \simeq SU(2) \times U(1)$ is diffeomorphic to a Hopf manifold $(\mathbb{R}^4 \setminus \{0\})/\Gamma$, where Γ is an infinite cyclic group generated by the homothety $z \mapsto \lambda z$ for some $\lambda \in \mathbb{R}_{>0}$. The vector field on G_0 corresponding to $\mathcal{E} \in \mathfrak{g}_0$ lifts to the ordinary Euler vector field on $\mathbb{R}^4 \setminus \{0\}$ which generates the flow of homotheties. It is remarkable that the Euler vector field \mathcal{E} on $SU(3)$ retains some useful properties, as we show in the following proposition. It should be mentioned that the vector field \mathcal{E} appeared in [PPS], but the notation in that paper slightly differs from ours.

Proposition 4.2. *The vector field \mathcal{E} possesses the following properties:*

1. \mathcal{E} is holomorphic with respect to I, J, K ;
2. $\nabla \mathcal{E} = Id$, where Id is understood as a section of $\Lambda^1 G \otimes TG \simeq \text{End}(TG)$;
3. $\nabla^2 \mathcal{E} = 0$;
4. If we denote by h the Killing form on \mathfrak{g} , then

$$\nabla_{\mathcal{E}} h = -2h, \quad \nabla_{I\mathcal{E}} h = \nabla_{J\mathcal{E}} h = \nabla_{K\mathcal{E}} h = 0.$$

Proof. 1. We have $(\mathcal{L}_\mathcal{E}I)X = [\mathcal{E}, IX] - I[\mathcal{E}, X]$ which obviously equals zero when $X \in \mathfrak{g}_0$ since \mathcal{E} lies in the center of \mathfrak{g}_0 . If $X \in \mathfrak{g}_1$ then $IX = [X, I\mathcal{E}]$ and $[\mathcal{E}, [X, I\mathcal{E}]] = [[\mathcal{E}, X], I\mathcal{E}] = I[\mathcal{E}, X]$, so again $(\mathcal{L}_\mathcal{E}I)X = 0$. The same argument applies to J and K .

2. For $X \in \mathfrak{g}_0$ we have $\nabla_X \mathcal{E} = -X \cdot \mathcal{E} = X$ (see Remark 4.1).

Now suppose that $X \in \mathfrak{g}_1$. It follows from 1 that $\bar{\partial}\mathcal{E} = 0$ and in view of (2.4) and (2.5)

$$\nabla_X \mathcal{E} = \frac{1}{2}(-J[X, J\mathcal{E}] + K[IX, J\mathcal{E}]) = \frac{1}{2}(-J^2X + KJIX) = X.$$

3. Immediately follows from 2.

4. A straightforward computation using the bi-invariance of the Killing form:

$$\begin{aligned} (\nabla_\mathcal{E}h)(X, Y) &= -h(\nabla_\mathcal{E}X, Y) - h(X, \nabla_\mathcal{E}Y) \\ &= -h(\nabla_X \mathcal{E} + [\mathcal{E}, X], Y) - h(X, \nabla_Y \mathcal{E} + [\mathcal{E}, Y]) \\ &= -2h(X, Y). \end{aligned}$$

The last three equalities are obtained analogously using the fact that h is quaternionic Hermitian. □

Remark 4.3. Note that if M is a compact manifold with a torsion-free connection ∇ then the existence of a vector field \mathcal{E} with $\nabla \mathcal{E} = Id$ has some strong implications for ∇ . Namely, observe that for any vector field X we have $\nabla_\mathcal{E}X = X + \mathcal{L}_\mathcal{E}X$ and $\nabla_\mathcal{E}\alpha = -\alpha + \mathcal{L}_\mathcal{E}\alpha$ for any 1-form α . Next, take a tensor field of type (k, m) : $T \in \Gamma((TM)^{\otimes k} \otimes (T^*M)^{\otimes m})$. Representing T locally as a sum of the elements of the form $X_1 \otimes \dots \otimes X_k \otimes \alpha_1 \otimes \dots \otimes \alpha_m$, we obtain $\nabla_\mathcal{E}T = (k - m)T + \mathcal{L}_\mathcal{E}T$. Suppose that ∇ preserves T ; then $\mathcal{L}_\mathcal{E}T = (m - k)T$. If T is non-zero at some point, take an integral curve of \mathcal{E} through this point and observe that unless $m = k$ the norm (with respect to an arbitrary metric) of T restricted to this integral curve will tend to infinity which is impossible for compact M . This means that ∇ can preserve tensor fields only of type (k, k) , as opposed to, say, Levi-Civita connection. Note also that the vector field \mathcal{E} is always unique when it exists, for if $\nabla \mathcal{E}' = Id$ then ∇ preserves $\mathcal{E} - \mathcal{E}'$ and therefore $\mathcal{E} - \mathcal{E}' = 0$.

4.2 Computation of the holonomy

We will need the following technical lemma.

Lemma 4.4. *Denote by R the curvature of the Obata connection.*

1. $R(X, IX)X + JR(X, KX)X - KR(X, JX)X = 0$ for all X ;
2. *Suppose that \mathcal{Z} is a vector field such that $R(X, Y)\mathcal{Z} = 0$ for any vector fields X and Y . Then $R(\mathcal{Z}, X)X = 0$ for all X .*

Proof. We will use the first Bianchi identity (which is true for any torsion-free connection) and the fact that the curvature of the Obata connection is an $SU(2)$ -invariant 2-form with coefficients in \mathbb{H} -linear endomorphisms by Proposition 2.2. We have:

$$\begin{aligned} R(X, IY)Z &= R(Z, IY)X + R(X, Z)IY \\ &= R(Z, IY)X + IR(Y, Z)X + IR(X, Y)Z, \end{aligned}$$

where the second equality follows from \mathbb{H} -linearity of $R(X, Z)$ and the first Bianchi identity. Similarly

$$R(X, IY)Z = R(Y, IX)Z = R(Z, IX)Y + IR(Y, Z)X,$$

and we obtain the following identity for any vector fields X, Y, Z :

$$R(Z, IX)Y = R(Z, IY)X + IR(X, Y)Z.$$

Substituting $Y = JX, Z = X$ yields the first claim of the lemma. It also follows that $R(\mathcal{Z}, IX)IX = -R(\mathcal{Z}, X)X$ and the same is true for J and K . Thus, $R(\mathcal{Z}, X)X = R(\mathcal{Z}, IJKX)IJKX = -R(\mathcal{Z}, X)X$ which proves the second claim. \square

Let us make a few remarks about the curvature of the Obata connection on $SU(3)$. Recall that we have the decomposition $\mathfrak{su}(3) = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ where \mathfrak{g}_0 and \mathfrak{g}_1 are one-dimensional \mathbb{H} -subspaces spanned by \mathcal{E} and W respectively. Note that it is possible to choose W in such a way that $\nabla_W W \neq 0$: if not we would have $\nabla_W W = 0$ for all $W \in \mathfrak{g}_1$. Since the Obata connection is \mathbb{H} -linear this would also imply $\nabla_{IW} W = \nabla_{JW} W = \nabla_{KW} W = 0$. Since \mathfrak{g}_1 is one-dimensional we would have $\nabla_X Y = 0$ and consequently $[X, Y] = 0$ for all X, Y in \mathfrak{g}_1 which is obviously not true.

Recall that we have the following expression for the curvature: $R(X, Y)Z = \text{Alt}(\nabla^2 Z)(X, Y)$, where $\nabla^2 Z \in \Lambda^1 G \otimes \Lambda^1 G \otimes TG$ is a bilinear form with values in vector fields and Alt means antisymmetrization of this form. From the third part of Proposition 4.2 we obtain $R(X, Y)\mathcal{E} = \text{Alt}(\nabla^2 \mathcal{E})(X, Y) = 0$, thus \mathfrak{g}_0 lies in the kernel of all the endomorphisms $R(X, Y)$.

We claim that $R(X, Y)\mathfrak{g}_1 = \mathfrak{g}_1$. Suppose that $X, Y \in \mathfrak{g}_0$, then the first Bianchi identity implies $R(X, Y)\mathfrak{g}_1 = 0$. Next take $X \in \mathfrak{g}_0$ and $Y \in \mathfrak{g}_1$; since the subspace \mathfrak{g}_1 is one-dimensional and the curvature is $SU(2)$ -invariant, it follows from the second part of Lemma 4.4 that $R(X, Y)Z = 0$ for any $Z \in \mathfrak{g}_1$. Note that the Obata connection respects the grading on \mathfrak{g} , consequently if $X, Y \in \mathfrak{g}_1$ then $R(X, Y)\mathfrak{g}_1 \subset \mathfrak{g}_1$. We remark that the image of $R(X, Y)$ must be nontrivial for some X, Y , because otherwise the Obata connection would be flat, which is not the case. We will need the following statement.

Proposition 4.5. *The holonomy group of the Obata connection contains an element that acts identically on \mathfrak{g}_0 and multiplies \mathfrak{g}_1 by a non-zero non-real quaternion.*

Proof. By Ambrose-Singer theorem (see e.g. [Bes]) the Lie algebra of the holonomy group contains all the endomorphisms $R(X, Y)$. If $X, Y \in \mathfrak{g}_1$, then the endomorphism $R(X, Y)$ acts trivially on \mathfrak{g}_0 and preserves \mathfrak{g}_1 . Recall that \mathfrak{g}_1 is one-dimensional over \mathbb{H} and is generated by W . Put $Z_1 = R(W, IW)W$, $Z_2 = R(W, JW)W$, $Z_3 = R(W, KW)W$. It follows from the first part of Lemma 4.4 that the subspace generated by Z_1 , Z_2 and Z_3 is at least two-dimensional. Indeed, otherwise we would have $Z_i = \alpha_i Z_0$, $i = 1, 2, 3$, for some $\alpha_i \in \mathbb{R}$ and $Z_0 \in \mathfrak{g}_1$. Then by Lemma 4.4, $(\alpha_1 + \alpha_3 J - \alpha_2 K)Z_0 = 0$, and this would imply $Z_i = 0$ meaning that the connection is flat, which is not true. Thus the subalgebra generated by the endomorphisms $R(X, Y)$ with $X, Y \in \mathfrak{g}_1$ is at least two-dimensional. The claim of the proposition follows. \square

The proof of the main theorem will be based on the following.

Proposition 4.6. *The holonomy of the Obata connection on $SU(3)$ is irreducible.*

Proof. The proof will consist of two parts. First, we will show that there exist no left-invariant subbundles of TG that are preserved by the holonomy. Second, we will prove that there exist no holonomy-invariant subbundles at all.

Suppose that $\mathfrak{h} \subset \mathfrak{g}$ is a subspace corresponding to a left-invariant subbundle preserved by the holonomy. The left-invariance implies $\nabla_X Y \in \mathfrak{h}$ for all $X \in \mathfrak{g}$ and $Y \in \mathfrak{h}$. Let $V \in \mathfrak{h}$ and $V = V_0 + V_1$ where $V_0 \in \mathfrak{g}_0$, $V_1 \in \mathfrak{g}_1$. Then

$$\nabla_{\mathcal{E}} V = \nabla_V \mathcal{E} + [\mathcal{E}, V] = V + [\mathcal{E}, V_1],$$

because \mathcal{E} lies in the center of \mathfrak{g}_0 . We conclude that $[\mathcal{E}, V_1] \in \mathfrak{h}$. Note that under identification (3.1) of \mathfrak{g} with skew-Hermitian matrices, $\mathcal{E} \in \mathfrak{g}$ corresponds to a diagonal matrix with $D = -(b/2)Id$. It is easy to check that $(\text{ad}_{\mathcal{E}})^2$ acts on \mathfrak{g}_1 by real scalar multiplication, so we have $V_1 \in \mathfrak{h}$ and consequently $V_0 \in \mathfrak{h}$. If there exists some $V \in \mathfrak{h}$ with $V_0 \neq 0$, it follows that the Euler vector field \mathcal{E} lies in \mathfrak{h} and this implies $\mathfrak{h} = \mathfrak{g}$. Otherwise $\mathfrak{h} \subset \mathfrak{g}_1$. But this can happen only if $\mathfrak{h} = 0$: it was remarked above that \mathfrak{g}_1 is \mathbb{H} -spanned by W with $\nabla_W W \neq 0$ and it follows from \mathbb{H} -linearity of ∇ that $\nabla_W V \neq 0$ and lies in \mathfrak{g}_0 for non-zero $V \in \mathfrak{g}_1$.

Now, we proceed to the second part of the proof. Let $L_g: G \rightarrow G$ denote the left translation $h \mapsto gh$. Suppose that there exists some (not left-invariant) proper subbundle B preserved by the holonomy. Then for any $g \in G$ the subbundle $L_g^* B$ is also preserved by the holonomy, thus there exists a continuous family of holonomy-invariant subbundles. We claim that it is possible to find a holonomy-invariant subbundle B with the following properties:

1. $\dim_{\mathbb{R}} B = 4$,
2. B is invariant with respect to some of the complex structures,
3. $\dim_{\mathbb{R}}(B \cap L_g^* B)$ is either 0 or 4 for all $g \in G$.

We will first find a subbundle that possesses the first two properties. Consider holonomy-invariant subbundle B of a minimal possible dimension. Then $\dim_{\mathbb{R}} B$ must be less or equal to 4, otherwise we could replace B with $B \cap L_g^* B$ which is a proper subbundle of B for some $g \in G$. Next, we consider the four possibilities. If $\dim_{\mathbb{R}} B = 1$, we can take the \mathbb{H} -span of B and obtain \mathbb{H} -invariant subbundle of real dimension 4. If $\dim_{\mathbb{R}} B = 2$, we can take $B + IB$. If $\dim_{\mathbb{R}} B = 3$, we can take $B + IB$ and obtain a subbundle of complex dimension 2 or 3. In the former case we are done, and in the latter case, we can intersect the subbundle with its left translation and decrease its dimension. Consider the case when $\dim_{\mathbb{R}} B = 4$. Then B is either I -invariant or $B \cap IB = 0$. In the latter case $B \oplus IB$ is a complex representation of

the holonomy group. Suppose that it is irreducible. Consider the operator \mathcal{C} that fixes B and multiplies IB by -1 . This operator is I -antilinear and is preserved by the holonomy, and so is the complex structure J . The composition $J\mathcal{C}$ is I -linear and thus by Schur's lemma must be equal to λId with $\lambda \in \mathbb{C}$. But since $\mathcal{C}^2 = Id$, we have $\lambda\mathcal{C} = J$ and $-Id = J^2 = \lambda\mathcal{C}\lambda\mathcal{C} = |\lambda|^2 Id$ which is impossible. Consequently, the representation $B \oplus IB$ is reducible. We can replace B with a proper I -invariant subbundle of TG preserved by the holonomy and of minimal dimension. The real dimension of B must be less or equal to 4, otherwise we could replace B with $B \cap L_g^*B$ for some $g \in G$. Since we are considering the case when the minimal dimension of such a subbundle is greater or equal to 4, $\dim_{\mathbb{R}} B = 4$ and we obtain a subbundle satisfying the first two requirements.

Now, if the subbundle B does not possess the third property, then there exists such $g \in G$ that $\dim_{\mathbb{R}}(B \cap L_g^*B) = 2$. We can then replace B with the \mathbb{H} -span of $B \cap L_g^*B$. Since $B \cap L_g^*B$ is I -invariant, its \mathbb{H} -span will have real dimension 4 and will satisfy all the three requirements.

Let the subbundle B possess all the three properties listed above. Since it can not be left-invariant, there exist $g_1, g_2 \in G$ such that $B, L_{g_1}^*B$ and $L_{g_2}^*B$ form a triple of pairwise complementary subbundles. Now we are going to use the following observation.

Lemma 4.7. *Let V be a $2n$ -dimensional vector space, and V_1, V_2, V_3 three pairwise complementary n -dimensional subspaces. Denote by P_{ij} the projection operator onto V_i along V_j . Then the algebra generated by P_{ij} is isomorphic to $\text{Mat}_2(\mathbb{R})$, the algebra of 2×2 matrices.*

Proof. Consider the operator $A = P_{12}P_{31}$. It maps V_2 isomorphically onto V_1 ; if we consider the decomposition $V = V_1 \oplus V_2$ and identify V_1 and V_2 via A then the operators $P_{12}, P_{21}, P_{12}P_{31}$ and $P_{21}P_{32}$ will have the block matrix forms $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ respectively. \square

The holonomy group preserves a triple of pairwise complementary subbundles. These subbundles are invariant with respect to some of the complex structures. We will fix this complex structure and consider TG as a complex vector bundle. The holonomy group must centralize the algebra generated by projections. Therefore we can choose an isomorphism of vector spaces $\mathfrak{g} \simeq \mathbb{C}^2 \otimes_{\mathbb{C}} \mathbb{C}^2$ with the holonomy acting trivially on the first factor and non-trivially on the second. In particular, each operator in the holonomy group must have at most two distinct eigenvalues. But Proposition 4.5 implies that

the holonomy group contains an operator with three distinct eigenvalues (one real, equal to 1, and two complex-conjugate). This contradiction ends the proof of irreducibility of the holonomy. \square

Now we are ready to prove the main theorem.

Theorem 4.8. *The holonomy group of the Obata connection on $SU(3)$ with the homogeneous hypercomplex structure is $GL(2, \mathbb{H})$.*

We will give two proofs of the theorem: one using the classification of irreducible holonomy groups, the other using one result of Kostant, [K].

Proof. By Proposition 4.6 the holonomy is irreducible. The statement of the theorem follows from the classification of irreducible holonomies from [MS]. Indeed, the Obata connection on $SU(3)$ does not preserve any metric (see Remark 4.3). Here is the list of non-metric holonomy groups with representation space \mathbb{R}^8 ($T_{\mathbb{F}}$ denotes any connected Lie subgroup of \mathbb{F}^*):

From Table 2 in [MS]	From Table 3 in [MS]
$T_{\mathbb{R}} \cdot SL(8, \mathbb{R})$	$SL(2, \mathbb{C})$ acting on $S^3\mathbb{C}^2$
$T_{\mathbb{C}} \cdot SL(4, \mathbb{C})$	$\mathbb{C}^* \cdot SL(2, \mathbb{C})$ acting on $S^3\mathbb{C}^2$
$T_{\mathbb{R}} \cdot SL(2, \mathbb{H})$	$\mathbb{C}^* \cdot Sp(2, \mathbb{C})$
$Sp(4, \mathbb{R})$	$SL(2, \mathbb{R}) \cdot SO(p, q), p + q = 4$
$Sp(2, \mathbb{C})$	$Sp(1) \cdot SO(2, \mathbb{H})$
$\mathbb{R}^* \cdot SO(p, q), p + q = 8$	
$T_{\mathbb{C}} \cdot SO(4, \mathbb{C})$	
$T_{\mathbb{R}} \cdot SL(m, \mathbb{R}) \cdot SL(n, \mathbb{R}), mn = 8$	
$T_{\mathbb{R}} \cdot SL(m, \mathbb{H}) \cdot SL(n, \mathbb{H}), mn = 2$	

The most of the entries in the list are obviously not contained in $GL(2, \mathbb{H})$ because of dimension reasons or because they do not preserve any complex structure. Note that the action of $SL(2, \mathbb{C})$ on $S^3\mathbb{C}^2$ does not preserve quaternionic structure, because it does not commute with any non-scalar \mathbb{R} -linear operator. Indeed, let $A \in \text{End}(\mathbb{R}^8)$ be a real endomorphism commuting with the action of $SL(2, \mathbb{C})$ on $S^3\mathbb{C}^2 \simeq \mathbb{R}^8$. Consider the weight decomposition $S^3\mathbb{C}^2 = \bigoplus_{\lambda} V_{\lambda}$ where V_{λ} is an eigenspace of $H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathfrak{sl}(2, \mathbb{C})$ with eigenvalue λ , and the sum runs over $\lambda = 3, 1, -1, -3$. Since the eigenvalues are real, A must preserve the eigenspaces V_{λ} . Moreover, A has to be \mathbb{C} -linear because it commutes with $\sqrt{-1}H \in \mathfrak{sl}(2, \mathbb{C})$ which has the same eigenspaces V_{λ} with eigenvalues $\sqrt{-1}\lambda$. By Schur's lemma A must be equal to a scalar

operator. Thus, the groups $SL(2, \mathbb{C})$ and $\mathbb{C}^* \cdot SL(2, \mathbb{C})$ can not occur as holonomy groups of the Obata connection.

The list contains only one proper subgroup of $GL(2, \mathbb{H})$, namely $SL(2, \mathbb{H})$. But if the holonomy was $SL(2, \mathbb{H})$, the Obata connection would preserve a holomorphic volume form, and this is impossible (see Remark 4.3). Thus, the holonomy group must coincide with $GL(2, \mathbb{H})$. \square

Alternative proof. We denote by $\mathfrak{hol} \subset \mathfrak{gl}(\mathfrak{g}_0 \oplus \mathfrak{g}_1)$ the holonomy algebra. Fixing one of the complex structures, we identify $\mathfrak{g}_0 \oplus \mathfrak{g}_1$ with \mathbb{C}^4 . Then \mathfrak{hol} is a real Lie subalgebra of $\mathfrak{gl}(4, \mathbb{C})$. Denote by $\mathfrak{hol}_{\mathbb{C}}$ the \mathbb{C} -subalgebra generated by \mathfrak{hol} . We need to prove that $\mathfrak{hol}_{\mathbb{C}} \simeq \mathfrak{gl}(4, \mathbb{C})$.

By Proposition (4.6) $\mathfrak{g}_0 \oplus \mathfrak{g}_1$ is a simple \mathfrak{hol} -module. This implies that it is also a simple $\mathfrak{hol}_{\mathbb{C}}$ -module. According to [K], it suffices to prove that $\mathfrak{hol}_{\mathbb{C}}$ contains a non-nilpotent endomorphism of rank one. Recall from Proposition (4.5) and its proof that \mathfrak{hol} contains a two-dimensional subspace \mathfrak{h} whose elements act trivially on \mathfrak{g}_0 . We identify \mathfrak{h} with a subspace in $\mathfrak{gl}(\mathfrak{g}_1)$. Let $\mathfrak{h}_{\mathbb{C}}$ be the \mathbb{C} -span of \mathfrak{h} in $\mathfrak{gl}(\mathfrak{g}_1) \simeq \mathfrak{gl}(2, \mathbb{C})$. A non-zero element $A \in \mathfrak{h}_{\mathbb{C}}$ has rank one if and only if its determinant is zero, and it is nilpotent if and only if both determinant and trace are zero. The determinant is a quadratic form on $\mathfrak{gl}(2, \mathbb{C})$, so $\mathfrak{h}_{\mathbb{C}}$ contains endomorphisms of rank one. The elements with trace zero form a one-dimensional \mathbb{R} -subspace in \mathfrak{h} . Since a non-zero $A \in \mathfrak{h}$ acts as quaternionic multiplication, it is invertible, hence its determinant is not zero. We see that the endomorphisms in $\mathfrak{h}_{\mathbb{C}}$ are not nilpotent, and the result follows. \square

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