Prof. Dr. Daniel Huybrechts
Dr. Andrey Soldatenkov

Winter term 2016/17

Intersection theory and pure motives, Exercises – Week 8

Exercise 32. *Cohomology of the projective line.*
Show that $H^1(\mathbb{P}^1) = 0$ for every Weil cohomology theory.

Exercise 33. *Universal families.*
Consider the universal family $\mathcal{X} \to |\mathcal{O}(d)|$ of hypersurfaces $X \subset \mathbb{P}^n$ of degree d. Show that $\mathrm{CH}^{n-1}(\mathcal{X}_\eta) \cong \mathbb{Z}$, where $\eta \in |\mathcal{O}(d)|$ is the generic point. Find an example where $\mathrm{CH}^{n-1}(\mathcal{X}_t) \not\cong \mathbb{Z}$ for some closed point t.

Exercise 34. *Projection formula.*
Fix a Weil cohomology theory for a field k with coefficient field K.

(i) Use the axioms of a Weil cohomology theory to prove the projection formula

$$f_*(\alpha \cdot f^*\beta) = f_*(\alpha) \cdot \beta$$

for a morphism $f\colon X \to Y$ of smooth projective varieties. Here, $f^*\colon H^*(Y) \to H^*(X)$ is the pull-back and $f_*\colon H^*(X)(\dim X) \to H^{*-2(\dim X - \dim Y)}(Y)(\dim Y)$ the induced *Gysin morphism*.

(ii) Let $f\colon X \to Y$ be a generically finite morphism of degree d. Show that $f_* f^* = d \cdot \mathrm{id}$ on $H^*(Y)$.

Exercise 35. *Lefschetz trace formula.*
Fix a Weil cohomology theory for a field k with coefficient field K of characteristic zero.

(i) Use the formula $\langle \delta, \mathrm{cl}[\Delta] \rangle = \sum (-1)^i \mathrm{tr}(\delta_*|_{H^i(X)})$ (see Friday lecture) for $\delta \in H^{\dim X}(X \times X)(\dim X)$ to prove its more general version

$$\langle \delta, {}^t\gamma \rangle = \sum (-1)^i \mathrm{tr}((\gamma \circ \delta)_*|_{H^i(X)}),$$

where $\delta \in H^{2\dim Y + i}(X \times Y)(\dim Y + n)$ and $\gamma \in H^{2\dim X - i}(Y \times X)(\dim X - n)$.

(ii) Convince yourself that this can be used to conclude the proof of Jannsen's theorem.

(iii) Let X be a smooth projective variety and $Y \subset X$ a smooth hyperplane section such that $\mathrm{cl}^1[Y] \in H^2(X)$ generates the K-algebra $H^*(X)$ (for simplicity trivialize the Tate twist). Show that every automorphism $f\colon X \xrightarrow{\sim} X$ has a fixed point.

Exercise 36. *Degree of the exceptional divisor.*
Let $X \to \mathbb{P}^n$ be the blow-up in a linear subspace $\mathbb{P}^m \subset \mathbb{P}^n$. Denote the exceptional divisor by $E \subset X$. Compute $\deg[E]^n$.

Due Wednesday 14 December, 2016.