Prof. Dr. Daniel Huybrechts
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Intersection theory and pure motives, Exercises – Week 7

Exercise 27. *Adequate equivalence relation.*
Show that rational equivalence $\sim_{\text{rat}}$ and numerical equivalence $\sim_{\text{num}}$ are adequate equivalence relations (assuming the moving lemma). Furthermore, prove that for any adequate equivalence relation $\sim$ one has:

$$\alpha_1 \sim_{\text{rat}} \alpha_2 \quad \Rightarrow \quad \alpha_1 \sim \alpha_2 \quad \Rightarrow \quad \alpha_1 \sim_{\text{num}} \alpha_2.$$

Exercise 28. *Finite group actions.*
Let G be a finite group acting on a smooth projective variety X of dimension d. For $g \in G$ let $[g]$ denote the class of the graph $\Gamma_g \subset X \times X$.

(i) Show that $p := (1/|G|) \sum_{g\in G}[g] \in \mathrm{CH}^d(X \times X)_{\mathbb{Q}}$ is a projector.

(ii) Assume that the group action is free with quotient $X \to Y$. Show that $(X, p) \cong \mathfrak{h}(Y)$. In other words, $(X, (1/|G|) \sum[g]) \cong \mathfrak{h}(X/G)$.

Exercise 29. *Riemann-Roch for surfaces.*
Let S be a smooth surface. For two divisors D, D' on S denote by (D, D') the intersection index. Deduce from Grothendieck-Riemann-Roch formula that for any divisor D

$$\chi(\mathcal{O}_S(D)) = \frac{1}{2}(D - K_S, D) + \chi(\mathcal{O}_S),$$

where K_S is the canonical divisor.

Exercise 30. *Hilbert polynomial.*
Let L be an ample line bundle on a smooth projective variety X of dimension n. Show that $h^0(X, L^k)$, $k \gg 0$ is a polynomial of degree n, i.e. $h^0(X, L^k) = \sum_{i=0}^n a_i k^i$, $a_i \in \mathbb{Q}$. Describe a_0, a_{n-1}, and a_n in terms of Chern numbers.

Exercise 31. *Lines in (cubic) surfaces.*
Consider the universal family of lines $\mathcal{L} \subset G(2, 4) \times \mathbb{P}^3$ in $\mathbb{P}^3$ with the two projections $p\colon \mathcal{L} \to G := G(2, 4)$ and $q\colon \mathcal{L} \to \mathbb{P}^3$. So, $\mathcal{L} \cong \mathbb{P}(\mathcal{S})$, where $\mathcal{S} \subset \mathcal{O}_G^{\oplus 4}$ is the universal subbundle. For $d > 0$ let $E_d := p_* q^* \mathcal{O}(d)$ which is a locally free sheaf of rank $d + 1$.

(i) Show that a hypersurface $X \subset \mathbb{P}^3$ defined by $s \in H^0(\mathbb{P}^3, \mathcal{O}(d))$ defines a section $\tilde{s} \in H^0(G, E_d)$ such that $\tilde{s}$ has a zero in $\ell \in G$ if and only if $\ell \subset X$.

(ii) Show that $\deg \mathrm{c}_4(E_3) = 27$ (which, at least morally, shows that every smooth cubic surface contains 27 lines).

Due Wednesday 7 December, 2016.