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Intersection theory and pure motives, Exercises – Week 5

Exercise 18. *Chern classes of dual bundles.*
Let E be a locally free sheaf on X. Assume that there exists a perfect pairing $E \otimes E \to \mathcal{O}_X$ (which could be symmetric or skew-symmetric or non of the two). Show that $2\mathrm{c}_i(E) = 0$ for i odd. Find an example where the 2 cannot be omitted.

Exercise 19. *Chern classes of subvarieties.*
Use the normal bundle sequence to compute the total Chern classes of the following bundles:

1. The normal bundle $\mathcal{N}_{\mathbb{P}^n/\mathbb{P}^m}$ of the d-fold Veronese embedding $\mathbb{P}^n \hookrightarrow \mathbb{P}^m$, $m = \binom{n+d}{d} - 1$.
2. The tangent bundle $\mathcal{T}_G$ of the Grassmannian $G = \mathrm{G}(2, 4)$.

Exercise 20. *Zero sets of regular sections.*
Let E be a locally free sheaf of rank r and $s \in H^0(X, E)$ a regular section with scheme of zeros $Z = Z(s) \subset X$ (which is then pure of codimension r).

1. Show that $\mathrm{c}_r(E) = [Z]$ in $\mathrm{CH}(X)$. See [F, Ex. 3.2.16].
2. Recall that there exists a locally free resolution $0 \to \bigwedge^r E^* \to \cdots \to E^* \to \mathcal{O}_X \to \mathcal{O}_Z \to 0$ (Koszul resolution). In view of the additivity of the Chern character it makes sense to define $\mathrm{ch}(\mathcal{O}_Z) := \sum (-1)^p \mathrm{ch}(\bigwedge^p E^*)$. Show that $[Z] = \mathrm{ch}(\mathcal{O}_Z)\mathrm{td}(E)$ in $\mathrm{CH}(X) \otimes \mathbb{Q}$. See [F, Ex. 3.2.5].
3. One can similarly define the total Chern class $\mathrm{c}(\mathcal{O}_Z)$ as $\prod_p \mathrm{c}(\bigwedge^p E^*)^{(-1)^p}$. Prove that $[Z] = (-1)^{r-1}(1/(r-1)!)\mathrm{c}_r(\mathcal{O}_Z)$.

Exercise 21. *Zero cycles on subvarieties.*
Let $Y \subset X$ be a closed subvariety of a variety X over k. The induced map $\mathrm{CH}_0(Y) \to \mathrm{CH}_0(X)$ is *universally surjective* if for all field extensions K/k the induced map $\mathrm{CH}_0(Y_K) \to \mathrm{CH}_0(X_K)$ is surjective (cf. Exercise 14).

(i) For $k = \bar{k}$ consider $X = \mathbb{P}^n$ and $\emptyset \neq Y \subset X$ any subvariety. Show that $\mathrm{CH}_0(Y) \to \mathrm{CH}_0(X)$ is universally surjective.

(ii) Show that a smooth projective curve C of genus $g(C) > 0$ does not contain a proper subvariety $Y \subset C$ with $\mathrm{CH}_0(Y) \to \mathrm{CH}_0(X)$ universally surjective. (But there are examples for which nevertheless $\mathrm{CH}_0(Y) \otimes \mathbb{Q} \to \mathrm{CH}_0(X) \otimes \mathbb{Q}$ is surjective.) What happens for $g(C) = 0$?

Due Wednesday 23 November, 2016.