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Winter term 2016/17

Intersection theory and pure motives, Exercises – Week 4

Exercise 14. *A lemma of Bloch-Srinivas*
Let X be a variety of dimension n over a field k, $V \subset X$ a proper closed subset and $U = X \backslash V$ its open complement. Assume that there exists $N \in \mathbb{Z}$, such that for any — not necessarily algebraic — field extension $k \subset K$ we have $\mathrm{CH}_0(U_K)$ is N-torsion, where U_K is the result of base change to K.

Let $\Delta \subset X \times X$ be the diagonal. Prove that there exists a proper closed subset $V' \subset X$, and cycles $W, W' \in \mathcal{Z}_n(X \times X)$ that satisfy the following conditions:

- $\mathrm{Supp(W)} \subset V \times X$ and $\mathrm{Supp(W')} \subset X \times V'$;
- $N[\Delta] = [W] + [W']$ in $\mathrm{CH}_n(X \times X)$.

Hint: use exercise 7.

Exercise 15. *Subring generated by Segre/Chern classes.*
Let X be a projective variety. Recall that for any vector bundle E we have defined the morphisms

$$s_i(E) \cap -\colon \mathrm{CH}^k(X) \to \mathrm{CH}^{k+i}(X). \tag{1}$$

We call them “intersection products with Segre classes”. Denote by $\mathrm{CH}^k_{\mathrm{vb}}(X)$ the subgroup generated by all cycles of the form $s_{i_1}(E_1) \cap \ldots \cap s_{i_p}(E_p) \cap [X]$ where $E_1, \ldots, E_p$ are vector bundles and $i_1 + \ldots + i_p = k$. Prove that the products (1) induce a natural commutative graded ring structure on $\mathrm{CH}^\bullet_{\mathrm{vb}}(X)$.

Note that since Segre classes are polynomials in Chern classes and visa versa, the same ring can be defined using Chern classes.

Exercise 16. *Chern classes under natural operations.*
Let E and F be vector bundles of rank r and s, and L be a line bundle on a variety X. Express the following in term of Chern classes of E, F and L:
1) $c_1(\Lambda^k E)$, $1 \leq k \leq r$; 2) $c_1(S^k E)$, $k \geq 1$; 3) $c_1(E \otimes F)$;
4) $c_i(E \otimes L)$, $i \geq 1$; 5) $c_2(\Lambda^2 E)$.

Exercise 17. *Examples of intersection products.*
Let X be a projective variety. For a cycle class $\alpha \in \mathrm{CH}_k(X)$ and line bundles $L_1, \ldots, L_k$ the intersection index of α with $L_1, \ldots, L_k$ is the degree of the zero-cycle $c_1(L_1) \cap \ldots \cap c_1(L_k) \cap \alpha$. It will be denoted by $L_1 \cdot \ldots \cdot L_k \cdot \alpha$. Compute:

1) $\mathcal{O}(d_1) \cdot \ldots \cdot \mathcal{O}(d_n)$ on $\mathbb{P}^n$;
2) $\mathcal{O}_X(\Delta) \cdot \mathcal{O}_X(\Delta)$, where $X = C \times C$, C is a smooth projective curve, Δ is the diagonal;
3) $\mathcal{O}_X(E) \cdot [l]$, where X is the blow-up of a smooth projective variety Y along a smooth subvariety Z, E is the exceptional divisor and l is a line in the fibre over some point of Z.

Due Wednesday 16 November, 2016.