Prof. Dr. Daniel Huybrechts
Dr. Andrey Soldatenkov

Winter term 2016/17

Intersection theory and pure motives, Exercises – Week 13

Exercise 56. *Finite-dimensionality of homological motives.*
In class we have seen that the sign conjecture $C^+(X)$ for a smooth projective variety X implies that its homological motive $\mathfrak{h}(X)$ in $\mathrm{Mot}_H(k)$ is finite-dimensional in the sense of Kimura–O'Sullivan. Now, assume that the sign conjecture $C^+(X)$ holds for all $X \in \mathrm{SmProj}(k)$ and deduce from it that every $M \in \mathrm{Mot}_H(k)$ is finite-dimensional.

Exercise 57. *Natural bundles on the Jacobian.*
Let C be a smooth projective curve with Poincaré bundle $\mathcal{P}$ on $J(C) \times C$. As in class, let $E_n \coloneqq p_*(\mathcal{P} \otimes q^*\mathcal{O}(nx_0))$ for some distinguished point $x_0 \in C(k)$. Recall that for $n \geq 2g(C)-1$ the sheaf E_n is locally free of rank $n+1-g(C)$.

(i) Prove that E_n is not a trivial locally free sheaf, i.e. $E_n \not\cong \mathcal{O}_{J(C)}^{\oplus n+1-g(C)}$.

(ii) Show that $\mathrm{c}(E_{n+1}) = \mathrm{c}(E_n)$, $n \geq 2g(C)-1$.

Exercise 58. *Finite-dimensionality of the universal hypersurface.*
Let $\mathcal{X} \to |\mathcal{O}_{\mathbb{P}^n}(d)|$ be the universal hypersurface of degree d in $\mathbb{P}^n$. Show that $\mathfrak{h}(\mathcal{X})$ is finite-dimensional in the sense of Kimura–O'Sullivan.

Exercise 59. *Finite-dimensionality under base change.*
Let $M \in \mathrm{Mot}(k)$ and let K/k be a field extension. Prove that M in $\mathrm{Mot}(k)$ is even/odd finite-dimensional in the sense of Kimura–O'Sullivan if and only if $M \otimes K$ in $\mathrm{Mot}(K)$ is. (Here, for $M = (X, p, m)$ one defines $M \otimes K \coloneqq (X \times_k K, p \times 1_K, m)$.)

Exercise 60. *Atiyah flop.*
Let V_1, V_2 be two-dimensional vector spaces. On $\mathbb{P}(V_1)$ consider the vector bundle $\mathcal{E}_1 = \mathcal{O} \oplus (\mathcal{O}(-1) \otimes V_2)$ and let $p_1 \colon \mathbb{P}(\mathcal{E}_1) \to \mathbb{P}(V_1)$; on $\mathbb{P}(V_2)$ consider the vector bundle $\mathcal{E}_2 = \mathcal{O} \oplus (\mathcal{O}(-1) \otimes V_1)$ and let $p_2 \colon \mathbb{P}(\mathcal{E}_2) \to \mathbb{P}(V_2)$. On $\mathbb{P}(V_1) \times \mathbb{P}(V_2)$ consider $\mathcal{F} = \mathcal{O} \oplus \mathcal{O}(-1,-1)$ and let $q \colon \mathbb{P}(\mathcal{F}) \to \mathbb{P}(V_1) \times \mathbb{P}(V_2)$. Denote the two projections from $\mathbb{P}(V_1) \times \mathbb{P}(V_2)$ to its two factors by π_1, π_2. Observe that there are natural inclusions of vector bundles $\mathcal{F} \hookrightarrow \pi_1^*\mathcal{E}_1$ and $\mathcal{F} \hookrightarrow \pi_2^*\mathcal{E}_2$. Composing the pull-backs of these inclusions by q with $\mathcal{O}_q(-1) \hookrightarrow q^*\mathcal{F}$ we embed $\mathcal{O}_q(-1)$ into $q^*\pi_1^*\mathcal{E}_1$ and $q^*\pi_2^*\mathcal{E}_2$. This defines two maps ε_1, ε_2 that fit into the following commutative diagram:

$$
\begin{array}{ccccc}
\mathbb{P}(\mathcal{E}_1) & \xleftarrow{\ \varepsilon_1\ } & \mathbb{P}(\mathcal{F}) & \xrightarrow{\ \varepsilon_2\ } & \mathbb{P}(\mathcal{E}_2) \\
\downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle p_2} \\
\mathbb{P}(V_1) & \xleftarrow{\ \pi_1\ } & \mathbb{P}(V_1) \times \mathbb{P}(V_2) & \xrightarrow{\ \pi_2\ } & \mathbb{P}(V_2)
\end{array}
$$

Denote by s_1, s_2 and t the sections of p_1, p_2 and q that correspond to the trivial summands of $\mathcal{E}_1$, $\mathcal{E}_2$ and $\mathcal{F}$ respectively. Prove that ε_1 is the blow-up of $\mathbb{P}(\mathcal{E}_1)$ in the image of s_1, and analogous statement for ε_2. The composition $\varepsilon_2 \circ \varepsilon_1^{-1}$ is a rational map from $\mathbb{P}(\mathcal{E}_1)$ to $\mathbb{P}(\mathcal{E}_2)$ which is called the Atiyah flop. Describe the induced action on the Chow ring (i.e. the morphism $\varepsilon_{2*} \circ \varepsilon_1^*$). Consider the morphism from $\mathbb{P}(\mathcal{E}_1)$ given by the full linear system $|\mathcal{O}_{p_1}(1)|$. Describe the image and the exceptional locus of this morphism. Analogous question for $\mathbb{P}(\mathcal{E}_2)$ and $\mathbb{P}(\mathcal{F})$. Observe that this construction works equally well when V_1 and V_2 have arbitrary dimensions.

Due Wednesday 1 February, 2017.