Prof. Dr. Daniel Huybrechts
Dr. Andrey Soldatenkov

Winter term 2016/17

Intersection theory and pure motives, Exercises – Week 12

Exercise 52. *Zero-dimensional cycles over finite fields.*
Let X be a smooth projective variety of dimension n over $k = \bar{k}$. The following two statements have been proved in class for $n = 1$. Show that they generalize to arbitrary dimension:

(i) $\mathrm{CH}^n(X)_0$ is divisible.

(ii) $\mathrm{CH}^n(X) \otimes \mathbb{Q} \cong \mathbb{Q}$ if $k = \bar{\mathbb{F}}_p$.

Exercise 53. *Voevodsky–Voisin for genus one.*
Reprove the theorem of Voevodsky and Voisin for curves of genus one. More precisely, let C be an elliptic curve over $k = \bar{k}$ and $\alpha \in \mathrm{CH}_0(C)_\mathbb{Q}$ of degree zero. Show that then $\alpha \times \alpha = 0$ in $\mathrm{CH}_0(C \times C)_\mathbb{Q}$. Generalize this to the statement $\alpha \times \alpha - d\Delta_* \alpha = 0$ in $\mathrm{CH}_0(C \times C)_\mathbb{Q}$ for any $\alpha \in \mathrm{CH}_0(C)_\mathbb{Q}$ of degree d. (*Note:* The latter is still true for curves of genus $g(C) = 2, 3$ (Faber–Pandharipande), but fails for $g(C) \geq 4$ and e.g. $k = \mathbb{C}$ (Green–Griffiths, Yin).)

Exercise 54. *Motives of curves.*
Let C_1, C_2 be smooth projective curves over a field k. Show that

$$\mathrm{Mor}(\mathfrak{h}^1(C_1), \mathfrak{h}^1(C_2)) \cong \mathrm{Mor}(J(C_1), J(C_2)) \otimes \mathbb{Q},$$

where on the left hand side the morphisms are in the category of Chow motives and on the right hand side in the category of abelian varieties over k. (The integral version has been shown in class.) This can be used to prove that $\mathfrak{h}(C_1) \cong \mathfrak{h}(C_2)$ if $J(C_1)$ and $J(C_2)$ are isogenous.

Exercise 55. *Poincaré bundle as a correspondence.*
Let C be a smooth projective curve over a field k and let $J(C)$ be its Jacobian. Denote by $\mathcal{P}$ the Poincaré bundle on $J(C) \times C$ and consider it as a class in $\mathrm{CH}^1(J(C) \times C)$. Show that the induced map $\mathrm{CH}^g(J(C)) \to \mathrm{CH}^1(C)$ composed with the natural map $J(C)(k) \to \mathrm{CH}^g(J(C))$ gives back $J(C)(k) \to \mathrm{CH}^1(C)$, $t = [L] \mapsto \mathcal{P}|_{C \times t} \cong L$ and thus induces the canonical isomorphism $J(C)(k) \cong \mathrm{Pic}^0(C) \subset \mathrm{CH}^1(C)$. (*Warning:* The map $\mathrm{CH}^g(J(C)) \to \mathrm{CH}^1(C)$ is not an isomorphism in general.)

Note that $\mathcal{P}$ can also be used to define a natural homomorphism $\mathrm{CH}^1(C) \to \mathrm{CH}^1(J(C))$, which on $C(k) \subset \mathrm{CH}^1(C)$ ($g(C) > 0$) is just $x \mapsto \mathcal{P}|_{J(C) \times x}$. As it turns out, this induces an isomorphism $J(C)(k) \cong \mathrm{Pic}^0(C) \cong \mathrm{Pic}^0(J(C))$. (The Jacobian is a principally polarized abelian variety and, hence, $J(C)$ is isomorphic to its dual.)

Due Wednesday 25 January, 2017.