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Winter term 2016/17

Intersection theory and pure motives, Exercises – Week 1

Exercise 1. *Linear equivalence on curves.*
(i) Recall that two distinct closed points $x, y \in C$ on a smooth projective non-rational curve C cannot be linearly (or, equivalently, rationally) equivalent. Could it happen that $2[x] \sim 2[y]$? Is the projectivity needed in this statement?

(ii) Consider the cubic $C \subset \mathbb{P}^2_{\mathbb{C}}$ given by $y^2z = x^3 + 3xz^2$. Denote $\omega = \frac{1+\sqrt{-3}}{2}$, $\bar{\omega} = \frac{1-\sqrt{-3}}{2}$. Show that the points $x_1 \coloneqq [1+\omega : \sqrt{3}(1+\omega) : 1]$, $x_2 \coloneqq [1+\bar{\omega} : \sqrt{3}(1+\bar{\omega}) : 1]$, $x_3 \coloneqq [1+\omega : -\sqrt{3}(1+\omega) : 1]$, $x_4 \coloneqq [1+\bar{\omega} : -\sqrt{3}(1+\bar{\omega}) : 1]$ are contained in C and satisfy $[x_1] + [x_2] \sim [x_3] + [x_4]$.

Is the following statement true: If C is a smooth projective curve and $x_1, x_2, x_3, x_4 \in C$ are (pairwise) distinct points with $[x_1] + [x_2] \sim [x_3] + [x_4]$, then $g(C) \leq 1$?

Exercise 2. *Rationality of conics.*
Consider the degree map $\deg\colon \mathrm{CH}_0(C) \to \mathbb{Z}$ for a a smooth conic $C \subset \mathbb{P}^2_k$. Show that C is rational if and only if deg is surjective, which in turn is equivalent to C admitting a line bundle of odd degree.

Exercise 3. *Fundamental classes of subschemes.*
Decide whether the following subschemes of $X \coloneqq V(xy^2) \subset \mathbb{A}^2_k = \mathrm{Spec}(k[x, y])$ define the same class in $Z(X)$:

(i) $V(\bar{x}^2)$ and $V(\bar{x})$; (ii) $V(\bar{y})$ and $V(\bar{y}^2)$; (iii) $V(\bar{x}, \bar{y})$ and $V(\bar{x}, \bar{y}^2)$.

Here, $\bar{x}, \bar{y}$ denote the images of x, y in the coordinate ring of X.

Exercise 4. *Order of vanishing.*
Consider the plane cubic $X \coloneqq V(y^2 - x^2(x+1)) \subset \mathbb{A}^2_k$. Compute $\mathrm{ord}_{(0,0)}(\bar{x})$ and $\mathrm{ord}_{(0,0)}(\bar{y})$.

Due Tuesday 25 October, 2016.