Kapitel 10

Seminar: Dirac Operators in Riemannian Geometry: The Theorems of Bochner and Lichnerowicz

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10.1 Introduction and some History

Today we want to talk about about the relations between Dirac operators or more precisely the induced Laplacian on a compact riemannian manifold and its geometry, i.e. the curvature and the Betti numbers of the manifold.
The methods I want to talk about were introduced by the jewish mathematician **Solomon Bochner** and is sometimes called the Bochner method.
Bochner was born in 1899 in Krakòv in Austria-Hungary (now Poland) and died in 1982 in Houston (Texas, USA). Solomon Bochner studied at the University of Warsaw. His Ph.D. from the University of Berlin in 1921 was on orthogonal systems of complex analytic functions. It was supervised by Schmidt. Bochner worked with Harald Bohr, Hardy and Littlewood in Copenhagen, Oxford and Cambridge respectively. Much of this work was on the zeta function. Bochner lectured in Munich from 1924 to 1933 and developed major results in harmonic analysis. His work developed into the theory of distributions. Driven out of Germany in 1933 he accepted a position at Princeton where he remained until he retired. He worked at this time on summation of Fourier series and was considered as one of the greatest experts on Fourier analysis. Bochner worked jointly with von Neumann for a while. His major books include Harmonic Analysis and the Theory of Probability (1955). In the 1960s he worked on the history and philosophy of mathematics. [1]
Using harmonic theory Bochner was thereby able to show that certain Betti numbers of X are vanishing under certain estimates on the curvature of a compact riemannian manifold. We are using in the first part the methods of Bochner. In the second part we will consider the formula by **André Lichnerowicz** (1915-1998) who follows Bochners way to compute the geometry of manifolds. The main part of the work of Lichnerowicz is about mathematical physics and riemannian geometry but he pusblished in different other areas, too.
There are a lot of different works continuing the work of Bochner. For example there are similar (stronger) results for non compact complete manifold with $\mathrm{Ric} \geq 0$ by Cheeger and Gromov. Today I

[1] Source: `http://www-groups.dcs.st-and.ac.uk/ history/Mathematicians/Bochner.html`

want to give an outlook to these methods.

10.2 Notation and Basic Properties

10.2.1 Notations

X	compact smooth riemannian manifold
$E \longrightarrow X$	any hermitian vector bundle on X with metric connection
∇	its associated covariant derivative
$\Gamma(E)$	smooth sections of E
$\Gamma_0(E)$	smooth sections of E with compact support
$e_1, e_2, \dots$	local orthonormal tangent frame field
S	any Dirac bundle over X (see p. 112 [Mic89])
$D : \Gamma(S) \longrightarrow \Gamma(S)$	Dirac operator
	$D = \sum_{j=1}^{n} e_j \cdot \nabla_{e_j}$

10.2.2 Curvature

I want to recall some definitions and relations from differential geometry without proof. Let $V, W \in TX$, we define an associated invariant 2nd derivative

$$\begin{array}{rcl} \nabla^2_{V,W} : \Gamma(E) & \longrightarrow & \Gamma(E) \\ \varphi & \longmapsto & \nabla_V \nabla_W \varphi - \nabla_{\nabla_V W} \varphi \end{array} \tag{10.1}$$

This operator depends at any point $x \in X$ only on the values V_x and W_x and we have:

$$\nabla^2_{V,W} - \nabla^2_{W,V} = R_{V,W} \tag{10.2}$$

If $\varphi \in \Gamma(E)$, then we see $\nabla^2_{\cdot,\cdot} \varphi \in \Gamma(T^* \otimes T^* \otimes E)$, that is, at each point it defines a bilinear form on the tangent space with values in E.

Definition 10.1 (Connection Laplacian). The **connection laplacian** is defined by

$$\begin{array}{rcl} \nabla^* \nabla : \Gamma(E) & \longrightarrow & \Gamma(E) \\ \nabla^* \nabla \varphi & \equiv & -\operatorname{trace}\left(\nabla^2_{\cdot,\cdot} \varphi\right) \end{array} \tag{10.3}$$

Definition 10.2 (Parallel Sections). A section $\sigma \in \Gamma(E)$ is *parallel* if σ satisfies $\nabla \sigma = 0$

We know the following properties of $\nabla^* \nabla$.

Lemma 10.3. *The connection laplacian $\nabla^* \nabla : \Gamma(E) \longrightarrow \Gamma(E)$ is non-negative and formally self-adjoint. In particular*

$$(\nabla^* \nabla \varphi, \psi) = (\nabla \varphi, \nabla \psi) \qquad \forall \varphi, \psi \in \Gamma_0(E) \tag{10.4}$$

The principal symbol of $\nabla^* \nabla$ is $\sigma_\xi(\nabla^* \nabla) = \|\xi\|^2$.
If the manifold X is compact, then $\nabla^* \nabla \varphi = 0$ if and only if $\nabla \varphi \equiv 0$, i.e. if and only if φ is a globally parallel section. The operator $\nabla^* \nabla$ is essentially self-adjoint on a complete riemannian manifold and hence there exists a unique self-adjoint extension on $L^2(E)$. In this case the kernel of $\nabla^* \nabla$ consists of all parallel sections of E.

Definition 10.4 (Curvature Tensor/Operator). Let $R_{V,W} : E_x \longrightarrow E_x$ be the **curvature transformation**. The **curvature tensor** $\langle R_{V_1,V_2} V_3, V_4 \rangle$ is antisymmetric in the pairs (V_1, V_2) and (V_3, V_4) and symmetric under interchange of these pairs(lemma 10.5). R can be considered as a symmetric endomorphism $R : \Lambda^2(X) \longrightarrow \Lambda^2(X)$. We want to call the operator R **curvature operator**. R is positive (non-negative) if all eigenvalues of R are < 0 (≤ 0).

Lemma 10.5 (Bianchi Identities). *Let R be the curvature tensor of a riemanian manifold X. Then R satisfies the following identities:*

$$
\begin{aligned}
R_{U,V}W + R_{V,W}U + R_{W,U}V &= 0 \qquad (10.5)\\
\langle R_{U,V}W, Y\rangle &= \langle R_{W,Y}U, V\rangle \qquad (10.6)
\end{aligned}
$$

This is proved in several book on differential geometry.

Definition 10.6 (Ricci Curvature). The **Ricci transform** of $T(X) = \Lambda^1(X)$ is defined by the formula

$$\mathrm{Ric}(\varphi) \equiv -\sum_{j=1}^{n} R_{e_j,\varphi}(e_j) \tag{10.7}$$

where R is the curvature transformation of $T(X)$. This determines a symmetric bilinear form

$$\mathrm{Ric}(\varphi, \psi) = \langle \mathrm{Ric}(\varphi), \psi\rangle \tag{10.8}$$

which is called **Ricci curvature form**.

Definition 10.7 (Scalar Curvature). The scalar curvature $\kappa : X \longrightarrow \mathbb{R}$ is defined by

$$\kappa \equiv \mathrm{trace}(\mathrm{Ric}) = -2\,\mathrm{trace}(R) \tag{10.9}$$

If we have an orthonormal tangent frame $(e_1, \ldots, e_n)$ at a point $x \in X$ we have for κ

$$\kappa = -\sum_{i,j=1}^{n} \left\langle R_{e_i,e_j}(e_i), e_j\right\rangle \tag{10.10}$$

If X has dimension 2, then κ coincides with the classical Gauss curvature.

10.3 The Bochner Formulas/Method

We want to define a canonical section $\mathfrak{R} \in \Gamma(\mathrm{Hom}(S, S))$ by the following formula

$$\mathfrak{R}(\varphi) := \frac{1}{2}\sum_{j,k=1}^{n} e_j \cdot e_k \cdot R_{e_j,e_k}(\varphi) \tag{10.11}$$

Where $\cdot$ is the Clifford multiplication.

Theorem 10.8 (General Bochner Identity). *Let S be any Dirac bundle and D the Dirac operator. Let $\nabla^*\nabla$ be the connection laplacian for S. Then*

$$D^2 = \nabla^*\nabla + \mathfrak{R} \tag{10.12}$$

Proof. Fix $x \in X$ and choose a local orthogonal frame field $(e_1, \ldots, e_n)$ such that $(\nabla e_j)_x = 0$ for all

j. Then using equation 10.2 we have at x that

$$
\begin{aligned}
D^2 &= \sum_{j,k=1}^{n} e_j \cdot \nabla_{e_j} (e_k \cdot \nabla_{e_k}) \qquad \text{because} \quad D = \sum_{\nu=1}^{n} e_\nu \cdot \nabla_{e_\nu} \\
&= \sum_{j,k} e_j \cdot e_k \cdot \nabla_{e_j} \nabla_{e_k} \\
&\overset{\text{Eq. 10.1}}{=} \sum_{j,k} e_j \cdot e_k \cdot (\nabla^2_{e_j,e_k} - \underbrace{\nabla_{\nabla_{e_j} e_k}}_{=0}) \\
&= \sum_{j,k} e_j \cdot e_k \cdot \nabla^2_{e_j,e_k} \\
&= \underbrace{-\sum_j \nabla^2_{e_j,e_j}}_{=\nabla^*\nabla} + \underbrace{\sum_{j<k} e_j \cdot e_k \cdot \underbrace{\left(\nabla^2_{e_j,e_k} - \nabla^2_{e_k,e_j k}\right)}_{\overset{\text{Eq.10.2}}{=} \quad R_{e_j,e_k}}}_{\overset{\text{Eq.10.11}}{=} \quad \Re} \\
&= \nabla^*\nabla + \Re
\end{aligned} \tag{10.13}
$$

qed

Now we want to apply this theorem to the bundle $S = \mathrm{Cl}(X)$. For this we need the following lemma and lemma 10.5.

Lemma 10.9. *There is a canonical bundle mapping*

$$
\begin{aligned}
L : \mathrm{Cl}(X) &\longrightarrow \mathrm{Cl}(X) \\
\varphi &\longmapsto -\sum_{j=1}^{n} e_j \varphi e_j
\end{aligned} \tag{10.14}
$$

for any orthonormal basis $e_1, \ldots, e_n$ of $T_x(X)$. L has the following properties

1. *L is globally diagonalizable and yields the bundle decomposition $\mathrm{Cl}(X) = \bigoplus_{p=0}^{n} \Lambda^p(X)$*
2. *$L = (-1)^p(n-2p)$ on $\Lambda^p(X)$ for $p = 0, \ldots, n$*
3. *L considered as a section in $\hom(\mathrm{Cl}(X), \mathrm{Cl}(X))$ is globally parallel, i.e. $[\nabla, L] = 0$*
4. *For a Dirac operator D and $\Delta = D^2$ we have $[\Delta, L] = 0$.*

Proof. See [Mic89] (p. 128 formula 5.20-22 and p.130 lemma 5.18 and corollary 5.21) qed

Lemma 10.10 (Weitzenböck). *Let Δ be the Hodge laplacian and $\nabla^*\nabla$ the connection laplacian of the tangent bundle $T(X)$. Then*

$$
\Delta = \nabla^*\nabla + \mathrm{Ric} \tag{10.15}
$$

Proof. Consider $\mathrm{Cl}(X) \supset \Lambda^1(X) = T(X) = S$, so we have on the right hand side $D^2 = \Delta$ for vectors $\varphi \in \Lambda^1(X)$. We have from the previous lemma that $[\Delta, L] = [\nabla, L] = 0$. Furthermore L preserves the subbundle $\Lambda^1(X)$. Hence Δ and $\nabla^*\nabla$ preserve $\Lambda^1(X) = T(X)$ and because of the decomposition

$D^2 = \nabla^*\nabla + \mathfrak{R}$ (corollary 10.10) $\mathfrak{R}$ preserves $T(X)$, too.

$$
\begin{aligned}
\mathfrak{R}(\varphi) &= \frac{1}{2}\sum_{i,j} e_i e_j R_{e_i,e_j}(\varphi) \\
&= \frac{1}{2}\sum_{i,j,k} e_i e_j \left\langle R_{e_i,e_j}(\varphi), e_k\right\rangle e_k \quad \text{Split the sum and use symmetry properties of } R \\
&= \frac{1}{6}\sum_{i\neq j\neq k\neq i} \underbrace{\left\langle R_{e_i,e_j}(e_k) + R_{e_j,e_k}(e_i) + R_{e_k,e_i}(e_j), \varphi\right\rangle}_{\overset{\text{Eq.10.5}}{=} 0} e_i e_j e_k \\
&\quad + \overbrace{\frac{1}{2}\sum_{i,j} e_i e_j \left\langle R_{e_i,e_j}(\varphi), e_i\right\rangle e_i}^{\text{part for } k=i} + \overbrace{\frac{1}{2}\sum_{i,j} e_i e_j \left\langle R_{e_i,e_j}(\varphi), e_j\right\rangle e_j}^{\text{part for } k=j} \\
&= \frac{1}{2}\sum_{i,j} \underbrace{e_i e_j}_{=-1} \left\langle R_{e_i,e_j}(\varphi), e_i\right\rangle e_i + \frac{1}{2}\sum_{i,j} \underbrace{e_i e_j}_{=-1} \left\langle R_{e_i,e_j}(\varphi), e_j\right\rangle e_j \quad R \text{ is antisymmetric in some arguments} \\
&= -\sum_{i,j} \left\langle R_{e_i,e_j}(\varphi), e_j\right\rangle e_i \qquad \text{eq. 10.6} \\
&= -\sum_{i,j} \left\langle R_{\varphi,e_j}(e_i), e_j\right\rangle e_i \overset{\text{antisymmetry}}{=} -\sum_{i,j} \left\langle R_{e_j,\varphi}(e_j), e_i\right\rangle e_i = -\sum_{j} R_{e_j,\varphi}(e_j) \\
&= \mathrm{Ric}(\varphi) \qquad (10.16)
\end{aligned}
$$

qed

10.4 Bochner's Formulas and the Betti Numbers

As consequence we get the following important theorem of Bochner

Theorem 10.11 (Bochner). *Let X be a compact riemannian manifold with $\partial X = \emptyset$. If $\mathrm{Ric} > 0$, then the first Betti number $b_1(X) := \dim H^1(X, \mathbb{R}) = 0$. The conlusion also holds if $\mathrm{Ric} \geq 0$ and $\mathrm{Ric} > 0$ at one point.*

Proof. Suppose $b_1(X) = \dim H^1(X, \mathbb{R}) > 0$. Then there exists a non-zero harmonic 1-form $\varphi \in \mathcal{H}^1 \cong H^1(X, \mathbb{R})$ (by Hodge - De Rham theory). By lemma 10.3 and lemma 10.10 we have

$$\int_X \mathrm{Ric}(\varphi, \varphi) = -(\nabla^*\nabla\varphi, \varphi) \overset{\text{eq.10.4}}{=} -\|\nabla\varphi\|^2 \qquad (10.17)$$

If $\mathrm{Ric} \geq 0$ we can conclude that $\nabla\varphi \equiv 0$ (i.e. φ is parallel). This means that $\|\varphi\|$ is constant. If $\mathrm{Ric} > 0$ at one point the integral does not vanish. Hence we have a contradiction. qed

Note: We also proved that if $\mathrm{Ric} \geq 0$ every harmonic 1-form is parallel. We can conclude the following because under the metric correspondence $T^*X \cong TX$ parallel 1-forms become parallel vectorfields.

Theorem 10.12. *Let X be a compact riemannian manifold of non negative Ricci curvature. Then $b_1(X)$ equals the dimension of the space of parallel vectorfields. Thus in particular*

$$b_1(X) \leq \dim(X) \qquad (10.18)$$

with equality if and only if X is a flat torus.

Proof.

part 1 Let $k = b_1(X)$. Then by the argument above, there are k linearly independent parallel vectorfields on X. We know that parallel vector fields are linearly independent if and only if they are linearly independent at each point. It follows from a theorem in linear algebra that then $k \leq \dim(X)$.

part 2 Equality holds if and only if X has a globally parallel framing. X has to be a flat torus, we will see it as follows. Choose parallel vector fields which are pointwise orthonormal. Since

$$[E_i, E_j] = \nabla_{E_i} E_j - \nabla_{E_j} E_i = 0 \qquad \forall i, j \tag{10.19}$$

these vectorfields generate a free $\mathbb{R}^k$-action. Since $k = \dim(X)$, we see that X is an orbit of this action. So we can write $X \cong \mathbb{R}^k/\Lambda$ where Λ is a lattice in $\mathbb{R}^k$. X is flat because the metric agrees with the canonical flat metric on $\mathbb{R}^k$.

qed

With some different assumptions (curvature,...) one can also prove results for higher Betti numbers. I want to give one example.

Theorem 10.13 (Gallot,Meyer). *Let X be a compact riemannian manifold of dimension n without boundary. Assume that the curvature operator R is positive at every point of the manifold. Then all Betti numbers $b_p(X) = 0$ for $p = 1 \dots n-1$, i.e. X is a homology sphere. The same conclusion holds if the operator is non-negative everywhere and positive at one point.*

For the proof we need the following lemma.

Lemma 10.14. *Let $\varphi \in \Lambda^p(\mathbb{R}^n) \subset \mathrm{Cl}(\mathbb{R}^n)$ for $p = 1 \dots n-1$. If $\mathrm{ad}_\xi(\varphi) = 0$ for al $\xi \in \Lambda^2(\mathbb{R}^n)$, then $\varphi = 0$.*

Proof. **(lemma)**

Let $e_1, \dots, e_n$ be the standard basis of $\mathbb{R}^n$. We write $\varphi = \sum_{|I|=p} a_I e_I$. By assumption, $[e_i e_j, \varphi] = 0$ for all $i < j$. One can easily see that

$$[e_i e_j, e_I] = \begin{cases} 0 & \text{if } i \in I \text{ and } j \in I \\ 0 & \text{if } i \notin I \text{ and } j \notin I \\ 2 e_i e_j e_I & \text{otherwise} \end{cases} \tag{10.20}$$

If $i \notin I$ and $j \in I$, then $e_i e_j e_I = \pm e_{I \cup \{i\} - \{j\}}$. Therefore $[e_i e_j, \varphi] = 0$ implies that $a_I = 0$ whenever either $i \in I$ or $j \in I$. Applying this for all $i < j$ shows that $\varphi = 0$, provided that $p \neq 0$ or $p \neq n$. qed

Proof. **(Gallot, Meyer)**It will be sufficient to prove that the positivity of the curvature operator implies that

$$\langle \mathfrak{R}(\varphi), \varphi \rangle > 0 \qquad \varphi \in \Lambda^p(X) - \{0\},\ p = 1, \dots, n-1 \tag{10.21}$$

Now we proceed similarly to the proof of theorem 10.11. We have

$$\begin{aligned}
\langle \Re(\varphi), \varphi \rangle &= \frac{1}{2}\sum_{i,j=1}^{n} \left\langle e_i e_j R_{e_i,e_j}(\varphi), \varphi \right\rangle = \sum_{i<j} \left\langle e_i e_j R_{e_i,e_j}(\varphi), \varphi \right\rangle \\
&= \frac{1}{2}\sum_{i<j} \left\langle [e_i e_j, R_{e_i,e_j}(\varphi)], \varphi \right\rangle \qquad \text{Warum?} \\
&= -\frac{1}{2}\sum_{i<j} \left\langle R_{e_i,e_j}(\varphi), [e_i e_j, \varphi] \right\rangle \overset{\mathrm{ad}_X Y := [X,Y]}{=} -\frac{1}{2}\sum_{i<j} \left\langle R_{e_i,e_j}(\varphi), \mathrm{ad}_{e_i e_j}(\varphi) \right\rangle \\
&\qquad \text{We can write } R_{V,W} : \mathrm{Cl}(X) \longrightarrow \mathrm{Cl}(X) \\
&\qquad R_{V,W}(\varphi) = \frac{1}{2}\sum_{i<j} \langle R_{V,W}(e_i), e_j \rangle \, \mathrm{ad}_{e_i e_j}\ (\varphi) \qquad \text{Hence} \\
&= -\frac{1}{4}\sum_{i<j,k<l} \left\langle R_{e_i,e_j}(e_k), e_l \right\rangle \left\langle \mathrm{ad}_{e_i e_j(\varphi)}, \mathrm{ad}_{e_k e_l(\varphi)} \right\rangle \\
&= -\frac{1}{4}\sum_{i<j,k<l} \langle R(e_i e_j), e_k e_l \rangle \left\langle \mathrm{ad}_{e_i e_j}(\varphi), \mathrm{ad}_{e_k e_l}(\varphi) \right\rangle \\
&\qquad \{e_i e_j\} \text{ form an onb of } \Lambda^2(X) \subset \mathrm{Cl}(X). \text{ This expression is independent} \\
&\qquad \text{of the choice of an onb. Let } \{\xi_\alpha\}_\alpha \text{ be any onb of } \Lambda^2(X) \text{ which} \\
&\qquad \text{diagonalizes } R. \text{ Set } \lambda_\alpha = \tfrac{1}{4}\lambda'_\alpha \text{ where the } \lambda'_\alpha (< 0) \text{ are eigenvalues of } R. \\
&= -\frac{1}{4}\sum_{\alpha,\beta} \langle R(\xi_\alpha), \xi_\beta \rangle \left\langle \mathrm{ad}_{\xi_\alpha}(\varphi), \mathrm{ad}_{\xi_\beta}(\varphi) \right\rangle \\
&= \sum_{\alpha} \lambda_\alpha \left\| \mathrm{ad}_{\xi_\alpha}(\varphi) \right\|^2 \qquad \text{since } \lambda_\alpha > 0 \ \forall \alpha \\
&\geq 0 \qquad \text{For the case of } = 0 \text{ see lemma 10.14.} \quad \text{Hence} \\
&> 0 \qquad (10.22)
\end{aligned}$$

qed

10.5 The Formula of Lichnerowicz

Now we want to consider the case of spinor bundles. So from now on let X be a compact spin manifold with fixed spin structure on its tangent bundle and S be any spinor bundle on $T(X)$ endowed with its canonical metric connection.

Theorem 10.15 (A. Lichnerowicz). *Let X be a spin manifold and suppose, that S is any bundle of spinors over X endowed with the canonical riemannian connection. Let $\not{D}$ denote the Atiyah-Singer operator (i.e. the associated dirac operator) and $\nabla^*\nabla$ the connection laplacian on S. Then*

$$\not{D}^2 = \nabla^*\nabla + \frac{\kappa}{4} \tag{10.23}$$

Proof. In fact this proof is very similar to the proof of the Weitzenböck formula in theorem 10.10. We have to compute the curvature term in equation 10.12 for the canonical spinor connection. In this case for all $V, W \in T_x(X)$ curvature transformation $R^S_{V,W} : S_x \longrightarrow S_x$ is given (see [Mic89] p. 110 theorem 4.15) by a slightly different formula

$$\frac{1}{4}\sum_{k,l} \langle R_{V,W}(e_k), e_l \rangle \, e_k e_l \tag{10.24}$$

where $(e_1, \ldots, e_n)$ is any orthonormal base of $T_x(X)$. So we get

$$\begin{aligned}
\Re &= \frac{1}{2}\sum_{i,j} e_i e_j R^S_{e_i,e_j} \\
&= \frac{1}{8}\sum_{i,j,k,l} \langle R_{V,W}(e_k), e_l\rangle\, e_i e_j e_k e_l \qquad \text{split sum} \\
&= \frac{1}{8}\sum_{l}\left(\sum_{i\neq j\neq k\neq i} \underbrace{\big\langle R_{e_i,e_j}(e_k) + R_{e_j,e_k}(e_i) + R_{e_k,e_i}(e_j), e_l\big\rangle}_{\overset{\text{eq.10.5}}{=}0} e_i e_j e_k \right. \\
&\quad \left. + \overbrace{\sum_{i,j}\big\langle R_{e_i,e_j}(e_i), e_l\big\rangle\, e_i e_j e_i}^{i=k} + \overbrace{\sum_{i,j}\big\langle R_{e_i,e_j}(e_i), e_l\big\rangle\, e_i e_j e_j}^{j=k}\right) e_l \\
&= \frac{1}{4}\sum_{i,j,l}\big\langle R_{e_i,e_j}(e_i), e_l\big\rangle\, e_j e_l \\
&= -\frac{1}{4}\sum_{j,l}\mathrm{Ric}(e_j, e_l) e_j e_l = \frac{\kappa}{4} \qquad (10.25)
\end{aligned}$$

qed

This theorem has some important consequences. I want to talk about some of this.

Definition 10.16 (Harmonic Spinors). A spin manifold has no harmonic spinors if $\ker \not{D} = 0$ for any spinor bundle associated to $T(X)$.

Korollar 10.17. *Any compact spin manifold with $\kappa > 0$ ($\kappa \geq 0$ and $\kappa > 0$ for at least one point) admits no harmonic spinors.*

Proof. The proof is similar to the one of theorem 10.11 of Bochner qed

Korollar 10.18. *On a compact spin manifold with $\kappa \equiv 0$ evry harmonic spinor is globally parallel.*

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