HIGHER-ORDER TIME ASYMPTOTICS OF FAST DIFFUSION IN EUCLIDEAN SPACE: A DYNAMICAL SYSTEMS APPROACH

JOCHEN DENZLER, HERBERT KOCH, AND ROBERT J. MCCANN

Abstract. This paper quantifies the speed of convergence and higher-order asymptotics of fast diffusion dynamics on $\mathbf{R}^n$ to the Barenblatt (self similar) solution. Degeneracies in the parabolicity of this equation are cured by re-expressing the dynamics on a manifold with a cylindrical end, called the cigar. The nonlinear evolution becomes differentiable in Hölder spaces on the cigar. The linearization of the dynamics is given by the Laplace-Beltrami operator plus a transport term (which can be suppressed by introducing appropriate weights into the function space norm), plus a finite-depth potential well with a universal profile. In the limiting case of the (linear) heat equation, the depth diverges, the number of eigenstates increases without bound, and the continuous spectrum recedes to infinity. We provide a detailed study of the linear and nonlinear problems in Hölder spaces on the cigar, including a sharp boundedness estimate for the semigroup, and use this as a tool to obtain sharp convergence results toward the Barenblatt solution, and higher order asymptotics. In finer convergence results (after modding out symmetries of the problem), a subtle interplay between convergence rates and tail behavior is revealed. The difficulties involved in choosing the right functional spaces in which to carry out the analysis can be interpreted as genuine features of the equation rather than mere annoying technicalities.

2010 *Mathematics Subject Classification.* 35B40; 33C50, 35K61, 37L10, 58J50, 76S05.

The authors are pleased to acknowledge the hospitality of the Mathematical Sciences Research Institute at Berkeley, and of the Banff International Research Station, where parts of this work were performed. They are grateful to Juan-Luis Vázquez for conversations which helped to stimulate this project, and for his hospitality in Madrid, and to Thomas Hagen for providing an early opportunity to present these results in the May 2008 AIMS Conference on Dynamical Systems and Differential Equations. This research was supported in part by Natural Sciences and Engineering Research Council of Canada Grants 217006-03 and -08 and United States National Science Foundation Grant DMS-0354729 to Robert McCann; by a grant from the Simons Foundation (#208550) to Jochen Denzler; and by the DFG through SFB 611 (Herbert Koch).

1. Introduction

Long-time asymptotics of nonlinear diffusion processes have been a subject of much recent interest. The porous medium equation

$$\rho_\tau = \frac{1}{m}\Delta\rho^m \tag{1}$$

is a prototypical example; it governs the evolution of a nonnegative density $\rho(\tau, \boldsymbol{y})$ on $[0,\infty[\, \times \mathbf{R}^n$, as described in Vázquez book [45] and its references. The basin of attraction [26] [44] of its self-similar solution [49] [4] [39], and the rate of convergence of other solutions to it [11] [13] [19] [36] [20] [31] [37] [6] [7] [9] [8] [22] have attracted a steady stream of attention since such rates were first obtained by Carrillo-Toscani [12], Otto [38] and del Pino-Dolbeault [18]. Sharp results for convergence in entropy sense have since been extended to the full range of $m \in \mathbf{R}$ by the quintet consisting of Blanchet, Bonforte, Dolbeault, Grillo and Vazquez [7], the quartet [8] and the trio [9]. However, few results are known concerning higher asymptotics, beyond the two improvements accessible by choosing an appropriate translation in space (Carrillo, di Francesco, Kim, McCann, Slepcev and Toscani and the quartet [8] in various combinations and settings [43] [13] [31] [37] [10]) and in time by the Dolbeault-Toscani duo [22] (c.f. [3] [47]). In the one-dimensional porous medium regime $m > 1$, one has a full asymptotic expansion of Angenent [1] based on the spectral calculation of Zel'dovitch and Barenblatt [48]. Koch's habilitation thesis provides a potential framework for generalizing this to higher dimensions [32]. In the fast diffusion regime $m < 1$, a spectral calculation by Denzler and McCann [19] [20] has been used to derive the first two corrections [37] [8] [22] to the leading order asymptotics [18] [38], but the higher-order modes never been successfully related to the nonlinear dynamics. It is framework for achieving such a relation which we develop for the first time below — inspired by ideas from dynamical systems, and sticking to the mass-preserving range $m \in \,]\frac{n-2}{n}, 1[$.

To achieve this, several obstacles must be confronted. The spectrum of Denzler and McCann [20] contains only finitely many eigenvalues below the continuum threshold, so only finitely many modes are in principle accessible. Moreover, there is an incompatibility between the spaces in which the eigenfunctions live, and the spaces in which the dynamics (1) turn out to depend differentiably on their initial conditions. It is this differentiable dependency that we need to establish to justify the linearization which leads to the spectral problem.

Guessing a space in which it will hold is far from trivial, however. One of the technical devices we use to achieve this appears independently in Bonforte, Grillo and Vázquez' [9] entropy-based approach to the special case $m = \frac{n-4}{n-2}$: namely, after linearizing the rescaled evolution in relative variables, we restore uniform parabolicity to this apparently degenerate equation by carrying out the analysis on $\mathbf{R}^n$ viewed as a Riemannian manifold $\mathcal{M}$ with an asymptotically cylindrical metric, known as the cigar.

Reconciling the above-mentioned incompatibility of spaces for the linearized and nonlinear dynamics complicates our analysis and yields a richness to the statements of our theorems which goes beyond the intrinsic complexity of a spectrum whose features include a web of eigenvalue crossings as m is varied. To obtain higher-order asymptotics, we need to work in weighted spaces which discount information appropriately at spatial infinity. The further we wish to penetrate into the spectrum, the more severe the required discounting. The corresponding linearized operators are no longer self-adjoint; they act on weighted Banach rather than Hilbert spaces.

Since the qualitative behaviour of (1) varies considerable with m in different regimes, let us set

$$m_p := 1 - \frac{2}{n+p}, \tag{2}$$

where the parameter $p = \frac{2}{1-m} - n$ is the *moment-index* introduced in [19]. Thus $m_{-2} = \frac{n-4}{n-2}$, $m_0 = \frac{n-2}{n}$, $m_2 = \frac{n}{n+2}$, and $m_n = \frac{n-1}{n}$, for example.

Herrero and Pierre [29] obtained remarkable local estimates for solutions to the fast diffusion equations if $0 < m < 1$ and $m_0 < m < 1$, which they and Dahlberg and Kenig [15] used to prove that every solution has an initial trace, and it can be extended for all t, and vice versa, there exists a global solution for every initial data that is a local measure.

For $m > m_0$, the evolution (1) of compactly supported initial data is mass-preserving; this is the regime we shall explore. For $m < m_0$, the mass dwindles to zero in finite time; the basin of attraction and leading-order asymptotics describing this disappearance have been provided by Daskalopoulos-Sesum [16] and the quintet [7], respectively.

Although the dynamics (1) has no fixed point, a well-known rescaling using the self similar coordinates

$$\begin{aligned} \boldsymbol{x} &= (1+2p\tau)^{-\beta}\boldsymbol{y}, \qquad t = \tfrac{1}{2p}\ln(1+2p\tau)\,, \\ \beta &= (2-(1-m)n)^{-1} = \tfrac{1}{2}(1+\tfrac{n}{p})\,, \\ u(t,\boldsymbol{x}) &= e^{(n+p)nt}\rho((e^{2pt}-1)/(2p), e^{(n+p)t}\boldsymbol{x}) \end{aligned} \tag{3}$$

yields an alternate description

$$\frac{\partial u}{\partial t} = \frac{1}{m}\Delta u^m + \frac{2}{1-m}\boldsymbol{\nabla}\cdot(\boldsymbol{x}u) \tag{4}$$

of the same evolution, but with a fixed point in the new variables. This scaling coincides with the one used in our proceedings report [19] and by the quartet [8]. Conversely, we can express the density of the solution to the fast diffusion equation by

$$\rho(\tau,\boldsymbol{y}) = (1+2p\tau)^{-\beta n} u\left(\frac{1}{2p}\ln(1+2p\tau), (1+2p\tau)^{-\beta}\boldsymbol{y}\right)\,. \tag{5}$$

The character of this rewriting depends on the sign of β. If $m > m_0$, then $\tau \to \infty$ is equivalent to $t \to \infty$ and the $\tau \to \infty$ asymptotics for (1) translates to $t \to \infty$ asymptotics for (4). This remains true after a similar transformation for $m = m_0$. For $m < m_0$ solutions may extinguish in finite time and the asymptotics of (4) for $t \to \infty$ give information about the asymptotics of (1) at the extinguishing time, as in work of the quintet [7] and trio [9].

The stationary solution

$$u_B(\boldsymbol{x}) = (B + |\boldsymbol{x}|^2)^{-\frac{1}{1-m}} \tag{6}$$

to (4) is related to the Barenblatt solution [49] [4] [39]

$$\rho_B(\tau, \boldsymbol{y}) = (2p\tau + 1)^{-n\beta} u_B((2p\tau + 1)^{-\beta} \boldsymbol{y}), \tag{7}$$

where B is determined by a quantity $\int \rho_0 d\boldsymbol{x}$ called the *mass*, at least if this mass is bounded. It has been known to attract all solutions which share its mass since the work of Friedman and Kamin [26]; see also Vázquez [44]. The Barenblatt solution has finite moments of exactly those orders smaller than p.

It has been shown by Vázquez [44, Thm. 21.1] that weak assumptions like

$$\int \rho(0, \boldsymbol{y}) d\boldsymbol{y} = \int \rho_B(0, \boldsymbol{y}) d\boldsymbol{y}$$

and

$$\sup_{\boldsymbol{y}} |\rho(0, \boldsymbol{y})/\rho_B(0, \boldsymbol{y})| < \infty$$

imply

$$\limsup_{\tau \to \infty} \sup_{\boldsymbol{y}} \left| \frac{\rho(\tau, \boldsymbol{y})}{\rho_B(\tau, \boldsymbol{y})} - 1 \right| = 0 \ . \tag{8}$$

We investigate the sharp decay rate of the quantity $|\cdot|$ in (8), called the *relative* L^∞ distance. The known sharp rates of decay for integral expressions — such as $\|\rho - \rho_B\|_{L^1(\mathbf{R}^n)}$ or the relative entropy [12] [18] [38] [8] — imply a (non-sharp) rate of decay in relative L^∞ through the work of the quintet [7].

Because of the gradient structure with respect to the Wasserstein distance discovered by Otto [38], convergence questions can be attacked using displacement convexity; see also [37]. It is within this framework that the linearized problem for $m < 1$ had been studied in great detail [20]. Unfortunately, there is no clean way of passing from the linearized operator to the full nonlinear equation. However, by cleverly employing the entropy method, McCann-Slepcev [37], the quartet [8], and the duo [22] were able improve on the sharp integral rates of convergence of Carrillo-Toscani ($m > 1$) [12], Otto ($m \geq m_n$) [38] and del-Pino-Dolbeault ($m \geq m_n$) [18], to extract first [37] [8] and second [22] corrections. The first order correction depends on centering the data, and the second-order on choosing a suitable translation in τ. Our first result gives the sharp relative L^∞ rate of convergence for centered initial data. It improves on the rate found independently by the quintet [7] without centering, and implies the sharp entropy rate of convergence also found independently by the quartet for centered data [8].

Theorem 1.1 (Exact leading-order asymptotics in the relative L^∞ norm)**.** *Fix* $0 < m \in]m_0, 1[$ *with* $m_0 = \frac{n-2}{n}$*. Suppose* $\rho(\tau, \boldsymbol{y})$ *satisfies* (1) *and the condition* (8) *holds for some* $B > 0$*. If* $m = m_2 = \frac{n}{n+2}$*, we assume in addition*

$$\int_{\mathbf{R}^n} \left|1 - \frac{\rho(0, \boldsymbol{y})}{\rho_B(0, \boldsymbol{y})}\right|^2 (1 + |\boldsymbol{y}|^2)^{-n/2} d\boldsymbol{y} < \infty \,. \tag{9}$$

Then there exists $\boldsymbol{z} \in \mathbf{R}^n$ *such that*

$$\limsup_{\tau \to \infty} \sup_{\boldsymbol{y} \in \mathbf{R}^n} \tau \left|\frac{\rho(\tau, \boldsymbol{y} - \boldsymbol{z})}{\rho_B(\tau, \boldsymbol{y})} - 1\right| < \infty \,. \tag{10}$$

Without the additional condition (9) *in the case* $m = m_2$*, we still get* (10)*, except that the leading factor* τ *is replaced by* $\tau^{1-\varepsilon}$*, for any* $\varepsilon > 0$*.*

Let us note that obtaining the sharp estimate (10) without an ε in all cases other than $m = m_2$ is not automatic, but requires a detailed study of the linearized PDE, as given in the proof of Thm. 8.5. We rely on the manner in which the essential spectral radius depends on decay properties built in the space. The argument in case $m = m_2$ and assuming the moment condition (9) is of a different nature, basically a spin-off of the L^2 theory and relying on self-adjointness in this case.

The case $m \in]m_0, m_2]$ had already been proved by Kim-McCann [31], subject to a mild moment condition, which is not needed here, except for the similar moment-type condition (9) in the case $m = m_2$. The case $m \geq m_n$ had been handled to order $O(1/\tau^{1-\varepsilon})$ by McCann and Slepčev [37] in the somewhat weaker L^1 norm. Before this, Carrillo and Vázquez had given the sharp $1/\tau$ rate in the relative L^∞ norm for radially symmetric data [13], and a weaker $O(1/\tau^{1/2})$ rate in the L^1 norm over the whole parameter range $m > m_0$. The sharp rate for the case $m_2 < m < m_n$ remained open for a while, even though the role of the translation had been understood in the linearized setting by [20] and [13], before being resolved independently in different metrics through the work of the quartet [8] and the present manuscript. The work of the quintet [7] gave sharp but implicit integral rates for non-concentric initial data which extend even into the finite extinction range of parameters $m < m_0$. The work of the trio [9] addresses the exceptional case $m = m_{-2}$, in which the essential spectral gap (12) described below and found by Denzler and McCann [19] [20] vanishes. Whereas the works of the quintet and its subgroups are entropy-based, we use a dynamical systems approach that successfully bridges the functional analytic gap between the linearization argument of [20] and the nonlinear estimates, allowing higher modes to be accessed.

The proof, which is found in Sec. 9, involves preliminary results of fairly different flavors. A prominent role is played by a manifold $\mathcal{M}$ with one cylindrical end, which is called the cigar in the two dimensional case; the same manifold was employed independently by the trio [9]. We express the equation in self similar variables and turn to the relative size u/u_B as main dependent variable, as do the quintet [7] and their successors [8] [9] [22]. The equation for the quotient can be understood as a uniformly parabolic reaction/transport/nonlinear diffusion equation on the manifold $\mathcal{M}$. Well-posedness and smooth dependence on the initial data follow by suitable adaptations of known techniques. However, since our setting deviates from more standard situations we provide complete proofs for the convenience of the reader, and for the convenience of having results tailored to our needs. The (rescaled and time translated) Barenblatt solutions are barriers, and local existence immediately implies global existence if $m \geq m_0 = \frac{n-2}{n}$. Below $m < m_0$ the time translation corresponds to an unstable mode, which is obvious by looking at the Barenblatt solution, where a time shift is used to adjust the τ at which the solution extinguishes.

The asymptotics of Theorem 1.1 are determined by the spectrum of the linearized operator. Its largest eigenvalue λ_{00} is zero. It corresponds to the rescaling

$$\rho_\sigma(\tau, \boldsymbol{x}) = \rho(\tau/\sigma^2, \boldsymbol{x}/\sigma) .$$

The invariant manifold of this mode is given by the set of stationary solutions u_B, which is parametrized by B. For $m > \frac{n-2}{n}$, B is determined by the mass (L^1 norm) of the initial data, which is a constant of motion. Once we adjust the mass, this spectral value becomes irrelevant.

Equation (1) is spatially translation invariant. The corresponding eigenvalue of the linearization is $\lambda_{10} = -n - p$. The invariant manifold is determined by all translations of Barenblatt solutions. For $m > m_1 = \frac{n-1}{n+1}$, the first moments of the Barenblatt solutions are defined and conserved, and, by centering ρ_0 we get rid of this mode.

Equation (1) is also invariant under time translations (which are equivalent to rescalings of space). The corresponding spectral value is $\lambda_{01} = -2p$. This time there is no related conserved quantity, and there seems to be no direct way of determining the time translation parameter from the initial data [47]. This eigenvalue is responsible for the convergence rate given in Theorem 1.1. This suggests that we may improve the results of Theorem 1.1 by modding out the time shift as well, as was achieved in the entropy sense by the duo [22], and as we hereafter independently achieve in the stronger senses of Remark 1.9 and Theorem 11.1.

Indeed, having quantified a rate of contraction, higher asymptotics become accessible, as we now explain. Accounting for the time shift is merely the first in a series of many improvements which are possible for $m > m_2$, if we allow ourselves to measure convergence in weighted norms which suppress information in the tail region $|\boldsymbol{y}| \gtrsim \tau^\beta$ and reveal higher modes. How strongly this information is suppressed depends on the degree of asymptotic accuracy desired. It is necessary to introduce weights since λ_{01} turns out to coincide with the threshold of the essential spectrum in the unweighted Hölder space $C^\alpha(\mathcal{M})$ of functions on the cigar defined by (33). For $m < m_2$ on the other hand, any eigenvalues apart from λ_{01} turn out to be embedded in a spectral continuum and Theorem 10.1 shows we can obtain very rapid decay, but only for a more restricted class of initial data, since the weights in this case amplify the significance of tail information.

Given $p = 2(1-m)^{-1} - n$ and non-negative integers $k, \ell \in \mathbf{N}$, define

$$-\lambda_{\ell k} := (\ell + 2k)p + n\ell + 4k(1 - \ell - k) \tag{11}$$

and

$$-\lambda_0^{\text{cont}} := (\frac{p}{2} + 1)^2 \ . \tag{12}$$

In a certain critically weighted Hölder space, the $\lambda_{\ell k}$ for $\ell + 2k < 1 + p/2$ will turn out to be eigenvalues describing the exponential rate of contraction of $u(t, \boldsymbol{x})/u_B(\boldsymbol{x})$ towards the constant state 1, and λ_0^{cont} will turn out to be the threshold of the essential spectrum. For irrational $m \in \,]m_2, 1[$ and any $\Lambda \in \,]\lambda_0^{\text{cont}}, \lambda_{01}[$, varying the weights in the linearization carried out below suggests there exist $u_{\underline{\iota}}(\boldsymbol{x})$ depending on $u(0, \boldsymbol{x})$ such that

$$\left\| \frac{(B + |\boldsymbol{x}|^2)(u(t,\boldsymbol{x})/u_B(\boldsymbol{x}) - 1) - \sum_{\Lambda < \underline{\iota}\cdot\underline{\lambda} < 0} u_{\underline{\iota}}(\boldsymbol{x}) e^{\underline{\iota}\cdot\underline{\lambda} t}}{(B + |\boldsymbol{x}|^2)^{\left(p+2-\sqrt{(p+2)^2+4\Lambda}\right)/4}} \right\|_{C^\alpha(\mathcal{M})} = O(e^{\Lambda t}) \tag{13}$$

as $t \to \infty$, where the sum is over integer-valued multi-indices $\underline{\iota} = (i_{\ell k})_{\ell, k \in \mathbf{N}}$ constrained so that $i_{00} = 0 = i_{10}$ (assuming the mass and the center of mass of $u(0, \boldsymbol{x}) - u_B(\boldsymbol{x})$ vanish), and so that $\underline{\iota} \cdot \underline{\lambda} := \sum_{\ell,k=0}^\infty i_{\ell k}\lambda_{\ell k}$ lies strictly between Λ and zero. The presence of essential spectrum also suggests that incorporation of further terms $u(\boldsymbol{x})e^{\lambda t}$ into the sum cannot generally make the error term smaller than $O(e^{-(\frac{p}{2}+1)^2 t})$. In this conjecture we see clearly the trade-off introduced by the choice of weight between weaker norms and the faster rates Λ of decay. This is a new feature to emerge from the present work, at least relative to the existing entropy-based results of the quartet [8] and duo [22], and to the conjecture advanced in [20].

When m (or equivalently p) is rational, the coefficients $u_{\underline{\iota}}(\boldsymbol{x})$ would need to be replaced by a polynomial function $u_{\underline{\iota}}(\boldsymbol{x}, t)$ of time in case of an eigenvalue resonance, i.e. $\underline{\iota} \cdot \underline{\lambda} = \underline{\iota}' \cdot \underline{\lambda}$ with $\sum i_{\ell k} > 1$, as in Angenent [1]. The possibility of such resonances can, for example, be ruled out either by taking m irrational (in view of the rational dependence of the eigenvalues (11) on p), or by limiting ourselves to a $\Lambda > 2\lambda_{01}$ within a factor of two of the spectral gap, which is $\lambda_{01} = \max_{\underline{\iota}} \underline{\iota} \cdot \underline{\lambda}$ for $m > m_2$ and centered mass distributions. The restriction $\Lambda > 2\lambda_{01}$ also simplifies the conjecture in two other ways, allowing us to prove the following theorem. First, it forces the functions $u_{\underline{\iota}}(\boldsymbol{x})/(B + |\boldsymbol{x}|^2)$ to be eigenfunctions corresponding to the eigenvalue $\underline{\iota} \cdot \underline{\lambda}$ of the linearized operator

(23); this operator generates the long-time dynamics near the fixed point in the critically weighted Hölder space $C^\alpha_{\eta_{cr}}(\mathcal{M})$ defined at (79). Moreover, since the nonlinearity is analytic, its effects at this level are benign: the quadratic approximation of an orbit by its tangent is higher order than the accuracy $\Lambda/\lambda_{01} < 2$ accessible in the following theorem, one of our main results.

Theorem 1.2 (Higher-order asymptotics in weighted Hölder spaces)**.** *Fix $p = 2(1-m)^{-1} - n > 2$ (equivalently $m \in]m_2, 1[$) and $\Lambda \in [\lambda_0^{\text{cont}}, \lambda_{01}] = [-(\frac{p}{2}+1)^2, -2p]$ subject to the condition $2\lambda_{01} < \Lambda$. If $u(t,\boldsymbol{x})$ is a solution to* (4) *with center of mass and $\lim_{t\to\infty} \|u(t,\boldsymbol{x})/u_B(\boldsymbol{x}) - 1\|_{L^\infty(\mathbf{R}^n)} = 0$ both vanishing, then there exist a sequence of polynomials $(u_{\ell k}(\boldsymbol{x}))_{\ell k}$, each element of which either vanishes or has degree $\ell + 2k \in]1, \frac{p}{2}+1[$, such that*

$$(14) \quad \left\| \frac{(B+|\boldsymbol{x}|^2)(u(t,\boldsymbol{x})/u_B(\boldsymbol{x}) - 1) - \sum_{\Lambda<\lambda_{\ell k}<0} u_{\ell k}(\boldsymbol{x}) e^{\lambda_{\ell k} t}}{(B+|\boldsymbol{x}|^2)^{\left(p+2-\sqrt{(p+2)^2+4\Lambda}\right)/4}} \right\|_{C^\alpha(\mathcal{M})} = O(e^{\Lambda t})$$

as $t \to \infty$, where the sum is over non-negative integers $k, \ell \in \mathbf{N}$ for which $\lambda_{\ell k}$ defined by (11) *lies in the interval $]\Lambda, 0[$, and for which $\ell \le 1$ if $n = 1$. The functions $u_{\ell k}(\boldsymbol{x})/(B+|\boldsymbol{x}|^2)$ lie in the $\lambda_{\ell k}$ eigenspace of the linear operator* (23) *on $C^\alpha_{\eta_{cr}}(\mathcal{M})$, and the norm $\|\cdot\|_{C^\alpha(\mathcal{M})} \ge \|\cdot\|_{L^\infty(\mathbf{R}^n)}$ is defined by* (33).

Remark 1.3. *The coefficients $u_{\ell 0}(\boldsymbol{x})$ are given explicitly by Corollary 11.3; the invariant manifolds corresponding to these particular modes can be identified using the same idea.*

Remark 1.4. *The sum appearing in* (14) *contains up to eight non-zero terms, depending on n, m_p and on Λ. These correspond to eigenvalues from some subset of $\{\lambda_{01}, \lambda_{02}, \lambda_{03}, \lambda_{11}, \lambda_{12}, \lambda_{20}, \lambda_{21}, \lambda_{30}\}$.*

Proof of Remark 1.4. Formula (11) implies the monotone dependence of $\lambda_{\ell k} > \lambda_{\ell k+1}$ on k in the admissible range $\ell + 2k \in]1, \frac{p}{2}+1[$, so $\lambda_{\ell k}$ exceeds $]\Lambda, 0[$ unless $-\lambda_{\ell 0} = \ell(p+n) < 4p = -2\lambda_{01}$. This forces $\ell < 4$. On the other hand $\ell + 2k < \frac{p}{2} + 1$ implies $(p+n)\ell + 2k(\frac{p}{2}+1-\ell) < \lambda_{\ell k}$. For this to be within $2\lambda_{10} = -4p$ of the origin we need $k < 4 - \ell - \frac{2\ell^2+(n-6)\ell+8}{p-2\ell+2} < 4 - \ell$ which establishes the remark (except for the case $n = 1$ which needs to be argued separately). If $n = 3$ and $m = m_{11}$ and $\Lambda/\lambda_{10} \sim 2$ all eight terms may appear. □

These conclusions translate back to the original variables easily:

Corollary 1.5. *Fix $p = 2(1-m)^{-1} - n > 2$ and $2\lambda_{01} < \Lambda \in [\lambda_0^{\text{cont}}, \lambda_{01}]$. If $\rho(\tau, \boldsymbol{y})$ is a solution to* (1) *with vanishing center of mass and satisfying* (8), *then*

$$(15) \qquad \left\| \frac{\left(\dfrac{\rho_B(\tau,\mathbf{0})}{\rho_B(\tau,\boldsymbol{y})}\right)^{\frac{2}{p+n}} \left(\dfrac{\rho(\tau,\boldsymbol{y})}{\rho_B(\tau,\boldsymbol{y})} - 1\right) - \displaystyle\sum_{\Lambda<\lambda_{\ell k}<0} \dfrac{u_{\ell k}\big((1+2p\tau)^{-\beta}\boldsymbol{y})\big)}{B(1+2p\tau)^{-\lambda_{\ell k}/2p}}}{(\rho_B(\tau,\mathbf{0})/\rho_B(\tau,\boldsymbol{y}))^{\left(p+2-\sqrt{(p+2)^2+4\Lambda}\right)/(2p+2n)} \times \tau^{\Lambda/2p}} \right\|_{L^\infty(\mathbf{R}^n)}$$

remains bounded as $\tau \to \infty$, the notation being the same as in Theorem 1.2.

Proof. Evaluate (14) using $2p\tau = e^{2pt} - 1$ to get

$$\left\| \frac{\frac{B+|\boldsymbol{x}|^2}{B}\left(\frac{u(\frac{1}{2p}\log|1+2p\tau|,\boldsymbol{x})}{u_B(\boldsymbol{x})} - 1\right) - \frac{1}{B}\sum u_{\ell k}(\boldsymbol{x})(1+2p\tau)^{\lambda_{\ell k}/2p}}{((B+|\boldsymbol{x}|^2)/B)^{\left(p+2-\sqrt{(p+2)^2+4\Lambda}\right)/4}} \right\| = O(\tau^{\Lambda/2p})$$

as $\tau \to \infty$. Setting $\boldsymbol{x} = (1+2p\tau)^{-\beta}\boldsymbol{y}$, comparison with (3)–(7) establishes (15). □

Remark 1.6. *From* (11) *and $\beta = \frac{1}{2}(1+\frac{n}{p})$, only products of powers of $\tau^{-1/2}$ and of $\tau^{-1/2p}$ can arise as cofactors of the polynomials appearing in* (15)*. Since $\Lambda/2p > -2$, we can actually replace both occurrences of $1+2p\tau$ by $2p\tau$ in* (15)*.*

Taking $\Lambda = \lambda_{01} = -2p$, the sum in (15) disappears and we recover an estimate in an unweighted norm, which is Theorem 1.1. By taking $\Lambda < \lambda_{01}$, we now concretely identify the leading order correction appearing in the weighted asymptotic expansion. This yields a corollary analogous in our framework to the entropic improvement found by Dolbeault and Toscani [22].

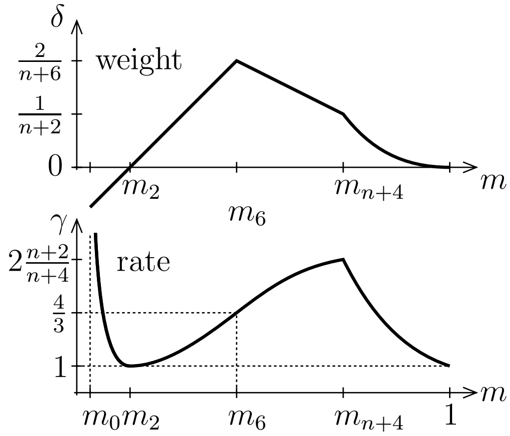


FIGURE 1. *Rate γ and weight δ from Corollary 1.7 and Theorem 10.1.*

Corollary 1.7 (Second order asymptotics modulo translations and dilations). *Suppose* $1 > m > m_2 = \frac{n}{n+2}$ *and* $\rho(\tau, \boldsymbol{y})$ *satisfies the assumptions of Theorem 1.1, and the center of mass of* $\rho(0, \boldsymbol{y})$ *is at the origin. Then there exists* $\tau_0 \in \mathbf{R}$ *such that*

$$\limsup_{\tau\to\infty} \sup_{\boldsymbol{y}} \tau^\gamma \left(\frac{\rho_B(\tau, \boldsymbol{y})}{\rho_B(\tau, 0)}\right)^\delta \left|\frac{\rho(\tau, \boldsymbol{y})}{\rho_B(\tau - \tau_0, \boldsymbol{y})} - 1\right| < \infty$$

where, for $n \geq 2$*, we have*

$$\gamma = \tfrac{(p+2)^2}{8p} = \tfrac{[2-(n-2)(1-m)]^2}{8(1-m)[2-n(1-m)]} \quad \text{if } m_2 < m \leq m_6\,,$$
$$(16) \qquad \gamma = \tfrac{2(p-2)}{p} = \tfrac{4-2(n+2)(1-m)}{2-(1-m)n} \quad \text{if } m_6 \leq m \leq m_{n+4}\,,$$
$$\gamma = \tfrac{n+p}{p} = \tfrac{2}{2-(1-m)n} \quad \text{if } m_{n+4} \leq m < 1\,,$$

and

$$\delta = \tfrac{1}{n+p}(\tfrac{p}{2} - 1) = \tfrac{1}{4}(2m - n(1-m)) \quad \text{if } m_2 < m \leq m_6\,,$$
$$(17) \qquad \delta = \tfrac{2}{n+p} = 1 - m \quad \text{if } m_6 \leq m \leq m_{n+4}\,,$$
$$\delta = \tfrac{1}{n+p}\left(\tfrac{p}{2} - 1 - \sqrt{(\tfrac{p}{2} - 1)^2 - 2n}\right) \quad \text{if } m_{n+4} \leq m < 1\,.$$

(Recall $m_2 = \frac{n}{n+2}$*,* $m_n = \frac{n-1}{n}$*,* $m_{n+4} = \frac{n+1}{n+2}$*.)*
For $n = 1$*, the first case applies to* $m_2 < m \leq m_{p_*}$*, and the third case to* $m_{p_*} \leq m < 1$*, with* $p_* = 2(\sqrt{2} + 1)$*, the middle case being omitted.*

Remark 1.8. *On compact sets, or sets that grow at rate no faster that* τ^β*, we therefore get the improved convergence rate of* $O(\tau^{-\gamma})$*, whereas an unweighted estimate that is uniform over all of* $\mathbf{R}^n$ *cannot be obtained from this theorem.*

Remark 1.9 (Even higher asymptotics). *Here* $\gamma = \Lambda/\lambda_{01}$ *and* δ *correspond to the choice* $\Lambda = \max\{\lambda_0^{\text{cont}}, \lambda_{02}, \lambda_{20}\} =: \lambda$ *in Corollary 1.5. Corollary 1.7 asserts that with this choice, a suitable translation of the solution* $\rho(\tau, \boldsymbol{y})$ *in time makes the summation in* (15) *vanish. Assuming* $\rho(\tau, \boldsymbol{y})$ *itself denotes this translation and letting* $u(t, \boldsymbol{x})$ *be the rescaled solution* (3)*, hypothesis* (103) *of Theorem 11.1 is then satisfied, which allows us to access even more terms in the asymptotic expansion than provided by Theorem 1.2: up to* $\Lambda > \max\{\lambda + \lambda_{01}, \lambda_0^{\text{cont}}\}$*.*

Looking ahead at Theorem 8.1, it will be noticed that m_n is the value where the eigenvalues λ_{10} and λ_{01} cross, and that m_{n+4} is where the eigenvalues $\lambda_{20} = -2p - 2n$, $\lambda_{11} = -3p - n + 4$ and $\lambda_{02} = -4p + 8$ cross.

Corollary 1.10 (Third order asymptotics and affinely self similar solutions). *Fix* $m = m_p = 1 - 2/(p + n)$ *with* $p > 4$*,* $n \geq 2$*, and let* $\rho(\tau, \boldsymbol{y})$ *satisfy the assumptions of Theorem 1.1 with the center of mass of* $\rho(0, \boldsymbol{y})$ *at the origin. Then a traceless* $n \times n$ *symmetric matrix* Σ_0 *and function* $\sigma(\tau)$ *exist satisfying*

$(d\sigma/d\tau)^{p+n} = c_B \det \Sigma(\tau)$, where $\Sigma(\tau) = \Sigma_0 + \sigma(\tau) I \geq 0$ and c_B is a constant depending on (n, p, B), such that $\tilde\rho(\tau, \boldsymbol{y}) = u_B(\Sigma^{-1/2}(\tau)\boldsymbol{y}) \det \Sigma^{-1/2}(\tau)$ solves (1) for large τ and

$$\limsup_{\tau\to\infty} \sup_{\boldsymbol{y}} \tau^\gamma \left(\frac{\rho_B(\tau,\boldsymbol{y})}{\rho_B(\tau,0)}\right)^\delta \left|\frac{\rho(\tau,\boldsymbol{y})}{\tilde\rho(\tau,\boldsymbol{y})} - 1\right| < \infty\,, \tag{18}$$

where $\gamma = -\lambda_{11}/2p$ and $\delta = \left(\frac{p}{2} - 1 - \sqrt{(\frac{p}{2}+1)^2 + \lambda_{11}}\right)/(p+n)$.

A more detailed interpretation of Theorem 1.2 will be given in the next section, after an exposition of the proofs with some more technical details. The full proof (relative to all preceding technical lemmas) will be given in Section 11. Let it suffice to say here that there is an intricate interplay between tail behavior δ and convergence rate γ that is responsible for the occurrence of the weights in the theorem.

A partner theorem for the case $m \in]m_0, m_2[$ is available below (Thm. 10.1).

Although the results contained herein fall short of a full-blown invariant manifold theory for the fast diffusion equation, they represent a significant advance in that direction. Invariant manifolds have a rich history of successful applications in partial differential equations, as in e.g. Wayne [46] and the references there. However, much of that work has been devoted to semilinear heat equations, and cannot be directly adapted to the quasilinear equation now confronted. Wayne himself raises the question of whether it is possible to extend such an analysis to nonlinearities which, though still semilinear, depend on derivatives. The present work develops the relevant ideas and techniques, and applies them to an example which may provide insights and a blueprint for the general problem. It complements the studies of the porous medium equation $m > 1$ by Angenent [1] [2] in one-dimension, and in higher dimensions by Koch [32]. Still there are some significant ways in which our analysis resembles that of Wayne [46] or his results on the convergence of $2D$ Navier-Stokes dynamics to Oseen vortices with Gallay [28]: we study the size of the solution relative to the fixed point of an appropriately rescaled dynamics (3) as they do (namely we work with $(u - u_B)/u_B$) , and like them we use weighted spaces to shift the essential spectrum and reveal higher asymptotics in (14). However, we are forced to work in Hölder spaces (like Angenent [1], but on an unbounded domain), rather than the Hilbert spaces of Gallay-Wayne — or for that matter of McCann-Slepcev [37], the duo [22], or the quartet [8] — and with a spectral theory for non-self-adjoint operators, which is more involved than the linear problems analyzed in those works. Furthermore, there is a limit $\Lambda \in [\lambda_0^{\text{cont}}, 0]$ to the resolution which we can hope to attain from this linear analysis, in sharp contradistinction to the complete asymptotics accessible in the problems addressed by Gallay and Wayne, and by Angenent. Except in the special case $m = m_{-2}$ [9], we do not know how dynamical information beyond the threshold imposed by our continuous spectrum could be resolved.

2. Overview of Obstructions and Strategies, and Notation

The basic idea to prove the asymptotic results is taken from dynamical systems as outlined in [20], [46], or the references there: the eigenvalues of the linearization 'ought' to determine the rate of convergence to the equilibrium u_B. For example, when the analogous smooth finite-dimensional evolution

$$\frac{d}{dt}x(t) = -V(x(t)) \in \mathbf{R}^n$$

is linearized around a fixed point $V(x_\infty) = 0$ we get:

$$\frac{d}{dt}(x(t) - x_\infty) = -DV(x_\infty)(x(t) - x_\infty) + O(x(t) - x_\infty)^2 .$$

If $\sigma(DV(x_\infty)) = \{\lambda_1 \leq \cdots \leq \lambda_n\}$ with eigenvectors $\hat{\phi}_i$, a naive expectation (albeit not entirely correct due to the possibility of resonances) is

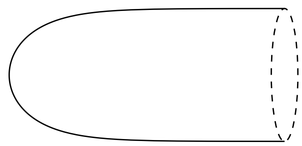
$$x(t) = \sum_{i=1}^{n} c_i \hat{\phi}_i e^{-\lambda_i t} + \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}\hat{\phi}_i\hat{\phi}_j e^{-(\lambda_i+\lambda_j)t} + \sum_i\sum_j\sum_k \cdots .$$

Notice however, that we may neglect the iterated summations and ignore resonances if we are content with asymptotics to order $O(e^{-2\lambda_1 t})$ as $t \to \infty$. This is the strategy employed in our theorems. *Differentiability* of $V(x_0)$ or $x(t) = X(t, x_0)$ with respect to $x_0 \in \mathbf{R}^n$ is crucial. However for the infinite dimensional dynamics of u, the choice of the right space becomes an issue. Even more than in the semilinear context [27], the issue is plagued by conflicting requirements from PDE theory and from functional analysis.

The Hilbert space setting (a weighted Sobolev space $W^{1,2}_\rho$, which will be called $W^{1,2}_{u_B}$ here) in which the spectral analysis was originally carried out in [20] is (at best) inconvenient to deal with the nonlinearity. Its geometric interpretation in terms of mass transport also restricts the parameter range for m to the case where Barenblatt has second moments.

The singularity of the equation near 0 is an obstruction to getting a *smooth* semigroup in many Banach spaces: While the parabolic evolution moves u pointwise away from the singularity 0 immediately, it does not necessarily move it out of any norm neighborhood of 0, unless weights are built into the norm that are adjusted to the Barenblatt profile. On the other hand, as in Angenent [1], smoothness, rather than mere continuity, of the semiflow is crucial for the dynamical systems approach via spectrum of a linearization to have a significance.

The 'relative' variable $v = u/u_B$ introduces the appropriate weight, and Vázquez' estimate (8) guarantees that after finite time, every solution is bounded away from 0 in the L^∞ norm of v. The singularity still shows up in the fact that the spatial differential operator fails to be *uniformly* elliptic in $\mathbf{R}^n$, due to the decaying Barenblatt whose power multiplies the Laplacian. But now $\mathbf{R}^n$ can be endowed with a conformally flat Riemannian metric, such that the Laplace–Beltrami operator with respect to this metric coincides with the linearized evolution operator in its principal part. Working consistently with respect to this metric restores uniform parabolicity of the equation. $\mathbf{R}^n$ becomes a manifold $\mathcal{M}$ that can be illustrated by isometrically embedding it into $\mathbf{R}^{n+1}$ like the surface of a cigar.



The natural space in which to get a well-posed initial value problem for the nonlinear equation is a Hölder space, and Hölder quotients need to be referred to the geodesic distance on the cigar manifold. Likewise, for derivatives, unit vectors with respect to the Riemannian metric should be used to get appropriately weighted higher Hölder norms. It is in these $C^\alpha(\mathcal{M})$ spaces that we get smoothness of the evolution for the nonlinear equation. The detailed discussion of the cigar manifold and the Hölder spaces are found in Sections 4 and 5, respectively.

Section 6 develops some auxiliary heat estimates which are used in Section 7 to carry over the parabolic Schauder theory for linear and nonlinear equations from local results in flat space to the uniform setting on the cigar. (Self-contained proofs for the estimates of Section 6 are deferred to an appendix.) The key point is that the cigar can be covered with coordinate charts in which the distortion between its Riemannian metric and a (local) flat Euclidean metric is bounded with all derivatives, uniformly over all coordinate charts, even though the global distortion between the a priori Euclidean metric on $\mathbf{R}^n$ and the cigar metric is of course unbounded. Thereafter, the analytic local semiflow for the nonlinear equation follows just like the linear estimates by contraction mapping, exactly as it would be done in the textbook model cases on a bounded domain. Iterative usage of the local estimates can be controlled by a priori estimates via the maximum principle to get the global semiflow. Let it be stressed that the existence of a semiflow for the FDE has long been established by Herrero and Pierre [29] for L^1_{loc} data, our concern here is to get a flow map which depends *differentiably* on the initial data.

Before comparing the nonlinear flow with its linearization, let us remember the obstruction given by Vázquez [44, Thm. 1.3]: There cannot be a quantitative convergence rate to Barenblatt for general L^1 data. The idea behind this result is that one can start with small lumps of mass arbitrarily far apart, and the time until the spreading Barenblatts that arise from these lumps overlap significantly depends on their distance. So an infinity of lumps (with finite total mass) can produce arbitrarily slow convergence rates out of arbitrarily large distances. The spectral analysis two of us developed in [20] is unaffected by this obstruction, because it is based on a linearization within Otto's formalism [38], which takes the distance between lumps into account in the form of transport costs. Of course, now we have to rephrase the spectral theory within the framework of the Hölder space $C^\alpha(\mathcal{M})$, which will introduce some modifications. However, the key eigenvalues whose geometric meaning was understood from [20] are still in place, and so is their interpretation. We do have to cope with tail behavior issues, however, in particular for small m where the Barenblatt solution has few moments. The spectral analysis is done in Section 8. Much can be understood in terms of qualitative arguments of asymptotic analysis, only the precise values of the eigenvalues determining the convergence rates require the same explicit calculation as in [20]. We find it illustrative to get also the precise spectrum in various weighted Hölder spaces out of the previously established formulas.

Like for the L^1 norm, bounding the Hölder space norm does not prevent us from starting with lumps that are far away from each other. We therefore cannot expect a global estimate of the form $\|u(t) - u_{\scriptscriptstyle B}\|_1 \leq C(\|u(0)\|_2) \exp[-at]$ even if we were to attempt to control the constant in terms of a rather strong norm $\|u(0)\|_2$. In contrast to results of the quartet [8], our results are inherently local. It would be interesting to see if Wasserstein distance bounds could combine our spectral theoretic approach with more global information, but the functional analytic difficulties to make the 'formal infinite dimensional Riemannian manifold' approach work for the nonlinear problem have eluded us and appear to be pretty insurmountable. The proof of Thm. 1.1 is carried out as a rather immediate consequence of the flow properties and spectral analysis developed; see Sec. 9.

Finally, we need to address the influence of tail behavior on the spectral theory. The Hilbert space in which our first spectral analysis was carried out carries a power of $u_{\scriptscriptstyle B}$ as a weight, and allows a certain growth rate of functions in this space if $m > m_2$, while enforcing decay if $m < m_2$. The growth rates are powers of $|\boldsymbol{x}|$, which is exponential in the geodesic distance s from the center on $\mathcal{M}$; and this affects the essential spectral radius. We introduce weighted Hölder spaces C^α_η, which allow a spatial growth $|\boldsymbol{x}|^\eta$ (or equivalently $\cosh^\eta s$ in said geodesic distance from Sec 3). The critical growth rate allowed in the Hilbert space is $\eta_{cr} = \frac{p}{2} - 1$ where p is defined by (2). In terms of spectral theory, $C^\alpha(\mathcal{M})$ is therefore closest to the Hilbert space when $m = m_2$, because in this case $\eta_{cr} = 0$. The eigenfunctions of the linearization have their own characteristic growth rates (independent of m), which do not automatically coincide with the 'no growth' requirement that came from the need to keep the influence of the singularity $u = 0$ of the nonlinear equation outside a ball around the Barenblatt in the Banach space. There is a trade-off between growth hypotheses and convergence rates, and once the obvious geometric invariances (space and time translation) of the unrescaled fast diffusion equation are modded out, finer asymptotics (like, e.g., from the effect that the diffusion brings anisotropic initial data closer to the isotropic distribution u_B) need to measured in norms whose weight adjusts to the tail behavior of the linearization in the corresponding direction in function space, a tail behavior different from the one of Barenblatt itself. This is what makes Thm. 1.2 and its corollaries look somewhat technical. But there is good reason to believe that this displays a genuine phenomenon, not an artefact of the method. The good news is that, in fact, this discrepancy in tail behavior can be captured solely in terms of an adjusted weight in the norm in which the convergence rate is measured, but does not change the convergence rate itself. This fact arises as a consequence of the maximum principle for the linearized equation.

Getting precise convergence rates requires an estimate for the semigroup of the form $O(\exp[\lambda t])$ rather than merely $O(\exp[(\lambda+\varepsilon)t])$. It is in this step that the tail behavior discrepancy (and its resolution by juggling the weights) become significant. This is the contents of Thm. 8.5 and its proof. The finer asymptotics provided by Thm. 1.2 and its corollaries are then proved in Section 11 as a consequence in much the same way as Thm. 1.1, after a Lyapunov–Schmidt decomposition of the function space, with the exponential dichotomy property provided by the spectral gap.

In the following, we will need to rewrite equation (4) in various forms, bringing to light the various aspects outlined above; different variable names help to distinguish the different versions and navigate between them. The notations collected in Table 1 will serve as a reference in this endeavour.

3. The nonlinear and linear equations in cigar coordinates

We approach the proof of Theorem 1.1 by an analysis in self-similar coordinates.

To simplify the discussion we work with the relative density $v := u/u_B$ and calculate, using the PDEs for u and u_B,

$$\begin{aligned} v_t = u_B^{-1}u_t &= \tfrac{1}{m}u_B^{-1}\Delta(u_B^m v^m) + \tfrac{2}{1-m}u_B^{-1}\boldsymbol{\nabla}\cdot(\boldsymbol{x}u_B v)\,, \\ u_B^{-1}\Delta(u_B^m v^m) &= u_B^{m-1}\Delta(v^m) + 2u_B^{-1}\boldsymbol{\nabla}u_B^m\cdot\boldsymbol{\nabla}v^m + u_B^{-1}(\Delta u_B^m)\,v^m \\ &= u_B^{m-1}\Delta(v^m) - 2\tfrac{m}{1-m}\boldsymbol{\nabla}u_B^{m-1}\cdot\boldsymbol{\nabla}v^m - \tfrac{2m}{1-m}u_B^{-1}\boldsymbol{\nabla}\cdot(\boldsymbol{x}u_B)\,v^m\,, \\ u_B^{-1}\boldsymbol{\nabla}\cdot(\boldsymbol{x}u_B v) &= v u_B^{-1}\boldsymbol{\nabla}\cdot(\boldsymbol{x}u_B) + \boldsymbol{x}\cdot\boldsymbol{\nabla}v\,. \end{aligned}$$

Now we use Equation (6) to get

$$u_B^{-1}\boldsymbol{\nabla}\cdot(\boldsymbol{x}u_B) = n - \frac{2}{1-m}u_B^{1-m}|\boldsymbol{x}|^2 = n - \frac{2}{1-m} + \frac{2}{1-m}\frac{B}{B+|\boldsymbol{x}|^2}\,,$$

ρ	solution of the unscaled FDE (1) – arguments τ and $\boldsymbol{y}$
u	solution of the rescaled FDE (4) – arguments t and $\boldsymbol{x}$
	generic name for function on uniform manifold (Sec. 7)
u_B	Barenblatt solution (6), with parameter B determining its mass
v	$= u/u_B$ solves (19), (21); generic function in heat equation estimates Sec. 6
$\bar{v}$	Variable of linearisation with respect to v about 1
w	$w = v - 1 = u/u_B - 1$; generic Hölder function in Sec. 5
v_0, $\bar{v}_0$	initial data at time 0
$\tilde{v}$	tilde refer to conjugation with a power of $\cosh s$
$\tilde{u}$	exception: $u = u_0 + \tilde{u}$ in Sec. 7
v_l, u_l	local functions on uniform manifold (pulled back to coordinate patch) $v_l = \eta_l u_l$
l	index for coordinate patches (distinct from ℓ!)
η_l	partition of unity
χ_l	coordinate maps
m_p	value of m when u_B has moments below order p: $m_p = \frac{n+p-2}{n+p}$
p	moment parameter: $n + p = 2/(1 - m)$
$\mathbf{L}$	linearization operator (22)
$\mathbf{L}_\eta$	conjugated linearization operator, see (25)
$\mathbb{L}$, **Ł**, $\text{\L}_\eta$	various operators with same principal part as $\mathbf{L}$, $\mathbf{L}_\eta$
$\mathcal{M}$	cigar manifold
$\mathbf{M}$	generic uniform manifold (of which $\mathcal{M}$ is a special case)
r	$= \|\boldsymbol{x}\|$
s	geodesic distance from origin: $B^{1/2}\sinh s = r$; $B\cosh^2 s = u_B^{-(1-m)}$
C^α	Hölder spaces (space or space-time) – Sec. 4
C^α_η	Hölder space with weight allowing growth; see (79)
C_b	space of bounded and continuous functions
α	Hölder exponent
β	see (3)
$\mathbf{Q}$	spectral projection for $\mathbf{L}$ or $\mathbf{L}_\eta$ on a finite dimensional space
$\mathbf{P}$	$= 1 - \mathbf{Q}$
$\mathbf{S}$	(analytic) linear semigroup generated by $\mathbf{L}$ – analogous for $\mathbf{S}_\eta$
ℓ	angular momentum quantum number – except for (28)
$\lambda_{\ell k}, \lambda_\ell^{\text{cont}}, \lambda$	eigenvalue, onset of essential spectrum, spectral parameter
$\psi_{\ell k}, v_{\ell k}$	eigenfunctions in various settings: Thms. 8.1 and 8.2 respectively
$u_{\ell k}$	$= v_{\ell k} u_B^{m-1}$
$Y_{\ell\mu}$	spherical harmonics
η, η_{cr}	parameter for conjugation of operator (growth in space); $\eta_{cr} = \frac{p}{2} - 1$
$b^\infty(\eta)$	coefficient of 1st order term of $\mathbf{L}_\eta$ at ∞
$c^\infty(\eta)$	coefficient of 0th order term of $\mathbf{L}_\eta$ at ∞
T	finite time step; we consider time-T maps
T	related to spectral parameter λ, see Fig. 2 on page 51

TABLE 1. *Overview of notations used*

and consequently

(19)

$$v_t = \frac{1}{m} u_B^{m-1} \Delta v^m + \frac{2}{1-m}\left(\boldsymbol{x}\cdot\boldsymbol{\nabla}(v - 2v^m) + \left(n - \frac{2u_B^{1-m}}{1-m}|\boldsymbol{x}|^2\right)(v - v^m)\right) .$$

This reaction / transport / nonlinear diffusion equation will play a central role in our analysis. It will turn out that it is a uniformly parabolic problem (for $c^{-1} \le v \le c$) on an unbounded manifold with a cylindrical end, known as the cigar. Its structure can be seen more easily in polar coordinates. We may and do normalize the parameter B to $B = 1$.

With $|\boldsymbol{x}| =: r$, we can write $\Delta = \partial_r^2 + \frac{n-1}{r}\partial_r + \frac{1}{r^2}\Delta_{\mathbf{S}}$, where $\Delta_{\mathbf{S}}$ is the Laplace-Beltrami operator on the unit sphere. So we want to rewrite (19) as

$$\begin{aligned} v_t &= \tfrac{1}{m}\Big(1+r^2\Big)\Big(\tfrac{\partial^2}{\partial r^2}v^m + \tfrac{n-1}{r}\tfrac{\partial}{\partial r}v^m\Big) + \tfrac{1}{m}\Big(\tfrac{1}{r^2}+1\Big)\Delta_{\mathbf{S}}v^m \\ &\quad + \tfrac{2}{1-m}r\tfrac{\partial}{\partial r}(v-2v^m) + \big(\tfrac{2}{1-m}\big)^2\Big(\tfrac{n(1-m)}{2} - 1 + (1+r^2)^{-1}\Big)(v-v^m) \\ &= \tfrac{1}{m}\tfrac{\partial}{\partial r}\Big[(1+r^2)\tfrac{\partial}{\partial r}v^m\Big] + \tfrac{1}{m}\Big(\tfrac{1}{r^2}+1\Big)\Delta_{\mathbf{S}}v^m \\ &\quad + \tfrac{1}{m}\Big[-\tfrac{2(1+m)}{1-m}r + \tfrac{n-1}{r} + (n-1)r\Big]\tfrac{\partial}{\partial r}v^m \\ &\quad + \tfrac{2}{1-m}r\tfrac{\partial}{\partial r}v + \big(\tfrac{2}{1-m}\big)^2\Big(\tfrac{n(1-m)}{2} - 1 + (1+r^2)^{-1}\Big)(v-v^m)\,. \end{aligned} \tag{20}$$

We set $r = \sinh s$, hence $\frac{\partial}{\partial r} = \frac{\partial s}{\partial r}\frac{\partial}{\partial s} = (\cosh s)^{-1}\frac{\partial}{\partial s}$ and

$$\begin{aligned} \tfrac{1}{m}\tfrac{\partial}{\partial r}\Big[(1+r^2)\tfrac{\partial}{\partial r}v^m\Big] &= \tfrac{1}{m}(\cosh s)^{-1}\tfrac{\partial}{\partial s}\Big((1+\sinh^2 s)(\cosh s)^{-1}\tfrac{\partial}{\partial s}v^m\Big) \\ &= \tfrac{1}{m}\tfrac{\partial^2}{\partial s^2}v^m + \tfrac{1}{m}\tanh s\,\tfrac{\partial}{\partial s}v^m\,. \end{aligned}$$

This s will be the geodesic distance from the origin in the Riemannian metric of the cigar manifold introduced below. We can rewrite equation (20) as

$$\begin{aligned} v_t &= \frac{1}{m}\left(\frac{\partial^2}{\partial s^2}v^m + \frac{2(n-1)}{\sinh(2s)}\frac{\partial}{\partial s}v^m + (\tanh s)^{-2}\,\Delta_{\mathbf{S}}v^m\right) \\ &\quad + \frac{1}{m}\Big(n - \frac{2(m+1)}{1-m}\Big)\tanh s\frac{\partial}{\partial s}v^m \\ &\quad + \frac{2}{1-m}(\tanh s)\frac{\partial}{\partial s}v + \frac{4}{(1-m)^2}\Big(n\frac{1-m}{2} - 1 + (\cosh s)^{-2}\Big)(v-v^m)\,. \end{aligned} \tag{21}$$

It is convenient to introduce the operator $\mathbf{L}$ as the linearization about 1 of the operator in (21). Namely,

$$\begin{aligned} \mathbf{L} &:= \partial_s^2 + \frac{2(n-1)}{\sinh 2s}\partial_s + (\tanh s)^{-2}\,\Delta_{\mathbf{S}} \\ &\quad + \Big(n - \frac{2m}{1-m}\Big)\tanh s\,\partial_s + 2n - \frac{4}{1-m}\tanh^2 s\,. \end{aligned} \tag{22}$$

For later reference, we also calculate $\mathbf{L}$ in cartesian coordinates from (19):

$$\begin{aligned}\mathbf{L}\bar v &= u_B^{m-1}\Delta \bar v + \frac{2(1-2m)}{1-m}\boldsymbol{x}\cdot\boldsymbol{\nabla}\bar v + \Big(n - u_B^{1-m}\frac{2}{1-m}|\boldsymbol{x}|^2\Big)2\bar v \\ &= u_B^{-1}\boldsymbol{\nabla}\cdot\left(u_B\boldsymbol{\nabla}(u_B^{m-1}\bar v)\right) .\end{aligned} \tag{23}$$

Using $\mathbf{L}$, we will also find it convenient to rewrite (19) for $w = v - 1$ as

$$w_t = \mathbf{L}h(w) + \frac{2}{1-m}\boldsymbol{x}\cdot\boldsymbol{\nabla}(w - h(w)) + \frac{2}{1-m}\Big(n - \frac{2u_B^{1-m}}{1-m}|\boldsymbol{x}|^2\Big)(w - h(w))\,, \tag{24}$$

where $h(w) = \frac{(1+w)^m - 1}{m} = w + O(w^2)$.

Later on we will conjugate operator (22) by $\cosh^\eta s$. In order to keep these calculations at one place we calculate

$$\cosh^{-\eta} s \circ \partial_s \circ \cosh^\eta s = \partial_s + \eta \tanh s$$

and

$$\cosh^{-\eta} s \circ \partial_s^2 \circ \cosh^\eta s = \partial_s^2 + 2\eta \tanh s\, \partial_s + \eta(\eta - 1)\tanh^2 s + \eta\,,$$

which gives after some calculation

$$\begin{aligned}\mathbf{L}_\eta &:= \cosh^{-\eta} s \circ \mathbf{L} \circ \cosh^\eta s \\ &= \left(\partial_s^2 + \frac{2(n-1)}{\sinh 2s}\partial_s + (\tanh s)^{-2}\,\Delta_{\mathbf{S}}\right) \\ &\quad - 2\left(\tfrac{1}{1-m} - \tfrac{n}{2} - 1 - \eta\right)\tanh s\,\partial_s \\ &\quad + \left(\tfrac{1}{1-m} - \tfrac{n}{2} - 1 - \eta\right)^2 - \left(\tfrac{1}{1-m} - \tfrac{n}{2} + 1\right)^2 \\ &\quad + \left(\left(\tfrac{1}{1-m} + 1\right)^2 - \left(\tfrac{1}{1-m} - 1 - \eta\right)^2\right)\frac{1}{\cosh^2 s}\,.\end{aligned} \tag{25}$$

Since the leading order terms coincide with the Laplace-Beltrami operator (29), we can see $-\mathbf{L}_\eta$ as a Schrödinger operator for a quantum particle moving in a potential-well on the cigar manifold, perturbed by a transport term which spoils self-adjointness. The potential-well has a universal $(\cosh s)^{-2}$ profile of depth

$$d(\eta) = (2(1-m)^{-1} - \eta)(2 + \eta) = (p + n - \eta)(2 + \eta)\,,$$

and is asymptotic to the constant

$$-c^\infty(\eta) = (2(1-m)^{-1} - n - \eta)(2 + \eta) = (p - \eta)(2 + \eta) \tag{26}$$

at the $s \to \infty$ end of the cigar. The transport term shifts mass along the cigar with outward velocity asymptotic to

$$-b^\infty(\eta) = 2((1-m)^{-1} - \eta - 1 - n/2) = p - 2\eta - 2 \tag{27}$$

as $s \to \infty$, making it harder for the operator to support eigenstates unless $\eta \geq \frac{p}{2} - 1$.

The value $\eta_{cr} = \frac{m}{1-m} - \frac{n}{2} = \frac{p}{2} - 1$ plays a special role, causing the transport term to disappear and restoring the symmetry property

$$\int_0^\infty \int_{\mathbb{S}^{n-1}} (\mathbf{L}_{\eta_{cr}} \bar{v}_1)\, \bar{v}_2 \tanh^{n-1} s\, d\omega_{n-1} ds = \int_0^\infty \int_{\mathbb{S}^{n-1}} \bar{v}_1\, (\mathbf{L}_{\eta_{cr}} \bar{v}_2) \tanh^{n-1} s\, d\omega_{n-1} ds$$

of the operator $\mathbf{L}_{\eta_{cr}}$ with respect to the volume element (30) on the cigar manifold. Equivalently, $\mathbf{L}_0$ is symmetric with respect to u_B^m times the Euclidean volume element in $\mathbf{R}^n$, which is the analog of the self-adjointness in [20].

This same value $\eta_{cr} = \frac{m}{1-m} - \frac{n}{2} = \frac{p}{2} - 1$ minimizes $c^\infty(\eta)$, producing a quantity

$$c^\infty(\eta_{cr}) = -\left(\frac{1}{1-m} - \frac{n}{2} + 1\right)^2 = -\left(\frac{p}{2} + 1\right)^2$$

which gives the onset of the continuous spectrum, the negative of what was called λ_0^{cont} in [20], see Thm. 8.1 below. It combines with depth $d(\eta_{cr})$ to produce a vanishing ground state energy for the Schrödinger operator $\mathbf{L}_{\eta_{cr}}$ on $L^2(\mathcal{M})$; c.f. Thm. 8.2(1). We note that $\eta_{cr} = 0$ iff $m = m_2 = \frac{n}{n+2}$, it is positive for $m > m_2$ and negative for $m < m_2$. This is the reason for the case distinction between Thm. 1.2 (proved in Sec. 11) and Thm. 10.1.

In the remaining sections of this paper we will explore the consequences of our calculations. We shall see next section that the metric defined by the leading part of our operators has a very natural interpretation as a Riemannian metric on a well-known manifold, the cigar manifold.

The nonlinear differential equation (21) is (for v bounded above, and bounded away from 0) a uniformly parabolic equation on the cigar manifold, and well-posedness is a standard consequence of estimates for the linearized equations, which is basically standard parabolic regularity theory. Therefore we expect that the asymptotic behaviour is controlled by the spectral properties of the linearized differential equation. But since the cigar manifold is not compact, there is nontrivial essential spectrum for the linear semigroups $\mathbf{S}(t) = \exp t\mathbf{L}$ and $\mathbf{S}_\eta(t) = \exp t\mathbf{L}_\eta$ defined through (22) and (25). One may easily read off the essential spectral radius: It is $e^{c^\infty(\eta)t}$ and it depends on η.

We are mainly interested in modes related to eigenvalues outside the essential spectrum. All eigenvalues and eigenfunctions have been determined by [20], and we can use these results, with due adaptation, which is done in Sec. 8. In case $\eta = \eta_{cr}$, we can already anticipate that the number of eigenvalues will be finite and increase with the depth $d(\eta_{cr}) = (\frac{p}{2} + 1 + n)(\frac{p}{2} + 1)$ of the potential-well. In one-dimension the Schrödinger operator $\mathbf{L}_{\eta_{cr}}$ has been studied in connection with transparent scattering, and $k(k+1)(\cosh s)^{-2}$ for $k \in \mathbf{N}$ are known as the Bargmann potentials in that context; a reference for these is Sec. 2.6, Exercise 11 of [35], as well as Secs. 2.5 and 3.5 there. In our case $k = \frac{p}{2}$. As mentioned in Section 2, we will need the conjugated operator to accommodate the growth of relevant eigenfunctions. The dependence of the essential spectral radius on η will turn out to be useful in estimating sharp decay for the linear semigroup.

4. The cigar as a Riemannian manifold

The cigar is an analytic Riemannian manifold which we denote by $\mathcal{M}$. It can be described as $\mathbf{R}^n$ equipped with the metric

$$d\ell^2_{\mathcal{M}} := \left(1+|\boldsymbol{x}|^2\right)^{-1} \sum dx_i^2 = \left(ds^2 + \tanh^2 s \, d\ell^2_{\mathbf{S}}\right) , \tag{28}$$

where $d\ell^2_{\mathbf{S}}$ is the length element on the unit sphere and $\sinh s = |\boldsymbol{x}|$ was already introduced above. It is immediate from this formula that s is the geodesic distance from the origin.

In two dimensions, the cigar is a soliton solution for the Ricci flow (see ch.2 of [14]). For the fast diffusion problem, the cigar metric is appropriate for various reasons: The leading part of $\mathbf{L}$ (22) coincides with the Laplace-Beltrami operator

$$\begin{aligned} \Delta_{\mathcal{M}} &= \left(1+|\boldsymbol{x}|^2\right)^{n/2} \circ \partial_i \circ \left(1+|\boldsymbol{x}|^2\right)^{-n/2+1} \circ \partial_i \\ &= \partial_s^2 + \frac{2(n-1)}{\sinh 2s}\partial_s + (\tanh s)^{-2}\Delta_{\mathbf{S}} , \end{aligned} \tag{29}$$

which is also the leading part of the operator $\sum_{i=1}^n X_i^2$, where X_i are the obvious orthonormal vector fields

$$X_i := \left(1+|\boldsymbol{x}|^2\right)^{1/2} \boldsymbol{e}_i$$

and $\boldsymbol{e}_i := \partial/\partial x_i$ is the standard basis. Specifically,

$$\sum_{i=1}^n X_i^2 = \Delta_{\mathcal{M}} + (n-1)\tanh s \, \partial_s .$$

As a consequence, we shall see that the parabolic estimates become uniform when distances are measured with respect to the cigar metric.

The Riemannian volume element is

$$d\mu := \tanh^{n-1} s \, ds d\omega_{n-1} . \tag{30}$$

The vector fields X_i do not commute, but their commutators of any order can be written as a bounded linear combination of these vector fields. For instance,

$$[X_i, X_j] = \frac{x_i}{(1+|\boldsymbol{x}|^2)^{1/2}} X_j - \frac{x_j}{(1+|\boldsymbol{x}|^2)^{1/2}} X_i .$$

5. Uniform manifolds and Hölder spaces

We will need uniform Schauder estimates for parabolic equations on $\mathcal{M}$. The possibility of such *uniform* estimates relies on the fact that *locally*, $\mathcal{M}$ can be mapped into $\mathbf{R}^n$ with bounded distortion, the bound being global, even though a global map with bounded distortion is not possible between $\mathcal{M}$ and $\mathbf{R}^n$. It seems expedient to elaborate on this principle in its natural generality. So we will prove these estimates for parabolic equations on a uniform manifold $\mathbf{M}$, of which the cigar $\mathcal{M}$ is the example we are interested in: We study manifolds $\mathbf{M}$ (not necessarily Riemannian) with a distance d which turns $\mathbf{M}$ into a geodesic space: A metric space $\mathbf{M}$ is called a geodesic space if, given two points $x, y \in \mathbf{M}$ there exists a path $\gamma : [0,1] \to \mathbf{M}$ from x to y such that $d(\gamma(s), \gamma(t)) = |s - t| d(x,y)$. This requirement is in particular satisfied for the geodesic distance of a connected closed Riemannian manifold.

Definition 5.1 (Uniform manifold)**.** *Let* $\mathbf{M}$ *be a manifold with a metric* d *which turns* $\mathbf{M}$ *into a geodesic space. We say* $(\mathbf{M}, d)$ *has a uniform* C^k *structure (resp. uniform analytic structure) if there exist two constants* $R > 0$ *and* $C > 0$ *and coordinate maps* $\chi_x : B_R(x) \to \mathbf{R}^n$ *for each* $x \in \mathbf{M}$ *such that*

$$C^{-1} d(y,z) \le |\chi_x(y) - \chi_x(z)| \le C d(y,z) \qquad \text{for all } x \in \mathbf{M},\ y, z \in B_R(x) \tag{31}$$

with C^k *(resp analytic) coordinate changes* $\chi_y \circ \chi_x^{-1}$ *satisfying*

$$|\partial^\beta (\chi_y \circ \chi_x^{-1})| \le C^{|\beta|+1} \beta! \qquad \text{in } \chi_x(B_R(x) \cap B_R(y)) \tag{32}$$

for any multi-index β *of length* $|\beta| \le k$ *(resp. for any multi-index). We call* R *(and* C*) a radius (and constant) of uniformity for* $\mathbf{M}$*.*

The first condition says that balls of radius B_R are uniformly bilipschitz equivalent to subsets of $\mathbf{R}^n$. The second condition (32) implies that the coordinate maps are uniformly C^k (resp. analytic).

For our purposes, C^3 would sufficient regularity, but the cigar manifold is even analytic; and it is not hard to see that the exponential map provides such coordinate maps for the cigar manifold with $R = 1$ and a suitable constant C. Observe that we do not require a Riemannian structure on the manifold. But if the manifold carries a Riemannian metric then we use the associated metric, which turns the manifold into a geodesic space.

The following lemma implies that the volume of balls is at most exponential in the radius. Given an open subset A of $\mathbf{M}$ and $r > 0$, we denote by $N(A, r)$ the maximal number of disjoint balls of radius r in A (if such a maximum exists; else $N(A, r) := \infty$).

Lemma 5.2 (Packing r-balls into ρ-balls)**.** *Suppose that* $\mathbf{M}$ *is a uniform manifold of dimension* n *with a radius and constant of uniformity* R *and* C*. Then* *the following bound holds*

$$N(B_\rho(x_0), r) \leq \begin{cases} \left(\frac{C^2\rho}{r}\right)^n & \text{if } r < \rho \leq R\,, \\ (\max\{C^2R/r, 1\})^n (5C^2)^{5n\rho/R} & \text{if } \max\{r, R\} < \rho\,.. \end{cases}$$

Proof. Suppose first that $r < \rho \leq R$ and consider a ball $B_r(x_1)$ contained in $B_\rho(x_0) \subset B_R(x_0)$. Then

$$B_{r/C}(\chi_{x_0}(x_1)) \subset \chi_{x_0}(B_r(x_1)) \subset \chi_{x_0}(B_\rho(x_0)) \subset B_{C\rho}(\chi_{x_0}(x_0))\,.$$

Comparing the volumes of the images we see that there can be at most $(C^2\rho/r)^n$ such disjoint balls in $B_\rho(x_0)$.

Secondly, let's suppose $r = \frac{R}{5}$, and ρ arbitrary. The previous case $\rho \leq R$ will serve as the beginning of an induction with steps from ρ to $\rho + r$. So suppose $\{B_r(y_j) \mid j = 1, \ldots, N\}$ is a maximal packing of r-balls in $B_\rho(x_0)$. Then the $B_{2r}(y_j)$ will cover $B_{\rho-r}(x_0)$ as a consequence of the maximality of the packing. But then the $B_{4r}(y_j)$ will cover $B_{\rho+r}(x_0)$. This is because for any $z \in B_{\rho+r}(x_0)$, we can find a $z' \in B_{\rho-r}(x_0)$ that has distance $2r$ from z; and then $z \in B_{4r}(y_j)$ whenever $z' \in B_{2r}(y_j)$. Having constructed this covering, we now take a family of disjoint balls $\{B_r(z_i) \mid i \in I\}$ in $B_{\rho+r}(x_0)$. As each z_i must be in some $B_{4r}(y_j)$, the $B_r(z_i)$ lies in $B_{5r}(y_j)$. As there are at most $(5C^2)^n$ such $B_r(z_i)$ in each given $B_{5r}(y_j)$, and there are only N many y_j's, we conclude $N(B_{\rho+r}(x_0), r) \leq (5C^2)^n N(B_\rho(x_0), r)$.

We have thus proved inductively that $N(B_\rho(x_0), R/5) \leq (5C^2)^{5n\rho/R}$.

Now we turn to the general case: If $r > \frac{R}{5}$, the estimate for $r = \frac{R}{5}$ is still valid trivially. If $r < \frac{R}{5}$, suppose we have a maximal collection of $B_r(x_i)$ in $B_\rho(x_0)$. We also place a maximal collection of $B_{R/5}(y_j)$ in $B_\rho(x_0)$. As before, the maximality then ensures that the collection of $B_{3R/5}(y_j)$ covers all of $B_\rho(x_0)$ and each $B_r(x_i)$ is contained in some $B_{r+3R/5}(y_j) \subset B_{4R/5}(y_j)$.

By our second estimate, there are no more than $(5C^2)^{5n\rho/R}$ of the $B_{R/5}(y_j)$ fitting in $B_\rho(x_0)$, thus bounding the number of y_j, and by the first estimate no more than $(C^2R/r)^n$ of the $B_r(x_i)$ fitting in each $B_{4R/5}(y_j)$. □

As a consequence, given $r < R/3$ there is a sequence of points (x_j) with distance at least r such that $\mathbf{M}$ is covered by the balls $B_{3r}(x_j)$. There are coordinate maps χ_{x_j} defined on these balls since $r < R/3$. No point lies in more than $(3C^2)^n$ of the balls $B_{3r}(x_j)$. We fix r and the points x_j in the sequel. There is a partition of unity (η_l^2) subordinate to this covering with uniform bounds on derivatives (up to fixed order) of $\eta_l \circ \chi_x^{-1}$ on $\chi_{x_l}(B_R(x_l))$; we refer to this property as 'uniform smoothness'.

We may choose such a uniformly smooth partition of unity. The structure of uniform manifolds allows to construct useful functions, specifically a smooth approximation to the radial coordinate $d(\cdot, x_0)$:

Lemma 5.3 (Approximate radial coordinate on a uniform manifold)**.** *Choose* $x_0 \in \mathbf{M}$*. There exists a function* $\rho : \mathbf{M} \to \mathbf{R}$ *with bounded derivatives such that* $\rho(x) - d(x, x_0)$ *is bounded.*

Proof. We define the functions ρ_l on the l^{th} coordinate chart $\chi_{x_l}(B_R(x_l))$ as regularizations of $(\eta_l \circ \chi_{x_l}^{-1}) \cdot (d(\chi_{x_l}^{-1}(\cdot), x_0) - d(x_l, x_0))$, and then we can take $\rho = \sum \eta_l \cdot (\rho_l \circ \chi_{x_l} + d(x_l, x_0))$. ☐

Given a uniformly smooth partition of unity η_l^2 (as explained before the lemma) on the uniform manifold, we define Hölder norms

$$\|f\|_{C^\alpha(\mathbf{M})} := \sup_l \|(\eta_l f) \circ \chi_{x_l}^{-1}\|_{C^\alpha(\chi_{x_l}(B_R(x_l)))} ,$$

where on subsets $X \subset \mathbf{R}^n$,

$$\|f\|_{C^\alpha(X)} := \max\{[f]_\alpha, \|f\|_{L^\infty}\} \tag{33}$$

with

$$[f]_\alpha := \sup_{0<|x'-x|\leq 1} \frac{|f(x') - f(x)|}{|x' - x|^\alpha} .$$

This definition depends on the choice of the partition of unity, but different choices lead to equivalent norms. The presence of the L^∞-norm in (33) permits to drop the constraint $|x' - x| \leq 1$ altogether in the definition of $[f]_\alpha$, or else to replace it with a different bound $|x' - x| \leq R$, again providing equivalent norms. Likewise, it is easy to see that an equivalent norm can be defined without reference to a partition of unity by

$$\|f\|^\circ_{C^\alpha(\mathbf{M})} := \max\{\|f\|_{L^\infty(\mathbf{M})}, [f]^\circ_\alpha\} \tag{34}$$

with

$$[f]^\circ_\alpha := \sup_{0<d(x',x)} \frac{|f(x') - f(x)|}{d(x', x)^\alpha} .$$

We will use the variant (33) for the general proof of the Schauder estimates, but the equivalent variant (34), when dealing with $\mathcal{M}$ specifically. In this case, d obviously refers to the cigar metric on $\mathcal{M}$, not the euclidean metric on $\mathbf{R}^n$.

We can similarly define the spaces $C^{k,\alpha}$ by the maximum of the C^α norms (33) of all the derivatives up to order k.

Hölder spaces $C^\alpha([0, \infty) \times \mathbf{M})$ on space-time cylinders can be understood using the parabolic metric

$$d_P\big((t_1, x_1)\,;\ (t_2, x_2)\big) = \max\big\{|t_2 - t_1|^{\frac{1}{2}}, d(x_1, x_2)\big\} .$$

This leads to

$$\|f\|_{C^\alpha([0,T]\times\mathbf{M})} := \sup_l \|(\eta_l f) \circ \chi_{x_l}^{-1}\|_{C^\alpha([0,T]\times\chi_{x_l}(B_R(x_l)))} ,$$

where on subsets $X \subset \mathbf{R}^n$

$$\|f\|_{C^\alpha([0,T]\times X)} := \max\{[f]_{x;\alpha}, [f]_{t;\alpha/2}, \|f\|_{L^\infty}\} \tag{35}$$

with

$$[f]_{x;\alpha} := \sup_t \sup_{0<|x'-x|\le 1} \frac{|f(t,x')-f(t,x)|}{|x'-x|^\alpha} \ ,$$
$$[f]_{t;\alpha/2} := \sup_x \sup_{0<|t'-t|\le 1} \frac{|f(t',x)-f(t,x)|}{|t'-t|^{\alpha/2}} \ .$$

Similar comments about equivalent norms in the style of (34) apply in an obvious manner. In the literature, these spaces are sometimes denoted as $C^{\alpha/2,\alpha}$, and slight differences in the definitions lead to different, but trivially equivalent, norms.

We will henceforth find the abbreviations

$$\mathbf{M}_T := [0,T]\times\mathbf{M} \ , \quad \mathcal{M}_T := [0,T]\times\mathcal{M} \ , \quad \mathbf{R}^n_T := [0,T]\times\mathbf{R}^n \tag{36}$$

useful.

Using multi-index notation $\beta := (\beta_0, \beta_1, \ldots, \beta_n)$ with the parabolic weight $|\beta| := 2\beta_0 + \beta_1 + \ldots + \beta_n$ and $\partial^\beta := \partial_t^{\beta_0} X_1^{\beta_1} \cdots X_n^{\beta_n}$, we let, for functions on $\mathbf{R}^n$,

$$\|f\|_{C^{k,\alpha}} := \max\left\{ \sup_{|\beta|\le k} \|\partial^\beta f\|_{C^\alpha}, \sup_{|\beta|\le k-1} \sup_{x,|t_1-t_2|\le 1} [\partial^\beta f]_{t;(1+\alpha)/2} \right\}$$

and can obtain $C^{k,\alpha}$ norms on $\mathbf{M}_T$ or on $[0,\infty[\,\times\,\mathbf{M}$ by partitions of unity as before.

We note that not all authors include the term $[\partial^\beta f]_{t;(1+\alpha)/2}$ in their definition of the parabolic Hölder norm (e.g., Krylov [33] and Friedman [25] don't, but Ladyzhenskaya et al. [34] do). For $k = 2$, this term is indeed controlled by the other terms, as can be seen in Krylov's book from his Ex. 8.8.6 in connection with the proof of Thm. 8.8.1. The argument will generalize for even k. The non-equivalence of the two styles of parabolic Hölder norms for odd k will not be an issue for our purposes. We find it convenient to repeat the simple norm equivalence argument here, for easy reference. It suffices to do the argument, which is local, in flat space because of the local equivalence of the metrics. For the rest of the section, we simplify notation by giving the arguments for one space dimension (writing a scalar x), with the generalization to higher dimensions being obvious. We estimate

$$\begin{aligned}
&|w_x(t,x) - w_x(s,x)| \le \\
&\quad \le \left| w_x(t,x) - \frac{w(t,x+h)-w(t,x)}{h} - w_x(s,x) + \frac{w(s,x+h)-w(s,x)}{h} \right| \\
&\quad\quad + \left| \frac{w(t,x+h)-w(t,x)}{h} - \frac{w(s,x+h)-w(s,x)}{h} \right| \\
&\quad \le h \int_0^1 \int_0^1 |w_{xx}(t,y+lrh) - w_{xx}(s,y+lrh)|\, l\, dl\, dr \\
&\quad\quad + |t-s| \left| \frac{w_t(\theta,x+h)-w_t(\theta,x)}{h} \right| ,
\end{aligned}$$

where in the second term the mean value theorem was applied to the function $g(t) := \frac{w(t,x+h)-w(t,x)}{h}$. Using the Hölder estimates for w_{xx} and w_t, we conclude

$$|w_x(t,x) - w_x(s,x)| \le h|t-s|^{\alpha/2}\,[w_{xx}]_{t;\alpha/2} + |t-s|h^{\alpha-1}\,[w_t]_{x;\alpha}\ .$$

We now obtain $|w_x(t,x)-w_x(s,x)| \le |t-s|^{(1+\alpha)/2}([w_{xx}]_{t;\alpha/2}+[w_t]_{x;\alpha})$ by letting $h := |t-s|^{1/2}$.

We will heavily rely on spaces of Hölder continuous functions. They have nice algebraic properties: products and composition have good properties in these spaces and there are optimal estimates for linear parabolic equations in these function spaces. We begin with the algebraic side, for use in the next section.

Lemma 5.4 (Estimates for some nonlinearities in Hölder spaces)**.** *For $D \subset \mathbf{R}^2$ open and convex, $h \in C^{k+1}(D)$ and a pair of functions (f,g) with range in a compact subset of D, the following estimates hold on $\mathcal{M}$ and on $\mathcal{M}_T$:*

$$\|h \circ (f,g)\|_{C^{k,\alpha}} \le C\left(\|f\|_{C^{k,\alpha}}, \|g\|_{C^{k,\alpha}}\right) \tag{37}$$

where bounds on h and its derivatives on the convex hull of the range of f and g enter into the dependence of C on the norms $\|f\|$ and $\|g\|$;

$$\|fg\|_{C^{k,\alpha}} \le c\|f\|_{L^\infty}\|g\|_{C^{k,\alpha}} + c\|g\|_{L^\infty}\|f\|_{C^{k,\alpha}}\ ; \tag{38}$$

$$\|h\circ(f,g)g^2\|_{C^{k,\alpha}} \le C(\|f\|_{C^{k,\alpha}}, \|g\|_{C^{k,\alpha}})\|g\|_{L^\infty}\|g\|_{C^{k,\alpha}}\ . \tag{39}$$

If h is analytic, then the map

$$C^{k,\alpha} \times C^{k,\alpha} \ni (f,g) \longmapsto h(f,g)g^2 \in C^{k,\alpha} \tag{40}$$

is an analytic map of Banach spaces.

Remark 5.5. *The lemma will be applied in a slightly more general situation with h depending also on x and t. This however follows from the lemma above because the statement remains correct if we consider vector valued functions f and g, and we may replace f by the vector (t,x,f) for bounded functions x and t. Typically we will consider local coordinates with uniform estimates for all quantities which occur.*

Remark 5.6 (Notions of Analyticity)**.** *There are different equivalent notions of analyticity of maps defined on open sets in real or complex Banach spaces. We refer to section 15.1 in chapter 4 of* [17] *for a detailed exposition of the facts stated below. Let X and Y be complex Banach spaces and $U \subset X$ an open subset. A map $f : U \to Y$ is called weakly holomorphic if it is locally bounded and if for every one dimensional affine subspace $L \subset X$ and every $l \in Y^*$ the map $L \cap U \ni x \longmapsto l(f(x)) \in \mathbf{C}$ is complex differentiable. It is an immediate consequence (see exercise 2 in section 15 of* [17]*) of the Cauchy integral formula that a weakly holomorphic function is continuously differentiable and complex differentiable. Locally bounded complex differentiable functions are analytic in the sense that the Taylor expansion converges in a ball. Real analyticity can be defined in terms of convergent Taylor expansions. Real analytic functions* *can be extended to complex analytic functions, and conversely, any restriction to a real subspace (if X is the complexification of a real Banach space) of the real part of a holomorphic function (assuming that Y is the complexification of a real Banach space) is analytic. Moreover both the real analytic and complex inverse function theorem holds. In the sequel we do not distinguish between different definitions of analyticity and use whatever is convenient. The results of this paper do not depend heavily on the notion of analyticity. Its main purpose is a nontechnical approach to differentiability and higher differentiability in the previous lemma.*

Proof. The first inequality (37) requires a standard calculation. We may restrict ourselves to $h \circ f$ and verify

$$\|h \circ f\|_{C^{k,\alpha}} \leq C(\|f\|_{C^{k,\alpha}})$$

by allowing vector valued functions f. It is a routine induction over $|\beta|$ that for any multiindex β with $|\beta| \leq k$ all terms of $\partial^\beta h(f)$ can be estimated in the desired way.

We get the $k = 0$ case of (38) (which is the most important for us) from

$$f(x)g(x) - f(x')g(x') = f(x)(g(x) - g(x')) + (f(x) - f(x'))g(x') \ .$$

The case $k \geq 1$ of (38) is more subtle and is of an interpolation type, since naturally occurring intermediate norms as, e.g., $\|f\|_{C^1}\|g\|_{C^{0,\alpha}}$ must implicitly be controlled by the extreme norms on the right hand side. It suffices to prove (38) for fixed compact support, since the general case can be reduced to this special one by means of a locally finite partition of unity. In that case however, due to the local equivalence of metrics, one can prove the estimate in flat space. A tedious, but elementary, proof can be given inductively. For instance, to estimate a term like $\|f'\|_{L^\infty}[g]_\alpha$, one can use interpolation inequalities as Krylov's [33] Thm. 3.2.1 in the elliptic case, or Thm. 8.8.1 in the parabolic case; the scaling weight ε contained therein will be chosen in terms of the norms, such as to remove one higher norm from products like $\varepsilon^{1+\alpha}\|f\|_{C^{1,\alpha}}\|g\|_{C^{1,\alpha}}$.

A less tedious, but not as elementary, proof in the case of $\mathbf{R}^n$ can be found in the book of Runst and Sickel [41], Sec. 4.6.4, and the argument there generalizes to our setting. (Note that their setting in Besov spaces includes Hölder spaces $C^{k,\alpha} = B^{k+\alpha}_{\infty,\infty}$.) The Littlewood-Paley decomposition (dyadic partition of unity in Fourier space) that enters implicitly already in the definition of Besov spaces (pg. 8 of [41]) needs to be made anisotropic to reflect the parabolic scaling. We omit tedious details, which would only distract from the ideas of the present paper.

Inequality (39) is a consequence of the previous bounds:

$$\|h \circ (f,g)g^2\|_{C^{k,\alpha}} \leq c\|h \circ (f,g)\|_{L^\infty}\|g\|_{L^\infty}\|g\|_{C^{k,\alpha}} + c\|h \circ (f,g)\|_{C^{k,\alpha}}\|g\|^2_{L^\infty} \ .$$

Now suppose that h is analytic. It can be extended to a holomorphic complex map. We complexify the Hölder spaces by considering functions with complex values. Left composition with h is clearly a weakly holomorphic map, hence holomorphic. The restriction to real valued functions is therefore real analytic. $\square$

Remark 5.7. *The reason why we insist on* (38) *rather than the simpler algebra property* $\|fg\| \le c\|f\|\,\|g\|$ *in* $C^{k,\alpha}$ *is the following: While the Hölder spaces are needed for a clean PDE theory, we want to prove a theorem whose natural hypothesis is about the* L^∞ *norm, rather than having to introduce an unnatural smallness condition in a Hölder norm (see the proof of Thm. 7.8). In particular, when estimating the nonlinearity, we need to use a priori smallness in the* L^∞ *norm to avoid the singularity at* $u = 0$*, but may need to work in a subset of* C^α *that does not impose smallness of the* C^α *norm.*

We will also use a variant of Lemma 5.4 that deals with certain weight functions:

Lemma 5.8 (Weighted product estimate)**.** *Let* $f \in C^1(]-a,a[\, \to \mathbf{R})$ *with* $f(0) = 0$ *and let* $0 < h \in C^1(\mathbf{R}^n \to \mathbf{R})$ *be a positive weight function with* $|\boldsymbol{\nabla} \ln h|$ *bounded. Then*

$$\|(f \circ w)wh\|_{C^\alpha(\mathbf{R}^n)} \le C\|w\|_{L^\infty(\mathbf{R}^n)}\|wh\|_{C^\alpha(\mathbf{R}^n)}$$

where the constant C *depends on the* C^1 *norm of* f *and* $|\boldsymbol{\nabla} \ln h|$*. The same estimate holds with space-time norms over* $\mathbf{R}^n_T$*, provided* $|\partial_t \ln h|$ *is also bounded.*

The crucial issue here is that there is no second term in which the Hölder norm would apply to the unweighted function. Specifically, we are after the following

Corollary 5.9. *Let* $h(x) := (\cosh s)^{-\eta}$ *on the cigar manifold* $\mathcal{M}$*. Under the same hypotheses as in Lemma 5.8, we have*

$$\|(f \circ w)wh\|_{C^\alpha(\mathcal{M}_T)} \le C\|w\|_{L^\infty(\mathcal{M}_T)}\|wh\|_{C^\alpha(\mathcal{M}_T)}\,.$$

Proof of Lemma 5.8 and Cor. 5.9. Clearly $\|(f \circ w)wh\|_{L^\infty}$ is estimated in the claimed way; so we only need to estimate the Hölder quotient. We estimate, abbreviating $w(x_i) =: w_i$ and $h(x_i) =: h_i$,

$$(w_1 - w_2)(f(w_1)w_1h_1 - f(w_2)w_2h_2) = \\ (f(w_1)w_1 - f(w_2)w_2)(w_1h_1 - w_2h_2) + (h_2 - h_1)w_1w_2(f(w_1) - f(w_2))\,.$$

With no loss of generality we assume $h_2 \le h_1$. At least if $w_1 \ne w_2$, we can obtain

$$\begin{aligned}
&\frac{f(w_1)w_1h_1 - f(w_2)w_2h_2}{|x_1-x_2|^\alpha} = \\
&\qquad \frac{f(w_1)w_1 - f(w_2)w_2}{w_1-w_2}\,\frac{w_1h_1 - w_2h_2}{|x_1-x_2|^\alpha} \\
&\qquad\quad + h_1w_1w_2\frac{f(w_1)-f(w_2)}{w_1-w_2}\,\frac{\exp[\ln h_2 - \ln h_1]-1}{|x_1-x_2|^\alpha}\,.
\end{aligned}$$

We then get the estimate

$$\begin{aligned}
&\frac{|f(w_1)w_1h_1 - f(w_2)w_2h_2|}{|x_1-x_2|^\alpha} \le \\
&\qquad \left(\sup_{|r|\le\|w\|_{L^\infty}}\left|\frac{d}{dr}(f(r)r)\right|\right)\frac{|w_1h_1-w_2h_2|}{|x_1-x_2|^\alpha} \\
&\qquad\quad + \|wh\|_{L^\infty}\|w\|_{L^\infty}\left(\sup_{|r|\le\|w\|_{L^\infty}}|f'(r)|\right)\frac{\min\{1,|\ln h_2 - \ln h_1|\}}{|x_1-x_2|^\alpha} \\
&\quad \le C\|w\|_{L^\infty}\,\|wh\|_{C^\alpha} + Ca\|w\|_{L^\infty}\,\|wh\|_{L^\infty}\,\min_{d>0}\left\{d^{-\alpha}, \|\boldsymbol{\nabla}\ln h\|_{L^\infty}d^{1-\alpha}\right\}.
\end{aligned}$$

The estimate persists by continuity if $w_1 = w_2$. If space-time norms are desired, the same estimate applies to time Hölder quotients. This proves the lemma.

For the corollary on the cigar manifold, we only have to replace $|x_1-x_2|$ with $d(x_1,x_2) \ge |s_1-s_2|$, according to (28). □

6. Schauder estimates for the heat equation

The results and methods in this chapter are basically well-known. However, we need to elaborate on some details since we are also relying on some less familiar weighted norms.

Also the impact of low regularity inhomogeneities needs to be taken into account cleanly to cope with applications to the quasilinear case. We will use these estimates in the next section to get local existence for parabolic equations on uniform manifolds. Again, this will be done along traditional lines; however to control the noncompactness of the domain by means of uniform estimates requires recalling the proof details from the traditional case for inspection and reference.

Some comments concerning the semigroup point of view and the issue of big vs. little Hölder spaces will be given in Rmk. 7.2 below; suffice it to say here that these issues are not of any real significance for our purposes.

So we begin by studying the heat equation

$$v_t - \Delta v = \partial^2_{ij}f^{ij}(t,x) + \partial_i b^i(t,x) + c(t,x), \qquad v(0,x) = v_0(x) \tag{41}$$

and use the following lemma, which is basically standard; but need to implement some modifications to the usual versions. Namely, in order to prove a regularization estimate, we will also work with space-time norms that deteriorate as $t \to 0$: We define the space $C^\alpha_*(\mathbf{R}^n_T)$ to consist of those functions for which the norm

$$\|u\|^*_{C^\alpha(\mathbf{R}^n_T)} := \|u\|_{C^\alpha_*(\mathbf{R}^n_T)} := \max\Big\{\sup_{0<t\leq T} t^{\alpha/2}\|u\|_{C^\alpha([t,T]\times\mathbf{R}^n)}, \|u\|_{L^\infty(\mathbf{R}^n_T)}\Big\}$$

is finite. We prefer the first notation for typographical clarity even when referring to functions in the larger space $C^\alpha_*(\mathbf{R}^n_T)$. These norms reflect the short-time scaling $\|v(t)\|_{C^\alpha(\mathbf{R}^n)} \leq Ct^{-\alpha/2}\|v_0\|_{L^\infty(\mathbf{R}^n)}$, which is satisfied by the homogeneous heat equation. The L^∞ norm is included in the definition in order to salvage the algebra properties $\|uv\|^*_{C^\alpha(\mathbf{R}^n_T)} \leq C\|u\|^*_{C^\alpha(\mathbf{R}^n_T)}\|v\|^*_{C^\alpha(\mathbf{R}^n_T)}$ and also the stronger

$$\|uv\|^*_{C^\alpha(\mathbf{R}^n_T)} \leq C\left(\|u\|^*_{C^\alpha(\mathbf{R}^n_T)}\|v\|_{L^\infty} + \|u\|_{L^\infty}\|v\|^*_{C^\alpha(\mathbf{R}^n_T)}\right) , \tag{42}$$

which is a straightforward consequence of (38). The first component alone would not control the L^∞ norm as the function $u(t,x) = \ln t$ shows.

Lemma 6.1 (Hölder smoothing of rough sources by the heat flow)**.** *There exists a unique solution to the heat equation* (41) *in the space* $C^\alpha(\mathbf{R}^n_T)$. *It satisfies*

$$\begin{aligned}\|v\|_{C^\alpha(\mathbf{R}^n_T)} \leq C\Big(&\|v_0\|_{C^\alpha(\mathbf{R}^n)} + \|f\|_{C^\alpha(\mathbf{R}^n_T)} \\ &+ T^{\frac{1}{2}}\|b\|_{C^\alpha(\mathbf{R}^n_T)} + T^{1-\frac{1}{2}\alpha}\|c\|_{C^\alpha(\mathbf{R}^n_T)}\Big) ,\end{aligned} \tag{43}$$

and

$$\begin{aligned}\|v\|_{C^\alpha(\mathbf{R}^n_T)} \leq C\Big(&\|v_0\|_{C^\alpha(\mathbf{R}^n)} + \|f\|_{C^\alpha(\mathbf{R}^n_T)} \\ &+ T^{\frac{1-\alpha}{2}}\|b\|_{L^\infty(\mathbf{R}^n_T)} + T^{1-\frac{1}{2}\alpha}\|c\|_{L^\infty(\mathbf{R}^n_T)}\Big) ,\end{aligned} \tag{44}$$

and the regularization estimate

$$\begin{aligned}\|v\|^*_{C^\alpha(\mathbf{R}^n_T)} \leq C\Big(&\|v_0\|_{L^\infty(\mathbf{R}^n)} + \|f\|^*_{C^\alpha(\mathbf{R}^n_T)} \\ &+ T^{\frac{1}{2}}\|b\|^*_{C^\alpha(\mathbf{R}^n_T)} + T^{1-\frac{1}{2}\alpha}\|c\|^*_{C^\alpha(\mathbf{R}^n_T)}\Big) ,\end{aligned} \tag{45}$$

where the constant C may depend on an upper bound for T.

Going through the proofs, we will also see:

Corollary 6.2. *The very same estimates as in Lemma 6.1 hold in the half space with Dirichlet (or Neumann) boundary conditions.*

Proof of Lemma 6.1. By superposition, the estimates are assembled from four cases, in each of which exactly one of the quantities f, b, c, v_0 is non-zero. The estimate (43) for v_0 is classical, see for instance [34], IV, (2.2), with the critical part being (2.13) there. Likewise we can refer to ([34], IV, (2.1), and (2.8),(2.9) for the estimating $w^{ij}_t - \Delta w^{ij} = f^{ij}$, and obtain our case with $v = \partial^2_{ij} w^{ij}$.

We also note that the arguments for b^i and c in (43) can be given as (simpler) modifications of the estimates for the f^{ij}, thus concluding the proof of (43). However, as we will use the strengthened version (44), we will give these estimates explicitly in a moment. At the same time, we will later refer to (43) in order to derive (45) from it. For the convenience of the reader, we will also redo the detailed calculations pertaining to (43)–(45) in a self-contained manner in an appendix, with no-pretense to novelty.

By Γ, we denote the heat kernel: $\Gamma(t,x) := (4\pi t)^{-n/2}\exp[-|x|^2/4t]$.

We estimate <u>the contributions of $\|b\|_{L^\infty}$ to the alternate bound (44):</u>

In terms of $\|b\|_{L^\infty}$, we can estimate the spatial Hölder quotient as

$$\frac{|v(t,x') - v(t,x)|}{|x'-x|^\alpha} \le \int_0^t \int_{\mathbf{R}^n} \frac{|\boldsymbol{\nabla}\Gamma(\tau, x'-y) - \boldsymbol{\nabla}\Gamma(\tau, x-y)|}{|x'-x|^\alpha} \|b\|_{L^\infty}\, dy\, d\tau \ .$$

With $z := (x-y)/\sqrt{4\tau}$ and $h(z) := -2z\exp(-|z|^2) = \boldsymbol{\nabla}\Gamma(\frac14, z)$, this becomes

$$\frac{|v(t,x') - v(t,x)|}{|x'-x|^\alpha} \le c\int_0^t \tau^{-(1+\alpha)/2} \int_{\mathbf{R}^n} \frac{|h(z+\zeta) - h(z)|}{|\zeta|^\alpha}\, dz\, d\tau\, \|b\|_{L^\infty} \ ,$$

where $\zeta = (x'-x)/\sqrt{4\tau}$.

Now for $|s| \le 1$, we can estimate the inner integrand by the integrable quantity $\sup_{B_1(z)} |Dh|$; for $|s| \ge 1$, we can estimate the inner integral by $2\int |h(z)|\, dz$, and combining the two cases, we get $[v]_{x,\alpha} \le CT^{(1-\alpha)/2}\|b\|_{L^\infty(\mathbf{R}^n_T)}$. The estimate of $\|v\|_{L^\infty}$ is similar, but simpler, providing a $T^{1/2}$ factor.

For the time Hölder quotient, we estimate

$$\begin{aligned}
|v(t',x) - v(t,x)| &\le \int_t^{t'} \int_{\mathbf{R}^n} |\boldsymbol{\nabla}\Gamma(t'-\tau, y)|\, |b(\tau, x-y)|\, dy\, d\tau \\
&\quad + \int_0^t \int_{\mathbf{R}^n} |\boldsymbol{\nabla}\Gamma(t'-\tau, y) - \boldsymbol{\nabla}\Gamma(t-\tau, y)|\, |b(\tau, x-y)|\, dy\, d\tau \\
&\le C(\sqrt{t'} - \sqrt{t})\|b\|_{L^\infty} + \|b\|_{L^\infty} \int_0^t \int_{\mathbf{R}^n} \Big| \int_{t-\tau}^{t'-\tau} \boldsymbol{\nabla}\Gamma_s(s,y)\, ds \Big|\, dy\, d\tau \ .
\end{aligned}$$

We only have to estimate the last integral yet. Carrying out the y integration first gives $O(s^{-3/2})$, then the s integration yields

$$C\int_0^t [(t'-\tau)^{-1/2} - (t-\tau)^{-1/2}]\, d\tau = C(t'-t)^{1/2} \ .$$

Hence $[v]_{t;\alpha/2} \le CT^{(1-\alpha)/2}\|b\|_{L^\infty}$.

We estimate <u>the contributions of $\|c\|_{L^\infty}$ to the alternate bound (44):</u>

Finally, the c-term in (44), can be estimated in a similar way, only with Γ instead of $\boldsymbol{\nabla}\Gamma$. For the spatial Hölder quotient we obtain $O(T^{1-\frac{\alpha}{2}})$. The L^∞ estimate gives a factor T in a straightforward way.

For the time Hölder quotient we can easily estimate

$$|v(t',x)-v(t,x)|/\|c\|_{L^\infty} \le C\int_0^t [\ln(t'-\tau)-\ln(t-\tau)]\,d\tau\,,$$

which gives already the desired result, except for an extra logarithmic term; and this result would be sufficient for our purposes. But for good measure, let's prove the claimed, optimal, estimate:

$$\begin{aligned}|v(t',x)-v(t,x)| &\le \int_t^{t'}\int_{\mathbf{R}^n}\Gamma(t'-\tau,y)\,|c(\tau,x-y)|\,dy\,d\tau\\ &\quad+\int_0^t\int_{\mathbf{R}^n}|\Gamma(t'-\tau,y)-\Gamma(t-\tau,y)|\,|c(\tau,x-y)|\,dy\,d\tau\\ &\le (t'-t+I)\|c\|_{L^\infty}\end{aligned}$$

with

$$I := \int_0^t\int_{\mathbf{R}^n}|\Gamma(t'-\tau,y)-\Gamma(t-\tau,y)|\,dy\,d\tau\;. \tag{46}$$

Now for $t_1>t_0$, we have

$$\Gamma(t_1,y)\gtrless\Gamma(t_0,y) \iff |y|^2\gtrless r_*^2 := \frac{2n(\ln t_1-\ln t_0)}{t_1-t_0}t_0t_1\;.$$

Hence

$$\begin{aligned}&\int_{\mathbf{R}^n}|\Gamma(t_1,y)-\Gamma(t_0,y)|\,dy\\ &\qquad=\frac{|S^{n-1}|}{\pi^{n/2}}\,2\int_0^{r_*}\Big((4t_0)^{-n/2}e^{-r^2/4t_0}-(4t_1)^{-n/2}e^{-r^2/4t_1}\Big)\,r^{n-1}\,dr\\ &\qquad=\frac{|S^{n-1}|}{\pi^{n/2}}\left(\gamma\big(\frac{n}{2}\,\frac{\ln t_1-\ln t_0}{t_1-t_0}\,t_1\big)-\gamma\big(\frac{n}{2}\,\frac{\ln t_1-\ln t_0}{t_1-t_0}\,t_0\big)\right)\end{aligned}$$

where $\gamma(u):=\int_0^u e^{-s}s^{n/2-1}\,ds$ is the incomplete gamma function, whose second argument $\frac{n}{2}$ we suppress, and of which we only need that it is increasing with finite limit at $+\infty$, and with bounded derivative. With $t_1:=t'-\tau$, $t_0:=t-\tau$, $t_1-t_0=t'-t=:d$, and $(t-\tau)/d=:\sigma$ as a new integration variable, we get from (46)

$$I = Cd\int_0^{t/d}\Big(\gamma\big(\tfrac{n}{2}(s+1)\ln\tfrac{s+1}{s}\big)-\gamma\big(\tfrac{n}{2}s\ln\tfrac{s+1}{s}\big)\Big)\,ds\;.$$

As $\sup|\gamma'|$ is finite, the integrand is bounded by $C\ln\frac{s+1}{s}=O(\frac{1}{s})$ as $s\to\infty$. Therefore $I\le Cd(1+\ln(1+\frac{t}{d}))\le Cd+Cd\ln(1+\frac{T}{d})$. From this, it follows

$$[v]_{t;\alpha/2}\le\Big(CT^{1-\alpha/2}+C\sup_{d>0}d^{1-\alpha/2}\ln(1+\tfrac{T}{d})\Big)\|c\|_{L^\infty(\mathbf{R}^n_T)}\;.$$

The sup is taken on when $d = c_0 T$ with c_0 determined by a transcendental equation, and the value of the sup is therefore $CT^{1-\alpha/2}$, which proves the estimate for the time Hölder quotient.

We have thus concluded the proof of (44) and now turn to (45).

For the homogeneous heat equation with initial data v_0, we have first

$$|v(t,x)| \le \|v_0\|_{L^\infty} \ . \tag{47}$$

Then we have

$$|v(t,x') - v(t,x)| \le C\frac{|x'-x|}{t^{1/2}}\|v_0\|_{L^\infty} \tag{48}$$

because of the estimate

$$\begin{aligned} v(t,x') - v(t,x) &= \int_{\mathbf{R}^n} (\Gamma(t,x'-y) - \Gamma(t,x-y))\, v_0(y)\, dy \\ &= \int_{\mathbf{R}^n}\int_0^1 (x'-x)\cdot \boldsymbol{\nabla}\Gamma(t, x-y+s(x'-x))\, v_0(y)\, ds\, dy\,, \end{aligned}$$

$$|v(t,x') - v(t,x)| \le |x'-x|\, \|v_0\|_{L^\infty} \int_{\mathbf{R}^n} |\boldsymbol{\nabla}\Gamma(t,z)|\, dz \ .$$

(48) and (47) combined provide for $t \ge \tau$,

$$\tau^{\alpha/2}\frac{|v(t,x') - v(t,x)|}{|x'-x|^\alpha} \le \min\left\{ C\Big(\frac{|x'-x|}{\tau^{1/2}}\Big)^{1-\alpha}\ ,\ 2\frac{\tau^{\alpha/2}}{|x'-x|^\alpha}\right\}\, \|v_0\|_{L^\infty} \le C'\|v_0\|_{L^\infty} \ .$$

Similarly we can prove, for $t' > t \ge \tau$, that

$$|v(t',x) - v(t,x)| \le C\frac{t'-t}{t}\|v_0\|_{L^\infty} \tag{49}$$

and combine it with (47) to estimate the time Hölder quotient in the same manner.

Taking the supremum over $\tau \in\,]0,T]$, we conclude that $\|v\|^*_{C^\alpha(\mathbf{R}^n_T)} \le \|v_0\|_{L^\infty}$.

Now let's look at the estimate of $(\partial_t - \Delta)v = \partial^2_{ij} f^{ij}$ with initial data 0. We decompose $v = v_1 + v_2$ where

$$\begin{aligned} (\partial_t - \Delta)v_1 &= \chi_{[0,\tau/2]}(t)\, \partial^2_{ij} f^{ij} \quad v_1|_{t=0} = 0\,, \\ (\partial_t - \Delta)v_2 &= \chi_{[\tau/2,T]}(t)\, \partial^2_{ij} f^{ij} \quad v_2|_{t=0} = 0 = v_2|_{t=\tau/2} \ . \end{aligned}$$

For v_2, the classical Schauder estimate (43) applies on the time interval $[\tau/2, T]$ and gives

$$\tau^{\alpha/2}\|v_2\|_{C^\alpha([\tau,T]\times\mathbf{R}^n)} \le \tau^{\alpha/2}\|v_2\|_{C^\alpha([\tau/2,T]\times\mathbf{R}^n)} \le C\tau^{\alpha/2}\|f\|_{C^\alpha([\tau/2,T]\times\mathbf{R}^n)} \ .$$

For v_1, we note that

$$v_1(\tfrac{3}{4}\tau, x) = \int_0^{\tau/2}\int_{\mathbf{R}^n} \partial^2_{ij}\Gamma(\tfrac{3}{4}\tau - s, x-y) f^{ij}(s,y)\, dy\, ds \ .$$

Hence

$$(\tfrac14\tau)^{\alpha/2}\|v_1\|_{C^\alpha([\tau,T]\times\mathbf{R}^n)} \le C\|v_1(\tfrac34\tau,\cdot)\|_{L^\infty(\mathbf{R}^n)} \le \int_0^{\tau/2} (\tfrac34\tau - s)^{-1}\|f(s,\cdot)\|_{L^\infty(\mathbf{R}^n)}\,ds$$
$$\le C\|f\|_{L^\infty([0,\tau/2]\times\mathbf{R}^n)}\ .$$

In conclusion

$$\tau^{\alpha/2}\|v\|_{C^\alpha([\tau,T]\times\mathbf{R}^n)} \le C\Big(\tau^{\alpha/2}\|f\|_{C^\alpha([\tau/2,T]\times\mathbf{R}^n)} + \|f\|_{L^\infty(\mathbf{R}^n_T)}\Big)$$

If we can also estimate $\|v\|_{L^\infty}$ in like manner, taking the supremum over τ establishes the f part of (45) immediately. And indeed, we can estimate

$$\begin{aligned}
v(t,x) &= \int_0^t\int_{\mathbf{R}^n} \partial^2_{ij}\Gamma(t-s,x-y) f^{ij}(s,y)\,dy\,ds\\
&= \int_0^t\int_{\mathbf{R}^n} \partial^2_{ij}\Gamma(t-s,x-y)(f^{ij}(s,y)-f^{ij}(s,x))\,dy\,ds\,,\\
|v(t,x)| &\le \int_0^t\int_{\mathbf{R}^n} \frac{C}{(t-s)^{n/2+1}} e^{-|x-y|^2/5(t-s)} s^{-\alpha/2}|x-y|^\alpha \|f\|^*_{C^\alpha}\,dy\,ds\\
&\le C\int_0^t s^{-\alpha/2}(t-s)^{\alpha/2-1}\,ds\,\|f\|^*_{C^\alpha} \le C\|f\|^*_{C^\alpha}\ .
\end{aligned}$$

This concludes the proof of the f part of (45). The impact of b and c is obtained in a similar manner. □

Proof of Cor. 6.2. The proof for Dirichlet boundary conditions is essentially the same: In (114) and similar estimates, we merely have to replace $\Gamma(t-\tau,x-y)$ with $\Gamma(t-\tau,x-y)-\Gamma(t-\tau,x-y^*)$, where y^* is the reflection of y at the hyperplane bounding the half space; and of course we integrate only over the half space instead of $\mathbf{R}^n$. This half-space heat kernel obeys the same Gaussian estimates, and no further changes are needed. In (121) and similar equations, where we had conveniently put the y into Γ and the $x-y$ into the right hand side term, we can again swap them and now get expressions under the integral like $\big(\partial_i\Gamma(t-\tau,x-y)-\partial_i\Gamma(t-\tau,x-y^*)\big)(b^i(\tau,y)-b^i(\tau,y+x'-x))$, supporting the same estimates. The integrand within (46) now becomes

$$\begin{aligned}
&|\Gamma(t'-\tau,x-y)-\Gamma(t'-\tau,x-y^*)-\Gamma(t-\tau,x-y)+\Gamma(t-\tau,x-y^*)|\\
&\quad\le |\Gamma(t'-\tau,x-y)-\Gamma(t-\tau,x-y)| + |\Gamma(t'-\tau,x-y^*)-\Gamma(t-\tau,x-y^*)|
\end{aligned}$$

and we may enlarge the integration over all of $\mathbf{R}^n$ and re-use the old estimates. The same applies to Neumann boundary conditions. □

7. Quantitative global well-posedness of the linear and nonlinear equations in Hölder spaces

Consider a smooth function on a uniform manifold $\mathbf{M}$. Its derivative is a section in the cotangent bundle and its second derivative is a section in a suitable vector bundle. We consider differential equations

$$u_t = F(t, x, u, Du, D^2u) \tag{50}$$

governing functions u on $\mathbf{M}$. The precise structure of F is not of importance in this section. We only need that there is such a formulation. In local coordinates, it reduces to an equation of the same type, but now with standard derivatives in $\mathbf{R}^n$.

We assume that the equation can be written in local coordinates (and with the Einstein summation convention being in effect) as

$$u_t = \partial_{ij}^2(f^{ij}(x, u)) + \partial_i(b^i(x, u)) + c(x, u) \tag{51}$$

with nonlinearities defined for u in an open interval $U \subset \mathbf{R}$.

It is not difficult to check that this property is independent of our choice of local coordinates and the partition of unity. We say that the equation is uniformly parabolic if there exist positive constants λ, Λ such that

$$\lambda|\xi|^2 \leq \partial_u f^{ij}(x, u)\xi_i\xi_j \leq \Lambda|\xi|^2 \tag{52}$$

holds in such local coordinates as in Def. 5.1, for u in a compact subinterval $V \subset\subset U$, where λ, Λ may depend on V. We call the equation uniformly analytic if in addition for any compact $V \subset U$, all $k \geq 0$ and all multiindices β

$$|\partial_u^k \partial^\beta f^{ij}(x, u)| + |\partial_u^k \partial^\beta b^i(x, u)| + |\partial_u^k \partial^\beta c(x, u)| \leq Cr^{k+|\beta|}(k + |\beta|)^{k+|\beta|} \tag{53}$$

with r depending on V. Again this property is independent of the chosen points x_j and R – up to changing C and r.

To address (51), we will also be interested in linear equations, which can be written in local coordinates as

$$u_t - \partial_{ij}^2(a^{ij}u) - \partial_i(b^i u) - cu = f + \partial_i g^i \ , \qquad u(x, 0) = u_0(x) \tag{54}$$

assuming that the coefficients a^{ij} are uniformly elliptic and continuous.

Let us tackle the linear case first:

Theorem 7.1 (Inhomogeneous linear parabolic flows on uniform manifolds)**.** *Suppose that $u_0 \in C^\alpha(\mathbf{M})$, $0 < \alpha < 1$. Then there exists a unique weak solution $u \in C^\alpha([0, T] \times \mathbf{M})$ to* (54)*, and it satisfies*

$$\|u\|_{C^\alpha(\mathbf{M}_T)} \leq C\left(\|u_0\|_{C^\alpha(\mathbf{M})} + \|f\|_{L^\infty(\mathbf{M}_T)} + \|g\|_{L^\infty(\mathbf{M}_T)}\right) . \tag{55}$$

where the constant C depends on $T, \lambda, \Lambda, n, \|a^{ij}\|_{C^\alpha}, \|b^i\|_{L^\infty}, \|c\|_{L^\infty(\mathbf{M}_T)}$.

Likewise, a unique weak solution to (54) *exists in $C_*^\alpha(\mathbf{M}_T)$ for $u_0 \in L^\infty(\mathbf{M})$, and it satisfies the similar estimate* (61) *below.*

In the homogeneous case ($f = 0$, $g = 0$), the equation satisfies a comparison principle ($u \geq 0$ if $u_0 \geq 0$) and, provided the a^{ij}, b^i, c are independent of time, defines an analytic semigroup on $C^\alpha(\mathbf{M})$ in a sense made precise in the following Remark 7.2. The equation also defines an analytic (in the sense of the following remark) semigroup on $C_b(\mathbf{M})$ (the space of bounded continuous *functions with the $L^\infty(\mathbf{M})$ norm) as well, and it satisfies, for $0 < t \leq T$, the regularization estimate $\|u(t)\|_{C^\alpha(\mathbf{M})} \leq Ct^{-\alpha/2}\|u_0\|_{L^\infty(\mathbf{M})}$, where C may depend on T.*

Remark 7.2 (Analytic Semigroups)**.** *(a) By an analytic semigroup in C^α, we mean a family $t \in [0,\infty[\longrightarrow S(t)$ of linear operators on C^α satisfying $S(t_1+t_2) = S(t_1)S(t_2)$ for all $t_1, t_2 \geq 0$, which extends to a complex sector and is holomorphic in the interior of that sector, satisfies a bound $\|S(t)\| \leq Ce^{C\,\mathrm{Re}\,t}$, and for which $(t,x) \mapsto S(t)u_0(x)$ is in C^α with respect to t and x, including at $t = 0$. (As $t \to 0$, this implies $S(t)u_0 \to u_0$ uniformly, but not necessarily in the C^α norm as would be required for the standard definition of analytic semigroup, which includes strong continuity at $t = 0$.)*

The analogous definition applies for an analytic semigroup in C_b; the continuity requirement in this case is that the mapping $(t,x) \mapsto S(t)u_0(x)$ is in C_b with respect to both variables, including $t = 0$. This still implies $S(t)u_0 \to u_0$ pointwise (but not necessarily uniformly) as $t \to 0+$.

(b) Indeed, strong continuity at $t = 0$ cannot hold in the space C^α because, for u_0 not *in the closure of C^∞ within C^α, smoother functions like $S(t)u_0$ cannot converge to u_0 in the C^α-norm. Our estimates do imply that $\|S(t)u_0 - u_0\|_{L^\infty} \to 0$ as $t \to 0$. Moreover, if $u_0 \in C^\beta$ with $\beta > \alpha$, then $\|S(t)u_0 - u_0\|_{C^\alpha} \to 0$ as $t \to 0$, because by two independent easy estimates,*

$$\frac{|u(t,x') - u_0(x') - u(t,x) + u_0(x)|}{|x'-x|^\alpha} \leq \min\left\{ \begin{array}{c} ([u(t)]_{x;\beta} + [u_0]_{x;\beta})|x'-x|^{\beta-\alpha}\,, \\ 2\|u(t) - u_0\|_{L^\infty}/|x'-x|^\alpha \end{array} \right\}$$

and this minimum tends to 0 uniformly in space as $t \to 0$.

(c) Alternatively we could work in the 'little Hölder spaces' oC^α, the closure of C^∞ in C^α instead of C^α itself. The estimates are the same, and the changes in the proofs are marginal. In oC^α, the strong continuity of the semigroup is restored, because $u_0 \in oC^\alpha$ can be approximated by $\phi \in C^\beta$ for arbitrary $\beta \in]\alpha, 1[$, and using the standard estimate

$$\|S(t)u_0 - u_0\|_{C^\alpha} \leq \|S(t)\phi - \phi\|_{C^\alpha} + \|S(t)(u_0-\phi)\|_{C^\alpha} + \|u_0 - \phi\|_{C^\alpha}\ .$$

(d) These observations ensure that solutions are continuous up to $t = 0$ and the initial condition is understood in the obvious sense, albeit (for the classical Hölder spaces) not in the functional analytic sense of norm continuity. Basically, these subtleties at $t = 0$ are of no concern to us as we are interested in asymptotics for $t \to \infty$; on a technical level, this is borne out by taking iterative discrete time steps, and the finite time flow map has all the continuity, and even smoothness, we want, as elaborated upon at the end of the proof of Thm. 7.1.

Proof of Thm. 7.1. We turn to Equation (54) on the uniform manifold, i.e.,

$$u_t - \partial^2_{ij}(a^{ij}u) - \partial_i(b^i u) - cu = f + \partial_i g^i\ , \qquad u(x,0) = u_0(x)\ ,$$

where the data a^{ij}, b^i, c, f, u_0 are Hölder continuous, and introduce a small parameter γ to be chosen later. By the Hölder continuity of the inital data u_0 there exists δ such that $|a^{ij}(x) - a^{ij}(y)| \le \gamma$ if $d(x, y) < \delta$. We may decrease δ such that 3δ is still a radius of uniformity.

We use a partition of unity $\sum \hat{\eta}_l^2$ subordinate to a locally finite covering with balls of diameter δ, with local coordinate maps χ_l from these balls into $\mathbf{R}^n$. As the a^{ij} are almost constant on these balls, we can, after an affine change of coordinates, achieve that $a^{ij} - \delta^{ij}$ is L^∞-small on these balls. This introduces constants that can be bounded in terms of the ellipticity parameters λ, Λ. We decompose u by

$$v_l := (\hat{\eta}_l u) \circ \chi_l^{-1}, \qquad u = \textstyle\sum_l \hat{\eta}_l (v_l \circ \chi_l)\ .$$

In order to avoid clogging up the calculation with coordinate maps we define $\eta_l := \hat{\eta}_l \circ \chi_l^{-1}$ and $u_l := u \circ \chi_l^{-1}$ for the pullbacks of the partition of unity and of the function u to the respective coordinate patches in $\mathbf{R}^n$. This way, in the equations below, the coefficient functions as well as the quantities indexed with l live on $\mathbf{R}^n_T$ (or, in the case of u_l, a subset thereof), whereas the function u lives on $\mathbf{M}_T$.

Then v_l satisfies the initial condition $v_l(0) = \eta_l u_{0,l}$ and

$$\begin{aligned}\partial_t v_l - \Delta v_l &= \eta_l \partial_t u_l - \Delta(\eta_l u_l) \\ &= \partial^2_{ij}\big[(a^{ij} - \delta^{ij})\eta_l u_l\big] + \partial_i\big[b^i \eta_l u_l - 2a^{ij}(\partial_j \eta_l) u_l + g^i \eta_l\big] \\ &\quad + c\eta_l u_l + f\eta_l - g^i(\partial_i \eta_l) + \big[a^{ij}\partial^2_{ij}\eta_l - b^i(\partial_i \eta_l)\big] u_l\ .\end{aligned} \tag{56}$$

On the other hand, given u, we can use this equation to define the v_l, and hence a mapping $\mathcal{T} : u \mapsto \sum (\eta_l v_l) \circ \chi_l$, for which we want to find a fixed point by means of the Banach fixed point theorem (i.e., contraction mapping principle) and Lemma 6.1. On the support of η_l, $a^{ij} - \delta^{ij}$ is small in the L^∞ norm, and otherwise all coefficients are bounded in C^α. The η_l are bounded in the L^∞ norm, but their C^α norm will be large for small δ. Given u, (43) yields, for each l,

$$\begin{aligned}\|v_l\|_{C^\alpha(\mathbf{R}^n_T)} \le C\Big(&\|\eta_l u_{0,l}\|_{C^\alpha} + \gamma\|\eta_l u_l\|_{C^\alpha} + \|a^{ij} - \delta^{ij}\|_{C^\alpha}\|\eta_l u_l\|_{L^\infty} \\ &+ T^{1/2}\|\eta_l\|_{C^{2,\alpha}}\|(a^{ij}, b^i, c)\|_{C^\alpha}\|u_l\|_{C^\alpha} \\ &+ T^{1-\frac{\alpha}{2}}\|\eta_l\|_{C^\alpha}\|f\|_{C^\alpha} + T^{\frac{1}{2}}\|\eta_l\|_{C^{1,\alpha}}\|g^i\|_{C^\alpha}\Big)\,,\end{aligned} \tag{57}$$

where we have used the fine algebra estimate (38) on the term with the highest derivatives, but the simpler estimate $\|uw\|_{C^\alpha} \le C\|u\|_{C^\alpha}\,\|w\|_{C^\alpha}$ suffices for the other terms since their coefficients can be made small by choosing T small. Since u_l is the restriction of $\sum_k \eta_k^2 u_k$ to the l^{th} coordinate patch, and at most a fixed number N of terms (given by the ball packing Lemma 5.2) contributes to this sum for each l, we can estimate $\|u_l\|_{C^\alpha} \le NC_\delta\|u\|_{C^\alpha(\mathbf{M}_T)}$.

We want this estimate (57) for later reference, but we can strengthen it immediately: namely, a similar estimate based on (44) is possible with only the L^∞ norms of b, c instead of the Hölder norm; and also only the L^∞ norms of f, g, and correspondingly smaller exponents for T.

Similarly, given two sources u and $\bar{u}$ for (56), with the same initial trace u_0, we get for the difference $v_l - \bar{v}_l$ of the solutions the same estimate with u replaced by $u - \bar{u}$ on the right hand side and $u_0 = 0$, $f = 0 = g$. Thus

$$\begin{aligned} \|\eta_l(v_l - \bar{v}_l)\|_{C^\alpha(\mathbf{R}^n_T)} &\le \|v_l - \bar{v}_l\|_{C^\alpha(\mathbf{R}^n_T)} + \|\eta_l\|_{C^\alpha(\mathbf{R}^n)}\|v_l - \bar{v}_l\|_{L^\infty(\mathbf{R}^n_T)} \\ &\le (1 + T^{\alpha/2}\|\eta_l\|_{C^\alpha(\mathbf{R}^n)})\|v_l - \bar{v}_l\|_{C^\alpha(\mathbf{R}^n_T)} \end{aligned} \tag{58}$$

and similar with products $\eta_l\eta_k$ of cutoff functions. Recall that $\|\mathcal{T}u\|_{C^\alpha(\mathbf{M}_T)} := \sup_k \|\sum_l \eta_l v_l \eta_k\|_{C^\alpha(\mathbf{R}^n_T)}$. Here, for each k, the sum consists of at most N terms, where N describes the *uniform* local finiteness of the partition $\{\hat{\eta}_l^2\}$ as given by Lemma 5.2.

Therefore, in order to show that the map $u \mapsto \mathcal{T}u := \sum(\eta_l v_l) \circ \chi_l$ is a contraction, with contraction constant $\vartheta < 1$, we need to prove a similar contraction estimate (with smaller contraction constant ϑ/N) for the maps $u \mapsto v_l$. Choosing $\gamma = 1/(2CN)$ we determine the necessary δ and obtain a (large but fixed) bound for $\sup_l \|\eta_l\|_{C^{k,\alpha}}$. All terms in (57) that contain potentially large norms of η_l are multiplied by a power of T, which we choose small to compensate for the norm of the cutoff (u_0 doesn't occur in the difference, and we again note that $\|\bar{u} - u\|_{L^\infty(\mathbf{M}_T)} \le T^{\alpha/2}\|\bar{u} - u\|_{C^\alpha(\mathbf{M}_T)}$ because $(\bar{u} - u)|_{t=0} = 0$). So we conclude, for T sufficiently small (independent of u_0), that $\mathcal{T}$ is a contraction (globally on all of $C^\alpha(\mathbf{M}_T)$). An iterative procedure gives existence and uniqueness.

An estimate

$$\|u\|_{C^\alpha(\mathbf{M}_T)} \le C\left(\|u_0\|_{C^\alpha(\mathbf{M})} + \|f\|_{C^\alpha(\mathbf{M}_T)} + \|g\|_{C^\alpha(\mathbf{M}_T)}\right) \tag{59}$$

similar to (55) in Theorem 7.1 holds because Equation (57) implies

$$\begin{aligned} \|\mathcal{T}u\|_{C^\alpha(\mathbf{M}_T)} \le \tfrac{1}{2}\|u\|_{C^\alpha(\mathbf{M}_T)} &+ CNT^{1-\frac{\alpha}{2}}C_\delta\|f\|_{C^\alpha} + CNT^{\frac{1}{2}}C_\delta\|g\|_{C^\alpha} + \\ &+ CN\big(C_\delta\|u_0\|_{C^\alpha} + C_\delta\|u\|_{L^\infty} + T^{1/2}C_\delta\|u\|_{C^\alpha}\big)\,, \end{aligned} \tag{60}$$

with constants now depending on norms of a^{ij}, b^i, c. We again estimate $\|u\|_{L^\infty} \le \|u_0\|_{L^\infty} + T^{\alpha/2}\|u - u_0\|_{C^\alpha} \le (1 + T^{\alpha/2})\|u_0\|_{C^\alpha} + T^{\alpha/2}\|u\|_{C^\alpha}$ and solve for $\|\mathcal{T}u\|_{C^\alpha} = \|u\|_{C^\alpha}$. Likewise, by using the strengthened version of (57), based on (44), we get (55).

Next, we prove the regularization estimate. Using (45) instead of (43), we can construct a local solution by Banach's fixed point theorem as before, with the analog of (57) involving $\|\eta_l u_0\|_{L^\infty(\mathbf{M})} \le \|u_0\|_{L^\infty(\mathbf{M})}$ instead of $\|\eta_l u_0\|_{C^\alpha(\mathbf{M})}$, and otherwise $\|\cdot\|^*_{C^\alpha(\mathbf{M}_T)}$ instead of $\|\cdot\|_{C^\alpha(\mathbf{M}_T)}$. A slight modification of (58) is needed: we use that $\|\eta_l\|^*_{C^\alpha(\mathbf{R}^n_T)} \le \max\{1, T^{\alpha/2}\|\eta_l\|_{C^\alpha(\mathbf{R}^n)}\}$. Similarly, in the analog of (57), the term $\|a^{ij}-\delta^{ij}\|^*_{C^\alpha(\mathbf{R}^n_T)}\|\eta_l u_l\|_{L^\infty}$ can be estimated by $\max\{\gamma, T^{\alpha/2}\|a^{ij}-\delta^{ij}\|_{C^\alpha(\mathbf{R}^n)}\}\, \|\eta_l u_l\|^*_{C^\alpha(\mathbf{R}^n_T)}$.

This argument proves the short term existence and uniqueness for L^∞ initial data; it also provides, as in (60), an analog to (55):

$$\|u\|^*_{C^\alpha(\mathbf{M}_T)} \le C\left(\|u_0\|_{L^\infty(\mathbf{M})} + \|f\|^*_{C^\alpha(\mathbf{M}_T)} + \|g\|^*_{C^\alpha(\mathbf{M}_T)}\right) , \tag{61}$$

from which $\|u(t)\|_{C^\alpha(\mathbf{M})} \le Ct^{-\alpha/2}\|u_0\|_{L^\infty(\mathbf{M})}$ follows immediately when $f=0$, $g=0$.

Let us briefly note that these same estimates can also be used for differences $u_n - u_m$ of solutions with, say, the same initial data u_0 and with $f=0=g$, but different coeeficients a^{ij}_n, b^i_n, c_n and a^{ij}_m, b^i_m, c_m respectively. They will then provide in analogy to (57) that

$$\begin{aligned}
&\|v_{ln}-v_{lm}\|_{C^\alpha(\mathbf{R}^n_T)} \le \\
&\qquad \le C\Big(\gamma\|\eta_l(u_{ln}-u_{lm})\|_{C^\alpha} + \|a^{ij}_n-\delta^{ij}\|_{C^\alpha}\|\eta_l(u_{ln}-u_{lm})\|_{L^\infty} \\
&\qquad\qquad + T^{1/2}\|\eta_l\|_{C^{2,\alpha}}\|(a^{ij}_n, b^i_n, c_n)\|_{C^\alpha}\|u_{ln}-u_{lm}\|_{C^\alpha} \\
&\qquad\qquad + 2\gamma\|\eta_l u_{lm}\|_{C^\alpha} + \|a^{ij}_n - a^{ij}_m\|_{C^\alpha}\|\eta_l u_{lm}\|_{L^\infty} \\
&\qquad\qquad + T^{1/2}\|\eta_l\|_{C^{2,\alpha}}\|(a^{ij}_n-a^{ij}_m, b^i_n-b^i_m, c_n-c_m)\|_{C^\alpha}\|u_{lm}\|_{C^\alpha}\Big) .
\end{aligned}$$

Then, our estimate, applied to the respective fixed points u_n and u_m, provides continuous dependence of the solution on the coefficients:

$$\|u_n-u_m\|_{C^\alpha(\mathbf{R}^n_T)} \le C\|u_m\|_{C^\alpha(\mathbf{R}^n_T)}\,\|(a^{ij}_n-a^{ij}_m, b^i_n-b^i_m, c_n-c_m)\|_{C^\alpha} . \tag{62}$$

The comparison statement (nonnegative initial data give nonnegative solutions) would follow by commuting multiplication and differentiation and classical maximum principles if the coefficients were twice continuously differentiable, and if the manifold were compact. The first obstacle can easily be dealt with by regularization: Approximate the a^{ij}, b^i uniformly by smooth $\tilde a^{ij}_n, \tilde b^i_n$ with bounded C^α norms, and such that the $\tilde a^{ij}_n, \tilde b^i_n$ converge to a^{ij}, b^i in a (weaker) C^β norm, with $0<\beta<\alpha$. The corresponding solutions $\tilde u_n$ then converge to u in the C^β norm by the continuous dependence estimate (62).

So we now write, for smooth coefficients in local coordinates

$$u_t - a^{ij}\partial^2_{ij}u - \tilde b^i\partial_i u - \tilde c u = 0 .$$

Replacing u by $e^{-\lambda t}u$ if necessary, we may assume that $\tilde c \le 0$. Let ρ be the radius function constructed in Lemma 5.3. Then $-\varepsilon(\mu t+\rho)$ is a nonpositive subsolution, provided μ is sufficiently big. Hence, if u_0 is nonnegative and $f=0$, $g=0$, then $u+\varepsilon(\mu t+\rho)$ is a supersolution which tends to ∞ as $x\to\infty$ and which is nonnegative at $t=0$. By the maximum principle it does not assume its infimum for $0<t\le T$ and hence it is nonnegative. We let $\varepsilon\to 0$ to arrive at the desired conclusion.

The construction for short time T can be iterated to get global existence in time, and hence a semigroup. The fact that we have an *analytic* semigroup follows from a standard argument: We introduce a complex parameter τ and study, in a space of complex functions,

$$u_t = \tau\Big(\partial^2_{ij}(a^{ij}u) + \partial_i(b^i u) + cu\Big)\ , \qquad u(x,0) = u_0(x)\ . \tag{63}$$

The previous arguments work for all $\tau > 0$. We rewrite the equation as

$$u_t - \tau_0\big(\partial^2_{ij}(a^{ij}u) - \partial_i(b^i u) - cu\big) = (\tau - \tau_0)\big(\partial^2_{ij}(a^{ij}u) - \partial_i(b^i u) - cu\big)\ .$$

The right hand side is analytic in τ. We obtain an analytic dependence on τ, which is equivalent to having an analytic semigroup – as here we don't require norm continuity at time $t = 0$. Specifically, the same estimates we have made for $\tau = 1$ apply to the present case, for $\tau \in \mathbf{C}$, as long as $|\tau - \tau_0|/\tau_0$ is sufficiently small (with the right hand side not going into f, but distributing over all the other terms in (57)). So our estimates extend to a small sector in the complex time-plane.

Our estimates using the space C^α_* for L^∞ initial data will swiftly imply that the solution of (54) defines an analytic semigroup on $C_b(\mathbf{M})$, again in the sense of Remark 7.2. (We make the statement for C_b rather than L^∞ to avoid technicalities in which sense initial data are taken on, technicalities that would not pertain to the focus of our problem.)

To see this, let us first assume that our initial data u_0 are *uniformly* continuous. Then they can be approximated uniformly by C^α initial data $u_0^{[k]}$. Using the estimate (61), we infer that the corresponding solutions $u^{[k]}$ converge in C^α_* and hence uniformly to a (the) solution u for initial dat u_0. But the $u^{[k]}$ themselves were continuous (actually they are in the unmodified $C^\alpha(\mathbf{M}_T)$), so their limit is (uniformly) continuous.

Non-uniform continuity of u_0 can be handled by noting that for any continuous weight function w vanishing at infinity, wu_0 will be *uniformly* continuous if $u_0 \in C_b(\mathbf{M})$ is continuous. Such a weight function can be constructed in a smooth manner based on Lemma 5.3, and then rewriting (54) for the corresponding wu_0 results in an equation of the same type, so that the result with uniformly continuous data for the conjugated equation gives the continuity result for the original equation with (not necessarily uniformly) continuous data u_0. The boundedness is of course maintained from the argument with the unconjugated equation. (In the specific situation $\mathbf{M} = \mathcal{M}$ that interests us for the fast diffusion equation, we can take $w = (\cosh s)^{-\eta}$.) ☐

Basically the same proof yields

Corollary 7.3 (Variant for Dirichlet boundary data)**.** *For equation* (54) *on a smoothly bounded domain on* $\mathbf{M}$*, with Dirichlet boundary conditions* $u = 0$

and C^α initial data, there exists a unique solution in C^α, and estimate (55) *continues to apply, as well as all the other conclusions from Thm. 7.1.*

Now we can tackle the *short-term* estimates for the nonlinear equation by means of a slight modification of the proof of the linear theorem:

Theorem 7.4 (Short time smooth dependence on data)**.** *Suppose equation* (50) *can be written in local coordinates as* (51)*, namely*

$$u_t = \partial^2_{ij}(f^{ij}(x,u)) + \partial_i(b^i(x,u)) + c(x,u)$$

with nonlinearities defined for u in an open interval $U \subset \mathbf{R}$, and that this equation is uniformly both parabolic and analytic. Let $0 < \alpha < 1$, and choose two subintervals $V \subset\subset W \subset\subset U$. There exists $T > 0$ depending on V and W, $\|u_0\|_{C^\alpha(\mathbf{M})}$, and the nonlinearities of the operator, such that for $u_0 \in C^\alpha(\mathbf{M})$ with $u_0(\mathbf{M}) \subset V$ there exists a unique weak solution $u \in C^\alpha(\mathbf{M}_T)$ to (51)*. Its values are in W. For every $u_0 \in C^\alpha(\mathbf{M})$ with range in V, there exists an R-ball in $C^\alpha(\mathbf{M})$ and a T such that the map*

$$C^\alpha(\mathbf{M}) \supset B_R(u_0) \ni u_0 \longmapsto u \in C^\alpha(\mathbf{M}_T)$$

is analytic.

As stated, as of the present theorem, the existence time might depend on the Hölder norm of the initial data. However, we will see soon that this is not the case and that the existence time is influenced by u_0 only through its supremum norm.

Proof. The first step for the nonlinear problem, namely Equation (51),

$$u_t = \partial^2_{ij}(f^{ij}(x,u)) + \partial_i(b^i(x,u)) + c(x,u)$$

requires few changes. We rewrite the equation again as fixed point problem, this time in an L^∞ ball about u_0, within C^α. We choose ε_0 such that a $2\varepsilon_0$ neighbourhood of V still lies in W. We search a small solution in the form $u = u_0 + \tilde{u}$, where $\tilde{u}$ satisfies an equation of the same type; $\|\tilde{u}\|_{L^\infty}$ will be assumed to be $\leq 2\varepsilon_0$.

Just as in the linear case, the equation for $\tilde{u}$ can be localized using the partition of unity with

$$v_l = (\hat{\eta}_l \tilde{u}) \circ \chi_l^{-1}\,, \qquad \tilde{u} = \textstyle\sum_l \eta_l (v_l \circ \chi_l)\,.$$

Again $\tilde{u}_l := u \circ \chi_l^{-1}$ and $\eta_l := \hat{\eta}_l \circ \chi_l^{-1}$. One easily computes in local coordinates

$$\partial_t v_l - \Delta v_l = \partial^2_{ij} \tilde{f}^{ij}_l + \partial_i \tilde{b}^i_l + \tilde{c}_l \tag{64}$$

(with initial data 0) where, by a slight abuse of notation,

$$\begin{aligned}
\tilde{f}^{ij}_l &= \eta_l \left(f^{ij}(x, u_0 + \tilde{u}) - \delta^{ij}\tilde{u}\right),\\
\tilde{b}^i_l &= \eta_l b^i(x, u_0 + \tilde{u}) - 2(\partial_j \eta_l)\, f^{ij}(x, u_0 + \tilde{u})\,,\\
\tilde{c}_l &= \eta_l c(x, u_0 + \tilde{u}) - (\partial_i \eta_l)\, b^i(x, u_0 + \tilde{u}) + (\partial^2_{ij}\eta_l)\, f^{ij}(x, u_0 + \tilde{u})\,.
\end{aligned}$$

Using the considerations for the heat equation (41), we again aim to construct the solution up to short time T, by means of the Banach fixed point theorem applied in a closed subset $S := \{\tilde{u} \mid \|\tilde{u}\|_{L^\infty} \le 2\varepsilon_0\,,\ \|\tilde{u}\|_{C^\alpha} \le M\}$ of $C^\alpha(\mathbf{M}_T)$. Here M is a possibly large constant that will be chosen shortly and will depend on $\|u_0\|_{C^\alpha(\mathbf{M})}$. The mapping is again $\mathcal{T} : \tilde{u} \mapsto \sum_l (\eta_l v_l) \circ \chi_l^{-1}$. We first set a target contraction constant $\vartheta \in]\frac{1}{2}, 1[$. We also commit a priori to an upper bound T_0 for T and calculate $\|\mathcal{T}0\|_{C^\alpha(\mathbf{M}_T)} \le \|\mathcal{T}0\|_{C^\alpha(\mathbf{M}_{T_0})} =: A$. Then the choice $M := A/(1-\vartheta)$ will turn out to be expedient.

The estimates for the heat equation from Lemma 6.1 imply:

$$\|v_l\|_{C^\alpha(\mathbf{R}^n_T)} \le C\left(\|\tilde{f}_l\|_{C^\alpha} + T^{1/2}\|\tilde{b}_l\|_{C^\alpha} + T^{1-\alpha/2}\|\tilde{c}_l\|_{C^\alpha}\right) \tag{65}$$

and for differences of solutions

$$\|v_l - \bar{v}_l\|_{C^\alpha} \le C\left(\|\tilde{f}_l - \bar{\tilde{f}}_l\|_{C^\alpha} + T^{1/2}\|\tilde{b}_l - \bar{\tilde{b}}_l\|_{C^\alpha} + T^{1-\alpha/2}\|\tilde{c}_l - \bar{\tilde{c}}_l\|_{C^\alpha}\right) \tag{66}$$

with

$$\tilde{f}_l - \bar{\tilde{f}}_l = \eta_l \int_0^1 \left[\partial_u f^{ij}(x, u_0 + \bar{\tilde{u}} + s(\tilde{u} - \bar{\tilde{u}})) - \delta^{ij}\right]\, ds\,(\tilde{u} - \bar{\tilde{u}})\,. \tag{67}$$

The constants C in (65), (66) are uniform as long as u_0, $u_0 + \tilde{u}$, $u_0 + \bar{\tilde{u}}$ have range within W. All norms in (65), (66) are $C^\alpha(\mathbf{R}^n_T)$ norms.

Smallness of the oscillation of $\partial_u f^{ij}$ over all pertinent arguments $(x, u) \in \mathbf{M} \times W$ is crucial for the contraction estimate. Given $\gamma > 0$, there exist δ, ε_1 such that

$$|\partial_u f^{ij}(x, u) - \partial_u f^{ij}(y, \bar{u})| \le \gamma$$

if $d(x, y) < \delta$ and $|u - \bar{u}| \le \varepsilon_1$. By an affine change of local coordinates in a ball, we can actually assume δ^{ij} to be the value of $\partial_u f^{ij}$ at some (x, u). This will make our constants dependent on the ellipticity parameters λ, Λ, but does otherwise not affect the estimates. We choose $\gamma \le \frac{1}{2CN}$ with N bounding the number of δ-balls that can be packed in a 3δ-ball, a quantity independent of δ by Lemma 5.2. Then, (67) implies, with K a bound for $\|\partial_u f^{ij}\|_{L^\infty(\mathbf{M}\times W)}$ and $\|\partial_u^2 f^{ij}\|_{L^\infty(\mathbf{M}\times W)}$, that

$$\begin{aligned}
&\|\tilde{f}_l - \bar{\tilde{f}}_l\|_{C^\alpha(\mathbf{R}^n_T)} \le \gamma\|\eta_l(\tilde{u}_l - \bar{\tilde{u}}_l)\|_{C^\alpha(\mathbf{R}^n_T)} + \\
&\quad + \max\nolimits_{s\in[0,1]} \|\partial_u f^{ij}(x, u_0 + s\tilde{u} + (1-s)\bar{\tilde{u}}) - \delta^{ij}\|_{C^\alpha}\|\eta_l(\tilde{u}_l - \bar{\tilde{u}}_l)\|_{L^\infty} \\
&\quad \le \left(\gamma + K\|u_0 + s\tilde{u} + (1-s)\bar{\tilde{u}}\|_{C^\alpha} + 1)CT^{\alpha/2}\right)\|\tilde{u} - \bar{\tilde{u}}\|_{C^\alpha(\mathbf{M})}\,,
\end{aligned} \tag{68}$$

provided the partition of unity is made with balls of diameter $< \delta$. With δ thus fixed, we have bounds for the C^α norms of η_l and its derivatives and can use the smallness of T, dependent also on $K(\|u_0\|_{C^\alpha(\mathbf{M})} + M)$, to ensure from (66) and (68) that $\mathcal{T} : \tilde{u} \mapsto \sum(\eta_l v_l) \circ \chi_l^{-1}$ is a contraction from $S \cap C^\alpha(\mathbf{M}_T)$ to $C^\alpha(\mathbf{M}_T)$ with contraction constant ϑ.

From (65), we get a fixed (possibly large) bound for $\mathcal{T}\tilde{u} := \|\sum(\eta_l v_l)\circ\chi_l^{-1}\|_{C^\alpha}$, and again the smallness of T ensures that $\|\mathcal{T}\tilde{u}\|_{L^\infty} \le 2\varepsilon_0$. We also estimate

$$\|\mathcal{T}\tilde{u}\|_{C^\alpha(\mathbf{M}_T)} \le \|\mathcal{T}\tilde{u} - \mathcal{T}0\|_{C^\alpha(\mathbf{M}_T)} + \|\mathcal{T}0\|_{C^\alpha(\mathbf{M}_T)} \le \vartheta M + A = M \ .$$

This guarantees the applicability of Banach's fixed point theorem, and hence warrants local existence.

The same solution can also be obtained by means of the implicit function theorem, because Thm. 7.1 guarantees a bounded inverse for the linearization operator. The analytic dependence of the equation on the data then leads to the analytic dependence of the solution on the initial data and parameters. □

We will want to control the deviation of the nonlinear flow from the linear flow, at least for short times. Contingent upon second order differentiability of the flow in the appropriate function space C^α, this deviation should be of quadratic order in the norm of the space. But we will need an estimate that involves the weaker L^∞-norm. So we claim:

Lemma 7.5 (Linear approximation of nonlinear semiflow)**.** *Let $\bar{w}$ solve the homogeneous linear equation* (54)*, namely $(\partial_t - \mathbf{L})w = 0$ for initial data w_0, where in local coordinates, $\mathbf{L}w = \partial_{ij}^2(a^{ij}w) + \partial_i(b^i w) + cw$. Let w solve the quasilinear equation $(\partial_t - \mathbf{L})w = \tilde{\mathbf{L}}(f(w)w)$ for the same initial data w_0, where in local coordinates $\tilde{\mathbf{L}} = \mathbf{L} + \partial_i \circ \tilde{b}^i + \tilde{c}$ and f is a smooth function from an interval about 0 into $\mathbf{R}$ satisfying $f(0) = 0$. Assume the coefficients are smooth.*

Then, for sufficiently short time T and sufficiently small $\|w_0\|_{L^\infty}$ (dependent on the same quantities as in Thm. 7.4), there exists a constant K (uniform as $T \to 0$) such that we have the estimate

$$\|w - \bar{w}\|_{C^\alpha(\mathbf{M}_T)} \le K\|w\|_{C^\alpha(\mathbf{M}_T)}\|w\|_{L^\infty(\mathbf{M}_T)} \tag{69}$$

and from it the time-step estimate (with a different K):

$$\|\bar{w}(T) - w(T)\|_{C^\alpha(\mathbf{M})} \le K\|w_0\|_{C^\alpha(\mathbf{M})}\,\|w_0\|_{L^\infty(\mathbf{M})} \ . \tag{70}$$

Proof. The proof follows the Banach fixed point argument used in proving Theorems 7.1 and 7.4. We decompose, as before, the solution $\bar{w}$ as $\bar{w} = \sum \eta_l \bar{w}_l$ with $\bar{w}_l = \eta_l \bar{w}$ and $\{\eta_l^2\}$ a partition of unity. Similarly, we decompose $w = \sum \eta_l w_l$ in the same way as $\bar{w}$, and refer to the proof of Thm. 7.1, specifically Eqn. (57).

Now from Equation (56), we can copy

$$\begin{aligned}\partial_t \bar{w}_l - \Delta \bar{w}_l = \partial_{ij}^2\big[(a^{ij} - \delta^{ij})\eta_l \bar{w}_l\big] + \partial_i\big[b^i \eta_l \bar{w}_l - 2a^{ij}(\partial_j \eta_l)\bar{w}_l\big] \\ + c\eta_l \bar{w}_l + \big[a^{ij}(\partial_{ij}^2 \eta_l) - b^i(\partial_i \eta_l)\big]\bar{w}_l\end{aligned} \tag{71}$$

and likewise from (64)

$$
\begin{aligned}
\partial_t w_l - \Delta w_l = {} & \partial^2_{ij}\big[(a^{ij}-\delta^{ij})\eta_l w_l\big] + \partial_i\big[b^i\eta_l w_l - 2a^{ij}(\partial_j\eta_l)w_l\big] \\
& + c\eta_l w_l + \big[a^{ij}(\partial^2_{ij}\eta_l) - b^i(\partial_i\eta_l)\big]w_l \\
& + \partial^2_{ij}\big[a^{ij}\eta_l f(w_l)w_l\big] + \partial_i\big[b^i\eta_l f(w_l)w_l - 2a^{ij}(\partial_j\eta_l)f(w_l)w_l\big] \\
& + c\eta_l f(w_l)w_l + \big[a^{ij}(\partial^2_{ij}\eta_l) - b^i(\partial_i\eta_l)\big]f(w_l)w_l \\
& + \partial_i[\tilde b^i\eta_l f(w_l)w_l] + [\tilde c\eta_l - \tilde b^i(\partial_i\eta_l)]f(w_l)w_l\ .
\end{aligned} \tag{72}
$$

Applying the heat equation estimate to the difference (which has initial data 0), we conclude that

$$
\begin{aligned}
\|w_l - \bar w_l\|_{C^\alpha(\mathbf{R}^n_T)} \le C\Big(&\gamma\|w-\bar w\|_{C^\alpha(\mathbf{M}_T)} + \gamma\|\eta_l\|_{C^\alpha}\|w-\bar w\|_{L^\infty} \\
&+ \|a^{ij}-\delta^{ij}\|_{C^\alpha}\|w-\bar w\|_{L^\infty} + T^{1/2}\|\eta_l\|_{C^{2,\alpha}}\|w-\bar w\|_{C^\alpha}\Big) \\
&+ C\|(a^{ij}, b^i, c, \tilde b^i, \tilde c)\|_{C^\alpha}\|\eta_l\|_{C^{2,\alpha}}\|f(w)w\|_{C^\alpha}\ .
\end{aligned}
$$

(All norms of w on the right hand side are space-time norms, even if notationally suppressed for conciseness.) Combining contributions from all coordinate patches, and using the contraction estimate for small times, we conclude that

$$
\|w-\bar w\|_{C^\alpha(\mathbf{M}_T)} \le K\|f(w)w\|_{C^\alpha(\mathbf{M}_T)} \le K\|w\|_{C^\alpha(\mathbf{M}_T)}\,\|w\|_{L^\infty(\mathbf{M}_T)}\,,
$$

i.e. (69).

From the maximum principle, we can estimate, similarly as in the proof of Thm. 7.1, briefly postponing details, that $\|w\|_{L^\infty(\mathbf{M}_T)} \le C\|w_0\|_{L^\infty(\mathbf{M})}$. Now we assume $\|w_0\|_{L^\infty} < 1/(2KC)$, and we obtain, still in the space-time norms, $\|w\| \le \|\bar w\| + \|w-\bar w\| \le \|\bar w\| + \frac12\|w\|$, hence $\|w\|_{C^\alpha(\mathbf{M}_T)} \le 2\|\bar w\|_{C^\alpha(\mathbf{M}_T)}$. From Theorem 7.1, we know that $\|\bar w\|_{C^\alpha(\mathbf{M}_T)} \le C\|w_0\|_{C^\alpha(\mathbf{M})}$. Combining these estimates, we conclude, with a new constant $K' = 2KC$,

$$
\|w-\bar w\|_{C^\alpha(\mathbf{M}_T)} \le K'\|w_0\|_{C^\alpha(\mathbf{M})}\,\|w_0\|_{L^\infty(\mathbf{M})}\,,
$$

and this implies (70) immediately.

Let us now clarify the details of the comparison principle argument: After commuting the smooth coefficients in front of the derivatives, w satisfies (with appropriate new $b^i, c, \tilde b^i, \tilde c$)

$$
w_t - a^{ij}\partial^2_{ij}(w + f(w)w) - b^i\partial_i w - \tilde b_i\partial_i(f(w)w) - (c + \tilde c f(w))w = 0\ . \tag{73}
$$

We assume $w \mapsto w + f(w)w =: g(w)$ to be monotonic on an interval $|w| \le 2\delta_1$ and choose $\delta_0 < \delta_1$, assuming $\|w_0\|_{L^\infty} \le \delta_0$. Without loss of generality we truncate f smoothly to a constant outside the interval $[-2\delta_1, 2\delta_1]$, leaving f unchanged on $[-\delta_1, \delta_1]$, so that the nonlinearity is globally defined and the equation with the modified nonlinearity is still uniformly parabolic. The modification of the nonlinearity will not affect solutions whose range stays in $[-\delta_1, \delta_1]$. A smallness assumption on T dependent on δ_1/δ_0 will be imposed.

Namely, with $\max_{[-2\delta_1,2\delta_1]}|f(w)| =: M$, we want $\lambda \geq 0$ to be an upper bound for $|c| + M|\tilde{c}|$. For consistency, we require $\lambda T \leq \ln(\delta_1/\delta_0)$.

We now compare w with $\underline{w} := e^{\lambda t}(-\|w_0\|_{L^\infty} - \varepsilon(\mu t + \rho(x))$, where ρ is the smooth radial function from Lemma 5.3 and $\lambda \geq 0$ as given above, and μ, ε are positive constants. Given $\varepsilon > 0$, we are assured that $\underline{w} \leq w$ outside a compact set, and also for $t = 0$. We need to show that the same operator as in (73) applied to $\underline{w}$ is ≤ 0. Then Thm. 12 in Sec 3.7 of Protter-Weinberger [40] guarantees $w \geq \underline{w}$. Indeed,

$$\begin{aligned}&\underline{w}_t - a^{ij}\partial^2_{ij}g(\underline{w}) - b^i\partial_i\underline{w} - \tilde{b}^i\partial_i g(\underline{w}) - (c + \tilde{c}f(\underline{w}))\underline{w} = \\ &\quad = \underline{w}_t - a^{ij}g'(\underline{w})\partial^2_{ij}\underline{w} - g''(\underline{w})a^{ij}(\partial_i\underline{w})(\partial_j\underline{w}) - (b^i + \tilde{b}^i g'(\underline{w}))\partial_i\underline{w} - (c + \tilde{c}f(\underline{w}))\underline{w} = \\ &\quad = e^{\lambda t}\Big([\lambda - (c + \tilde{c}f(\underline{w}))](-\|w_0\| - \varepsilon(\mu t + \rho(x))) - \varepsilon\mu \pm \varepsilon O(1)\Big)\,.\end{aligned}$$

By choosing μ larger than the constant in $O(1)$, independent of ε, this quantity is indeed ≤ 0. Now we can let $\varepsilon \to 0$ and conclude $w \geq -e^{\lambda t}\|w_0\|_{L^\infty}$. A similar argument can be made with $\bar{w} := e^{\lambda t}(\|w_0\|_{L^\infty} + \varepsilon(\mu t + \rho(x))$, with the inequalities reversed and an upper bound proved. □

We now return to the fast diffusion equation formulated in the relative L^∞ norm on the cigar manifold $\mathcal{M}$, which is the motivating example for the equations studied so far in this chapter.

In order to get a priori control over the behavior of the dynamics for long time in the nonlinear case, we use a comparison principle with Barenblatt solutions to gain such control in the L^∞ norm, and a parabolic Nash-Moser-DeGiorgi result to upgrade this control to a Hölder norm. The a priori control gained from these arguments will allow to iterate the short-term existence from Thm. 7.4 and gain global well-posedness.

It is of course well known that the initial value problem for the fast diffusion equation has unique solutions, and hence the initial value problem for v is well-posed in suitable function spaces. We need however a more precise result that establishes, in particular, differentiable dependence of the semiflow on its initial data. We now prove this in the Hölder spaces C^α, remarking afterwards that the same proof extends to smoother spaces $C^{k,\alpha}$ and to spaces C^α_η with more restrictive weights $\eta < 0$ (but not with the more permissive weights $\eta > 0$, which will be discussed in Chapter 11).

Lemma 7.6 (Relative L^∞ bounds; cf [44])**.** *Suppose* $\frac{n-2}{n} = m_0 < m \in]0,1[$ *and the initial data* $u_0 \in C(\mathbf{R}^n)$ *of a solution* u *to* (4) *satisfies* $c^{-1}u_B(\boldsymbol{x}) \leq u_0(\boldsymbol{x}) \leq cu_B(\boldsymbol{x})$ *for all* $\boldsymbol{x} \in \mathbf{R}^n$ *and some constant* $c \geq 1$*. Then there exists* $C = C(n,m,c,B) \in [1,\infty[$ *such that* $(t,\boldsymbol{x}) \in [0,\infty[\times\mathbf{R}^n$ *implies*

$$C^{-1}u_B(\boldsymbol{x}) \leq u(t,\boldsymbol{x}) \leq Cu_B(\boldsymbol{x}). \tag{74}$$

Proof. This follows from the same comparison argument used in Vázquez [44, Thm. 21.1]. Consider a quotient of Barenblatt solutions:

$$\left(\frac{\rho_{B_+}(\tau,\boldsymbol{y})}{u_B(\boldsymbol{y})}\right)^{-(1-m)} = (1+2p\tau)^{-1}\,\frac{B_+\,(1+2p\tau)^{2\beta}+|\boldsymbol{y}|^2}{B+|\boldsymbol{y}|^2}\,. \tag{75}$$

Given any number $c \geq 1$ we can make the quotient $\rho_{B_+}(\tau,\boldsymbol{y})/u_B(\boldsymbol{y})$ larger than c (equivalently, the right hand side of (75) sufficiently small) by first choosing τ large and next $B_+ > 0$ small. This yields the inequality

$$\rho_{B_+}(\tau_+,\boldsymbol{y}) \geq c u_B(\boldsymbol{y}) \geq u_0(\boldsymbol{y})$$

for some $B_+ > 0$ small enough and τ_+ large enough. By the comparison principle, the solution of (1) with initial condition $\rho_0 := u_0$ satisfies $\rho(\tau,\boldsymbol{y}) \leq \rho_{B_+}(\tau+\tau_+,\boldsymbol{y}) \leq C_+\rho_{B_+}(\tau,\boldsymbol{y})$ for all $(\tau,\boldsymbol{y}) \in [0,\infty[\times\mathbf{R}^n$ and some explicitly computable $C_+ = C_+(n,m,\tau_+) < \infty$. The corresponding transformed solutions to equation (4) satisfy $u(t,\boldsymbol{x}) \leq C_+u_{B_+}(\boldsymbol{x})$. Since $1 \leq u_{B_+}/u_B \leq (B_+/B)^{-1/(1-m)}$ this completes the proof of the second assertion in (74).

The same estimate with B_- large and $\tau_- > -1/2p$ sufficiently negative can be used to get $\rho(\tau,\boldsymbol{y}) \geq \rho_{B_-}(\tau+\tau_-,\boldsymbol{y}) \geq C_-\rho_{B_-}(\tau,\boldsymbol{y})$ and complete the proof that $u(t,\boldsymbol{x}) \geq u_B(\boldsymbol{x})/C$ for some $C \in [1,\infty[$ and all $(t,\boldsymbol{x}) \in [0,\infty[\times\mathbf{R}^n$. □

In order to use the DeGiorgi, Nash and Moser result, we note that in local coordinates we can write equation (51) in divergence form as

$$u_t - \partial_i\big((\partial_u f^{ij})(x,u)\partial_j u\big) - \big((\partial_u b^j)(x,u) + (\partial_i\partial_u f^{ij})(x,u)\big)\partial_j u = \tilde{c}(x,u)$$

where

$$\tilde{c}(x,u) = c(x,u) + (\partial_i b^i)(x,u) + (\partial^2_{ij} f^{ij})(x,u)$$

is bounded if u is, i.e., if $u(\mathbf{M}_T) \subset W$. Let us clarify the notation here: expressions of the form $\partial_i(g(x,u))$ refer to a partial with respect to x_i in all locations including implicit in u, whereas $(\partial_i g)(x,u)$ would refer only to the explicit occurrence of x_i in the first argument of g.

We consider this equation as a linear equation of the type

$$u_t - \partial_i(\tilde{a}^{ij}\partial_j u) - \tilde{b}^i\partial_i u = \tilde{c} \tag{76}$$

where $\tilde{a}^{ij}$, $\tilde{b}^i$ and $\tilde{c}$ are bounded measurable functions and there exist $\lambda, \Lambda > 0$ with

$$\lambda\delta^{ij} \leq \tilde{a}^{ij} \leq \Lambda\delta^{ij}$$

in the sense of quadratic forms. Now we can apply

Theorem 7.7 (DeGiorgi-Nash-Moser)**.** *There exists* $0 < \alpha_0 < 1$ *depending only on* λ *and* Λ *so that any bounded weak solution* u *to* (76) *in the cylinder* $[0,2R^2]\times B_{2R}$ *lies in* $C^{\alpha_0}([R^2,2R^2]\times B_R)$ *and*

$$\|u\|_{C^{\alpha_0}([R^2,2R^2]\times B_R)} \leq c(\lambda,\Lambda,R,\|\tilde{b}\|_{L^\infty})\left(\|u\|_{L^\infty([0,2R^2]\times B_{2R})} + \|\tilde{c}\|_{L^\infty([0,2R^2]\times B_{2R})}\right).$$

Proof and statement can be found in Ladyženskaja, Solonnikov and Ural'ceva [34], III§10.

We can now state:

Theorem 7.8 (Long-time smooth dependence in C^α of the FDE on data)**.** *Let $\alpha \in]0,1[$, $m \in]0,1[$ and $m > m_0 = \frac{n-2}{n}$. Then there exists R such that for $v_0 \in C^\alpha$ with $\|v_0 - 1\|_{L^\infty(\mathbf{R}^n)} < R$ there exists a unique solution $v \in C^\alpha(\mathcal{M}_T)$ to* (21) *for each T. The map*

$$C^\alpha(\mathcal{M}) \ni v_0 \longmapsto v \in C^\alpha(\mathcal{M}_T)$$

has continuous derivatives of all orders. It is bounded in the sense that $C^{-1} \leq v(t, \boldsymbol{x}) < C$ for all $(t, \boldsymbol{x})$.

Proof. The local existence part is an immediate consequence of Theorem 7.4.

We are now in a position to see that the local existence time asserted in Thm. 7.4 is controlled only by the L^∞ norm of the initial data, and not by an otherwise conceivable deterioration of the Hölder norms.

Comparison of $\tilde{u}$ with $K \pm (mt + \varepsilon_0)$ (with m large enough to control c, ∂b and $\partial^2 f$) shows that there exists T_0 independent of the Hölder norm of the initial data so that u cannot leave W before that time. We again turn to the local uniformly parabolic equation which we write as

$$\tilde{u}_t = \sum_i \partial_i(\tilde{a}^{ij}(x, \tilde{u})\partial_i \tilde{u}) + \tilde{b}^i(x, \tilde{u})\partial_i \tilde{u} + \tilde{c}(x, \tilde{u}).$$

By Theorem 7.7 and uniformity of the manifold there exists $\alpha_0 > 0$ and C so that $\|u\|_{C^{\alpha_0}([\tau, \min\{T, T_0\}] \times \mathbf{M})} \leq C$ with C depending only on W (via the bounds of derivative of the nonlinearities, and the ellipticity of the operator) for all $\tau > 0$.

Localizing again on sufficiently small spatial and temporal scale we obtain again local equations

$$\tilde{u}_t - \Delta \tilde{u} = \partial^2_{ij}\Big(f^{ij}(x, u_0 + \tilde{u}) - \delta^{ij}(u_0 + \tilde{u})\Big) + \partial_i b^i(x, u_0 + \tilde{u}) + c(x, u_0 + \tilde{u}).$$

Now

$$\partial^2_{ij}(f^{ij}(x, u_0 + \tilde{u}) - \delta^{ij} u) = \partial^2_{ij}(f^{ij}(x, u_0 + \tilde{u}) - f^{ij}(x, u_0) - \delta^{ij} u) + \partial^2_{ij} f^{ij}(x, u_0).$$

Multiplying by a cutoff function we obtain

$$\|u\|_{C^\alpha([t_0 + R^2, t_0 + 2R^2] \times B_R(x_0))} \leq c\, \|u\|_{L^\infty([t_0, t_0 + 2R^2] \times B_{2R}(x_0))}$$

which gives the bound of Thm. 7.4 as long as u assumes values in W.

Therefore the only way that a solution may cease to exist is by v tending to zero or infinity. This is impossible by the pointwise bounds of Theorem 7.6. This proves the global semiflow property. □

Remark 7.9 (Smoother spaces). *We can carry out the same proof in $C^{k,\alpha}$ spaces, using the analog of* (43) *for these spaces, or by looking at systems of equations for the derivatives.*

We observe that without any change in the argument we obtain a similar global result in the function space C^α_η for $\eta \le 0$. This space is defined by the norm

$$\|f\|_{C^\alpha_\eta} := \|(\cosh s)^{-\eta} f\|_{C^\alpha} \; . \tag{77}$$

Let us be more precise. We define $\check{v} = (\cosh s)^{-\eta} v$, and then $\check{v}$ satisfies an equation similar to (21), for which the same proof carries over. In this process, $\eta \le 0$ is needed to guarantee that C^α_η still embeds into L^∞. Otherwise quadratic and higher order terms accumulate positive powers of $\cosh s$ and cannot be controlled anymore.

We will also obtain certain analogs of Thm. 7.4 and Lemma 7.5 in the case $\eta > 0$ in Chapter 11. They are of a mixed nature inasmuch as the hypotheses and statements still need to enforce the boundedness of w that is not enforced by the C^α_η norm itself.

8. The spectrum of the linearized equation

Since the spectral calculations of [19] and [20] have been done in a different framework, some explanations are at hand to obtain a valid comparison. Here we have adopted the normalization conventions of the announcement [19], which differ from those of [20].

Our fast diffusion equation (1), (4) differs from the one in [20] by a factor $1/m$ in front of the Laplacian and a factor $\frac{2}{1-m}$ in front of the rescaling term. This difference is explained by a correspondence $\boldsymbol{x}_{[20]} = \boldsymbol{x}\sqrt{\frac{2m}{1-m}}$, $t_{[20]} = \frac{2}{1-m}\,t$, $C_{[20]} = \frac{2m}{1-m}B$. τ coincides between [20] and here, but $\alpha_{[20]} = 1/\beta$. The linearization operator in [20] was

$$\mathbf{H}_{[20]}\Psi = -m u_{B}^{m-1}\Delta\Psi + \boldsymbol{x}\cdot\boldsymbol{\nabla}\Psi$$

(of which we should drop the m in front of the Laplacian and add a factor of $\frac{2}{1-m}$ in front of the rescaling term to account for our space variable, and which was defined there as the *negative* linearization operator). Let us import the spectral results from [20], but *adapted to the notation conventions of the present paper, which coincide with those of* [19]*:*

Theorem 8.1 (Spectral theory of the linearized operator [20]). *The operator*

$$\mathbf{H} : \Psi \mapsto u_{B}^{m-2}\boldsymbol{\nabla}\cdot[u_{B}\boldsymbol{\nabla}\Psi] = u_{B}^{m-1}\Delta\Psi - \frac{2}{1-m}\boldsymbol{x}\cdot\boldsymbol{\nabla}\Psi \tag{78}$$

is essentially self-adjoint on the Hilbert space defined by the norm $(\int u_{B}|\boldsymbol{\nabla}\Psi|^2\,d\boldsymbol{x})^{1/2}$ *(with constant functions Ψ modded out). The restriction of $\mathbf{H}$ to the eigenspaces* *of the spherical Laplacian* $\Delta_{\mathbf{S}}$ *with eigenvalue* $-\ell(\ell+n-2)$ *(where* $\ell = 0,1,2,\dots$*) has continuous spectrum for*

$$\lambda \le \lambda_\ell^{\rm cont} := -\left[\frac{1}{(1-m)^2} + \left(\frac{n}{2}+\ell-1\right)^2 - \frac{n-2}{1-m}\right] = -\ell(\ell+n-2) - \left(\frac{p}{2}+1\right)^2$$

and finitely many eigenvalues

$$\lambda_{\ell k} = -\tfrac{2}{1-m}(\ell+2k) + 4k(k+\ell+\tfrac{n}{2}-1) = -(\ell+2k)p - n\ell - 4k(1-\ell-k)$$

for integers k *satisfying*

$$0 \le k < \tfrac12\left[\tfrac{1}{1-m} - \tfrac{n}{2} + 1 - \ell\right] = \tfrac12\left[\tfrac{p}{2}+1-\ell\right],$$

with corresponding eigenfunctions $\psi_{\ell k}(r) Y_{\ell\mu}(\boldsymbol{x}/|\boldsymbol{x}|)$ *where the* $Y_{\ell\mu}$ *are spherical harmonics and*

$$\psi_{\ell k}(r) = r^\ell {}_2F_1\left(\begin{matrix} k+\ell-1-p/2,\ -k \\ \ell+n/2\end{matrix};-\frac{r^2}{B}\right),$$

which are polynomials of degree $\ell+2k$ *in* r.

Moreover, the operator $\mathbf{L}$ studied here is still not exactly the same as the operator $\mathbf{H}$ just imported from [19], [20]. This is due to a different linearization. In [19], [20], the linearization was defined in terms of the pushforward of a measure, $(I+\varepsilon\boldsymbol{\nabla}\Psi)_\# u_B$, whereas here we have the linearization $u_B(1+\varepsilon\bar v)$. At least on a formal level,

$$\begin{aligned}\big((I+\varepsilon\boldsymbol{\nabla}\Psi)_\# u_B\big)(\boldsymbol{x}) &= u_B\left((I+\varepsilon\boldsymbol{\nabla}\Psi)^{-1}\boldsymbol{x}\right)/\det(I+\varepsilon\boldsymbol{\nabla}\Psi) = \\ &= \big(u_B(\boldsymbol{x}) - \varepsilon\boldsymbol{\nabla}u_B(\boldsymbol{x})\cdot\boldsymbol{\nabla}\Psi(\boldsymbol{x})\big)\big(1-\varepsilon\,\mathrm{trace}\,D(\boldsymbol{\nabla}\Psi)\big) + O(\varepsilon^2) \\ &= u_B(\boldsymbol{x}) - \varepsilon\big(\boldsymbol{\nabla}u_B(\boldsymbol{x})\cdot\boldsymbol{\nabla}\Psi(\boldsymbol{x}) + u_B(\boldsymbol{x})\Delta\Psi(\boldsymbol{x})\big) + O(\varepsilon^2) \\ &= \big(u_B - \varepsilon\boldsymbol{\nabla}\cdot[u_B\boldsymbol{\nabla}\Psi] + O(\varepsilon^2)\big)(\boldsymbol{x}),\end{aligned}$$

so $-\bar v = u_B^{-1}\boldsymbol{\nabla}\cdot(u_B\boldsymbol{\nabla}\Psi) =: \Lambda\Psi$. So we should have $\mathbf{L}\circ\Lambda = \Lambda\circ\mathbf{H}$. This can be confirmed by a look at the 'factorized' forms in (78) and (23). Since $\Lambda = u_B^{1-m}\circ\mathbf{H}$, there is also an easier conjugacy: $\mathbf{L}\circ u_B^{1-m} = u_B^{1-m}\circ\mathbf{H}$. (In the notation (25), this means $\mathbf{H} = \mathbf{L}_{\eta=-2}$.)

In Theorem 8.1, an eigenvalue λ_{00} with eigenfunction a constant, does, strictly speaking, not exist, because constants were modded out in the Hilbert space $W^{1,2}_{u_B}$ defined by the norm $(\int u_B|\boldsymbol{\nabla}\Psi|^2 d\boldsymbol{x})^{1/2}$. In our context however, the constant function ψ_{00} (which becomes proportional to u_B^{1-m} under conjugacy) will correspond to the derivative of the Barenblatt with respect to mass: indeed, from (6), $(\partial_B u_B)/u_B = \frac{-1}{1-m}u_B^{1-m}$. The mass was of course fixed a priori in the (mass-transport based) linearization formalism of [19][20].

We can also introduce the Hilbert space $L^2_{u_B^{2-m}}$, which is defined by the norm $(\int u_B^{2-m}|\Psi|^2 d\boldsymbol{x})^{1/2}$. Now first by abstract functional analysis, then by explicit integration by parts, we have

$$\langle \mathbf{H}^{1/2}\Psi_1; \mathbf{H}^{1/2}\Psi_2\rangle_{W^{1,2}_{u_B}} = \langle \Psi_1; \mathbf{H}\Psi_2\rangle_{W^{1,2}_{u_B}} = \langle \mathbf{H}\Psi_1; \mathbf{H}\Psi_2\rangle_{L^2_{u_B^{2-m}}}.$$

Thus, as the quartet [8] also found, we get a Hilbert space isomorphism $\mathbf{H}^{1/2}: W^{1,2}_{u_B} \to L^2_{u_B^{2-m}}$ and $\mathbf{H}$ is self-adjoint with the same spectrum and eigenfunctions in $L^2_{u_B^{2-m}}$ as well, except that $\lambda_{00} = 0$ with the constant eigenfunction is genuine on $L^2_{u_B^{2-m}}$.

For $\mathbf{L}$, which is self-adjoint in the Hilbert space $L^2_{u_B^m}$, this means (at a formal level, since we want to consider $\mathbf{L}$ in a Hölder space setting) that the eigenfunctions for $\lambda_{\ell k}$ are now $u_B^{1-m}\psi_{\ell k}(r)Y_{\ell\mu}(\boldsymbol{x}/|\boldsymbol{x}|)$. They will lie in the Hölder space C^α only if they are bounded. Different weights in the Hölder space can however be used to incorporate these formal eigenfunctions into the space, or, equivalently, we can conjugate the operators by an appropriate weight and analyze the conjugated operators in the unweighted space C^α. In any case, the critical growth threshold for a function to be in $L^2_{u_B^m}$ differs from the growth threshold for a function to be in C^α. This affects the spectral theory. We define

$$\begin{aligned} &C^{k,\alpha}_\eta(\mathcal{M}) := \{g := (\cosh s)^\eta f \mid f \in C^{k,\alpha}(\mathcal{M})\} \\ &\|g\|_{C^{k,\alpha}_\eta(\mathcal{M})} := \|(\cosh s)^{-\eta} g\|_{C^{k,\alpha}(\mathcal{M})}. \end{aligned} \tag{79}$$

Also note that $\eta_{cr} = \frac{p}{2} - 1$, combined with (28) and (30), yields the isometry

$$\|f\|_{L^2_{u_B^m}(\mathbf{R}^n)} = \|(\cosh s)^{-\eta_{cr}} f\|_{L^2(\mathcal{M})} =: \|f\|_{L^2_{\eta_{cr}}(\mathcal{M})}$$

and thus displays that η_{cr} gives the critical growth for selfadjointness on $\mathcal{M}$.

Given an unbounded operator $\mathbf{L} : dom\mathbf{L} \longrightarrow C^\alpha$ defined on a linear subspace $dom\mathbf{L} \subset C^\alpha$, recall that $\lambda \in \mathbf{C}$ is said to be in the spectrum of $\mathbf{L}$ if $(\mathbf{L} - \lambda I)$ does not have a bounded inverse on C^α. There are two subclassifications of spectra that we also refer to: into *point, continuous*, and *residual* on the one hand, and into *essential* and *inessential* on the other. More specifically, λ is said to be *point* spectrum if $\mathbf{L} - \lambda I$ fails to be injective, *continuous* spectrum if $\mathbf{L} - \lambda I$ is injective and its range is dense but not closed (in which case $(\mathbf{L} - \lambda I)^{-1}$ fails to be bounded), and *residual* spectrum if $\mathbf{L} - \lambda I$ is injective but its range fails to be dense. We say $\mathbf{L} - \lambda I$ is *essential* spectrum if range$(\mathbf{L} - \lambda I)$ fails to be closed (finiteness of its codimension, when closed, and of the dimension of its kernel, always being satisfied in the examples below; c.f. [30, IV §5.6]).

The calculations from [20] can be reused, to give the following result:

Theorem 8.2 (Spectrum of $\mathbf{L}$ in Hölder spaces)**.** *Given $\eta \in \mathbf{R}$, the formula $\mathbf{L}v := u_B^{-1}\boldsymbol{\nabla} \cdot [u_B \boldsymbol{\nabla}(u_B^{m-1}v)]$ defines an operator in the space $C^\alpha_\eta(\mathcal{M})$ with domain $C^{2,\alpha}_\eta(\mathcal{M})$. Let $\mathbf{L}_\ell$ be its restriction to the eigenspace of $\Delta_{\mathbf{S}}$ for eigenvalue $-\ell(\ell+n-2)$ (where $\ell = 0, 1, 2, \ldots$), i.e., $\mathbf{L}(f(r)Y_{\ell\mu}(\omega)) = (\mathbf{L}_\ell f(r))Y_{\ell\mu}(\omega)$.*

The domain of $\mathbf{L}_\ell$ *is*

$$C^{2,\alpha}_{\eta,\ell} := \left\{ f \in C^{2,\alpha}[0,\infty[\;\middle|\; \begin{bmatrix} f(0)=f'(0)=f''(0)=0 & \text{if } \ell \geq 3 \\ f(0)=f'(0)=0 & \text{if } \ell = 2 \\ f(0)=f''(0)=0 & \text{if } \ell = 1 \\ f'(0)=0 & \text{if } \ell = 0 \end{bmatrix} \right\},$$

considered as an operator in

$$C^{\alpha}_{\eta,\ell} := \{ f \in C^{\alpha}[0,\infty[\mid f(0)=0 \text{ if } \ell \neq 0 \}\,.$$

Let $\eta_{cr} = \frac{p}{2} - 1$*, the critical value for* η *introduced after* (26)*. The spectrum of* $\mathbf{L}_\ell$ *consists of finitely many eigenvalues, plus an unbounded connected component consisting of a filled-in parabola (which degenerates to a ray if* $\eta = \eta_{cr}$*). The boundary of this parabola forms the essential spectrum. More precisely:*

(1) If $\eta = \eta_{cr}$*, then* $\mathbf{L}_\ell$ *has only point spectrum. This consists of essential spectrum for* $\lambda \leq \lambda^{\text{cont}}_\ell$ *(as in Thm. 8.1) in addition to discrete eigenvalues* $\lambda_{\ell k}$ *of finite multiplicity for integers* $0 \leq k < \frac{1}{2}[\frac{p}{2} + 1 - \ell]$*.*

(2) If $\eta > \eta_{cr}$*, the spectrum is still only point, with the discrete eigenvalues exactly the* $\lambda_{\ell k}$ *for* $0 \leq k < \frac{1}{2}[\frac{p}{2} + 1 - \ell - |\eta - \eta_{cr}|]$ *and the parabolic region*

$$\text{(80)} \qquad \operatorname{Re}\lambda \leq -(\frac{p}{2}+1)^2 - \ell(\ell+n-2) + (\eta-\eta_{cr})^2 - \left(\frac{\operatorname{Im}\lambda/2}{\eta-\eta_{cr}}\right)^2,$$

i.e., $\operatorname{Re}\sqrt{\mathrm{T}} \leq \eta + 1 - \frac{p}{2} = |\eta - \eta_{cr}|$*, where* $\mathrm{T} = \ell(\ell+n-2) + (\frac{p}{2}+1)^2 + \lambda$*. The largest* $\operatorname{Re}\lambda$ *occurring in the essential spectrum is precisely at* $\lambda^{\text{cont}}_{\ell,\eta} := (\eta-\eta_{cr})^2 - (\frac{p}{2}+1)^2 - \ell(\ell+n-2)$*. (This parabolic region includes now those eigenvalues* $\lambda_{\ell k}$ *that, in comparison with the case* $\eta = \eta_{cr}$*, have been 'lost' from the original list of eigenvalues.) For* $\ell = 0$ *and* $\eta = 0$ *(unweighted* C^α*), this threshold* $\lambda^{\text{cont}}_{0,0}$ *coincides with* λ_{01}*.*

(3) If $\eta < \eta_{cr}$*, the same formula* $\operatorname{Re}\sqrt{\mathrm{T}} \leq |\eta - \eta_{cr}|$ *characterizes now the residual spectrum (*$\mathbf{L} - \lambda$ *has closed range with codimension 1), and the same formulas as in (2) apply for the eigenvalues* $\lambda_{\ell k}$*, which now make up the only point spectrum.*

In cases (2) and (3), the boundary of the parabolic region, namely $\operatorname{Re}\sqrt{\mathrm{T}} = \eta + 1 - \frac{p}{2} = |\eta - \eta_{cr}|$*, is essential spectrum (range* $\mathbf{L} - \lambda$ *not closed). The remaining spectrum is non-essential: the kernel of* $\mathbf{L} - \lambda$ *is finite dimensional, and its range is closed with finite codimension.*

The eigenfunction for eigenvalue $\lambda_{\ell k}$ *is* $v_{\ell k}(|\boldsymbol{x}|) Y_{\ell\mu}(\boldsymbol{x}/|\boldsymbol{x}|)$ *where*

$$v_{\ell k}(r) = u_{\scriptscriptstyle B}^{1-m} \psi_{\ell k}(r) = (\cosh s)^{-2} \psi_{\ell k}(\sinh s)$$

$$= \left(\frac{\sinh^\ell s}{\cosh^2 s}\right) \times {}_2F_1\left(\begin{matrix} k+\ell-1-p/2,\ -k \\ \ell+n/2 \end{matrix} ; -\sinh^2 s \right).$$

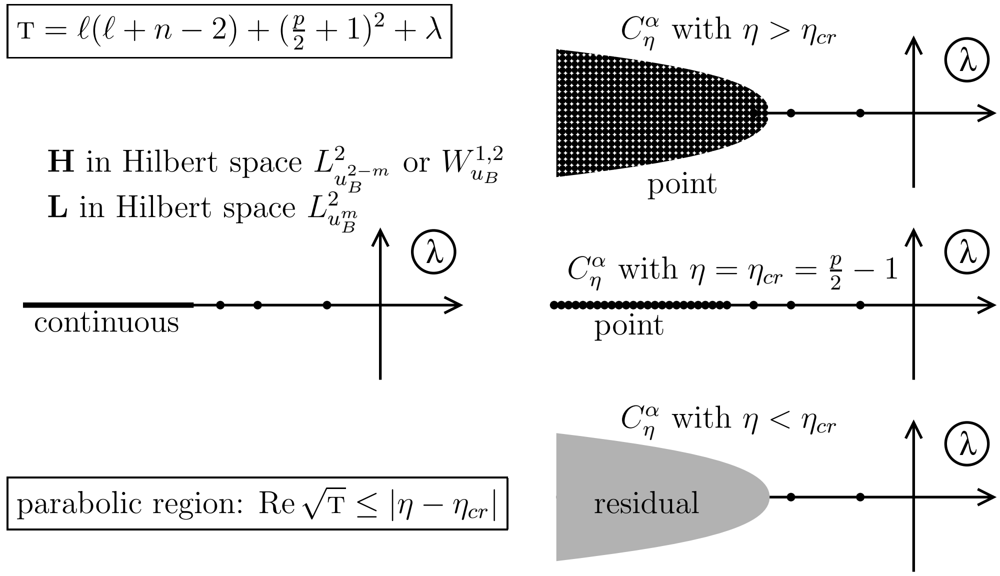


FIGURE 2. *The spectrum of* $\mathbf{L}_\ell$, *in various spaces, schematically.*

Remark 8.3. *The domain of* $\mathbf{L}$ *is dense in the little, but not the large, Hölder spaces* C^α. *The latter fact seems a bit annoying from the point of view of abstract operator theory, but will not cause us any trouble.*

Remark 8.4 (Relevant parts of spectrum)**.** *We will rely only on the value of the essential spectral radius of the semigroup generated by* $\mathbf{L}_\eta$ *(essential spectral abscissa of* $\mathbf{L}_\eta$*), and the eigenvalues outside the essential spectral radius, but not on any other aspects of the spectrum.*

After decomposition into spherical harmonics, the value of the essential spectral radius could be obtained qualitatively by asymptotic methods (see, eg., Ch. 2 of Fedoryuk [24]*) without reference to explicit solutions in terms of special functions. The precise nature of the spectrum is instructive to know, but not relevant for our arguments. Note that in contrast to the weighted space used in* [20]*, in the present, unweighted space, the onset of the essential spectrum of* $\mathbf{L}_0$ *is precisely at* λ_{01}.

Proof of Thm. 8.2. Lest there be any doubt, we point out beforehand that the sign of η, η_{cr} will not be relevant for the proof of this theorem (but will become relevant later). The claim that $\mathbf{L}$ defines an operator in C^α_η with the domain $C^{2,\alpha}_\eta$ is equivalent to the claim that the conjugated operator $\mathbf{L}_\eta = (\cosh s)^{-\eta} \circ \mathbf{L} \circ (\cosh s)^\eta$ is a mapping $C^{2,\alpha} \to C^\alpha$, and this is clear from the definition, see (25); or the equation in cartesian coordinates to see that there are actually no singularities at the origin. By slight abuse of notation, we let $\mathbf{L}_\ell$ operate on functions $f(s)$ rather than $f(s)Y_{\ell\mu}(\boldsymbol{x}/|\boldsymbol{x}|)$. Recalling (25), we have

$$
\begin{aligned}
\mathbf{L}_{\ell,\eta} :=& (\cosh s)^{-\eta} \circ \mathbf{L}_\ell \circ (\cosh s)^{\eta} \\
=& \Big(\partial_s^2 + \frac{2(n-1)}{\sinh 2s}\partial_s - (\tanh s)^{-2}\,\ell(\ell+n-2)\Big) \\
&+ \Big(n - \frac{2m}{1-m} + 2\eta\Big)\tanh s\,\partial_s \\
&+ n(\eta+2) + \Big(\eta^2 - \frac{2m}{1-m}\eta - \frac{4}{1-m}\Big)\tanh^2 s\ .
\end{aligned} \tag{81}
$$

For the claimed domains $C^{2,\alpha}_{\eta,\ell}$, notice that if $f(r)Y_{\ell\mu}(\boldsymbol{x}/|\boldsymbol{x}|)$ is in $C^{2,\alpha}(\mathcal{M})$, then f is $C^{2,\alpha}$ by restriction to rays. Moreover, any $C^{2,\alpha}(\mathcal{M})$ function can be written, near the origin, in the form $c_0 + \boldsymbol{c}_1 \cdot \boldsymbol{x} + c_2 r^2 + h_2(\boldsymbol{x}) + o(r^2)$, with h_2 a harmonic polynomial of homogeneous degree 2. (Readers looking for details on harmonic polynomials can find them on pp. 159-163 of [5]). If this function is also of the form $f(r)Y_{\ell\mu}(\boldsymbol{x}/|\boldsymbol{x}|)$, then we can multiply with various $Y_{\ell'\mu'}$ and integrate over the unit sphere. Unless $\ell = 0$, choosing $\ell' = 0$ gives $0 = c_0 + c_2 r^2 + o(r^2)$, hence $c_0 = c_2 = 0$. Unless $\ell = 1$, choosing $\ell' = 1$ gives $0 = \boldsymbol{c}_1$. Unless $\ell = 2$, testing with $\ell' = 2$ gives $h_2 = 0$. The claims about f and its derivatives at 0 follow.

Now we need to see, conversely, that when f is in the claimed domains, then $fY_{\ell\mu}$ is in $C^{2,\alpha}(\mathcal{M})$. It is only in a neighbourhood of the origin that this is nontrivial. Using Schauder theory, it suffices to show that $\Delta(f(r)Y_{\ell\mu}(\omega) = (\frac{d^2}{dr^2} + \frac{n-1}{r}\frac{d}{dr} - \frac{\ell(\ell+n-2)}{r^2})f(r)Y_{\ell\mu}(\omega)$ is (uniformly) C^α in a punctured neighborhood of the origin. Use of the initial conditions at 0 in each case establishes this in a straightforward manner.

The analysis of solutions to $(\mathbf{L}_\ell - \lambda)\psi = 0$ in terms of hypergeometric functions carried out in Sec. 4.2 of [20] carries over with only trivial modifications, and the discussion whether such solutions lie in $C^\alpha(\mathcal{M})$ reduces to growth at infinity (the smoothness being trivial). This establishes the precise point spectrum claimed for all cases, and the eigenfunctions $v_{\ell k}$.

We now assume we are not in the eigenvalue case. Translating the resolvent formula (4.36) from [20], we can write any solution to $(\mathbf{L}_\ell - \lambda)u = w$ as

$$
\begin{aligned}
u(r) &= v_2(r)\left(A_0 + \textstyle\int_0^r w v_1 u_B^m r^{n-1} dr\right) + v_1(r)\left(A_1 + \textstyle\int_r w v_2 u_B^m r^{n-1} dr\right) \\
u'(r) &= v_2'(r)\left(A_0 + \textstyle\int_0^r w v_1 u_B^m r^{n-1} dr\right) + v_1'(r)\left(A_1 + \textstyle\int_r w v_2 u_B^m r^{n-1} dr\right)\ .
\end{aligned} \tag{82}
$$

As $r \to \infty$, we have $u_B^m r^{n-1} \sim c r^{-p/2-\eta_{cr}}$ and

$$
v_2(r) \sim c r^{\eta_{cr} - \sqrt{\mathrm{T}}}\ , \qquad v_1(r) \sim c r^{\eta_{cr} + \sqrt{\mathrm{T}}}\ .
$$

As $r \to 0$, we have $u_B^m r^{n-1} \sim c r^{n-1}$ and

$$
v_2(r) \sim c r^{2-n-\ell}\ , \qquad v_1(r) \sim c r^{\ell}\ .
$$

First assume $\operatorname{Re}\sqrt{\mathrm{T}} > |\eta - \eta_{cr}|$ and let $w = O(r^\eta)$ as $r \to \infty$. Then the second integrand converges near ∞ and we may specify its upper limit as ∞. A growth estimate for u requires now $A_1 = 0$, because else the dominant term $A_1 r^{\eta_{cr}+\sqrt{\mathrm{T}}}$ is not in the space. The dominant term as $r \to 0$ is $A_0 r^{2-n-\ell}$, so we need $A_0 = 0$ to get $u(0) = 0$ in the case $\ell \geq 1$. In the case $\ell = 0$, we still need $A_0 = 0$ for $u'(0)$ to vanish. With these choices we do indeed get a solution u that satisfies the bounds near 0 and infinity required for it to be in the domain of $\mathbf{L}_\ell$ ($u''(0) = 0$ in the cases $\ell \geq 3$ uses $w(0) = 0$); the smoothness is trivial. So these cases are not in the spectrum.

Next we assume $\operatorname{Re}\sqrt{\mathrm{T}} < |\eta - \eta_{cr}|$. We are then in the case $\eta < \eta_{cr}$ because the other cases have been dealt with as point spectrum. If $w \in \operatorname{range}(\mathbf{L}_\ell - \lambda)$, then $\int_0^\infty w v_1 u_B^m r^{n-1}\, dr = 0$ by symmetry of $\mathbf{L}_\ell$ with respect to the inner product in $L^2(\mathbf{R}^+, u_B^m r^{n-1} dr)$, which *is* defined in this particular case because the integrand is bounded by $O(r^{\eta+\operatorname{Re}\sqrt{\mathrm{T}}-\eta_{cr}-1})$ for large r. Let us assume w satisfies this orthogonality condition, which restricts w to a closed subspace of codimension 1. We also have the convergence that allows us to have the upper limit ∞ in the second integral of the resolvent formula (82). Moreover, if $w = (\mathbf{L}_\ell - \lambda)u$ we need $A_1 = 0$ in the resolvent formula for the same growth reasons. We can then write

$$u(r) = A_0 v_2(r) - v_2(r) \int_r^\infty w v_1 u_B^m r^{n-1} dr + v_1(r) \int_r^\infty w v_2 u_B^m r^{n-1} dr\ .$$

We now need $A_0 = 0$, because the first term is the only one that would grow faster than $O(r^\eta)$. As before, we now confirm that u does have the required behavior as $r \to 0$ to be in the domain of $\mathbf{L}_\ell - \lambda$. We have (non-essential) residual spectrum in this case (injective, range closed and with codimension 1).

Let us now return to the point spectrum in $\operatorname{Re}\sqrt{\mathrm{T}} < |\eta - \eta_{cr}|$ in the case $\eta > \eta_{cr}$ to see that this spectrum is non-essential. Indeed, we want to show that $\mathbf{L}_\ell - \lambda$ is onto in this case. We cannot choose ∞ as upper limit of integration in (82) and choose 1 instead. A_1 is arbitrary anyways, since v_1 is a bona-fide eigenfunction. The choice $A_0 = 0$ is required for the same reasons as before, and with this choice we do get a solution u in the domain for every choice of w in the space $C^\alpha_{\eta,\ell}$.

Finally, we consider the cases on the boundary of the parabolic region. Let first $\operatorname{Re}\sqrt{\mathrm{T}} = \eta - \eta_{cr} \geq 0$. We know we have point spectrum, and we want to show that the range is *not* closed. Not being assured of convergence at ∞, we write

$$u(r) = v_2(r)\left(A_0 + \int_0^r w v_1 u_B^m r^{n-1} dr\right) + v_1(r)\left(A_1 - \int_1^r w v_2 u_B^m r^{n-1} dr\right)$$

For w to be in the range, it is necessary that the function $r \mapsto \int_1^r w v_2 u_B^m r^{n-1} dr$ remains bounded as $r \to \infty$, which is not automatic: $w \in C^\alpha_\eta$ would allow for logarithmic divergence. Assume this hypothesis is verified. Then the choice

$A_0 = 0$ does give a solution that satisfies the right bounds (and automatically the needed smoothness, too). With this characterization of the range, it can be seen that $w_* = r^{2(\eta-\eta_{cr})}\bar{v}_2\chi(r)/\ln r$ (where χ is a truncation at 0, constant 1 for $r \geq 1$) is not in the range, but $w_N = r^{2(\eta-\eta_{cr})}\bar{v}_2\chi(r)/(\ln r + \frac{r}{N})$ is in the range, for every N, and $w_N \to w_*$ uniformly, and also in the Hölder norm.

Now, for $\eta < \eta_{cr}$ and $\operatorname{Re}\sqrt{\mathrm{T}} = \eta_{cr} - \eta$, we can write the second integral as $\int_r^\infty$ again, and need $A_1 = 0$, $A_0 = 0$. We get a similar conclusion, namely that w is in the range if and only if $r \mapsto \int_0^r wv_1 u_B^m r^{n-1} dr$ is bounded as $r \to \infty$. With the same argument as before, the range is not closed. Is the range dense or not? We claim it is not dense, thus identifying the spectrum as residual rather than continuous. Indeed, if $\|g\|_{C^\alpha_\eta} < \varepsilon$, then $w(r) := r^{2(\eta-\eta_{cr})}\bar{v}_1(r)\chi(r) + g(r)$ parametrizes an open set of functions that is not in the range, where the implied smallness of the weighted L^∞ norm of g prevents it from compensating for the logarithmic divergence of the integral caused by the first term. □

The spectrum of $\mathbf{L}$ in C^α_η is the same as the spectrum of $\mathbf{L}_\eta$ (defined in (25)) in C^α. So we can now study the differential equation $v_t - \mathbf{L}_\eta v = 0$. It follows from Rmk. 7.9 that the linear semigroup $\mathbf{S}_\eta(t)$ generated is analytic on $C^\alpha(\mathcal{M})$. We want to obtain the usual semigroup estimates on the complement of the space spanned by the eigenfunctions $v_{\ell k}$. Major work goes into the boundedness result: $\|\exp[t\mathbf{L}_\eta]\| \leq C\exp[c_\infty(\eta)t]$ *without* an extra ε. The loss of self-adjointness makes this result non-trivial, in particular since we have established non-zero Fredholm index for the onset of the essential spectrum of $\mathbf{L}_\eta$. We claim

Theorem 8.5 (Semigroup Estimates)**.** *Equation* (25) *defines an analytic semigroup* $\mathbf{S}_\eta(t) = \exp t\mathbf{L}_\eta$ *on* $C^\alpha(\mathcal{M})$ *or on* $L^\infty(\mathbf{M})$*, in the sense of Remark 7.2.*

The essential spectrum of $\mathbf{S}_\eta(t)$ *is contained in* $\overline{B(0, e^{c^\infty(\eta)t})}$ *and in no smaller ball, where* $c^\infty(\eta)$ *is given by* (26) *and equals the* $\lambda^{\text{cont}}_{0,\eta}$ *from Thm. 8.2. Outside this ball there are only finitely many eigenvalues* $e^{\lambda_j t}$ *with* $c_\infty(\eta) < \lambda_1 \leq \lambda_2 \ldots$ *where the* λ_j *are the* $\lambda_{\ell k}$ *from Thm. 8.2*

The spectral projections onto the eigenspace corresponding to a set of eigenvalues $\lambda_{\ell k}$ *of the operator* $\mathbf{L}$ *in* $L^2_{u_B^m}(\mathbf{R}^n) = L^2_{\eta_{cr}}(\mathcal{M})$ *are, by the same formula, also well-defined mappings in* $C^\alpha_\eta(\mathcal{M})$*, or correspondingly weighted* $L^\infty_\eta(\mathcal{M})$*, provided the* $L^2_{\eta_{cr}}(\mathcal{M})$*-eigenfunctions for* $\lambda_{\ell k}$ *are still in the space* $C^\alpha_\eta(\mathcal{M})$*.*

Let v_j *be the eigenfunctions corresponding to the* λ_j *in* $L^2(\mathcal{M})$*. Then, for* $\eta \neq \eta_{cr}$*, each solution* $\bar{v} = \mathbf{S}_\eta(t)\bar{v}_0$ *with initial data* $\bar{v}_0 \in L^\infty$ *can be written in the form*

$$\bar{v}(t, \boldsymbol{x}) = \sum c_j e^{\lambda_j t} v_j(\boldsymbol{x}) + \bar{v}_{res}(t, \boldsymbol{x}) \tag{83}$$

with

$$\sup_{t \geq 1} e^{-c^\infty(\eta)t} \|\bar{v}_{res}\|_{L^\infty(\mathcal{M})} < \infty. \tag{84}$$

For $\eta = \eta_{cr}$, we at least get the same conclusion with $c^\infty(\eta) + \varepsilon$ instead of $c^\infty(\eta)$, for arbitrary $\varepsilon > 0$. For $\eta = \eta_{cr}$, we also get the analog of (84) *with the $L^2(\mathcal{M})$ norm (and no ε).*

Proof. It has been proven in Theorem 7.1 that the semigroup is analytic. The spectral statements follow from Thm. 8.2 by the functional calculus for analytic semigroups (e.g., Cor. IV, 3.12 in Engel, Nagel [23]). In using this reference, note that strong continuity in 0 is assumed there; so we get the estimate of the spectral radius for the little Hölder spaces oC^α directly, in view of our Remark 7.2(c). However, the result also carries over to the big Hölder spaces C^α, because the semigroup maps C^α initial data continuously into $\mathbf{C}^{\alpha'} \subset oC^\alpha$ (for $\alpha' > \alpha$) in any short time t_0, as a consequence of the regularization estimate in Thm. 7.1.

Clearly the projections $C^\alpha(\mathcal{M}) \ni u(s,\omega) \longmapsto u_{\ell\mu}(s) = \int u(s,\omega) Y_{\ell\mu}(\omega)\, d\omega \in C^\alpha[0,\infty[$ are continuous. If $\ell > 0$, the functions $u_{\ell\mu}$ satisfy $u_{\ell\mu}(0) = 0$, and we denote the space of those $C^\alpha[0,\infty[$-functions that vanish at 0 by $C^\alpha_0[0,\infty[$.

Conversely, the imbeddings $C^\alpha_0[0,\infty[\ni u(s) \longmapsto u(s) Y_{\ell\mu}(\boldsymbol{x}/|\boldsymbol{x}|) \in C^\alpha(\mathcal{M})$ are continuous for $\ell > 0$, and likewise the analogous imbedding $C^\alpha[0,\infty[\to C^\alpha(\mathcal{M})$ in the case $\ell = 0$. The L^∞ estimate being trivial, we deal with the Hölder quotient:

$$\frac{|f(r_1)Y(\omega_1) - f(r_2)Y(\omega_2)|}{d((r_1,\omega_1),(r_2,\omega_2))^\alpha} \le \frac{|f(r_1) - f(r_2)|\,|Y(\omega_1)|}{d(r_1,r_2)^\alpha} + \frac{|f(r_2)|\,|Y(\omega_1) - Y(\omega_2)|}{d((r_1,\omega_1),(r_2,\omega_2))^\alpha}$$

The first term is bounded in terms of $[f]_\alpha$. The second term vanishes if $\ell = 0$. If however $\ell \ge 1$, we use that $f(0) = 0$ and therefore $|f(r)| \le O(d(0,r)^\alpha) = O(s^\alpha)$. Then the second term is bounded by

$$c \frac{s_2^\alpha d(\omega_1,\omega_2)}{\max\{|s_2 - s_1|^\alpha, (\tanh s_2)^\alpha\, d(\omega_1,\omega_2)^\alpha\}}$$

where $s_1 \ge s_2$ without loss of generality. The estimate follows immediately.

Finally, the inner product $\langle v, v_{\ell k}\rangle_{L^2_{u_B^m}}$ is well-defined for $v \in C^\alpha[0,\infty[$ provided $\eta + \ell + 2k < p$. By the constraint on k, we have indeed $\eta + \ell + 2k < \eta + \frac{p}{2} + 1 - |\eta - \eta_{cr}| = p + \eta - \eta_{cr} - |\eta - \eta_{cr}| \le p$. So again these projections are continuous in each C^α_η, and so are trivially the corresponding imbeddings.

So, since for given $m < 1$, there are only finitely many eigenvalues $\lambda_{\ell k}$, we conclude that the spectral projection $\mathbf{Q}$ onto the span of their eigenspaces that was naturally constructed within the $L^2_{u_B^m}$ Hilbert space framework, namely

$$\mathbf{Q}v := \sum_{\ell,\mu} \sum_{k \le \bar{k}(\ell)} \frac{\langle v; v_{\ell k} Y_{\ell\mu}\rangle_{L^2_{u_B^m}}}{\langle v_{\ell k} Y_{\ell\mu}; v_{\ell k} Y_{\ell\mu}\rangle_{L^2_{u_B^m}}} v_{\ell k} Y_{\ell\mu} \,,$$

defines, by the same formula, a spectral projection in the $C^\alpha_\eta(\mathcal{M})$ framework, too. Here, $\bar{k}(\ell) = \frac{1}{2}[\frac{p}{2} + 1 - \ell - |\eta - \eta_{cr}|]$ is from Thm. 8.2. Let $\mathbf{P} := 1 - \mathbf{Q}$ be the complementary projection in C^α_η.

We turn to the semigroup estimate and consider a solution $\bar{v}$ with bounded initial data. The key point is that we get estimate (84) with precisely $c^\infty(\eta)$ in the exponent, rather than $c^\infty(\eta) + \varepsilon$, an issue that has become nontrivial because we lost the notion of self-adjointness when abandoning the Hilbert space setting. There is nothing to do in the case $\eta = \eta_{cr}$, where we did not claim such an improvement. Also, the $L^2(\mathcal{M})$ estimate for $\eta = \eta_{cr}$ is clear because the operator is self-adjoint (with the Riemannian volume on the cigar as measure) in this case.

We will distinguish the cases $\eta < \eta_{cr}$ and $\eta > \eta_{cr}$. We rely on the fact that $c^\infty(\eta) > c^\infty(\eta_{cr})$ and a trade-off between time decay and spatial growth that will be exhibited during the proof details. The key idea is that for $s \to \infty$, the maximum principle 'almost' gives the correct growth rate $c^\infty(\eta)$ in the L^∞ norm. There is a remnant of the 0^{th} order term that decays like $O(1/\cosh^2 s)$ and that could give rise to secular terms; controlling this effect gives rise to the said trade-off between spatial growth and time decay. To control the estimates for small s, we use conjugacy and estimates with a better decay constant than $c^\infty(\eta)$ for data in the correct spectral space.

The $\mathbf{L}_\eta$ invariant projection $\mathbf{P}_\eta$ corresponding to those eigenvalues that are strictly larger than $c^\infty(\eta)$ immediately gives rise to the decomposition (83). In case $c^\infty(\eta)$ is itself among the $L^2_{u_B^m}$ eigenvalues $\lambda_{\ell k}$, components in its eigenspace immediately satisfy (84). So we can project this eigenvalue away, too, and need to prove (84) only for $\bar{v} := \bar{v}_{res}$ in the complementing space. Moreover, for the price of a small time shift (regularization estimate in Thm. 7.1), we may assume $\bar{v}_0 \in C^\alpha$.

Case 1: $\eta < \eta_{cr}$**.** As remarked after (79), multiplication by $(\cosh s)^{\eta_{cr}}$ gives an isometry between $L^2(\mathcal{M})$ and $L^2_{u_B^m}(\mathbf{R}^n)$. Let us first conjugate the problem with $(\cosh s)^{\eta_{cr}-\eta}$ to obtain the self-adjoint operator $\mathbf{L}_{\eta_{cr}}$ with $L^2(\mathcal{M})$ initial data $\tilde{v}_0 := (\cosh s)^{\eta-\eta_{cr}}\bar{v}_0$. We find some $c^\infty(\eta) - \varepsilon$ strictly above the next smaller of the $L^2(\mathcal{M})$ eigenvalues (or, if there are no such eigenvalues, strictly above $c^\infty(\eta_{cr})$).

By the spectral properties of $\mathbf{L}_{\eta_{cr}}$ in $C^\alpha(\mathcal{M})$, we conclude, uniformly in t,

$$\|\tilde{v}(t)\|_{C^\alpha(\mathcal{M})} \le c e^{(c^\infty(\eta)-\varepsilon)t} \|vc_0\|_{C^\alpha(\mathcal{M})} \;;$$

using $\|vc_0\|_{C^\alpha} \le c\|\bar{v}_0\|_{C^\alpha}$, this gives in particular the weighted L^∞ estimate

$$|\bar{v}(t, \boldsymbol{x})| \le c(\cosh s)^{\eta_{cr}-\eta}\, e^{(c^\infty(\eta)-\varepsilon)t}\|\bar{v}_0\|_{C^\alpha} \,.$$

So we choose $\delta = \varepsilon/(\eta_{cr} - \eta) > 0$ such that

$$\sup_{s\le\delta t} |\bar{v}(t, \boldsymbol{x})| \le c e^{c^\infty(\eta)t}\|\bar{v}_0\|_{C^\alpha} \,. \tag{85}$$

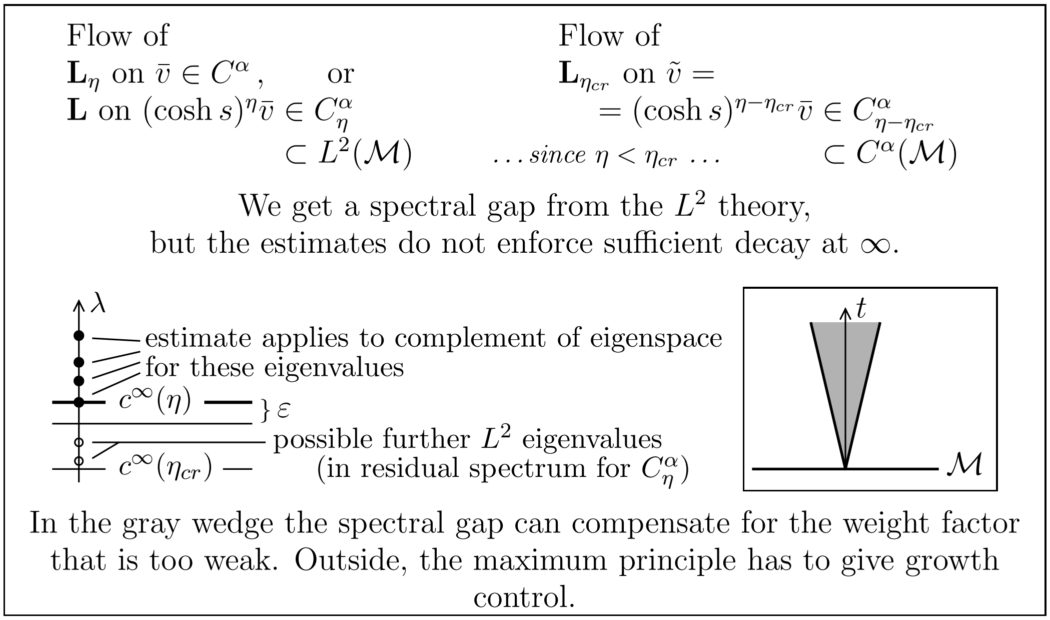


FIGURE 3. *Schematic outline of proof idea for $\eta < \eta_{cr}$: In the shaded cone, the spectral gap ε compensates for the 'too slow' decay. In the exterior region, the estimate carries over by the maximum principle.*

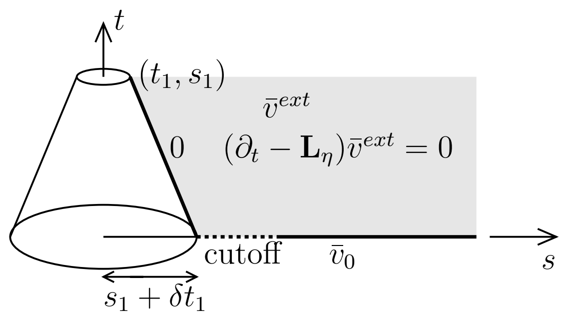
We want to apply the maximum principle to obtain the same kind of estimate in the set $s \geq \delta t$ as well. To this end, $\bar{\bar{v}} := e^{-c^\infty(\eta)t}\bar{v}$ satisfies $\big(\partial_t - (\mathbf{L}_\eta - c^\infty(\eta))\big)\bar{\bar{v}} = 0$. While $\mathbf{L}_\eta - c^\infty(\eta)$ barely fails to satisfy a straightforward maximum principle – its 0^{th} order term is positive according to (25) –, this term is $\leq c/\cosh^2 s \leq ce^{-2\delta t}$. So we let $a(t) := c^\infty(\eta)t + \int_0^t ce^{-2\delta t}\,dt$ and define instead the new $\bar{\bar{v}} := \exp[-a(t)]\bar{v}$, which solves $\big(\partial_t - (\mathbf{L}_\eta - a(t))\big)\bar{\bar{v}} = 0$ and satisfies a maximum principle. We get therefore, in the domain $s \geq \delta t$:

$$|\bar{v}(t,\boldsymbol{x})| = e^{a(t)}|\bar{\bar{v}}(t,\boldsymbol{x})| \leq e^{a(t)}c\|\bar{v}_0\|_{L^\infty} \leq Ce^{c^\infty(\eta)t}\|\bar{v}_0\|_{L^\infty} \ .$$

Together we have shown $\|\bar{v}(t)\|_{L^\infty} \leq Ce^{c^\infty(\eta)t}\|\bar{v}_0\|_{C^\alpha}$. With the regularization estimate $\|\bar{v}(t+1)\|_{C^\alpha} \leq C\|\bar{v}(t)\|_{L^\infty}$ from Thm. 7.1, the desired conclusion follows.

Case 2: $\eta > \eta_{cr}$. Here we let $\bar{v}^{ext}$ be the solution of an exterior initial boundary value problem that can be estimated with the maximum principle; then the remainder $\bar{v} - \bar{v}^{ext}$ has compactly supported initial data, which allow for using the conjugacy again. This decomposition idea requires the use of a cutoff function χ, which in turn entails writing some explicit projections to get back into the spectral spaces destroyed by the cutoff. Moreover, the remainder $\bar{v} - \chi\bar{v}^{ext}$ satisfies the differential equation with an inhomogeneity created by the cutoff; however the effect of the inhomogeneity can be controlled, see (89)–(91) below.

In detail, we first choose ε such that $c^\infty(\eta)-2\varepsilon$ is still larger than $c^\infty(\eta_{cr})$, and any larger pertinent L^2 eigenvalues buried in the essential spectrum for C^α_η as well. and we choose δ such that $\delta(\eta-\eta_{cr})=\varepsilon$. We then we choose $(t_1,\boldsymbol{x}_1)$ (with the geodesic radial coordinate s_1) and let $\bar{v}^{ext}$ solve $(\partial_t-\mathbf{L}_\eta)\bar{v}^{ext}=0$ in $s>s_1+\delta(t_1-t)$, $0\le t\le t_1$, with boundary data $\bar{v}^{ext}=0$ if $s=s_1+\delta(t_1-t)$, and initial data $\bar{v}_0^{ext}=\bar{v}_0$ for $s>s_1+\delta t_1+1$, and smoothly cut off to 0 for $s\in[s_1+\delta t_1,s_1+\delta t_1+1]$.



To prove the existence and uniqueness of the solution to this exterior boundary value problem, we first map the exterior of the cone to the exterior of a cylinder by means of the map $s+\delta t-(s_1+\delta t_1)=\sigma$, with σ the new radial coordinate, and then refer to Cor. 6.2. The estimates obtained there are uniform in (t_1,s_1) as long as s_1 is bounded away from 0. Indeed, with $\bar{v}(s,\omega,t)=V(\sigma,\omega,t)$, the equation $(\partial_t-\mathbf{L}_\eta)\bar{v}=0$ transforms into $(\partial_t-\delta\partial_\sigma-\tilde{\mathbf{L}}_\eta)V=0$, and $\tilde{\mathbf{L}}_\eta$ is the same as $\mathbf{L}_\eta$ in (25), except with ∂_σ replacing ∂_s.

With $a(t):=c^\infty(\eta)t+c\int_0^t e^{-2(s_1+\delta(t_1-t))}dt$, the function $\bar{\bar{v}}^{ext}:=e^{-a(t)}\bar{v}^{ext}$ satisfies a maximum principle again, and we get

$$|\bar{v}^{ext}(t,\boldsymbol{x})|=e^{a(t)}|\bar{\bar{v}}^{ext}(t,\boldsymbol{x})|\le e^{a(t)}\|\bar{v}_0\|_{L^\infty}\le Ce^{c^\infty(\eta)t}\|\bar{v}_0\|_{L^\infty} \tag{86}$$

with a constant $C=C(\delta)$ that is easily calculated to be uniform in (t_1,s_1) as long as s_1 is bounded away from 0. In particular, $\|\bar{v}^{ext}(t-1)\|_{L^\infty}\le Ce^{c^\infty(\eta)(t-1)}\|\bar{v}_0\|_{C^\alpha}$. Having been constructed by means of a cutoff function, $\bar{v}^{ext}$ may not be in the appropriate spectral space. But with a regularization estimate and successive projection, we can conclude that

$$\|\mathbf{P}_\eta\bar{v}^{ext}(t_1)\|_{C^\alpha}\le C\|\bar{v}^{ext}(t_1)\|_{C^\alpha}\le Ce^{c^\infty(\eta)t_1}\|\bar{v}_0\|_{C^\alpha}\,. \tag{87}$$

Now take a cutoff function $\chi\in C^\infty(\mathbf{R})$ for which $\chi(\sigma)=0$ if $\sigma\le 1$ and $\chi(\sigma)=1$ if $\sigma\ge 2$. Let

$$\bar{v}^{int}:=\bar{v}-\mathbf{P}_\eta\big(\bar{v}^{ext}\chi(s-s_1-\delta(t_1-t))\big)=\mathbf{P}_\eta(\bar{v}-\chi\bar{v}^{ext})\,,$$

where we have used in the last step that $\bar{v}$ is assumed to be in the range of $\mathbf{P}_\eta$. The initial data of $\check{v}^{int}:=\bar{v}-\chi\bar{v}^{ext}$ are compactly supported. So we can conjugate, letting

$$\tilde{\breve{v}}^{int} := (\cosh s)^{\eta-\eta_{cr}}\breve{v}^{int} \quad \text{and} \quad w^{int} = \mathbf{P}_{\eta_{cr}}\breve{w}^{int} = (\cosh s)^{\eta-\eta_{cr}}\bar{v}^{int} \,. \tag{88}$$

We have

$$\begin{aligned}(\partial_t - \mathbf{L}_{\eta_{cr}})\tilde{\breve{v}}^{int} &= (\cosh s)^{\eta-\eta_{cr}}(\partial_t - \mathbf{L}_\eta)(\bar{v} - \chi\bar{v}^{ext}) \\ &= -(\cosh s)^{\eta-\eta_{cr}}(\partial_t - \mathbf{L}_\eta)(\chi\bar{v}^{ext}) \\ &= -(\cosh s)^{\eta-\eta_{cr}}[\partial_t - \mathbf{L}_\eta, \chi]\bar{v}^{ext} \\ &= (\cosh s)^{\eta-\eta_{cr}}\big(\hat{\chi}(\sigma)\bar{v}^{ext} + 2\chi'(\sigma)\partial_s\bar{v}^{ext}\big) =: f + \partial_s g =: h\end{aligned} \tag{89}$$

where

$$\begin{aligned}g &:= 2(\cosh s)^{\eta-\eta_{cr}}\chi'(\sigma)\bar{v}^{ext} \quad \text{and} \\ f &:= (\cosh s)^{\eta-\eta_{cr}}\hat{\chi}(\sigma)\bar{v}^{ext} - 2\partial_s\big(\chi'(\sigma)\,(\cosh s)^{\eta-\eta_{cr}}\big)\,\bar{v}^{ext}\end{aligned}$$

and, using (25) and (27), $\hat{\chi} = (\frac{2(n-1)}{\sinh 2s} - b^\infty(\eta)\tanh s)\chi' + \chi'' \in C_0^\infty]1,2[$, and $\sigma := s - s_1 - \delta(t_1 - t)$.

We'd like to argue that

$$\begin{aligned}\tilde{\breve{v}}^{int}(t) &= \mathbf{S}_{\eta_{cr}}(t)\tilde{\breve{v}}^{int}(0) + \textstyle\int_0^t \mathbf{S}_{\eta_{cr}}(t-\tau)h(\tau)\,d\tau \,, \\ \tilde{v}^{int}(t) &= \mathbf{S}_{\eta_{cr}}(t)\tilde{v}^{int}(0) + \textstyle\int_0^t \mathbf{S}_{\eta_{cr}}(t-\tau)\mathbf{P}_{\eta_{cr}}h(\tau)\,d\tau \,.\end{aligned}$$

However, we have proved insufficient regularity for $h(\tau)$ to be an initial value for the semigroup estimate. While this could be fixed by stronger regularity estimates, we can more easily average over estimates for initial times $t_0 \in [0,1]$ and use Thm 7.1 directly. At a formal level the argument is written as follows: for $t \geq 1$ and all $t_0 \in [0,1]$, we have

$$\tilde{\breve{v}}^{int}(t) = \mathbf{S}_{\eta_{cr}}(t-t_0)\tilde{\breve{v}}^{int}(t_0) + \int_0^{t-t_0} \mathbf{S}_{\eta_{cr}}(t-t_0-\tau)h(t_0+\tau)\,d\tau \,; \tag{90}$$

so by averaging over t_0, we have

$$\begin{aligned}\tilde{\breve{v}}^{int}(t) &= \textstyle\int_0^1 \mathbf{S}_{\eta_{cr}}(t-t_0)\tilde{\breve{v}}^{int}(t_0)\,dt_0 \\ &\quad + \textstyle\int_0^1\int_0^{t-1} \mathbf{S}_{\eta_{cr}}(t-t_0-\tau)h(t_0+\tau)\,d\tau\,dt_0 \\ &\quad + \textstyle\int_0^1\int_{t-1}^{t-t_0} \mathbf{S}_{\eta_{cr}}(t-t_0-\tau)h(t_0+\tau)\,d\tau\,dt_0 \\ &= \mathbf{S}_{\eta_{cr}}(t-1)\textstyle\int_0^1 \mathbf{S}_{\eta_{cr}}(1-t_0)\tilde{\breve{v}}^{int}(t_0)\,dt_0 \\ &\quad + \textstyle\int_0^{t-1} \mathbf{S}_{\eta_{cr}}(t-1-\tau)\Big(\int_0^1 \mathbf{S}_{\eta_{cr}}(1-t_0)h(\tau+t_0)\,dt_0\Big)\,d\tau \\ &\quad + \textstyle\int_0^1\Big(\int_0^{1-t_0} \mathbf{S}_{\eta_{cr}}(1-t_0-\tau')h(t-(1-t_0)+\tau')\,d\tau'\Big)\,dt_0 \\ &=: \mathbf{S}_{\eta_{cr}}(t-1)\tilde{V}_0 \\ &\quad + \textstyle\int_0^{t-1} \mathbf{S}_{\eta_{cr}}(t-1-\tau)\tilde{V}_1(\tau)\,d\tau \\ &\quad + \textstyle\int_0^1 \tilde{V}_2(1-t_0)\,dt_0 \,.\end{aligned} \tag{91}$$

The fact that the integrals in (91) still manifestly apply the semigroup to a potentially singular distribution h is of no concern for the argument. The $\tilde{V}_{0,1,2}$

terms are simply solutions of the inhomogeneous equation with homogeneous initial data, and they can be estimated by Theorem 7.1.

$\tilde{V}_0$ is the solution at time 1 of $(\partial_t - \mathbf{L}_{\eta_{cr}})\tilde{V} = \tilde{\tilde{v}}^{int}$ with initial data 0, and so we have, using (89), (86) along with Thm. 7.1,

$$\begin{aligned}\|\tilde{V}_0\|_{C^\alpha(\mathcal{M})} &\leq C\|\tilde{\tilde{v}}^{int}\|_{L^\infty([0,1]\times\mathcal{M})} \ ,\\ \|\tilde{\tilde{v}}^{int}\|_{C^\alpha([0,1]\times\mathcal{M})} &\leq C(\|\tilde{\tilde{v}}^{int}(0)\|_{C^\alpha(\mathcal{M})} + \|f,g\|_{L^\infty([0,1]\times\mathcal{M})})\\ &\leq Ce^{(s_1+\delta t_1)(\eta-\eta_{cr})}(\|\bar{v}_0\|_{C^\alpha(\mathcal{M})} + \|\bar{v}^{ext}\|_{L^\infty([0,1]\times\mathcal{M})})\\ &\leq Ce^{(s_1+\delta t_1)(\eta-\eta_{cr})}\|\bar{v}_0\|_{C^\alpha(\mathcal{M})} \ .\end{aligned}$$

$\tilde{V}_1(\tau)$ is the solution at time 1 of $(\partial_t - \mathbf{L}_{\eta_{cr}})\tilde{V} = h(\tau + \cdot)$ with initial data 0. We have

$$\begin{aligned}\|\tilde{V}_1(\tau)\|_{C^\alpha(\mathcal{M})} &\leq C\|f,g\|_{L^\infty([\tau,\tau+1]\times\mathcal{M})}\\ &\leq Ce^{(s_1+\delta(t_1-\tau))(\eta-\eta_{cr})}\,\|\bar{v}^{ext}\|_{L^\infty([\tau,\tau+1]\times\mathcal{M})} \ .\end{aligned}$$

$\tilde{V}_2(1-t_0)$ is the solution at time $1-t_0 \leq 1$ of $(\partial_t - \mathbf{L}_{\eta_{cr}})\tilde{V} = h(t-(1-t_0)+\cdot)$ with initial data 0. We have

$$\begin{aligned}\|\tilde{V}_2(1-t_0)\|_{C^\alpha(\mathcal{M})} &\leq C\|f,g\|_{L^\infty([t-(1-t_0),t]\times\mathcal{M})}\\ &\leq Ce^{(s_1+\delta(t_1-t))(\eta-\eta_{cr})}\,\|\bar{v}^{ext}\|_{L^\infty([t-1,t]\times\mathcal{M})} \ .\end{aligned}$$

With these estimates established, we can now apply the spectral projection and conclude

$$\tilde{v}^{int}(t) = \mathbf{S}_{\eta_{cr}}(t-1)\mathbf{P}_{\eta_{cr}}\tilde{V}_0 + \int_0^{t-1} \mathbf{S}_{\eta_{cr}}(t-1-\tau)\mathbf{P}_{\eta_{cr}}\tilde{V}_1(\tau)\,d\tau + \int_0^1 \mathbf{P}_{\eta_{cr}}\tilde{V}_2(1-t_0)\,dt_0 \ ,$$

hence

$$\begin{aligned}\|\tilde{v}^{int}(t)\|_{C^\alpha(\mathcal{M})} \leq\ & Ce^{t(c^\infty(\eta)-2\varepsilon)}\|\tilde{V}_0\|_{C^\alpha(\mathcal{M})}\\ &+ C\textstyle\int_0^{t-1} e^{(t-1-\tau)(c^\infty(\eta)-2\varepsilon)}\|\tilde{V}_1(\tau)\|_{C^\alpha(\mathcal{M})}\\ &+ C\sup_{t_0\in[0,1]}\|\tilde{V}_2(1-t_0)\|_{C^\alpha(\mathcal{M})} \ .\end{aligned}$$

The first and third term can immediately be estimated as $C\exp[t(c^\infty(\eta)-2\varepsilon)+(s_1+\delta t_1)(\eta-\eta_{cr})]\|\bar{v}_0\|_{C^\alpha(\mathcal{M})}$ and $C\exp[(s_1+\delta(t_1-t))(\eta-\eta_{cr})+c^\infty(\eta)t]\|\bar{v}\|_{L^\infty(\mathcal{M})}$ respectively, and for $t = t_1$, they are controlled (with one ε to spare in the first term) by $C\exp[t_1c^\infty(\eta) + s_1(\eta-\eta_{cr})]\|\bar{v}_0\|_{C^\alpha(\mathcal{M})}$. The second term is estimated similarly to be

$$\leq C\int_0^{t-1} \exp[(c^\infty(\eta)-2\varepsilon)(t-1-\tau)+(s_1+\delta(t_1-\tau))(\eta-\eta_{cr})+c^\infty(\eta)\tau]\,d\tau\|\bar{v}_0\|_{C^\alpha(\mathcal{M})} \ ,$$

which, for $t = t_1$, is $\leq C\exp[t_1c^\infty(\eta)+s_1(\eta-\eta_{cr})]\int_0^{t_1-1} e^{-\varepsilon(t_1-1-\tau)}\,d\tau\,\|\bar{v}_0\|_{C^\alpha(\mathcal{M})}$, and the integral of order $1/\varepsilon$ can be absorbed in the constant.

So we have proved

$$\|\tilde{v}^{int}(t_1)\|_{C^\alpha(\mathcal{M})} \leq Ce^{t_1c^\infty(\eta)+s_1(\eta-\eta_{cr})}\|\bar{v}_0\|_{C^\alpha(\mathcal{M})} \ ;$$

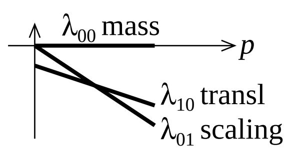
and in particular $|\tilde{v}^{int}(t_1, \boldsymbol{x}_1)| \le Ce^{t_1 c^\infty(\eta)+s_1(\eta-\eta_{cr})}$. With (88), this implies $|\bar{v}^{int}(t_1, \boldsymbol{x}_1)| \le Ce^{t_1 c^\infty(\eta)}$. The constant is uniform in (s_1, t_1), as long as s_1 is bounded away from 0, say $s_1 \ge 1$. But for $s < 1$, the estimate with $s_1 = 1$ still gives $|\bar{v}^{int}(t_1, \boldsymbol{x})| \le Ce^{t_1 c^\infty(\eta)}$. We review (86), which guaranteed $\|\bar{v}^{ext}(t_1)\|_{L^\infty} \le Ce^{c^\infty(\eta)t_1}\|\bar{v}_0\|_{L^\infty}$, and since the projection operator is bounded from L^∞ into itself, we also have $\|(\mathbf{P}_\eta(\chi\,\bar{v}^{ext}))(t_1)\|_{L^\infty} \le Ce^{c^\infty(\eta)t_1}\|\bar{v}_0\|_{L^\infty}$. In view of $\bar{v}(t_1, \boldsymbol{x}_1) = \bar{v}^{int}(t_1, \boldsymbol{x}_1) + (\mathbf{P}_\eta(\chi\bar{v}^{ext}))(t_1, \boldsymbol{x}_1)$, which we have from (90), we conclude

$$|\bar{v}(t_1, \boldsymbol{x}_1)| \le Ce^{c^\infty(\eta)t_1}\|\bar{v}_0\|_{C^\alpha}$$

uniformly in $(\boldsymbol{x}_1, t_1)$. This establishes $\|\bar{v}(t)\|_{L^\infty} \le Ce^{c^\infty(\eta)t}\|\bar{v}_0\|_{C^\alpha}$; and with another invocation of regularization, we can get the same estimate for the C^α norm on the left hand side. □

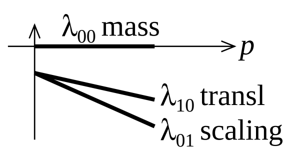
9. Proof of Theorem 1.1

Let ρ_0 be an initial density which satisfies the assumptions of Theorem 1.1. After scaling we may assume that its mass agrees with that of $u_B(\boldsymbol{x})$ for $B = 1$. We shall actually prove the asserted decay (10) in a norm $\|\cdot\|_{C^\alpha(\mathcal{M})}$ stronger than $\|\cdot\|_{L^\infty(\mathbf{R}^n)}$. As long as $m > m_1 = \frac{n-1}{n+1}$, the center of mass of ρ_0 is defined. We may shift ρ_0, such as to bring its center of mass to 0. We then claim that the statement of the theorem holds in particular by choosing $\boldsymbol{z} = 0$. The hypothesis $m > m_1$ is automatically satisfied for $n = 1$, and also whenever $m > m_n$. In the other cases, we either cannot define a center of mass or else will not bother to shift it, and we claim that in those cases, the theorem holds with any $\boldsymbol{z}$. For quick reference, here are the relevant eigenvalues of $\mathbf{L}$ in C^α (unweighted) in the case $n \ge 2$:



eigenvalues: $\lambda_{00} = 0$, $\lambda_{10} = -(p+n)$, $\lambda_{01} = -2p$

In the case $n = 1$ (where $m_n = 0$), the picture looks like this:



eigenvalues when $n = 1$: $\lambda_{00} = 0$, $\lambda_{10} = -(p+1)$, $\lambda_{01} = -2p$

In either case, the onset of the ($\ell = 0$ layer of the) essential spectrum is right at λ_{01}.

The evolution conserves mass and center of mass. In the variables of eqn. (21), these conservation laws amount to $\int(v-1)u_B\,d\boldsymbol{x} = 0$ and $\int(v-1)u_B\,\boldsymbol{x}\,d\boldsymbol{x} = 0$. In other words, using the inner product of $L^2_{u_B^m}$ makes $v-1$ orthogonal to the eigenfunction $v_{00} = u_B^{1-m}$ and, if $m > m_1$, to $v_{10} = ru_B^{1-m}Y_{1\mu}(\omega)$ for $\mu = 1, \dots, n$.

By (3), the statement of the theorem, with $\boldsymbol{z} = 0$, is equivalent to

$$\sup e^{-\lambda_{01} t} |v(\boldsymbol{x}, t) - 1| < \infty ,$$

recalling $\lambda_{01} = -2p$. Let us rewrite (21)–(22) as

(92)

$$\begin{aligned}(\partial_t - \mathbf{L})(v-1) = \ & \left(\frac{\partial^2}{\partial s^2} + \frac{2(n-1)}{\sinh 2s}\frac{\partial}{\partial s} + (\tanh s)^{-2}\Delta_{\mathbf{S}}\right)\left(\frac{v^m}{m} - v\right) \\ & + \left(n - \frac{2(m+1)}{1-m}\right)\tanh s \frac{\partial}{\partial s}\left(\frac{v^m}{m} - v\right) \\ & + \frac{4}{(1-m)^2}\left[n\frac{1-m}{2} - 1 + (\cosh s)^{-2}\right](mv - v^m + 1 - m) .\end{aligned}$$

With the trivial substitution $w = v - 1$, its structure becomes clearer in the form

$$(\partial_t - \mathbf{L})w = \tilde{\mathbf{L}}(f(w)w) \tag{93}$$

where $f(w) = ((1+w)^m - 1 - mw)/mw$ is analytic and vanishes at the origin, and where $\tilde{\mathbf{L}} = \mathbf{L} - \frac{2}{1-m}(\tanh s\, \partial_s + n - \frac{2}{1-m}\tanh^2 s)$ can be read off the right side of (92). We can apply Thm. 7.8. Let us write $w_j := w(jT) \in C^\alpha(\mathcal{M})$, for a constant T chosen sufficiently small for the contraction estimates of Chapter 7, specifically from Lemma 7.5, to apply. Also let $\mathbf{S}(t) := \exp[t\mathbf{L}]$ be the semigroup generated by the linear operator $\mathbf{L}$. We write

$$w_{j+1} = \mathbf{S}(T)w_j + g(w_j) , \tag{94}$$

where

$$C^\alpha(\mathcal{M}) \ni w \longmapsto g(w) \in C^\alpha(\mathcal{M})$$

is smooth and vanishes at the origin together with its derivative. More specifically, since the map $w_j \mapsto w_{j+1} =: F(w_j)$ is smooth according to Thm. 7.8, and $F(0) = 0$, $DF(0) = \mathbf{S}(T)$, we can write $w_{j+1} = \int_0^1 \frac{d}{d\sigma} F(\sigma w_j)\, d\sigma = \mathbf{S}(T)w_j + [\int_0^1 (DF(\sigma w_j) - \mathbf{S}(T))\, d\sigma]w_j =: \mathbf{S}(T)w_j + G(w_j)w_j$. Here G is smooth with values $G(w)$ being bounded linear maps $C^\alpha \to C^\alpha$, and $G(0) = 0$. This gives the claim.

In the case $m > m_n$, let $\mathbf{P}$ be the $\mathbf{S}$-invariant projection to the $L^2_{u_B^m}(\mathbf{R}^n)$ complement of the eigenspaces for $\lambda_{00} = 0$ and $\lambda_{10} = -(p+n)$. In the case $m \le m_n$, let $\mathbf{P}$ be the $\mathbf{S}$-invariant projection in $C^\alpha(\mathcal{M})$ to the complement of the eigenspace for $\lambda_{00} = 0$. It is well-defined according to Theorem 8.5. We have seen that the w_j are in the range of $\mathbf{P}$, and therefore $\|\mathbf{S}(t)w_j\| \le Ce^{\lambda_{01}t}\|w_j\|$ with a constant C independent of t, where $\|\cdot\|$ refers to $\|\cdot\|_{C^\alpha(\mathcal{M})}$.

We introduce a small quantity $r < 1$ and assume $\|w_0\|$ is less than r divided by a constant supplied from Lemma 7.6. This lemma then ensures that $\|w_j\|_{L^\infty} < r$ for all r. Lemma 7.5 applies to our operators $\mathbf{L}$, $\tilde{\mathbf{L}}$ and nonlinearity $f(w)w$, and it controls $\|g(w)\|_{C^\alpha} \le K\|w\|_{C^\alpha}\|w\|_{L^\infty}$ for $\|w\|_{L^\infty} < r$. Now K stays fixed, while we still may impose further smallness requirements on r. We denote by $\lambda < 0$ whatever spectral gap applies to the parameter m under consideration. Iterating (94) gives

$$(95)\qquad \begin{aligned} &w_k = \mathbf{S}(T)^k w_0 + \textstyle\sum_{j=1}^{k} \mathbf{S}(T)^{k-j} g(w_{j-1}) \\ &\|w_k\|_{C^\alpha} \le Ce^{\lambda kT}\|w_0\|_{C^\alpha} + \textstyle\sum_{j=1}^{k} CKe^{\lambda(k-j)T}\|w_{j-1}\|_{C^\alpha}\,\|w_{j-1}\|_{L^\infty}\ . \end{aligned}$$

Here, C is the constant obtained from Thm. 8.5. We know that

$$\|w_{j-1}\|_{L^\infty} \le \min\{r, \|w_{j-1}\|_{C^\alpha}\} \le r^{1/2}\|w_{j-1}\|_{C^\alpha}^{1/2}$$

and will prove inductively that $\|w_k\|_{C^\alpha} \le 2Ce^{\lambda kT}\|w_0\|_{C^\alpha}$. The start being trivial, the induction step in (95) follows if we can ascertain that

$$1 + \sum_{j=1}^{k} Ke^{-\lambda jT} r^{1/2} (2C)^{3/2} e^{3\lambda(j-1)T/2} \le 2\ .$$

By convergence of the geometric series $\sum^\infty e^{\lambda jT/2}$, this requirement can indeed be met by choosing r sufficiently small.

Apart from the adjustment of $\mathbf{P}$, the proof works the same in the full range of m (with $\lambda + \varepsilon$ in case $m = m_2$, per Thm. 8.5).

Finally, let's deal with the case $m = m_2$ under the moment hypothesis (9), which, in view of (28) and (30), amounts to $\int_{\mathcal{M}} w_0(y)^2\, d\mu < \infty$. (Recall $d\mu$ denotes the Riemannian volume element on $\mathcal{M}$.) We will argue shortly that

$$(96)\qquad \|w(t)\|_{L^2(\mathcal{M})} \le Ce^{\lambda t}\|w_0\|_{L^2(\mathcal{M})}\ .$$

We also note that $\|\mathbf{S}(T)w\|_{C^\alpha(\mathcal{M})} \le C\|w\|_{L^2(\mathcal{M})}$ by regularization: indeed, this is a regularization estimate for a uniformly parabolic linear PDE in each coordinate chart U_l separately, with the $\|w\|_{L^2(U_l)}$ all being trivially estimated by the global $\|w\|_{L^2(\mathcal{M})}$; recall that the distortion between the metric on the U_l and the euclidean metric is uniformly bounded.

From these two ingredients we conclude, in view of the close-to-sharp $O(e^{(\lambda+\varepsilon)t})$ decay already established, that

$$\|w(t+T)\|_{C^\alpha} \le \|\mathbf{S}(T)w(t)\|_{C^\alpha} + \|g(w(t))\|_{C^\alpha} \le Ce^{\lambda t}\|w_0\|_{L^2(\mathcal{M})} + Ce^{2(\lambda+\varepsilon)t}\ ,$$

and the claim follows since $2(\lambda+\varepsilon) < \lambda$. So all we have to establish is (96), and we do it by direct calculation from the PDE.

Letting $h(w) = \frac{(1+w)^m - 1}{m} = w + O(w^2)$ and $H(w) = \frac{1}{2}w^2 + O(w^3)$ its antiderivative, we use $E(t) := \int_{\mathcal{M}} H(w(t))\, d\mu$ as a proxy for the L^2-norm. For the technical reason of properly caring about a flux term at infinity, we will also need $E(r,t) := \int_{B(r)} H(w(t))\, d\mu$, with integration over the euclidean r-ball around the origin (equivalently the geodesic $\operatorname{arsinh} r$-ball). Formally, we calculate

$$\partial_t E(t) = \int_{\mathcal{M}} h(w) w_t \, d\mu =$$

$$= \int_{\mathcal{M}} h(w)\mathbf{L}h(w)\, d\mu + \frac{2}{1-m}\int_{\mathcal{M}} h(w)(\boldsymbol{x}\cdot\boldsymbol{\nabla})(w-h(w))\, d\mu + \tag{97}$$

$$+\frac{2}{1-m}\int_{\mathcal{M}}\Big(n - \frac{2u_B^{1-m}}{1-m}|\boldsymbol{x}|^2\Big)(w-h(w))\, h(w)\, d\mu \,.$$

Returning to the euclidean volume element $d^n\boldsymbol{x} = (1+|\boldsymbol{x}|^2)^{n/2} d\mu$, we want to integrate the second term of this sum by parts to avoid derivatives of w. With $h(w)\boldsymbol{\nabla}(w-h(w)) = \boldsymbol{\nabla}(H(w) - \frac{1}{2}h(w)^2)$, we obtain for $n \neq 2$ (with routine modifications in case $n = 2$)

$$\begin{aligned}\partial_t E(r,t) = \int_{B(r)} h(w) w_t \, d\mu = \int_{B(r)} h(w)\mathbf{L}h(w)\, d\mu + \\ +\frac{2}{(1-m)(2-n)}\int_{B(r)} \boldsymbol{\nabla}\cdot\Big(\big(H(w)-\tfrac{1}{2}h(w)^2\big)\,\boldsymbol{\nabla}(1+|\boldsymbol{x}|^2)^{-n/2+1}\Big)\, d^n\boldsymbol{x} \\ -\frac{2}{(1-m)(2-n)}\int_{B(r)} \big(H(w)-\tfrac{1}{2}h(w)^2\big)\,\Delta(1+|\boldsymbol{x}|^2)^{-n/2+1}\, d^n\boldsymbol{x} \\ +\frac{2}{1-m}\int_{B(r)}\Big(n - \frac{2u_B^{1-m}}{1-m}|\boldsymbol{x}|^2\Big)(w-h(w))\, h(w)\, d\mu \,.\end{aligned}$$

The last integrand is estimated as $O(w^3)$, or $O(wH(w)) = e^{(\lambda+\varepsilon)t}O(H(w))$. The second last term is integrable even over $\mathcal{M}$, or can be estimated in the same manner as the last term. We obtain (with dS the *Riemannian* surface element)

$$\partial_t E(r,t) \le \int_{B(r)} h\mathbf{L}h\, d\mu + C_1 e^{(\lambda+\varepsilon)t}\int_{\partial B(r)} |w(t)|^2\, dS + C_2 e^{(\lambda+\varepsilon)t} E(r,t)\,.$$

Letting $\gamma(t) := -\int_0^t C_2 e^{(\lambda+\varepsilon)t}\, dt$, we get

$$\begin{aligned}\partial_t(e^{\gamma(t)}E(r,t)) \le e^{\gamma(t)}\int_{B(r)} h(w(t))\mathbf{L}h(w(t))\, d\mu + \\ +C_1 e^{\gamma(t)+(\lambda+\varepsilon)t}\int_{\partial B(r)} |w(t)|^2\, dS\end{aligned} \tag{98}$$

and therefore

$$e^{\gamma(t)}E(r,t) - E(r,0) \le \int_0^t \int_{B(r)} h(w(t))\mathbf{L}h(w(t))\, d\mu\, dt + C\|w(t)\|^2_{L^\infty}\,.$$

Now from $\|w_0\|^2_{L^2(\mathcal{M})} < \infty$, we get $E(r,0) \to E(0) < \infty$ as $r \to \infty$. If $w(t) \in L^2$, we conclude $\int_{B(r)} h(w(t))\mathbf{L}h(w(t))\, d\mu \to \int_{\mathcal{M}} h(w(t))\mathbf{L}h(w(t))\, d\mu \le \lambda h(w(t))^2\, d\mu \le 0$. Without *assuming* $w(t) \in L^2$, we still maintain the inequality $\limsup_{r\to\infty}\int_{B(r)} h(w(t))\mathbf{L}h(w(t))\, d\mu \le 0$ by approximation, and this guarantees the finiteness of $\lim_{r\to\infty} e^{\gamma(t)}E(r,t)$, and therefore $w(t) \in L^2(\mathcal{M})$ for every t.

Knowing this, we can actually justify the formal calculation (97) and proceed with the same estimates and integration by parts done for $E(r,t)$ to obtain

$$\partial_t E(t) \le \lambda \int_{\mathcal{M}} h(w(t))\mathbf{L}h(w(t))\, d\mu + Ce^{(\lambda+\varepsilon)t}E(t) \le (2\lambda + Ce^{(\lambda+\varepsilon)t})E(t) \ .$$

The estimate $E(t) \le Ce^{2\lambda t}E(0)$ follows from this; and then immediately the same estimate for $\|w\|^2_{L^2(\mathcal{M})}$, hence (96).

10. Asymptotic estimates in weighted spaces: The case $m < \frac{n}{n+2}$

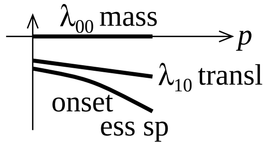
In the previous section, we have seen in particular that for $m < m_2 = \frac{n}{n+2}$, we get a convergence rate of $O(1/\tau)$, and no better, to the appropriately shifted Barenblatt with respect to the relative L^∞ norm. This is the same rate as was obtained by Kim and McCann [31] via Newton potentials; a worse rate which however extends to a wider range of m was later obtained by the quintet [7]. The slightly larger spectral gap in the weighted Hilbert space of [20] (where λ_{01} was not an eigenvalue any more for $m < m_2$ and the continuous spectrum started further below) was not actually expected to be dynamically effective from that setting, because the linearization formalism relied on the existence of second moments, in contrast to the reality in the case $m < m_2$.

In the present setting, the absence of a weight in C^α has changed the spectrum in comparison to [20]. To recover that spectrum, we need to study $\mathbf{L}$ in $C^\alpha_{\eta_{cr}}$, or equivalently, the operator $\mathbf{L}_{\eta_{cr}}$ in C^α. Since $\eta_{cr} = \frac{p}{2} - 1 < 0$ for $m < m_2$, this means we need to study the case where initial data deviate from Barenblatt less than what would automatically be achieved by Vázquez' result (8). For initial data with this special tail behavior, we can indeed get improved convergence rates from a study of the conjugated operator. These better convergence rates will apply to *appropriately weighted* norms.

We are careful to point out that the *nonlinear* evolution was constructed for unweighted norms only: this is in accordance with the fact that, in the relative L^∞ setting, unweighted norms keep a uniform distance from the singularity $u = 0$ of the Fast Diffusion Equation (1) and allow, on a technical level, uniformly parabolic estimates on the whole space. (However, in the present case $m < m_2$, Thm. 7.8 does carry over, as noted near the end of Section 7.) In contrast, the linearized flow can be studied with the same ease in weighted and unweighted spaces.

Let's have a quick look at the spectrum of $\mathbf{L}_{\eta_{cr}}$ in this range of parameters, where the only eigenvalues are $\lambda_{00} = 0$ and $\lambda_{10} = -(p+n)$, and the onset of the essential spectrum is $\lambda_0^{\text{cont}} = -(\frac{p}{2}+1)^2$:

In the case $n = 1$, one has $m_1 = 0$, so Barenblatt always has 1st moments and we can shift the center of mass. We also need to do this because λ_{10} is above the onset of the essential spectrum.



eigenvalues of $\mathbf{L}_{\eta_{cr}}$ when $n = 1$:

$$\lambda_{00} = 0$$
$$\lambda_{10} = -(p+1)$$
$$\lambda_0^{\text{cont}} = -(\tfrac{p}{2}+1)^2$$

In the case $n \geq 2$ the center of mass may or may not be defined (depending on whether $m > m_1$ or $m \leq m_1$), but there is no need to shift the center of mass, because the intersection of $-(\frac{p}{2}+1)^2$ with $\lambda_{10} = -(p+n)$ occurs at $p = 2\sqrt{n-1} \geq 2$, hence λ_{10} is below the onset of the essential spectrum (ranging from -1 to -4 monotonically) in the whole parameter range $m_0 < m < m_2$.

As we have resolved to work in the space $C^\alpha_{\eta_{cr}}$, we let $\tilde{w} = (\cosh s)^{-\eta_{cr}} w$, and now equation (93) becomes

$$(\partial_t - \mathbf{L}_{\eta_{cr}})\tilde{w} = \tilde{\mathbf{L}}_{\eta_{cr}}\big(f(1 + (\cosh s)^{\eta_{cr}}\tilde{w})\tilde{w}\big) \tag{99}$$

Since $\eta_{cr} < 0$, we can estimate $\|(\cosh s)^{\eta_{cr}}\tilde{w}\|_{C^\alpha(\mathcal{M})} \leq c\|\tilde{w}\|_{C^\alpha(\mathcal{M})}$, and this is why Theorem 7.8 carries over to this case along with its proof, as remarked at (77). We now can repeat the proof from Section 9 almost verbatim. We conclude: If the initial data w_0 satisfy

$$(\cosh s)^{-\eta_{cr}} w_0 \in L^\infty(\mathcal{M}) \; , \tag{100}$$

then

$$\sup_{t>1} e^{(-\lambda_0^{\text{cont}}-\varepsilon)t}\|(\cosh s)^{-\eta_{cr}} w(t)\|_{L^\infty} < \infty \tag{101}$$

for every $\varepsilon > 0$. (The same statement works also for the C^α-norm.) Here $-\lambda_0^{\text{cont}} = (\frac{p}{2}+1)^2 = (\frac{1}{1-m}+1-\frac{n}{2})^2$. If we put $L^\infty \cap L^2$ in the hypothesis (100), we get conclusion (101) without the ε.

We therefore get the following:

Theorem 10.1 (Fine Asymptotics for $m < m_2$, with restricted initial data)**.** *Assume $n \geq 2$ and $m \in]m_0, m_2[\, = \,]\frac{n-2}{n}, \frac{n}{n+2}[$, or else $n = 1$ and $0 < m < m_2 = \frac{1}{3}$. Further assume that the mass of ρ_0 is one, and if $n = 1$ also that the center of mass of ρ_0 is 0. Let $\delta := \frac{p/2-1}{p+n} = \frac{m}{2} - \frac{n(1-m)}{4}$ (note $\delta < 0$). If $\rho_B(0,\cdot)^\delta|1 - \rho_0/\rho_B(0,\cdot)| \in L^\infty$, then, for every $\varepsilon > 0$*

$$\limsup_{\tau\to\infty} \sup_{\boldsymbol{y}} \tau^{\gamma-\varepsilon} \left(\frac{\rho_B(\tau,\boldsymbol{y})}{\rho_B(\tau,0)}\right)^\delta \left|\frac{\rho(\tau,\boldsymbol{y})}{\rho_B(\tau,\boldsymbol{y})} - 1\right| < \infty \tag{102}$$

where $\gamma = \frac{1}{2p}(\frac{p}{2}+1)^2$. Likewise, we get the same conclusion without the ε, provided we add the hypothesis that $\rho_B(0,\cdot)^\delta|1 - \rho_0/\rho_B(0,\cdot)| \in L^\infty$ is also square integrable with respect to the measure $(1+|\boldsymbol{x}|^2)^{-n/2}\, d^n\boldsymbol{x}$.

A graph for γ, δ in dependence on m can be found in Figure 1. The tremendous improvement over the $O(1/\tau)$ convergence rate established by Kim and McCann [31] and in our present Thm. 1.1 depends on the strong decay assumptions for the relative deviation of the initial data.

Also note that our Theorem 10.1 is a generalization of the comparison estimates by Vázquez (of which we use a modified form as Lemma 7.6). Namely this comparison asserted that if a solution is close to Barenblatt in relative L^∞ norm, then it stays close to Barenblatt in this norm. Here we have shown the same in a stronger norm, showing that proximity to Barenblatt in a norm that enforces an even closer fit with Barenblatt in the tails, is also preserved under the dynamics.

11. Higher asymptotics in weighted spaces: The case $m > \frac{n}{n+2}$. Proof of Theorem 1.2 and its corollaries.

This section is devoted to proving Theorems 1.2 and 11.1. When applicable, the latter extends the conclusions of the former to decay rates Λ outside the range $]2\lambda_{01}, \lambda_{01}]$. For an example of its applicability, see Remark 1.9. The restriction $\Lambda > \lambda + \lambda_{01}$ appearing in this refinement arises from a limitation of our method, which relies on a weighted control of the quadratic term in the nonlinearity by a product of a weighted norm (103) decaying at rate λ and an unweighted norm decaying at the rate λ_{01} (given by Theorem 1.1).

It seems convenient to repeat Theorem 1.2 here:

Theorem 1.2 (Higher-order asymptotics in weighted Hölder spaces)**.** *Fix $p = 2(1-m)^{-1} - n > 2$ (equivalently $m \in]m_2, 1[$) and $\Lambda \in [\lambda_0^{\mathrm{cont}}, \lambda_{01}] = [-(\frac{p}{2}+1)^2, -2p]$ subject to the condition $2\lambda_{01} < \Lambda$. If $u(t, \boldsymbol{x})$ is a solution to* (4) *with center of mass and $\lim_{t\to\infty} \|u(t,\boldsymbol{x})/u_B(\boldsymbol{x}) - 1\|_{L^\infty(\mathbf{R}^n)} = 0$ both vanishing, then there exist a sequence of polynomials $(u_{\ell k}(\boldsymbol{x}))_{\ell k}$, each element of which either vanishes or has degree $\ell + 2k \in]1, \frac{p}{2}+1[$, such that*

$$\left\| \frac{(B+|\boldsymbol{x}|^2)(u(t,\boldsymbol{x})/u_B(\boldsymbol{x}) - 1) - \sum_{\Lambda<\lambda_{\ell k}<0} u_{\ell k}(\boldsymbol{x}) e^{\lambda_{\ell k} t}}{(B+|\boldsymbol{x}|^2)^{\left(p+2-\sqrt{(p+2)^2+4\Lambda}\right)/4}} \right\|_{C^\alpha(\mathcal{M})} = O(e^{\Lambda t}) \tag{14}$$

as $t \to \infty$, where the sum is over non-negative integers $k, \ell \in \mathbf{N}$ for which $\lambda_{\ell k}$ defined by (11) *lies in the interval $]\Lambda, 0[$, and for which $\ell \leq 1$ if $n = 1$. The functions $u_{\ell k}(\boldsymbol{x})/(B+|\boldsymbol{x}|^2)$ lie in the $\lambda_{\ell k}$ eigenspace of the linear operator* (23) *on $C^\alpha_{\eta_{cr}}(\mathcal{M})$, and the norm $\|\cdot\|_{C^\alpha(\mathcal{M})} \geq \|\cdot\|_{L^\infty(\mathbf{R}^n)}$ is defined by* (33).

Theorem 11.1 (Higher-order asymptotics in weighted Hölder spaces)**.** *Fix $p = 2(1-m)^{-1} - n > 2$ and $\Lambda \in [\lambda_0^{\mathrm{cont}}, \lambda_{01}]$, and choose $\lambda \leq \lambda_{01}$ such that $\lambda + \lambda_{01} < \Lambda$. If $u(t,\boldsymbol{x})$ satisfies the hypotheses of Theorem 1.2, and either*

$\lambda = \lambda_{01}$ *or else we have the stronger hypothesis*

$$\limsup_{t\to\infty} e^{-\lambda t} \left\| \frac{u(t,\boldsymbol{x})/u_B(\boldsymbol{x}) - 1}{(B + |\boldsymbol{x}|^2)^{\frac{p}{2}-1-\sqrt{(\frac{p}{2}+1)^2+\Lambda}}} \right\|_{C^\alpha(\mathcal{M})} < \infty\,, \tag{103}$$

then the conclusion (14) *of Theorem 1.2 remains true and* $u_{\ell k} = 0$ *for* $\lambda_{\ell k} > \lambda$.

Proof of Thm. 1.2. In case $\Lambda = \lambda_{01} = -2p$, the weights cancel in (14), and the result was established in Section 9 during the proof of Theorem 1.1. The improvement to $\Lambda < \lambda_{01}$ is based on the same principle as Section 10, except this time $\eta_{cr} > 0$, and the situation is dual to the previous one: we have a chance to obtain faster decay by introducing a weight which relaxes the strength of the norm as $s \to \infty$. We need not conjugate all the way to η_{cr}, but instead have the flexibility to choose $\eta \in [0, \eta_{cr}]$ to balance the error term in Theorem 1.2 against the severity of this relaxation. Indeed, the choice

$$\eta = \eta_{cr} - \sqrt{\Lambda - \lambda_0^{\mathrm{cont}}} \tag{104}$$

makes the radius $\lambda_{0,\eta}^{\mathrm{cont}}$ of the essential spectrum of $\mathbf{L}_\eta$ given by Theorem (8.2) coincide with $\Lambda \in [\lambda_0^{\mathrm{cont}}, \lambda_{01}]$. Choosing $\eta > \eta_{cr}$ would only prove the statement with a weaker norm, without giving better rates.

Conjugating with $\eta > 0$ however requires a reconsideration of the proof of Theorem 7.8, because a priori, uniform parabolicity cannot be expected for weights that allow growth relative to the Barenblatt. The redeeming feature will first be that we do retain the *unweighted* relative L^∞ hypothesis on the initial data; then since the nonlinearity enters through a term $f(w)w$, after conjugacy $\tilde{w} = (\cosh s)^{-\eta} w$, this term is still $f(w)\tilde{w}$, and the unweighted estimates continue to control the nonlinearity, whereas linear theory applies to the weighted estimates. We now carry this out in detail:

We again study equation (93) for initial data $w_0 \in C^\alpha$ (in particular bounded), but using weighted C^α_η norms with the more permissive weight $\eta > 0$ from (104). Letting $\tilde{w} := (\cosh s)^{-\eta} w$, Eq. (93) becomes

$$(\partial_t - \mathbf{L}_\eta)\tilde{w} = \mathbf{Ł}_\eta(f(w)\tilde{w}) \tag{105}$$

with $\mathbf{Ł}_\eta = \mathbf{L}_\eta - \frac{2}{1-m}(\tanh s\, \partial_s + n + (\eta - \frac{2}{1-m}) \tanh^2 s)$ and with the same $f(w)$ as before. We can use the fact, from the unweighted norm, that a solution $w \in C^\alpha(\mathcal{M}_T)$ to (93) exists; in particular the conjugated $\tilde{w}(\cdot, t)$ will also be in $C^\alpha(\mathcal{M}_T)$ still (even in $C^\alpha_{-\eta}$).

We obtain an analog to Lemma 7.5, but stated specifically for the FDE on the cigar manifold, because the a priori estimate on relative L^∞ norms from Lemma 7.6 is used: namely, we claim

Lemma 11.2 (Linear approximation of nonlinear semiflow; weighted norm). *Let* $\bar{\tilde{w}}$ *solve the homogeneous linear equation* (54)*, namely* $(\partial_t - \mathbf{L}_\eta)\tilde{w} = 0$ *for initial data* $\tilde{w}_0$*, where* $\mathbf{L}_\eta$ *is given by* (25) *and can be written, in local* *coordinates, as* $\mathbf{L}_\eta\tilde{w} = \partial^2_{ij}(a^{ij}\tilde{w}) + \partial_i(b^i\tilde{w}) + c\tilde{w}$*. Let* $\tilde{w}$ *solve the quasilinear equation* (105)*, for the same initial data* $\tilde{w}_0$*, and* $w = (\cosh s)^\eta\tilde{w}$*. In local coordinates, we can write* $\mathbf{L} = \mathbf{L} + \partial_i \circ \tilde{b}^i + \tilde{c}$*, and* f *is a smooth function from an interval about 0 into* $\mathbf{R}$ *satisfying* $f(0) = 0$*. The coefficients are smooth.*

Then, for sufficiently short time T*, there exists a constant* K *(uniform as* $T \to 0$*) such that we have the estimate*

$$\|\tilde{w} - \bar{\tilde{w}}\|_{C^\alpha(\mathcal{M}_T)} \leq K\|w\|_{L^\infty(\mathcal{M}_T)}\|\tilde{w}\|_{C^\alpha(\mathcal{M}_T)} \tag{106}$$

and from it the time-step estimate (with a different K*):*

$$\|\bar{\tilde{w}}(T) - \tilde{w}(T)\|_{C^\alpha(\mathcal{M})} \leq K\|\tilde{w}_0\|_{C^\alpha(\mathcal{M})}\, \|w_0\|_{L^\infty(\mathcal{M})}\ . \tag{107}$$

Proof of the Lemma. The proof is modeled right after the proof of Lemma 7.5. We can basically copy equations (71) and (72) almost verbatim, the only changes being that all w's have tildes now, except the ones inside $f(\cdot)$, and the insignificant fact that coefficients implicitly depend on the conjugacy parameter η (not to be confused with the partition of unity η_l). The proof ensues as before, with the estimate $\|f(w)\tilde{w}\|_{C^\alpha(\mathcal{M}_T)} \leq K\|f(w)\|_{L^\infty(\mathcal{M}_T)}\, \|\tilde{w}\|_{C^\alpha(\mathcal{M}_T)}$ provided by Corollary 5.9. □

Proof of Thm. 1.2 continued: As before in Ch. 9, we write $\tilde{w}_j := \tilde{w}(jT) \in C^\alpha(\mathcal{M})$, where T is sufficiently small for Lemma 11.2 to apply; and we write $\mathbf{S}_\eta(t) := \exp(t\mathbf{L}_\eta)$ for the semigroup. We continue to use Thm. 7.8 for the unconjugated flow and copy from (94) and the paragraph following it that

$$w_{j+1} = \mathbf{S}(T)w_j + G(w_j)w_j\ .$$

After conjugating, this becomes

$$\tilde{w}_{j+1} = \mathbf{S}_\eta(T)\tilde{w}_j + G_\eta(w_j)\tilde{w}_j\ , \tag{108}$$

where G_η is a smooth function on C^α with values $G_\eta(w) = (\cosh s)^{-\eta} \circ G(w) \circ (\cosh s)^\eta$ being bounded linear maps on $C^\alpha_{-\eta}(\mathcal{M})$. The estimate (70) guarantees that $G_\eta(w_j)$ is also a bounded linear map on $C^\alpha(\mathcal{M})$ (namely with norm $\leq K\|w_j\|_{L^\infty}$).

From Theorem 8.2, the only spectrum of $\mathbf{L}_\eta$ closer to zero than $\lambda^{\text{cont}}_{0,\eta} = \Lambda$ consist of eigenvalues (11) of finite multiplicity and indexed by non-negative integers $\ell, k \in \mathbf{N}$ such that $\ell + 2k < p/2 + 1 - |\eta - \eta_{cr}|$. Enumerate the eigenvalues $\lambda_{\ell k}$ which lie in the range $]\Lambda, 0]$ by $\Lambda_l \leq \Lambda_{l-1} \leq \cdots \leq \Lambda_1$, counting them with multiplicity. Set $\Lambda_{l+1} = \Lambda$ by convention, whether or not this is an eigenvalue.

Before continuing the analysis of (110), a brief remark about the heuristics of the proof strategy seems useful: We are not constructing a 'slow' or 'pseudo-center' manifold corresponding to the eigenvalues Λ_i using invariant manifold methods with respect to which one would naturally measure convergence rates of the transversal 'fast' dynamics. Rather we assess convergence rates in terms of distances to the linear eigenspaces using spectral projections. This is appropriate since the spectral ratio Λ/λ_{01} is less than 2, the order of the nonlinearity. For larger spectral ratios, the deviation of the slow manifold from its tangential space, rather than the eigenvalues alone, would indeed dominate the asymptotic behavior of the 'fast' dynamics transversal to the slow manifold. This is also the reason why products of eigenfunctions and linear combinations of eigenvalues do not occur in the statement of the theorem. (Fig. 4 explains this heuristics in a simple 2D model.)

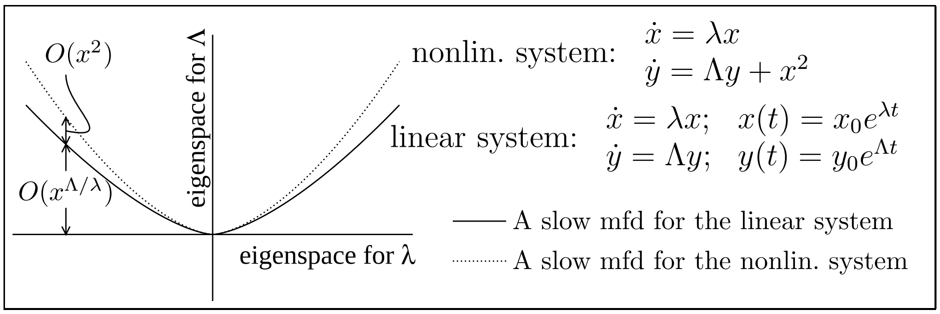


FIGURE 4. *A simple 2D model illustrating the heuristics of working with linear eigenspaces with $\Lambda < \lambda < 0$ in Thm. 1.2, rather than slow manifolds. Linear asymptotics is $x_0 e^{\lambda t} + O(e^{\Lambda t})$ (solid curve). Nonlinear asymptotics (dotted curve) should involve $e^{2\lambda t}$ etc., which gets absorbed into $O(e^{\Lambda t})$ if $\Lambda/\lambda < 2$.*

Returning to (108), let $\mathbf{Q}^{(i)}$ be the spectral projection onto the one-dimensional eigenspace corresponding to the eigenvalue Λ_i of $\mathbf{L}$. Take $\mathbf{Q} = \sum_{i=1}^{l} \mathbf{Q}^{(i)}$, and let $\mathbf{P}$ be the complementary projection.

Setting $\mathbf{Q}_\eta^{(i)} = (\cosh s)^{-\eta} \circ \mathbf{Q}^{(i)} \circ (\cosh s)^{\eta}$ and $\mathbf{P}_\eta = (\cosh s)^{-\eta} \circ \mathbf{P} \circ (\cosh s)^{\eta}$, we obtain

$$\mathbf{P}_\eta \tilde{w}_{j+1} = \mathbf{S}_\eta(T)(\mathbf{P}_\eta \tilde{w}_j) + (\mathbf{P}_\eta \circ G(w_j))\tilde{w}_j \ .$$

By iteration, we obtain inductively

$$\mathbf{P}_\eta \tilde{w}_k = \mathbf{S}_\eta(T)^k(\mathbf{P}_\eta \tilde{w}_0) + \sum_{j=0}^{k-1} \mathbf{S}_\eta(T)^{k-1-j}(\mathbf{P}_\eta \circ G(w_j))\tilde{w}_j \ .$$

We set $\vartheta_{l+1} := \exp[\Lambda_{l+1}T]$, so $\|\mathbf{S}_\eta(T)^k\| \le C\vartheta_{l+1}^k$ on the range of $\mathbf{P}_\eta$. We also use what we have already established in Thm. 1.1, namely that $\|w_j\|_{C^\alpha} \le C\vartheta^j$ with $\vartheta := \exp(\lambda_{01}T)$, with a similar estimate following for $\tilde{w}_j$. Then we conclude

$$\|\mathbf{P}_\eta \tilde{w}_k\|_{C^\alpha(\mathcal{M})} \le C\vartheta_{l+1}^k \Big(\|\mathbf{P}_\eta \tilde{w}_0\|_{C^\alpha(\mathcal{M})} + \sum_{j=0}^{k-1} \vartheta_{l+1}^{-j-1} K C^2 \vartheta^{2j} \Big) \ . \tag{109}$$

Since $\vartheta^2/\vartheta_{l+1} < 1$ (here we use $2\lambda_{01} < \Lambda$), the parenthesis is bounded independently of k. So we get

$$\|\mathbf{P}_\eta \tilde{w}_k\|_{C^\alpha(\mathcal{M})} \leq c_2 \vartheta_{l+1}^k \,. \tag{110}$$

A similar estimate can be made for the complementary projection

$$\mathbf{Q} = \sum_{i=1}^{l} \mathbf{Q}^{(i)} \,.$$

On the Λ_i eigenspace, $\mathbf{S}_\eta(T)$ operates as the scalar $\vartheta_i = \exp[\Lambda_i T]$, hence

$$\vartheta_i^{-k} \mathbf{Q}_\eta^{(i)} \tilde{w}_k = \mathbf{Q}_\eta^{(i)} \tilde{w}_0 + \sum_{j=0}^{k-1} \vartheta_i^{-j-1} \mathbf{Q}_\eta^{(i)} G(w_j) \tilde{w}_j$$

leads to a 2-sided Cauchy sequence estimate

$$\left\| \frac{\mathbf{Q}_\eta^{(i)} \tilde{w}_{k+q}}{\vartheta_i^{k+q}} - \frac{\mathbf{Q}_\eta^{(i)} \tilde{w}_k}{\vartheta_i^k} \right\|_{C^\alpha(\mathcal{M})} \leq \left(\vartheta^2/\vartheta_i\right)^k \sum_{j=0}^{q-1} c_3 \vartheta_i^{-1} \left(\vartheta^2/\vartheta_i\right)^j .$$

We conclude $\tilde{v}_i := \lim_{k\to\infty} \mathbf{Q}_\eta^{(i)} \tilde{w}_k / \vartheta_i^k$ exists, and $\tilde{v}_i$ must be an eigenvector of $\mathbf{L}_\eta$ in the Λ_i eigenspace. The convergence rate follows from this estimate:

$$\left\|\mathbf{Q}_\eta^{(i)} \tilde{w}_k - \tilde{v}_i \vartheta_i^k\right\|_{C^\alpha(\mathcal{M})} \leq c_3 (\vartheta^2)^k \,. \tag{111}$$

Returning to the unconjugated functions, with $v_i = (\cosh s)^\eta \tilde{v}_i$, we use from Theorems 8.1 and 8.2, that this eigenvector $v_i(\boldsymbol{x}) = u_i(\boldsymbol{x})/(B + |\boldsymbol{x}|^2)$ is given by a degree $\ell + 2k'$ polynomial $u_i(\boldsymbol{x})$, and $\Lambda_i = \lambda_{\ell k'}$. Since $u_B(\boldsymbol{x})$ has the same mass and the same center of mass as the initial data by hypothesis, we have $u_{00}(\boldsymbol{x}) = 0 = u_{10}(\boldsymbol{x})$ as remarked before equation (92), hence $\ell + 2k > 1$.

The last two inequalities combine with the identity $1 = \mathbf{P} + \sum_{i=1}^{l} \mathbf{Q}^{(i)}$ to yield

$$\left\|\frac{(B + |\boldsymbol{x}|^2) w_k - \sum_{i=1}^{l} \vartheta_i^k u_i}{(B + |\boldsymbol{x}|^2)^{1+\eta/2}}\right\|_{C^\alpha(\mathcal{M})} = \left\|\frac{w_k - \sum_{i=1}^{l} \vartheta_i^k v_i}{(\cosh s)^\eta}\right\|_{C^\alpha(\mathcal{M})} \leq c_4 \vartheta_{l+1}^k \,.$$

Our choices $\eta = \eta_{cr} - \sqrt{\Lambda - \lambda_0^{\text{cont}}} = \frac{p}{2} - 1 - \sqrt{\Lambda + (\frac{p}{2} + 1)^2}$ and $\vartheta_i = \exp[\Lambda_i T]$ for T sufficiently large convert this into the desired estimate (14). This concludes the proof of Thm. 1.2. □

Proof of Thm. 11.1. Since the case $\lambda = \lambda_{01}$ is Thm. 1.2, we assume $\lambda < \lambda_{01}$ and hypothesis (103). The proof of Thm. 1.2 carries over with only minor changes:

We keep conjugating with the η from (104) and instead of using $\|\tilde{w}_j\| \leq C \exp[\lambda_{01} T j] =: \vartheta^j$ obtained from Thm 1.1, we use the stronger (103), $\|\tilde{w}_j\| \leq C \exp[\lambda T j] =: \bar{\vartheta}^j \ll \vartheta^j$. As in the previous proof, we obtain (109), only with $(\vartheta\bar{\vartheta})^j$ instead of ϑ^{2j}, and use $(\vartheta\bar{\vartheta})/\vartheta_{l+1} < 1$ from $\lambda + \lambda_{01} < \Lambda$. The same change applies to the estimates of $\mathbf{Q}$.

It is easy to see that the largest eigenvalue $\lambda_{\ell k}$ for which the polynomial $u_{\ell k}(\boldsymbol{x}) \neq 0$ is non-vanishing cannot exceed λ without (14) contradicting either (103) (in case $\lambda < \lambda_{01}$) or Theorem 1.1 (in case $\lambda = \lambda_{01}$). $\square$

Corollary 11.3 (Coefficient formulae). *The coefficients appearing in the asymptotic expansion* (14) *from Theorems 1.2 and 11.1 are given by* $u_{\ell k}(\boldsymbol{x}) = \psi_{\ell k}(|\boldsymbol{x}|) \sum_\mu c_{\ell k \mu} Y_{\ell\mu}(\boldsymbol{x}/|\boldsymbol{x}|)$ *where* $\psi_{\ell k}$ *and* $Y_{\ell\mu}$ *are the polynomial eigenfunctions of* $\mathbf{H}$ *and spherical harmonics from Theorem 8.1 and*

$$c_{\ell k\mu} = \lim_{t\to\infty} e^{-\lambda_{\ell k} t} \langle u(t,\cdot) - u_B, \psi_{\ell k} Y_{\ell\mu} \rangle_{L^2(\mathbf{R}^n)} / \|\psi_{\ell k} Y_{\ell\mu} u_B^{1-m/2}\|^2_{L^2(\mathbf{R}^n)} \ . \tag{112}$$

For $k = 0 \neq \ell$*, the* $\psi_{\ell 0} Y_{\ell\mu}$ *are the homogeneous harmonic polynomials of degree* ℓ *and the expression under the limit* (112) *is independent of time, hence*

$$u_{\ell 0}(\boldsymbol{x}) = |\boldsymbol{x}|^\ell \sum_\mu Y_{\ell\mu}\Big(\frac{\boldsymbol{x}}{|\boldsymbol{x}|}\Big) \frac{\langle u(0,\cdot), \psi_{\ell 0} Y_{\ell\mu}\rangle_{L^2(\mathbf{R}^n)}}{\|\psi_{\ell 0} Y_{\ell\mu} u_B^{1-m/2}\|^2_{L^2(\mathbf{R}^n)}} \ . \tag{113}$$

Proof. Rewrite (14) in the form

$$\frac{u(t,\cdot) - u_B}{u_B^{2-m}} = \sum_{\Lambda < \lambda_{\ell k} < 0} u_{\ell k} e^{\lambda_{\ell k} t} + O(e^{\Lambda t})(B + |\boldsymbol{x}|^2)^{\left(p+2-\sqrt{(p+2)^2+4\Lambda}\right)/4} ,$$

where we have L^∞ control on the factor $O(e^{\Lambda t})$, and $u_{\ell k}$ lies in the $\lambda_{\ell k}$ eigenspace of the self-adjoint operator $\mathbf{H}$ on $L^2_{u_B^{2-m}}$ spanned by the family of degree $\ell + 2k < p/2 + 1$ polynomials $(\psi_{\ell k} Y_{\ell\mu})_\mu$. Multiplying this expression by $e^{-\lambda_{\ell k} t}\psi_{\ell k} Y_{\ell\mu}$ and by $u_B^{2-m} = (B + |\boldsymbol{x}|^2)^{-(n+p+2)/2}$, integration yields

$$e^{-\lambda_{\ell k} t} \langle u(t,\cdot) - u_B, \psi_{\ell k} Y_{\ell\mu}\rangle_{L^2(\mathbf{R}^n)} = \langle u_{\ell k}, \psi_{\ell k} Y_{\ell\mu}\rangle_{L^2_{u_B^{2-m}}} + O(e^{(\Lambda - \lambda_{\ell k})t}) \ .$$

The remainder term vanishes in the limit $t \to \infty$ to establish (112).

In case $k = 0 \neq \ell$ we have $\psi_{\ell 0}(|\boldsymbol{x}|) = |\boldsymbol{x}|^\ell$, hence $\{\psi_{\ell 0} Y_{\ell\mu}\}_\mu$ are the homogeneous harmonic polynomials of degree ℓ. These integrate to zero against the radial distribution u_B. It remains only to show time independence of the expression under the limit (112) to complete the proof of (113). Transforming back to the original variables (5), we see that the integral of $\rho(\tau, \boldsymbol{y})$ against any harmonic polynomial of degree less than p is independent of τ from the evolution equation (1); the spatial decay $|\boldsymbol{\nabla}\rho(\tau,\boldsymbol{y})| = O(1/|\boldsymbol{y}|^{n+p+1})$ in e.g. [31, Corollary 9] justifies the integration by parts. Thus

$$\int \rho(\tau, \boldsymbol{y}) |\boldsymbol{y}|^\ell Y_{\ell\mu}(\frac{\boldsymbol{y}}{|\boldsymbol{y}|}) d\boldsymbol{y} = \int u(\ln(1+2p\tau)^{1/2p}, \boldsymbol{x}) |(1+2p\tau)^\beta \boldsymbol{x}|^\ell Y_{\ell\mu}(\frac{\boldsymbol{x}}{|\boldsymbol{x}|}) d\boldsymbol{x}$$

is independent of $t = \frac{1}{2p}\ln(1+2p\tau)$. Since $\lambda_{\ell 0} = -2p\beta\ell$ from (3) and (11), this establishes (113). $\square$

Proof of Corollary 1.7. Let us quickly review the top of the spectrum from Theorem 8.2 (omitting the eigenvalues rendered ineffective by the mass / center of mass adjustments):

effective eigenvalues of $\mathbf{L}_\eta$ when $n \geq 3$:
$$\begin{aligned}\lambda_{01} &= -2((1-m)^{-1} - n) = -2p \\ \lambda_{02} &= -8((1-m)^{-1} - 1 - n/2) = -4p + 8 \\ \lambda_{20} &= -2p - 2n \\ \lambda_0^{\mathrm{cont}} &= -(p+2)^2/4\end{aligned}$$

The same picture, with the interval $[m_6, m_{n+4}]$ shrinking to a point, applies to $n = 2$, whereas in the case $n = 1$ we have the intersection of $\lambda_0^{\mathrm{cont}}$ and λ_{20} at $p = 2 + 2\sqrt{2}$.

Apart from $\lambda_{01} = -2p$, which forms the top of the spectrum of $\mathbf{L}_{\eta_{cr}}$ in the full range $p > 2$, the spectral gap is given by $\Lambda = \max\{\lambda_0^{\mathrm{cont}}, \lambda_{02}, \lambda_{20}\} = -\min\{(\frac{p}{2}+1)^2, 4(p-2), 2(p+n)\}$. Since $\lambda_0^{\mathrm{cont}}/\lambda_{01} < 2$, $\lambda_{02}/\lambda_{01} < 2$ and $\lambda_{20}/\lambda_{01} < 2$ are easily checked in the relevant ranges $p \in]2, 6]$, $p \in]6, n+4]$ and $p > n+4$, Corollary 1.5 yields an asymptotic expansion weighted by the denominator $\tau^{-\gamma}(\rho(\tau, \mathbf{0})/\rho(\tau, \boldsymbol{y}))^{\delta + 2/(n+p)}$ where $\gamma = \Lambda/\lambda_{01}$ is given by (16), and $\delta = \frac{1}{p+n}(\frac{p}{2} - 1 - \sqrt{(\frac{p}{2}+1)^2 + \Lambda})$ agrees with (17). The sum in this asymptotic expansion consists of only one term, and it corresponds to the eigenvalue λ_{01}. Thus

$$\frac{\rho(\tau, \boldsymbol{y})}{\rho_B(\tau, \boldsymbol{y})} = 1 + \frac{u_{11}((1+2p\tau)^{-\beta}\boldsymbol{y})}{B(1+2p\tau)} \left(\frac{\rho_B(\tau, \boldsymbol{y})}{\rho_B(\tau, 0)}\right)^{2/(p+n)} + O(1/\tau^\gamma)\,,$$

as $\tau \to \infty$, where $u_{11}(\boldsymbol{x})/(B + |\boldsymbol{x}|^2)$ belongs to the one-dimensional eigenspace of λ_{01}, and the error is measured in an L^∞ norm with the desired weight. Similarly Corollary 1.5 asserts

$$\frac{\rho_B(\tau - \tau_0, \boldsymbol{y})}{\rho_B(\tau, \boldsymbol{y})} = 1 + c(\tau_0)\frac{u_{11}((1+2p\tau)^{-\beta}\boldsymbol{y})}{B(1+2p\tau)} \left(\frac{\rho_B(\tau, \boldsymbol{y})}{\rho_B(\tau, 0)}\right)^{2/(p+n)} + O(1/\tau^\gamma)\,,$$

in the same norm, since the eigenspace is one-dimensional. If τ_0 can be chosen to make $c(\tau_0) = 1$, then subtracting these identities and using the fact (8) that $\|\rho_B(\tau - \tau_0, \boldsymbol{y})/\rho_B(\tau, \boldsymbol{y}) - 1\|_{L^\infty(\mathbf{R}^n)} \to 0$ will conclude the proof of Corollary 1.7.

On the other hand, from definition (7) we compute

$$\frac{\rho_B(\tau - \tau_0, \mathbf{0})}{\rho_B(\tau, \mathbf{0})} = \left(\frac{1 + 2p(\tau - \tau_0)}{1 + 2p\tau}\right)^{-n\beta} = 1 + n\beta\tau_0/\tau + o(1/\tau)\,,$$

from which we deduce that $c(\tau_0)$ does not generally vanish but depends linearly on τ_0. Thus a suitable choice of τ_0 yields $c(\tau_0) = 1$ to complete the proof. □

Proof of Corollary 1.10. In the range $m > m_{n+4}$ and $n \geq 2$, the next largest spectral value after λ_{20} is λ_{11} (and not $\lambda_0^{\mathrm{cont}}$). Taking $\Lambda = \lambda_{11}$ and observing that the spectral ratio $\lambda_{11}/\lambda_{01} = 2 - (p + 4 - n)/2p$ is strictly less than 2 in the range $p > n+4$ of interest, as in the preceding proof we can choose τ_0 to obtain

$$\frac{\rho(\tau-\tau_0,\boldsymbol{y})}{\rho_B(\tau,\boldsymbol{y})} = 1 + \frac{u_{20}((1+2p\tau)^{-\beta}\boldsymbol{y})}{(1+2p\tau)^{(3p+n-4)/2p}} \left(\frac{\rho_B(\tau,\boldsymbol{y})}{\rho_B(\tau,0)}\right)^{\frac{2}{p+n}} + O\left(\tau^{-\gamma}\right),$$

as $\tau \to \infty$, where the error is measured in an L^∞ norm with the desired weight according to Theorem 1.2, and $u_{20}(\boldsymbol{y})$ is a homogeneous harmonic polynomial of degree 2 as in Corollary 11.3.

A number of authors starting with Titov and Ustinov [42] and Tartar (circa 1986, unpublished but cited in [45]) and including two of us [21], have independently observed that the family of functions $u_B(\Sigma^{-1/2}\boldsymbol{y}) \det \Sigma^{-1/2}$ parametrized by positive definite symmetric matrices $\Sigma > 0$ form a invariant manifold of dimension $n(n+1)/2$ under the porous medium and fast diffusion dynamics (1). Each solution $\tilde{\rho}(\tau,\boldsymbol{y}) = u_B(\Sigma^{-1/2}(\tau)\boldsymbol{y}) \det \Sigma^{-1/2}(\tau)$ satisfies

$$\frac{\rho(\tau-\tilde{\tau}_0,\boldsymbol{y})}{\rho_B(\tau,\boldsymbol{y})} = 1 + \frac{\tilde{u}_{20}((1+2p\tau)^{-\beta}\boldsymbol{y})}{(1+2p\tau)^{(3p+n-4)/2p}} \left(\frac{\rho_B(\tau,\boldsymbol{y})}{\rho_B(\tau,0)}\right)^{\frac{2}{p+n}} + O\left(\tau^{-\gamma}\right)$$

as above. Since $\Sigma(\tau)$ is proportional to the moment of inertia tensor $\int_{\mathbf{R}^n} \boldsymbol{y} \otimes \boldsymbol{y}\rho(\tau,\boldsymbol{y})d\boldsymbol{y}$, Corollary 11.3 shows the traceless part Σ_0 of $\Sigma(\tau)$ is independent of τ and can be selected to make $\tilde{u}_{20} = u_{20}$. The evolution equation $(d\sigma/d\tau)^{p+n} = c_B \det \Sigma(\tau)$ for $\Sigma(\tau) = \Sigma_0 + \sigma(\tau)I > 0$ is from Corollary 4 of [21]. Subtracting the two equations above and using the fact that $\|\tilde{\rho}(\tau-\tilde{\tau}_0,\boldsymbol{y})/\rho_B(\tau,\boldsymbol{y}) - 1\|_{L^\infty(\mathbf{R}^n)} \to 0$ as in (8) yields the desired limit (18), after translating the solution $\tilde{\rho}$ in time by $\tau_0 - \tilde{\tau}_0$. □

12. Appendix: Pedestrian derivation of all Schauder Estimates

Although the announced results have now been established, for the sake of convenience, we include this appendix giving a self-contained derivation of all Schauder estimates for linear equations that we have relied on.

Self-contained proof of Lemma 6.1. By superposition, the estimates are assembled from four cases, in each of which exactly one of the quantities f, b, c, v_0 is non-zero.

We repeat and modify the arguments from [34], IV, appropriately to match our situation, denoting by Γ the heat kernel: $\Gamma(t,x) := (4\pi t)^{-n/2} \exp[-|x|^2/4t]$.

We estimate the <u>spatial Hölder quotients for the contribution from f:</u>

In this estimate, $B(2r,x)$ denotes the ball centered at x with radius $2r = 2|x-x'|$. We have

$$
\begin{aligned}
v(t,x)-v(t,x') &= \int_0^t \int_{\mathbf{R}^n} \partial_{ij}^2\Gamma(t-\tau,x-y)\big(f^{ij}(\tau,y)-f^{ij}(\tau,x)\big)\,dy\,d\tau \\
&\quad - \int_0^t \int_{\mathbf{R}^n} \partial_{ij}^2\Gamma(t-\tau,x'-y)\big(f^{ij}(\tau,y)-f^{ij}(\tau,x')\big)\,dy\,d\tau \\
= \int_0^t &\int_{B(2r,x)} \partial_{ij}^2\Gamma(t-\tau,x-y)\big(f^{ij}(\tau,y)-f^{ij}(\tau,x)\big)\,dy\,d\tau \\
&- \int_0^t \int_{B(2r,x)} \partial_{ij}^2\Gamma(t-\tau,x'-y)\big(f^{ij}(\tau,y)-f^{ij}(\tau,x')\big)\,dy\,d\tau \\
&+ \int_0^t \int_{B(2r,x)^c} \big(\partial_{ij}^2\Gamma(t-\tau,x-y)-\partial_{ij}^2\Gamma(t-\tau,x'-y)\big)\big(f^{ij}(\tau,y)-f^{ij}(\tau,x)\big)\,dy\,d\tau \\
&- \int_0^t \big(f^{ij}(\tau,x)-f^{ij}(\tau,x')\big)\int_{B(2r,x)^c} \partial_{ij}^2\Gamma(t-\tau,x'-y)\,dy\,d\tau\;.
\end{aligned}
$$

We note: If $|x-y|\le 2r$, then $|x'-y|\le 3r$. In the third term, if $|x-y|\ge 2r$, then $|\sigma x+(1-\sigma)x'-y|\ge \frac{1}{2}|x-y|$ for any $\sigma\in[0,1]$. Therefore

(114)

$$
\begin{aligned}
|v(t,x)-v(t,x')| &\le \int_0^t \int_{B(2r,x)} |\partial_{ij}^2\Gamma(t-\tau,x-y)|\,|x-y|^\alpha\,[f^{ij}(\tau)]_{x;\alpha}\,dy\,d\tau \\
&+ \int_0^t \int_{B(2r,x)} |\partial_{ij}^2\Gamma(t-\tau,x'-y)|\,|x'-y|^\alpha\,[f^{ij}(\tau)]_{x;\alpha}\,dy\,d\tau \\
&+ \int_0^t \int_{B(2r,x)^c} \big|\partial_{ij}^2\Gamma(t-\tau,x-y)-\partial_{ij}^2\Gamma(t-\tau,x'-y)\big|\,|x-y|^\alpha\,[f^{ij}(\tau)]_{x;\alpha}\,dy\,d\tau \\
&+ \int_0^t |x-x'|^\alpha\,[f^{ij}(\tau)]_{x;\alpha}\bigg|\int_{B(2r,x)^c} \partial_{ij}^2\Gamma(t-\tau,x'-y)\,dy\bigg|\,d\tau \\
&\le C\int_0^t \int_{B(2r,x)} (t-\tau)^{-\frac{n}{2}-1}\exp\big[-C\tfrac{(x-y)^2}{t-\tau}\big]\,|x-y|^\alpha\,[f^{ij}(\tau)]_{x;\alpha}\,dy\,d\tau \\
&+ C\int_0^t \int_{B(3r,x')} (t-\tau)^{-\frac{n}{2}-1}\exp\big[-C\tfrac{(x'-y)^2}{t-\tau}\big]\,|x'-y|^\alpha\,[f^{ij}(\tau)]_{x;\alpha}\,dy\,d\tau \\
&+ C\int_0^t \int_{B(2r,x)^c} |x-x'|(t-\tau)^{-\frac{n}{2}-\frac{3}{2}}\exp\big[-C\tfrac{(x-y)^2}{t-\tau}\big]\,|x-y|^\alpha\,[f^{ij}(\tau)]_{x;\alpha}\,dy\,d\tau \\
&+ C\int_0^t |x-x'|^\alpha\,[f^{ij}(\tau)]_{x;\alpha}\int_{\partial B(2r,x)} |\boldsymbol{\nabla}\Gamma(t-\tau,x'-y)|\,dS(y)\,d\tau\;.
\end{aligned}
$$

Now if we estimate $[f(\tau)]_{x;\alpha}\le \|f\|_{C^\alpha(\mathbf{R}^n_T)}$ and in the first three integrals evaluate the time integration first, using that, for $k>1$,

$$
(115)\quad \int_0^t (t-\tau)^{-k}\exp\big[-\tfrac{A^2}{t-\tau}\big]\,d\tau = (A^2)^{1-k}\int_0^{t/A^2} s^{-k}\exp\big[-\tfrac{1}{s}\big]\,ds \le C(A^2)^{1-k}\,,
$$

we conclude that each term comes to the same estimate, namely we get

$$|v(t,x) - v(t,x')| \le C\|f\|_{C^\alpha(\mathbf{R}^n_T)}\, |x-x'|^\alpha \,.$$

In comparison, if we estimate $[f(\tau)]_{x;\alpha} \le \tau^{-\alpha/2}\|f\|^*_{C^\alpha(\mathbf{R}^n_T)}$, we obtain the following time integral instead, and we split it in the middle, obtaining

$$\begin{aligned}
&\Big(\int_0^{t/2} + \int_{t/2}^t\Big)(t-\tau)^{-k}\exp\big[-\frac{A^2}{t-\tau}\big]\tau^{-\alpha/2}\,d\tau \\
&\quad\le \max_{\sigma\in[t/2,t]}\big(\sigma^{1-k}\exp[-A^2/\sigma]\big)\int_0^{t/2}(t-\tau)^{-1}\tau^{-\alpha/2}\,d\tau \\
&\qquad + (t/2)^{-\alpha/2}(A^2)^{1-k}\int_0^{t/2A^2} s^{-k}\exp[-1/s]\,ds \\
&\quad\le Ct^{-\alpha/2}(A^2)^{1-k}\,,
\end{aligned}\tag{116}$$

with the same estimate for each summand. This results in

$$t^{\alpha/2}\,|v(t,x) - v(t,x')| \le C\|f\|^*_{C^\alpha(\mathbf{R}^n_T)}\,|x-x'|^\alpha \,.$$

We now estimate the time Hölder quotients for the contribution from f:
We assume $t' > t$ and let $t' - t =: d$. Then

$$\begin{aligned}
v(t',x) - v(t,x) &= \int_0^{t'}\int_{\mathbf{R}^n}\partial^2_{ij}\Gamma(t'-\tau,x-y)\big(f^{ij}(\tau,y) - f^{ij}(\tau,x)\big)\,dy\,d\tau \\
&\quad - \int_0^{t}\int_{\mathbf{R}^n}\partial^2_{ij}\Gamma(t-\tau,x-y)\big(f^{ij}(\tau,y) - f^{ij}(\tau,x)\big)\,dy\,d\tau \\
= \int_{(t-d)_+}^{t'}&\int_{\mathbf{R}^n}\partial^2_{ij}\Gamma(t'-\tau,x-y)\big(f^{ij}(\tau,y) - f^{ij}(\tau,x)\big)\,dy\,d\tau \\
- \int_{(t-d)_+}^{t}&\int_{\mathbf{R}^n}\partial^2_{ij}\Gamma(t-\tau,x-y)\big(f^{ij}(\tau,y) - f^{ij}(\tau,x)\big)\,dy\,d\tau \\
+ \int_0^{(t-d)_+}&\int_{\mathbf{R}^n}\big[\partial^2_{ij}\Gamma(t'-\tau,x-y) - \partial^2_{ij}\Gamma(t-\tau,x-y)\big]\big(f^{ij}(\tau,y) - f^{ij}(\tau,x)\big)\,dy\,d\tau \,.
\end{aligned}$$

Hence

$$\begin{aligned}
&|v(t',x) - v(t,x)| \le \\
&\quad C\int_{(t-d)_+}^{t'}\int_{\mathbf{R}^n}(t'-\tau)^{-\frac{n}{2}-1}\exp\big[-C\tfrac{|x-y|^2}{t'-\tau}\big]|x-y|^\alpha\,dy\,[f(\tau)]_{x;\alpha}\,d\tau \\
&\qquad + C\int_{(t-d)_+}^{t}\int_{\mathbf{R}^n}(t-\tau)^{-\frac{n}{2}-1}\exp\big[-C\tfrac{|x-y|^2}{t-\tau}\big]|x-y|^\alpha\,dy\,[f(\tau)]_{x;\alpha}\,d\tau \\
&\qquad + C\int_0^{(t-d)_+}(t'-t)\int_{\mathbf{R}^n}(t^*-\tau)^{-\frac{n}{2}-2}\exp\big[-C\tfrac{|x-y|^2}{t^*-\tau}\big]|x-y|^\alpha\,dy\,[f(\tau)]_{x;\alpha}\,d\tau\,,
\end{aligned}\tag{117}$$

where $t^* \in [t, t']$; and therefore, in the last domain of integration, we have $\frac{1}{2}(t' - \tau) \le t_* - \tau \le t' - \tau$. Evaluating the space integrals directly first, we obtain

$$
\begin{aligned}
&|v(t',x) - v(t,x)| \le \\
&\quad C\int_{(t-d)_+}^{t'} (t'-\tau)^{\frac{\alpha}{2}-1}\,[f(\tau)]_{x;\alpha}\,d\tau + \text{same with } t \text{ instead of } t' \\
&\qquad + C\int_0^{(t-d)_+} (t'-t)(t'-\tau)^{\frac{\alpha}{2}-2}\,[f(\tau)]_{x;\alpha}\,d\tau\;.
\end{aligned}
\tag{118}
$$

Each term is dominated by $C(t'-t)^{\alpha/2}\sup_\tau \|f(\tau)\|_{C^\alpha(\mathbf{R}^n)} \le C(t'-t)^{\alpha/2}\|f\|_{C^\alpha(\mathbf{R}^n_T)}$, as desired.

If we use the weighted norms in (45) instead, we obtain

$$
\begin{aligned}
&|v(t',x) - v(t,x)| \le \\
&\quad C\int_{(t-d)_+}^{t'} (t'-\tau)^{\frac{\alpha}{2}-1}\tau^{-\alpha/2}\|f\|^*_{C^\alpha(\mathbf{R}^n_T)}\,d\tau \\
&\qquad + C\int_{(t-d)_+}^{t} (t-\tau)^{\frac{\alpha}{2}-1}\tau^{-\alpha/2}\|f\|^*_{C^\alpha(\mathbf{R}^n_T)}\,d\tau \\
&\qquad + C(t'-t)\int_0^{(t-d)_+} (t-\tau)^{\frac{\alpha}{2}-2}\tau^{-\alpha/2}\|f\|^*_{C^\alpha(\mathbf{R}^n_T)}\,d\tau \\
&\quad \le C(t'-t)^{\alpha/2}t^{-\alpha/2}\|f\|^*_{C^\alpha(\mathbf{R}^n_T)}\;.
\end{aligned}
\tag{119}
$$

To justify the last estimate for the first two integrals, we distinguish two cases: If $d \ge \frac{1}{2}t$, our estimate $[\frac{t'-t}{t}]^{\alpha/2} = (d/t)^{\alpha/2}$ is weaker than a constant, whereas the integrals can be extended to a lower limit $\tau = 0$ and readily estimated by a constant. If $d \le \frac{1}{2}t$, the term $\tau^{-\alpha/2}$ is $\le (\frac{1}{2}t)^{-\alpha/2}$, and the other factor can be integrated and found to be bounded by $O((t'-t)^{\alpha/2})$.

For the third integral, in the case that $d \ge \frac{1}{2}t$, we argue that $(t-\tau)^{\frac{\alpha}{2}-2} \le d^{\frac{\alpha}{2}-2} = (t'-t)^{\frac{\alpha}{2}-2}$, and the coefficient of $\|f\|^*$ from the third term is dominated by $C(t'-t)^{\frac{\alpha}{2}-1}(t-d)_+^{1-\frac{\alpha}{2}}$. This quantity is again bounded by a constant, whereas our claimed estimate is weaker than a constant. On the other hand, if $d < \frac{1}{2}t$, we split the integral in the middle, estimate the bounded factor under each integral by its maximum (at $\tau = \frac{1}{2}t$ for each integral) and integrate the remaining factor. Now $d\int_0^{t/2} \ldots d\tau \le Cd/t \le C(d/t)^{\alpha/2}$ and $d\int_{t/2}^{t-d} \ldots d\tau \le C(d/t)^\alpha/2$, hence the desired estimate.

We now estimate the supremum norm for the contribution from f:

This is the easy estimate

$$
(120)\qquad \begin{aligned}
|v(t,x)| &\le \int_0^t \int_{\mathbf{R}^n} |\partial_{ij}^2 \Gamma(t-\tau, x-y)|\,|f^{ij}(\tau,y) - f^{ij}(\tau,x)|\,dy\,d\tau \\
&\le C \int_0^t \int_{\mathbf{R}^n} (t-\tau)^{-\frac{n}{2}-1} \exp\bigl[-C\tfrac{|x-y|^2}{t-\tau}\bigr]\,|x-y|^\alpha\,dy\,[f(\tau)]_{x;\alpha}\,d\tau \\
&\le C \int_0^t (t-\tau)^{\frac{\alpha}{2}-1} [f(\tau)]_{x;\alpha}\,d\tau \le C t^{\alpha/2} \|f\|_{C^\alpha(\mathbf{R}^n_T)}\ .
\end{aligned}
$$

Since t is bounded, we have estimated $|v(t,x)|$ by the Hölder norm of f. In the same way, we can also estimate $t^{\alpha/2}|v(t,x)| \le C t^{\alpha/2} \|f\|^*_{C^\alpha(\mathbf{R}^n_T)} \le C \|f\|^*_{C^\alpha(\mathbf{R}^n_T)}$.

We now estimate the spatial Hölder quotient contributed from b:

They can be estimated as in (114), but without splitting the space integral, as

$$
(121)\qquad \begin{aligned}
\frac{|v(t,x) - v(t,x')|}{|x-x'|^\alpha} &\le \int_0^t \int_{\mathbf{R}^n} |\partial_i \Gamma(t-\tau, y)|\,[b^i(\tau)]_{x;\alpha}\,dy\,d\tau \\
&\le C \int_0^t (t-\tau)^{-\frac{1}{2}}\,[b(\tau)]_{x;\alpha}\,d\tau \\
&\le \min\Bigl\{C t^{\frac{1}{2}} \|b\|_{C^\alpha(\mathbf{R}^n_T)}\,,\ C t^{\frac{1}{2}-\frac{\alpha}{2}} \|b\|^*_{C^\alpha(\mathbf{R}^n_T)}\Bigr\}\ .
\end{aligned}
$$

We now estimate the time Hölder quotients contributed from b:

For the time Hölder quotients, we can argue as in (118), only with an extra power of $(t'-\tau)^{1/2}$ under the integrals. We get

$$
\begin{aligned}
|v(t',x) - v(t,x)| &\le C\Bigl((t'-t)^{\frac{\alpha}{2}+\frac{1}{2}} + (t'-t)(t'-t)^{\frac{\alpha}{2}-\frac{1}{2}}\Bigr) \sup_\tau [b(\tau)]_{x;\alpha} \\
&\le C (t'-t)^{\frac{\alpha}{2}} T^{\frac{1}{2}} \|b\|_{C^\alpha(\mathbf{R}^n_T)}\ .
\end{aligned}
$$

If we use the weighted norms $\|\cdot\|^*$ instead, we get by modification of (119):

$$
(122)\qquad \begin{aligned}
|v(t',x) - v(t,x)| \le\ & \\
& C\|b\|^*_{C^\alpha(\mathbf{R}^n_T)} \int_{(t-d)_+}^{t'} (t'-\tau)^{\alpha/2-1/2} \tau^{-\alpha/2}\,d\tau \\
& + C\|b\|^*_{C^\alpha(\mathbf{R}^n_T)} \int_{(t-d)_+}^{t} (t-\tau)^{\alpha/2-1/2} \tau^{-\alpha/2}\,d\tau \\
& + C(t'-t)\|b\|^*_{C^\alpha(\mathbf{R}^n_T)} \int_0^{(t-d)_+} (t-\tau)^{\frac{\alpha}{2}-\frac{3}{2}} \tau^{-\alpha/2}\,d\tau\ .
\end{aligned}
$$

The same splitting argument as for (119) proves that each term is dominated by $C(t'-t)^{\alpha/2+1/2} t^{-\alpha/2} \|b\|^*_{C^\alpha(\mathbf{R}^n_T)} \le C T^{1/2} (t'-t)^{\alpha/2} t^{-\alpha/2} \|b\|^*_{C^\alpha(\mathbf{R}^n_T)}$.

We now estimate the supremum norm contributed from b:

As in (120), we conclude $|v(t,x)| \le Ct^{\frac{\alpha}{2}+\frac{1}{2}}\|b\|_{C^\alpha(\mathbf{R}^n_T)}$ and $|v(t,x)| \le Ct^{\frac{1}{2}}\|b\|^*_{C^\alpha(\mathbf{R}^n_T)}$. An even simpler version of the same estimate yields $|v(t,x)| \le Ct^{\frac{1}{2}}\|b\|_{L^\infty(\mathbf{R}^n_T)}$ for later use in proving (44).

We now estimate the <u>space and time Hölder quotients contributed from c:</u>

In contrast to b and f, we cannot follow the paradigms of (114) and (117) here, because Γ carries no space derivative here. This impacts the results of $[v]_{t;\alpha/2}$ since we cannot benefit from using spatial Hölder quotients of c. We get

$$\frac{|v(t,x)-v(t,x')|}{|x-x'|^\alpha} \le \int_0^t \int_{\mathbf{R}^n} \Gamma(t-\tau,y)\,[c(\tau)]_{x;\alpha} dy\, d\tau \le t\|c\|_{C^\alpha(\mathbf{R}^n_T)}\ .$$

Similarly, $t^{\alpha/2}[v(t)]_{x;\alpha} \le t\|c\|^*_{C^\alpha(\mathbf{R}^n_T)}$.

For the time Hölder quotients, we have (with $t'>t$)

$$\begin{aligned}\frac{|v(t',x)-v(t,x)|}{|t'-t|^{\alpha/2}} &\le \frac{1}{|t'-t|^{\alpha/2}}\int_t^{t'}\int_{\mathbf{R}^n}\Gamma(\tau,y)c(t'-\tau,x-y)\,dy\,d\tau \\ &\quad + \int_0^t\int_{\mathbf{R}^n}\Gamma(\tau,y)\frac{|c(t'-\tau,x-y)-c(t-\tau,x-y)|}{|t'-t|^{\alpha/2}}\,dy\,d\tau \\ &\le (t'-t)^{1-\alpha/2}\|c\|_{L^\infty(\mathbf{R}^n_T)} + T[c]_{t;\alpha/2} \le T^{1-\alpha/2}\|c\|_{C^\alpha(\mathbf{R}^n_T)}\ .\end{aligned}$$

With $|v(t,x)| \le t^{\alpha/2}[v]_{t;\alpha/2}$, the c-estimate in (43) follows immediately. The example $c\equiv 1$, $v(t,x)=t$ shows that the estimate is optimal.

Estimating the time Hölder quotients in terms of the weighted norm, we get analogously

$$\begin{aligned}\frac{|v(t',x)-v(t,x)|}{|t'-t|^{\alpha/2}} &\le \frac{1}{|t'-t|^{\alpha/2}}\int_t^{t'}\|c(t'-\tau)\|_{L^\infty(\mathbf{R}^n)}\,d\tau \\ &\quad + \int_0^t (t-\tau)^{-\alpha/2}\|c\|^*_{C^\alpha(\mathbf{R}^n_T)}\,d\tau \\ &\le C(t'-t)^{1-\alpha}\|c\|^*_{C^\alpha(\mathbf{R}^n_T)} + Ct^{1-\alpha/2}\|c\|^*_{C^\alpha(\mathbf{R}^n_T)}\ .\end{aligned}$$

Hence $t^{\alpha/2}\frac{|v(t',x)-v(t,x)|}{|t'-t|^{\alpha/2}} \le CT^{1-\alpha/2}\|c\|^*_{C^\alpha(\mathbf{R}^n_T)}$; again, as a corollary, we get $|v(t',x)| \le CT\|c\|^*_{C^\alpha(\mathbf{R}^n_T)}$. This gives the c estimate in (45).

The estimates for the <u>contributions from v_0</u> have been given in Sec. 6, Equations (47),(48),(49) already.

Likewise, the estimates for the <u>contributions of $\|b\|_{L^\infty}$ and $\|c\|_{L^\infty}$ to the alternate bound (44)</u> have been given in Sec. 6 already.

This concludes the proof of Lemma 6.1. □

References

[1] Sigurd Angenent. Large time asymptotics for the porous media equation. In *Nonlinear diffusion equations and their equilibrium states, I (Berkeley, CA, 1986)*, volume 12 of *Math. Sci. Res. Inst. Publ.*, pages 21–34. Springer, New York, 1988.

[2] Sigurd Angenent. Local existence and regularity for a class of degenerate parabolic equations. *Math. Ann.*, 280(3):465–482, 1988.

[3] Sigurd B. Angenent and Donald G. Aronson. Optimal asymptotics for solutions to the initial value problem for the porous medium equation. In T.S. Angell et al, editor, *Nonlinear Problems in Applied Mathematics: in honor of Professor Ivar Stakgold on his 70th birthday*, pages 10–19. Society for Industrial and Applied Mathematics, Philadelphia, 1996.

[4] Grigory I. Barenblatt. On some unsteady motions of a liquid or gas in a porous medium. *Akad. Nauk. SSSR. Prikl. Mat. Mekh.*, 16:67–78, 1952.

[5] Marcel Berger, Paul Gauduchon, and Edmond Mazet. *Le spectre d'une variété riemannienne*. Lecture Notes in Mathematics, Vol. 194. Springer-Verlag, Berlin, 1971.

[6] Adrien Blanchet, Matteo Bonforte, Jean Dolbeault, Gabriele Grillo, and Juan-Luis Vázquez. Hardy-Poincaré inequalities and applications to nonlinear diffusions. *C. R. Math. Acad. Sci. Paris*, 344(7):431–436, 2007.

[7] Adrien Blanchet, Matteo Bonforte, Jean Dolbeault, Gabriele Grillo, and Juan Luis Vázquez. Asymptotics of the fast diffusion equation via entropy estimates. *Arch. Ration. Mech. Anal.*, 191(2):347–385, 2009.

[8] Matteo Bonforte, Jean Dolbeault, Gabriele Grillo, and Juan L. Vázquez. Sharp rates of decay of solutions to the nonlinear fast diffusion equation via functional inequalities. *Proc. Natl. Acad. Sci. USA*, 107(38):16459–16464, 2010.

[9] Matteo Bonforte, Gabriele Grillo, and Juan Luis Vázquez. Special fast diffusion with slow asymptotics: entropy method and flow on a Riemann manifold. *Arch. Ration. Mech. Anal.*, 196(2):631–680, 2010.

[10] José A. Carrillo, Marco Di Francesco, and Giuseppe Toscani. Strict contractivity of the 2-Wasserstein distance for the porous medium equation by mass-centering. *Proc. Amer. Math. Soc.*, 135(2):353–363 (electronic), 2007.

[11] José A. Carrillo, Claudia Lederman, Peter A. Markowich, and Giuseppe Toscani. Poincaré inequalities for linearizations of very fast diffusion equations. *Nonlinearity*, 15(3):565–580, 2002.

[12] José A. Carrillo and Giuseppe Toscani. Asymptotic L^1-decay of solutions of the porous medium equation to self-similarity. *Indiana Univ. Math. J.*, 49(1):113–142, 2000.

[13] José A. Carrillo and Juan L. Vázquez. Fine asymptotics for fast diffusion equations. *Comm. Partial Differential Equations*, 28(5-6):1023–1056, 2003.

[14] Bennett Chow and Dan Knopf. *The Ricci flow: an introduction*, volume 110 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2004.

[15] Björn E. J. Dahlberg and Carlos E. Kenig. Nonnegative solutions to fast diffusions. *Rev. Mat. Iberoamericana*, 4(1):11–29, 1988.

[16] Panagiota Daskalopoulos and Natasa Sesum. Eternal solutions to the Ricci flow on $\mathbb{R}^2$. *Int. Math. Res. Not.*, pages Art. ID 83610, 20, 2006.

[17] Klaus Deimling. *Nonlinear functional analysis*. Springer-Verlag, Berlin, 1985.

[18] Manuel Del Pino and Jean Dolbeault. Best constants for Gagliardo-Nirenberg inequalities and applications to nonlinear diffusions. *J. Math. Pures Appl. (9)*, 81(9):847–875, 2002.

[19] Jochen Denzler and Robert J. McCann. Phase transitions and symmetry breaking in singular diffusion. *Proc. Natl. Acad. Sci. USA*, 100(12):6922–6925 (electronic), 2003.

[20] Jochen Denzler and Robert J. McCann. Fast diffusion to self-similarity: complete spectrum, long-time asymptotics, and numerology. *Arch. Ration. Mech. Anal.*, 175(3):301–342, 2005.

[21] Jochen Denzler and Robert J. McCann. Nonlinear diffusion from a delocalized source: affine self-similarity, time reversal, and nonradial focusing geometries. *Ann. Inst. H Poincaré Anal. Non Lineaire*, 25:865–888, 2008.

[22] Jean Dolbeault and Giuseppe Toscani. Fast diffusion equations: matching large time asymptotics by relative entropy methods. *Kinet. Relat. Models*, 4(3):701–716, 2011.

[23] Klaus-Jochen Engel and Rainer Nagel. *One-parameter semigroups for linear evolution equations*, volume 194 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2000. With contributions by S. Brendle, M. Campiti, T. Hahn, G. Metafune, G. Nickel, D. Pallara, C. Perazzoli, A. Rhandi, S. Romanelli and R. Schnaubelt.

[24] Mikhail V. Fedoryuk. *Asymptotic analysis*. Springer-Verlag, Berlin, 1993. Linear ordinary differential equations, Translated from the Russian by Andrew Rodick.

[25] Avner Friedman. *Partial differential equations of parabolic type*. Prentice-Hall Inc., Englewood Cliffs, N.J., 1964.

[26] Avner Friedman and Shoshana Kamin. The asymptotic behavior of gas in an n-dimensional porous medium. *Trans. Amer. Math. Soc.*, 262(2):551–563, 1980.

[27] Thierry Gallay. A center-stable manifold theorem for differential equations in Banach spaces. *Comm. Math. Phys.*, 152(2):249–268, 1993.

[28] Thierry Gallay and C. Eugene Wayne. Invariant manifolds and the long-time asymptotics of the Navier-Stokes and vorticity equations on $\mathbf{R}^2$. *Arch. Ration. Mech. Anal.*, 163(3):209–258, 2002.

[29] Miguel A. Herrero and Michel Pierre. The Cauchy problem for $u_t = \Delta u^m$ when $0 < m < 1$. *Trans. Amer. Math. Soc.*, 291(1):145–158, 1985.

[30] Tosio Kato. *Perturbation theory for linear operators*. Springer-Verlag, Berlin, second edition, 1976. Grundlehren der Mathematischen Wissenschaften, Band 132.

[31] Yong Jung Kim and Robert J. McCann. Potential theory and optimal convergence rates in fast nonlinear diffusion. *J. Math. Pures Appl. (9)*, 86(1):42–67, 2006.

[32] Herbert Koch. Non-Euclidean Singular Integrals and the Porous Medium Equation. Habilitation Thesis, Universität Heidelberg, Germany, 1999.

[33] N. V. Krylov. *Lectures on elliptic and parabolic equations in Hölder spaces*, volume 12 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 1996.

[34] O. A. Ladyženskaja, V. A. Solonnikov, and N. N. Ural′ceva. *Linear and quasilinear equations of parabolic type*. Translated from the Russian by S. Smith. Translations of Mathematical Monographs, Vol. 23. American Mathematical Society, Providence, R.I., 1967.

[35] George L. Lamb, Jr. *Elements of soliton theory*. John Wiley & Sons Inc., New York, 1980. Pure and Applied Mathematics, A Wiley-Interscience Publication.

[36] Claudia Lederman and Peter A. Markowich. On fast-diffusion equations with infinite equilibrium entropy and finite equilibrium mass. *Comm. Partial Differential Equations*, 28(1-2):301–332, 2003.

[37] Robert J. McCann and Dejan Slepčev. Second-order asymptotics for the fast-diffusion equation. *Int. Math. Res. Not.*, pages Art. ID 24947, 22, 2006.

[38] Felix Otto. The geometry of dissipative evolution equations: the porous medium equation. *Comm. Partial Differential Equations*, 26(1-2):101–174, 2001.

[39] R.E. Pattle. Diffusion from an instantaneous point source with concentration dependent coefficient. *Quart. J. Mech. Appl. Math.*, 12:407–409, 1959.

[40] Murray H. Protter and Hans F. Weinberger. *Maximum principles in differential equations*. Prentice-Hall Inc., Englewood Cliffs, N.J., 1967.

[41] Thomas Runst and Winfried Sickel. *Sobolev spaces of fractional order, Nemytskij operators, and nonlinear partial differential equations*, volume 3 of *de Gruyter Series in Nonlinear Analysis and Applications*. Walter de Gruyter & Co., Berlin, 1996.

[42] S.S. Titov and V.A. Ustinov. Investigation of polynomial solutions of the two-dimensional Leĭbenzon filtration equation with an integral exponent of the adiabatic curve (in Russian). In A.F. Sidorov and S.V. Vershinin, editors, *Approximate methods for solving boundary value problems of continuum mechanics (in Russian)*, volume 91, pages 64–70. Ural. Nauchn. Tsentr, Sverdlovsk, 1985.

[43] Juan Luis Vázquez. Asymptotic behaviour and propagation properties of the one-dimensional flow of gas in a porous medium. *Trans. Amer. Math. Soc.*, 277(2):507–527, 1983.

[44] Juan Luis Vázquez. Asymptotic behaviour for the porous medium equation posed in the whole space. *J. Evol. Equ.*, 3(1):67–118, 2003. Dedicated to Philippe Bénilan.

[45] Juan Luis Vázquez. *The porous medium equation. Mathematical theory.* Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, Oxford, 2007.

[46] C. Eugene Wayne. Invariant manifolds for parabolic partial differential equations on unbounded domains. *Arch. Rational Mech. Anal.*, 138(3):279–306, 1997.

[47] Thomas P. Witelski and Andrew J. Bernoff. Self-similar asymptotics for linear and nonlinear diffusion equations. *Stud. Appl. Math.*, 100(2):153–193, 1998.

[48] Ya.B. Zel'dovich and G.I. Barenblatt. The asymptotic properties of self-modelling solutions of the nonstationary gas filtration equations. *Sov. Phys. Doklady*, 3:44–47, 1958.

[49] Ya.B. Zel'dovich and A.S. Kompaneets. Theory of heat transfer with temperature dependent thermal conductivity. In *Collection in Honour of the 70th Birthday of Academician A.F. Ioffe*, pages 61–71. Izdvo. Akad. Nauk. SSSR, Moscow, 1950.

E-mail address: `denzler@math.utk.edu`

E-mail address: `koch@math.uni-bonn.de`

E-mail address: `mccann@math.toronto.edu`

Jochen Denzler, Department of Mathematics, University of Tennessee, Knoxville, TN 37996, USA

Herbert Koch, Mathematisches Institut, Universität Bonn, Endenicher Allee 60, 53115 Bonn, Germany

Robert McCann, Department of Mathematics, University of Toronto, Toronto, Ontario M5S 2E4, Canada