NONLINEAR DISPERSIVE EQUATIONS

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1. Introduction

Non-linearly interacting waves are often described by asymptotic equations. The derivation typically involves an ansatz for an approximate solution where higher order terms - the precise meaning of higher order term depends on the context and the relevant scales - are neglected. Often a Taylor expansion of a Fourier multiplier is part of that process.

There is an immediate consequence: This type of derivation leads to a huge set of asymptotic equations, and one should search for a general understanding of interacting nonlinear waves by asking for precise results for specific equations.

The most basic asymptotic equation is probably the nonlinear Schrödinger equation, which describes wave trains or frequency envelopes close to a given frequency, and their self interactions. The Korteweg-de-Vries equation typically occurs as first nonlinear asymptotic equation when the prior linear asymptotic equation is the wave equation. It is one of the amazing facts that many generic asymptotic equations are integrable in the sense that there are many formulae for specific solutions, conserved quantities, Lax-Pairs and BiHamiltonian structures.

This text will focus on adapted function spaces and their recent application to a number of dispersive equations. They are build on functions of bounded p variation, and their companion, the atomic space U^p. Combined with stationary phase resp. Strichartz estimates and bilinear refinements thereof they provide an alternative to the Fourier restriction spaces $X^{s,b}$ which is better suited for scaling critical problems.

We discuss teh method of stationary phase and dispersive estimates in Section 2, the application to the nonlinear Schrödinger equation in Section 3, the spaces U^p and V^p in section 4, bilinear estimates in Section 5 and application to nonlinear dispersive equations in Section 6

In order to make these note reasonable self contained there are three appendix on Young's inequality, real and complex interpolation, on Bessel functions and on the Fourier transform.

2. Stationary phase and dispersive estimates

We begin with the evaluations of several integrals. Let m^d be the d dimensional Lebesgue measure and define

$$I_d = \int_{\mathbb{R}^d} e^{-|x|^2} dm^d(x).$$

Then, with Fubini,

$$\begin{aligned} I_{d_1+d_2} =& \int_{\mathbb{R}^{d_1}\times\mathbb{R}^{d_2}} e^{-|x_1|^2-|x_2|^2} dm^{d_1+d_2}(x) \\ =& \int_{\mathbb{R}^{d_1+d_2}} e^{-|x_1|^2} e^{-|x_2|^2} dm^{d_1+d_2}(x) \\ =& \int_{\mathbb{R}^{d_1}} e^{-|x_1|^2} \int_{\mathbb{R}^{d_2}} e^{-|x_2|^2} dm^{d_2} dm^{d_1} \\ =& I_{d_2} \int_{\mathbb{R}^{d_1}} e^{-|x_1|^2} dm^{d_1} \\ =& I_{d_1} I_{d_2} \end{aligned}$$

hence

$$I_d = I_1^d.$$

Applying Fubini twice we get

$$\begin{aligned} I_d =& m^{d+1}(\{(x,t) : 0 < t < e^{-|x|^2}\}) \\ =& \int_0^1 m^d(\{x : e^{-|x|^2} > t\}) dt \\ =& \int_0^1 m^d(B_{(-\ln(t))^{1/2}}(0)) dt \\ =& m^d(B_1(0)) \int_0^1 (-\ln(t))^{d/2} dt \\ =& m^d(B_1(0)) \int_0^\infty s^{d/2} e^{-s} ds \\ =& m^d(B_1(0)) \Gamma(\frac{d}{2}+1) \end{aligned}$$

and hence $I_2 = \pi$, $I_1 = \sqrt{\pi}$, $I_d = \pi^{d/2}$,

$$m^d(B_1(0)) = \frac{\pi^{d/2}}{\Gamma(\frac{d}{2}+1)}$$

and

$$\Gamma(\frac{1}{2}) = 2\Gamma(\frac{3}{2}) = \sqrt{\pi}.$$

We proceed with

$$I(\tau) := \int_{-\infty}^\infty e^{-\frac{\tau}{2}x^2} dx$$

for $\operatorname{Re}\tau > 0$. Then

$$\begin{aligned} \frac{d}{dt}\sqrt{t+is}I(t+is) =& \frac{1}{2(t+is)}\sqrt{t+is}I(t+is) - \frac{1}{2}\sqrt{t+is}\int_{-\infty}^\infty e^{-\frac{t+is}{2}x^2} x^2 dx \\ =& \frac{\sqrt{t+is}}{2(t+is)}\left(I(t+is) + \int_{-\infty}^\infty \frac{d}{dx} e^{-\frac{t+is}{2}} x dx \right) \\ =& \frac{\sqrt{t+is}}{2(t+is)}\left(I(t+is) - \int_{-\infty}^\infty e^{-\frac{t+is}{2}x^2} dx \right) \\ =& 0 \end{aligned}$$

and similarly

$$\frac{d}{ds}\sqrt{t+is}I(t+is)=0$$

Thus

$$\sqrt{\tau}I(\tau)=\sqrt{2}I(2)=\sqrt{2\pi}$$

and hence

$$\int e^{-\frac{\tau}{2}x^2}dx=\sqrt{\frac{2\pi}{\tau}}. \tag{2.1}$$

Now we fix τ and study

$$\int e^{-\frac{\tau}{2}x^2}x^k dx.$$

This vanishes when k is odd, since then the integrand is an odd function. Let

$$\begin{aligned}J(k)=\int e^{-\frac{\tau}{2}x^2}x^{2k}dx=&\frac{2k-1}{\tau}J(k-1)\\ =&1*3*\cdots*(2k-1)\tau^{-k}\sqrt{\frac{2\pi}{\tau}}\\ =&\frac{1}{2^k k!}(\tau^{-1}\frac{d^2}{dx})^k x^{2k}\Big|_{x=0}\sqrt{\frac{2\pi}{\tau}}.\end{aligned}$$

Let p be a polynomial. It is a sum of monomials and hence

$$\int e^{-\frac{\tau}{2}x^2}p(x)dx=\sqrt{\frac{2\pi}{\tau}}\sum_{k=0}^{\infty}\frac{1}{2^k k!}(\tau^{-1}\frac{d^2}{dx})^k p(x)\Big|_{x=0}$$

The higher dimensional case is contained in the following lemma. Let $A=A_0+iA_1$ be a real symmetric $d\times d$ matrix with A_0 positive definite. This is equivalent to all eigenvalues λ_j being in $\{\lambda:\operatorname{Re}\lambda>0\}$. Let (a_{ij}) be the inverse. By an abuse of notation we set

$$\det(A)^{-1/2}=\prod\lambda_j^{-1/2}$$

where the λ_j are the eigenvalues of A.

Lemma 2.1. *Let p be a polynomial. Then*

$$\int e^{-\frac{1}{2}x^TAx}p(x)\,dx=(2\pi)^{d/2}(\det A)^{-1/2}\sum_{k=0}^{\infty}\frac{1}{2^k k!}(\sum_{i,j=1}^{d}a_{ij}\partial_{ij}^2)^k p(x)\Big|_{x=0}. \tag{2.2}$$

The sum contains only finitely many non-vanishing terms.

Proof. We begin with a fact from linear algebra and claim that there exists a real $d\times d$ matrix B and a diagonal matrix D such that

$$A=BDB^T.$$

By the Schur decomposition there is an orthogonal matrix O and a diagonal matrix D_0 with non-negative entries such that

$$A_0=OD_0O^T.$$

We set $B_0=O\sqrt{D_0}$. Then

$$A_0+iA_1=B_0(1+iB_0^{-1}A_1B_0^{-T})B_0^T$$

Again by the Schur decomposition there is an orthogonal matrix U with and a diagonal matrix D_1 with

$$B_0^{-1} A_1 B_0^{-T} = U D_1 U^T$$

hence

$$A_0 + iA_1 = B(1 + iD_1)B^T$$

with $B = B_0 U$. We set $D = 1_{\mathbb{R}^d} + iD_1$.

We change coordinates to $y = B^T x$. Then

$$\int e^{-\frac{x^T A x}{2}} p(x) dm^d(x) = (\det B)^{-1} \int e^{-\frac{y^T(1+iD_1)y}{2}} p(B^{-T} y) dm^d(y)$$

and by Fubini and the previous calculations

$$\int e^{-\frac{y^T(1+iD_1)y}{2}} y^\alpha dm^d(y) = 0$$

if one of the indices is odd, and otherwise, with d_j the diagonal entries of D_1,

$$\begin{aligned}\int e^{-\frac{y^T D y}{2}} y^{2\alpha} dm^d(y) =& (2\pi)^{d/2} \det(D)^{-1/2} \frac{1}{2^{|\alpha|}\alpha!} \prod ((1+id_j)^{-1} \partial^2_{y_j y_j})^{\alpha_j} y_j^{2\alpha_j} \Big|_{y=0} \\ =& (2\pi)^{d/2} \det(D)^{-1/2} \frac{1}{2^{|\alpha|}|\alpha|!} \Big[\sum_{j=1}^d (1+id_j)^{-1} \partial_j^2 \Big]^{|\alpha|} y^\alpha \Big|_{y=0}.\end{aligned}$$

Thus, for any polynomial q,

$$\int e^{-\frac{y^T D y}{2}} q(y) dm^d(y) = (2\pi)^{d/2} \det(D)^{-1/2} \sum_{k=0}^\infty \Big[\sum_{j=1}^d (1+id_j)^{-1} \partial_j^2 \Big]^k q(y) \Big|_{y=0}.$$

We complete the calculation by

$$(\det A)^{1/2} = (\det D)^{1/2} |\det B|$$

and, by the chain rule,

$$\sum a_{ij} \partial^2_{x_i x_j} p(x) = \Big[\sum_{j=1}^d (1+id_j)^{-1} \partial_j^2 \Big] p(B^{-T} y).$$

□

Observe that the formulas on the right hand side have a limit as A tends to a purely imaginary invertible matrix. We call the integral on the left hand side oscillatory integral in that limit.

Oscillatory integrals play a crucial role when studying dispersive equations. We consider

$$I = \int_{\mathbb{R}^s} a(\xi) e^{i\tau\phi(\xi)} d\xi.$$

where a and ϕ are smooth functions. The simplest result is

Lemma 2.2. *Suppose that $a \in C_0^\infty(\mathbb{R}^d)$, $\phi \in C^\infty(\mathbb{R}^d)$ with $\operatorname{Im}\phi \geq 0$ and*

$$|\nabla \phi| + \operatorname{Im} \phi > 0$$

on supp a. Given $N > 0$ there exists c_N with

$$|I(\tau)| \leq c_N \tau^{-N}.$$

The constant c depends only on N, the lower bound above, and derivatives up to order N.

Proof. By compactness there is $\kappa > 0$ such that

$$|\nabla \phi| + \operatorname{Im} \phi > \kappa$$

on supp $a(\xi)$. Using a partition of unity we may restrict to the two cases:

(1) $\operatorname{Im} \phi > \kappa/2$ on supp a, in which case we get a bound $Ce^{-\kappa\tau/2}$
(2) $|\nabla\phi| \ge \kappa/2$ on supp a, which we consider now.

We write

$$\begin{aligned}\int a(\xi)e^{i\tau\phi(\xi)}d\xi =&(i\tau)^{-1}\int a(\xi)|\nabla\phi|^{-2}\nabla\phi\nabla e^{i\tau\phi(\xi)}d\xi \\ =&-(i\tau)^{-1}\int (\nabla\cdot(\frac{a(\xi)\nabla\phi}{|\nabla\phi|^2}))e^{i\tau\phi(\xi)}d\xi.\end{aligned}$$

which is again an integral of the same type. Induction implies the full statement. $\square$

In many cases these bounds hold even for non compactly supported a.

Lemma 2.3. *Suppose that $A = A_0 + iA_1$ be invertible with A_0 positive semi-definite. Let $\eta \in C_0^\infty(\mathbb{R}^d)$ be identically 1 in a ball of radius 1, and supported in $B_2(0)$, and let a be a smooth function with uniformly bounded derivatives of order $M > \frac{N}{1-s}$ for some $M, N > 0$ and $0 < s < \frac{1}{2}$. Then*

$$\left|\int e^{-\frac{\tau}{2}x^TAx}e^{-\varepsilon|x|^2}a(x)(1-\eta(x\tau^{-s}))dm^d(x)\right| \le c_N\tau^{-N}$$

with c_N depending only on N, the norm of A and its inverse, and derivatives up to some order M of a, but not on $\varepsilon > 0$. The limit $\varepsilon \to 0$ exists.

We will use the formula with $\varepsilon = 0$.

Proof. We argue similarly to above. Each integration by parts gains as a factor τ, unless the derivative falls upon η. In that case the gain is only τ^{1-s} and we also loose a power of $|x|^{-1}$. Otherwise we get a factor $|x|^{-1} + |x|^{-2}$.

On the support of $\nabla\eta$

$$\tau^{-1}|x|^{-2} + \tau^{s-1}|x|^{-1} \le c\tau^{2s-1}.$$

and integrations by parts gain us τ^{2s-1}. If no derivatives fall on η there remains an integration over whole space, but we gain a factor bounded by constant times $|x|^{-1}$ the derivatives falls on a, and $|x|^{-2}$ if the derivative falls on $\frac{Ax+2\frac{\varepsilon}{\tau}x}{x^tAx+2\frac{\varepsilon}{\tau}|x|^2}$. We integrate by parts (and split the summands) until either

(1) M derivatives fall on a or
(2) $\frac{N}{1-2s} + d$ derivatives fall on the other terms.

The integrand (after the integrations by parts) converges pointwise with a majorant as above. This implies the statement on the limit as $\varepsilon \to 0$. $\square$

Similar statements hold for more general phase functions if

$$|\nabla\phi| \ge c|x|^\delta \qquad \text{for } |x| \ge R$$

and

$$|\partial^\alpha\phi| \le |x|^{-\delta}|\nabla\phi| \qquad \text{for } |x| \ge R$$

some R and δ, and $|\alpha| \ge 2$.

Lemma 2.4. *Let A be invertible, symmetric, with real part positive semi-definite, and $\psi \in C^\infty$ with bounded derivatives of order $\geq M$. Given $N > 0$ there exist $L > 0$ and $C_N > 0$ such that for $\tau > 0$,*

$$\left| \int e^{-\frac{\tau}{2}x^T Ax}\psi(x)dx - (2\pi)^{d/2}\tau^{-d/2}(\det A)^{-1/2}\sum_{k=0}^{L}\tau^{-k}(\sum_{ij} a_{ij}\partial^2)^k\psi\Big|_{x=0}\right| \leq c_N\tau^{-N}. \tag{2.3}$$

Proof. We subtract the Taylor expansion p of ψ at 0 up to some order L. We choose $0 < s < \frac{1}{2}$ and decompose the integral into

$$\int e^{-\frac{1}{2}x^T Ax}\left[p(x) + \eta(x\tau^{-s})(\psi(x) - p(x)) + [1 - \eta(x\tau^{-s})](\psi(x) - p(x))\right] dx.$$

The integral over the first summand has been evaluated in 2.1 The integral over the third summand is small by Lemma 2.3, and the one over the second summand is bounded by

$$\tau^{s(d-L)}$$

by a direct estimate. □

Now we consider

$$I(\tau) = \int e^{i\tau\phi(x)}\psi(x)dx$$

where ψ is compactly supported, 0 is the only point in the support where the imaginary part of ϕ and $\nabla\phi$ vanish, the imaginary part of ϕ is non-negative and the Hessian of ϕ at 0 is invertible.

Lemma 2.5. *Let $\frac{1}{3} < s < \frac{1}{2}$. Then, with η as above, $\psi \in C_0^\infty$ and $N > 0$*

$$\left| \int e^{i\tau\phi(x)}(1 - \eta(x\tau^{-s}))\psi(x)dx \right| \leq c_N\tau^{-N}.$$

Proof. The proof is the same as for the quadratic phase. Again this formula the compact support assumption on ψ can be weakened. □

We write

$$\phi(x) = a_0 + \frac{i}{2}x^T Ax + \psi(x)$$

where A is invertible and ψ is smooth with $\psi(x) = O(|x|^3)$.

Theorem 2.6 (Stationary phase)**.** *Under the assumptions above, given $N > 0$ there exists c_N such that for $\tau > 1$*

$$\left| \int e^{i\tau\phi}a(x)dx - (2\pi)^{d/2}\tau^{-d/2}(\det A)^{-1/2}e^{\phi(0)}\sum_{k=0}^{N}\frac{1}{2^k k!\tau^k}(a_{ij}\partial^2)^k[e^{i\tau\psi(x)}a(x)]_{x=0}\right| \leq c_N\tau^{-d/2-\frac{N+1}{3}}.$$

Proof. We assume that the real part of A is positive definite. The general statement follows then by an obvious limit.

We choose M large and write $e^{i\tau\psi}\psi = p_M(x) + r_M(x)$ where p_M is the Taylor polynomial of degree M, and r_M is the remainder term. Clearly p_M depends on τ with typical terms of the type being polynomials in τx^α where α is a multi-index of length at least 3, and x_j. We write the term in the bracket as a sum of three terms,

$$\int e^{i\tau\phi}\psi(x)(1-\eta(x\tau^s))dm^d(x)$$

$$\int e^{-\frac{\tau}{2}x^TAx}p_M(x)(1-\eta(x\tau^s))dm^d(x)$$

and

$$\int \eta(x\tau^s)\left[e^{i\tau\phi}\psi(x)-e^{-\frac{\tau}{2}x^TAx}p_M(x)\right]dm^d(x).$$

Lemma 2.3 and Lemma 2.5 control the first and the second term.

The integrand of the third term is bounded by

$$\tau^{\frac{M}{3}}\tau^{-Ms}$$

and hence the third term is bounded by a constant times

$$\tau^{-ds+M(\frac{1}{3}-s)}.$$

We choose s between $\frac{1}{3}$ and $\frac{1}{2}$ and M large. Finally we check the bound for the sum from $N+1$ to M term by term using Lemma 2.1. □

In the one dimensional setting the situation the Lemma of van der Corput provides an extremely useful and simple estimate.

Lemma 2.7. *Suppose that $d=1$, ψ is of bounded variation with support in $[c,d]$, $\phi\in C^k(\mathbb{R})$ with $k\ge 1$, ϕ real, and $\phi^{(k)}(\xi)\ge\tau$ for $\xi\in[c,d]$. If $k=1$ we assume in addition that ϕ' is monotone. Then*

$$I=\left|\int\psi(x)e^{i\phi(x)}dx\right|\le 3k\tau^{-1/k}\int|\psi'|dx.$$

Proof. We begin with $k=1$, assuming that ϕ' is monotone. It suffices to consider the case when the support of a is a compact interval $[c,d]$.

$$\begin{aligned}\left|\int\psi e^{i\phi}dx\right| &= \left|\int\psi/\phi'\frac{d}{dx}e^{i\phi}dx\right|\\ &=\left|\int e^{i\phi}\frac{d}{dx}(\psi/\phi')\right|\\ &\le\sup|\psi|\left|\frac{1}{\phi'(d)}-\frac{1}{\phi'(d)}\right|+\tau^{-1}\int|\psi'|\\ &\le\frac{3}{2}\tau^{-1}\int|\psi'|dx\end{aligned}$$

We use induction on k on the inequality

$$\left|\int\psi(x)e^{i\phi(x)}dx\right|\le 2k\tau^{-1/k}\left(\|\psi\|_{sup}+\|\psi'\|_{L^1}\right).$$

Suppose that the estimate holds for $k-1\ge 1$ and we want to prove it for k. Suppose that there is point ξ_0 with $\phi^{(k-1)}(\xi_0)=0$. We decompose the interval $[c,d]$ into $[c,\xi_0-\delta]$, $[\xi_0-\delta,\xi_0+\delta]$ and $[\xi_0+\delta,d]$. Then, by induction

$$|I|\le 2\delta\|\psi\|_{sup}+2(k-1)(\delta\tau)^{-1/(k-1)}(\|\psi\|_{sup}+\|\psi'\|_{L^1}).$$

We choose $\delta = \tau^{-\frac{1}{k}}$. Then

$$|I| \le 2k\tau^{-\frac{1}{k}}\tau^{-\frac{1}{k}}\left(\|\psi\|_{sup} + \|\psi'\|_{L^1}\right)$$

which implies the desired inequality. The argument is easier if there is no such point ξ_0. $\square$

2.1. **Examples and dispersive estimates.**

2.1.1. *The Schrödinger equation.* We consider the linear Schrödinger equation

$$i\partial_t u + \Delta u = 0$$

A Fourier transform (see next section, which we denote by $\mathcal{F}_x$) gives

$$i\partial_t \mathcal{F}_x u - |\xi|^2 \mathcal{F}_x u = 0$$

and hence the unique solution in the space of tempered distributions is given by its Fourier transform

$$\mathcal{F}_x u(t,\xi) = e^{-it|\xi|^2}\mathcal{F}_x u(0,\xi)$$

Then

$$\frac{1}{(2\pi)^{d/2}} \int e^{-it|\xi|^2} d\xi = \frac{1}{\sqrt{2it}^d}.$$

Moreover a change of coordinates shows that

$$\frac{1}{(2\pi)^{d/2}} \int e^{-i(t|\xi|^2 - x\xi)} d\xi = e^{i\frac{x^2}{4t}} \int e^{it\xi^2} d\xi = \frac{1}{\sqrt{2it}^d} e^{i\frac{x^2}{4t}}. \tag{2.4}$$

Again we suppress the approximation by a positive definite real part, and the corresponding limit.

2.1.2. *The Airy function and the Airy equation.* We consider the Airy equation

$$u_t + u_{xxx} = 0.$$

The Fourier transform transforms the equation to

$$\mathcal{F}_x u_t = (ik)^3 \mathcal{F}_x u$$

and hence, as above

$$\mathcal{F}_x u(t,\xi) = e^{it\xi^3}\mathcal{F}_x u(0)(\xi)$$

The Airy function is defined by

$$\mathrm{Ai}(x) = \frac{1}{2\pi} \int e^{i\frac{1}{3}\xi^3 + ix\xi} d\xi$$

where the right hand side has to be understood (as usual) as

$$\lim_{\varepsilon\to 0} \frac{1}{2\pi} \int e^{i\frac{1}{3}\xi^3 - \varepsilon|\xi|^2 + ix\xi} d\xi.$$

As above for the quadratic phase function we see that the limit exists at every point.

The phase function is

$$\phi(\xi) = \frac{1}{3}\xi^3 + x\xi$$

has as critical points the ξ which satisfy

$$\xi^2 = -x.$$

If x is negative there are two real critical points.

We choose $\rho \in C^\infty(\mathbb{R})$, supported in $[-1,\infty)$ and identically 1 in $[1,\infty]$, with $\rho(\xi)+\rho(-\xi) = 1$. Then $\mathrm{Ai}(x)$ is the real part of

$$\frac{1}{2\pi}\int \rho(\xi) e^{i(\frac13 \xi^3 + x\xi)} d\xi$$

There is no harm from the non-compact interval of integration and we to apply the stationary phase, Theorem 2.6 for $x \to -\infty$. The Hessian at the stationary points is $2\tau := 2(-x)^{1/2}$ and we write

$$\phi(\xi) = \tau \phi_0(\xi - (-x)^{1/2})$$

where

$$\phi_0(\eta) = \frac{1}{3\tau}\eta^3 + \frac12 \eta^2$$

which satisfies

$$\phi_0'(0) = 0, \phi_0''(0) = 1, \phi_0'''(0) = 2[-x]^{-1/2}.$$

We write the integral as

$$\frac{1}{2\pi} e^{-i\frac23 |x|^{\frac32}} \int \rho(\eta + (-x)^{1/2}) e^{i\tau\phi_0(\eta)} d\eta$$

The application of the stationary phase theorem, 2.6, gives

$$\left| \mathrm{Ai}(x) - \frac{1}{\sqrt{\pi}} |x|^{-1/4} \cos(\frac23 |x|^{\frac32} - \frac{\pi}{4}) \right| \le c|x|^{-\frac74}$$

and there is even an asymptotic series. To see the error term we compute the next term, the sixth derivative of $e^{i\phi_0(\eta)}$, evaluated at 0. It gives an additional factor $\tau^{-3} = |x|^{-\frac32}$.

For large positive x we need a different idea. For positive x there is fast decay and we want to determine the leading term. In this case the two critical points are purely imaginary real, and we shift the contour of integration to

$$\xi + i\sqrt{x}.$$

To be more precise we define

$$\mathrm{Ai}_\sigma(x) = \frac{1}{2\pi}\int e^{i[\frac13(\xi+i\sigma)^3 + x(\xi+i\sigma)]} d\xi.$$

We expand

$$i[\frac13(\xi+i\sigma)^3 + x(\xi+i\sigma) = i(\frac13\xi^3 + x\xi - \xi\sigma^2) - \sigma(\xi^2 + x - \frac13\sigma^2).$$

We calculate, using the Cauchy Riemann equations

$$\frac{d}{d\sigma}\mathrm{Ai}_\tau(x) = \frac{1}{2\pi} = \int i\frac{\partial}{\partial\xi} e^{i(\frac13\xi^3 + x\xi - \xi\sigma^2) - \sigma(\xi^2 + x - \frac13\sigma^2)} d\xi = 0$$

and hence, with $\sigma = \sqrt{x}$,

$$\mathrm{Ai}(x) = \frac{1}{2\pi}\int e^{i\frac13\xi^3 - \sqrt{x}\xi^2 - \frac23 x^{\frac32}} d\xi$$

with the critical point $\xi = 0$, at which point the Hessian is $2\sqrt{x}$. We argue as above and obtain

$$\left| \mathrm{Ai}(x) - \frac{1}{2\sqrt{\pi}} |x|^{-1/4} e^{-\frac23 x^{\frac32}} \right| \le c|x|^{-\frac74} e^{-\frac23 x^{\frac32}}. \tag{2.5}$$

The lemma of van der Corput ensures that the function Ai is bounded. More is true: About half a derivative of the Airy function is bounded in the following sense:

Lemma 2.8.
$$\left| \int |\xi|^{1/2} e^{i(\frac{1}{3}\xi^3 + x\xi)} d\xi \right| \le C$$

This is left as an exercise.

The Airy function is the inverse Fourier transform of

$$\widehat{\mathrm{Ai}}(\xi) = (2\pi)^{-1/2} e^{i\frac{1}{3}\xi^3}$$

Clearly

$$(\xi^2 + i\partial_\xi) e^{i\frac{1}{3}\xi^3} = 0$$

and hence

$$\mathrm{Ai}'' + x\,\mathrm{Ai} = 0$$

This however implies

$$(\partial_t + \partial^3_{xxx})((t/3)^{-1/3}\,\mathrm{Ai}(x(t/3)^{-1/3})) = 0$$

and (as oscillatory integral)

$$\int \mathrm{Ai}(x) dx = (2\pi)^{-1/2}$$

The convolution be the Airy function gives a solution to the initial value problem

$$u_t + u_{xxx} = 0, \qquad u(0,x) = u_0(x),$$
$$u(t,x) = \int (t/3)^{-1/3}\,\mathrm{Ai}((x-y)(t/3)^{-1/3}) u_0(y) dy.$$

Again the equation defines unitary operators $S(t)$ which satisfy

$$\|S(t)u_0\|_{sup} \le ct^{-1/3} \|u_0\|_{L^1}$$

and, in the sense of (2.8)

$$\||D|^{\frac{1}{2}} S(t) u_0\|_{sup} \le ct^{-\frac{1}{2}} \|u_0\|_{L^1}. \tag{2.6}$$

2.1.3. *Laplacian and related operators.* Let $d > 2$. Then

$$\widehat{|x|^{2-d}} = \frac{1}{2^{(d-4)/2}\Gamma(\frac{d-2}{2})} |\xi|^{-2}$$

and

$$-\Delta \frac{(4\pi)^{d/2}}{\Gamma(\frac{d-2}{2})} \int |x-y|^{2-d} f(y) dy = f(y).$$

The Fourier transform transforms higher partial derivatives into multiplication by monomial functions. For example

$$\mathcal{F}(u - \Delta u) = (1 + |\xi|^2)\hat{u}$$

and hence

$$\hat{u} = (1 + |\xi|^2)^{-1} \hat{f}$$

is the Fourier transform of a Schwartz function u (if f is a Schwartz function) which satisfies

$$-\Delta u + u = f.$$

Here $(1+|\xi|^2)^{-1}$ is a smooth function with bounded derivatives, but not a Schwartz function. Its inverse Fourier transform k allows to define a solution for a given function f by

$$u = (2\pi)^{d/2} k * f$$

We compute k in one space dimension

$$\int_{-\infty}^{\infty} e^{ix\xi}(1+\xi^2)^{-1} d\xi = \pi e^{-|x|} \tag{2.7}$$

using the residue theorem: The singular points are the zeroes of the polynomial $1+\xi^2$, which are $\pm i$. Consider the case $x > 0$ first. By the residue theorem

$$\int_{C_R} e^{ix\xi}(1+\xi^2)^{-1} d\vec{\xi} = \pi e^{-|x|}$$

where C_R is the union of the path from $-R$ to R and the upper semi circle. The limit $R \to \infty$ implies the statement.

2.1.4. *Gaussians, heat and Schrödinger equation.*

Lemma 2.9. *Let $A = A_0 + iA_1$ be an invertible symmetric matrix (A_0 and A_1 real) with A_0 positive semi-definite. Then*

$$\mathcal{F} e^{-\frac{1}{2}x^T A x}(\xi) = \det(A)^{-1/2} e^{-\frac{1}{2}\xi^T A^{-1}\xi}.$$

Proof. The formula is correct at $\xi = 0$ by Lemma 2.1 . We assume first that A_0 is positive definite. The general statement follows then by continuity of both sides. By definition

$$\nabla e^{-\frac{1}{2}x^T A x} + e^{-\frac{1}{2}x^T A x} A x = 0$$

The Fourier transform g is a Schwarz function which then satisfies

$$g\xi + A\nabla g = 0.$$

This is an ordinary differential equation on lines through the origin. There is a unique solution with the given value at $\xi = 0$, which has to coincide with the function on the right hand side. $\square$

With $A = 2t1_{\mathbb{R}^d}$ we obtain the formula for the fundamental solution to the heat equation. The inverse Fourier transform of $e^{-it|\xi|^2}$ is - as computed twice -

$$(2it)^{d/2} e^{-\frac{|x|^2}{4it}}$$

A solution to the Schrödinger equation

$$iu_t + \Delta u = 0$$

with initial data u_0 is given by

$$u(t,x) = \int_{\mathbb{R}^d} (4i\pi t)^{-d/2} e^{-\frac{|x-y|^2}{4it}} u_0(y) dy \tag{2.8}$$

We denote the map $u(0,.) \to u(t,.)$ by $S(t)$. It is defined by the Fourier transform by

$$\widehat{S(t)u_0} = e^{-it|\xi|^2} \hat{u}_0(\xi).$$

It is a unitary operator:

$$\|S(t)u_0\|_{L^2} = \|\widehat{S(t)u_0}\|_{L^2} = \|e^{-it|\xi|^2}\hat{u}_0\|_{L^2} = \|\hat{u}_0\|_{L^2} = \|u_0\|_{L^2}$$

and it satisfies the socalled dispersive estimate

$$\|S(t)u_0\|_{sup} \le |4\pi t|^{-d/2}\|u_0\|_{L^1}.$$

2.1.5. *The half-wave equation.* The solution to the wave equation

$$u_{tt} - \Delta u = 0$$

with initial data

$$u(0,x) = u_0(x), \qquad u_t(0,x) = u_1(x)$$

is given by Kirchhoff's formula for $d = 3$:

$$u(t,x) = \frac{1}{4\pi t^2}\int_{\partial B_t(x)} u_0 d\mathcal{H}^2 + \frac{1}{4\pi t}\int_{\partial B_t(x)} \partial_\nu u_0 d\mathcal{H}^2 + \frac{1}{4\pi t}\int_{\partial B_t(x)} u_1 d\mathcal{H}^2.$$

There are similar formulas in odd dimensions, and slightly more complicated ones in even dimensions.

The Fourier transform transforms the PDE to the ODE

$$\hat{u}_{tt} + |\xi|^2\hat{u} = 0$$

which factorizes into

$$(\partial_t - i|\xi|)(\partial_t + i|\xi|) = 0.$$

This motivated the study of the half wave equation

$$(i\partial_t + |\xi|)\hat{u}(t,\xi) = 0$$

which can easily be solved in the form

$$\hat{u}(t,\xi) = e^{it|\xi|}\hat{u}(0,\xi).$$

As above we restrict to $t = 1$. Since $e^{it|\xi|}$ is radial

$$\int e^{i(|\xi|+x\xi)}d\xi = dm^d(B_1(0))|x|^{-\frac{d-2}{2}}\int_0^\infty r^{d/2}e^{ir}J_{\frac{d-2}{2}}(|x|r)dr$$

provided the integrals exist as oscillatory integrals. They do as we will see. By Lemma 8.1 we can write

$$z^{\frac{d-1}{2}}J(z) = \mathrm{Re}(e^{-iz}\phi(z))$$

for $z \ge 1$, with ϕ satisfying

$$|\phi^{(k)}(z)| \le c_k z^{-k}.$$

We begin to consider $|x| \ge 2$ We decompose the integral above into two parts with a smooth cutoff function, one over $r \ge |x|^{-1}$, and one over $2|x|^{-1}$. In the first integral we integrate by parts as often as we like:

$$\int_0^\infty (1-\eta(r|x|))e^{ir(1\pm|x|)}p(rx)dx = \frac{i}{1\pm x}\int_0^\infty e^{ir(1\pm|x|)}(\frac{d}{dr}((1-\eta(r|x|))p(rx))dx$$

which gains a factor r in the integration, as well as a power $|x|^{-1}$. We repeat this as often as necessary. The second integral is bounded by $|x|^d$.

The same arguments apply as for $|x| \ne 1$, given bounds which depend only on $|x| - 1$. A careful calculation gives the first part of the following estimate

Lemma 2.10. *Suppose that* $|x| \neq 1$ *then*

$$\left| \int e^{i|\xi|+ix\xi} d\xi \right| \leq \begin{cases} c_d |1-|x||^{-\frac{d+1}{2}} & \text{if } |x| \leq 2 \\ c_d |x|^{-d} & \text{if } |x| \geq 2 \text{ and } d \text{ even} \end{cases}$$

and

$$\left| \int |\xi|^{-\frac{d+1}{2}} e^{i|\xi|+ix\xi} d\xi - c \ln |1-|x|| \right| \leq \ c_d$$

if $|x| \leq 2$ *and*

$$\left| \int |\xi|^{-\frac{d+1}{2}} e^{i|\xi|+ix\xi} d\xi \right| \leq c_d |x|^{-\frac{d-1}{2}}.$$

for $|x| \geq 2$.

Proof. Only the second part remains to be shown. There is no difference in the argument for $|x| \leq 2$, unless $|x|$ is close to 1. In that case we decompose the integral into $r \leq 2$, $1 \leq r \leq |x|-1$ and $r \geq |x|-1$. The last part is bounded by the previous arguments. The first part is bounded because of the size $r \leq 1$. The second part is

$$\int_1^{||x|-1|} r^{-1} dr = \ln r$$

plus something bounded. □

There is an important difference compared to the previous two examples: the group velocity depends only on the direction of ξ, not on the amplitude.

2.1.6. *The Klein-Gordon half wave.* Let

$$g(t,x) = \int e^{it\sqrt{1+|\xi|^2}+ix\xi} d\xi.$$

As above we obtain

Lemma 2.11. *The following estimates hold for* $t \geq 1$,

$$|g(t,x)| \leq c \begin{cases} t^{-d/2}(1-|x|/t)^{-\frac{d+1}{2}} & \text{if } |x| < t \\ t^{-d}(|x|/t-1)^{-\frac{d+1}{2}} & \text{if } t < |x| \leq 2t \\ \frac{1}{|x|^d t^{d-1}} & \text{if } |x| \geq 2t \end{cases}$$

and if $0 < t < 1$

$$|g(t,x)| \leq c \begin{cases} t^{-d} & \text{if } |x| < t \\ t^{-d}(|x|/t-1)^{-\frac{d+1}{2}} & \text{if } t < |x| \leq 2t \\ \frac{1}{|x|^d t^{d-1}} & \text{if } |x| \geq 2t \end{cases}$$

Moreover

$$h = \int |\xi|^{-\frac{d+1}{2}} e^{it\sqrt{1+|\xi|^2}+ix\xi} d\xi$$

satisfies for $t \geq 1$ *and* $|x| \geq 2t$

$$|h(t,x)| \leq C \frac{1}{|x|^{\frac{d-1}{2}} t^{\frac{d}{2}-3}}$$

$$\left| h(t,x) - ct^{\frac{1}{2}} |\ln ||1-|x|/t|| \right| \leq ct^{\frac{1}{2}}$$

for $1 \leq t$, $|x| \leq 2t$. *Finally, if* $0 < t \leq 1$, *then*

$$\left| \int |\xi|^{-\frac{d+1}{2}} e^{it|\xi|+ix\xi} d\xi - ct^{-\frac{d-1}{2}} \ln |1-|x|| \right| \leq c_d t^{-\frac{d-1}{2}}$$

if $|x| \le 2t$ *and*

$$\left| \int |\xi|^{-\frac{d+1}{2}} e^{i|\xi| + ix\xi} d\xi \right| \le c_d \frac{1}{|x|^{\frac{d-1}{2}} t^{\frac{d}{2}-3}}.$$

for $|x| \ge 2$.

2.1.7. *The Kadomtsev-Petviashvili equation.* The linear parts of the Kadomtsev-Petviashvili equations are

$$u_t + u_{xxx} \pm \partial_x^{-1} u_{yy} = 0$$

where $+$ is the linear KP-II equation and $-$ the linear KP-I equation. The equation should be understood as

$$\partial_x u_t + u_{xxxx} \pm u_{yy} = 0,$$

We denote the Fourier variables by ξ (of x) and η (of y). As above (for $+$, the argument for $-$ is very similar),

$$\mathcal{F}_{x,y} u(t, \xi, \eta) = e^{it(\xi^3 - \xi^{-1}\eta^2)} \mathcal{F}_{x,y} u(0, \xi, \eta)$$

and

$$\int e^{i[(\xi^3 - \xi^{-1}\eta^2) + x\xi + y\eta]} d\xi d\eta = (4\pi)^{-1/2} \int (-i\xi)^{\frac{1}{2}} e^{i[\xi^3 + \xi x + \xi y^2/4]} d\xi.$$

The stationary points of the phase function satisfy

$$3\xi^2 + x + y^2/4 = 0$$

with zeroes

$$\xi = \pm \sqrt{-(x + y^2/4)/3}$$

provided

$$x < -\frac{1}{4} y^2.$$

The contribution from the Hessian compensates the factor $(-i\xi)^{\frac{1}{2}}$. A rigorous proof uses a smooth partition of unity, which decomposes the integral into one around $\xi = 0$, one over $\xi \ge 1$ and one with $\xi \le -1$. The first integral is handled by the lemma of van der Corput, and the other two by stationary phase.

Otherwise, by the non-degeneracy of the phase

$$\left| \int e^{i[(\xi^3 - \xi^{-1}\eta^2) + x\xi + y\eta]} d\xi d\eta \right| \le c_k |x + y^2/4|^{-k}$$

The t dependence below is obtained by scaling

Lemma 2.12.

$$\left| \int e^{it(\xi^3 \mp \eta^2/\xi) + ix\xi} d\xi d\eta \right| \le c_k |t|^{-1} (1 + (\frac{x}{t^{\frac{1}{3}}} \pm \frac{y^2}{t^{\frac{2}{3}}})_+)^{-k}.$$

There is an interesting interpretation:

- Waves move to left for Kadomtsev-Petviashvili II,
- and to both sides for Kadomtsev-Petviashvili I (with respect to x)

This makes the study of Kadomtsev-Petviashvili I considerably harder than the study of Kadomtsev-Petviashvili II.

We define

$$\rho(x, y) = 2\pi \mathcal{F}^{-1}(e^{i(\xi^3 - \eta^2/\xi)}).$$

Since $u(\lambda^3 t, \lambda x, \lambda^2 y)$ satisfies the linear KP equation for $\lambda > 0$ if and only if u does we obtain the representation

$$u(t,x,y) = g_t * u(0,.,.)(x,y)$$

where

$$g_t(x,y) = t^{-1}\rho(x/t^{1/3}, y/t^{2/3}).$$

Hence, with $S(t)$ denoting the evolution operator,

$$\|S(t)u_0\|_{L^2} = \|u_0\|_{L^2}$$

and

$$\|S(t)u_0\|_{sup} \le c|t|^{-1}\|u_0\|_{L^1(\mathbb{R}^2)}.$$

3. Strichartz estimates and small data for the Nonlinear Schrödinger equation

3.1. **Strichartz estimates for the Schrödinger equation.** We return to the linear Schrödinger equation

$$i\partial_t u + \Delta u = 0$$

and the unitary operators $S(t): u(0) \to u(t)$. They form a group: For $s, t \in \mathbb{R}$

$$S(t+s) = S(t)S(s).$$

We claim that for $2 \le p \le \infty$ and p' with $\frac{1}{p} + \frac{1}{p'} = 1$

$$\|S(t)\|_{L^p} \le (4\pi|t|)^{-\frac{d}{2}(1-\frac{2}{p})}\|u_0\|_{L^{p'}}, \tag{3.1}$$

which follows by complex interpolation from

$$\|S(t)u_0\|_{L^2} = \|u_0\|_{L^2}$$

and the dispersive estimate

$$\|S(t)u_0\|_{L^\infty} \le (4\pi|t|)^{-\frac{d}{2}}\|u_0\|_{L^1}.$$

Let us be more precise. We put $p_0 = q_0 = 2$ and $p_1 = 1$, $q_1 = \infty$, $2 < \tilde{p} < \infty$ and determine λ so that

$$\frac{1-\lambda}{2} = \frac{1}{p},$$

resp.

$$\lambda = 1 - \frac{2}{p}$$

define

$$\frac{1-\lambda}{2} + \lambda = \frac{1}{q}.$$

We check easily

$$\frac{1}{p} + \frac{1}{q} = 1,$$

and obtain by the complex interpolation theorem (7.6) of Riesz-Thorin (3.1).

The variation of constants formula resp. Duhamel's formula

$$u(t) = -i\int_{-\infty}^{t} S(t-s)f(s)ds$$

defines a solution to

$$i\partial_t u + \Delta u = f$$

at least for Schwartz functions f in $d+1$ variables.

From the $L^{p'}$ to L^p estimate (3.1) one obtains

$$\|u(t)\|_{L^p} \le (4\pi)^{-\frac{d}{2}(1-\frac{2}{p})} \int_{-\infty}^{t} |t-s|^{\frac{d}{2}-\frac{d}{p}} \|f(s)\|_{L^{p'}} ds.$$

The right hand side is a convolution $h * g$ where

$$h(t) = \begin{cases} 0 & \text{if } t \ge 0 \\ |4\pi t|^{-d(\frac{1}{2}-\frac{1}{p})} & \text{if } t < 0 \end{cases}$$

and

$$g(t) = \|f(t)\|_{L^{p'}(\mathbb{R}^d)}.$$

An immediate calculation gives $|t|^{-1/r} \in L^r_w(\mathbb{R})$ and by the weak Young inequality of Proposition 7.2

$$\|g * h\|_{L^q(\mathbb{R})} \le c\|g\|_{L^{q'}} \|h\|_{L^r_w} \tag{3.2}$$

where

$$\frac{1}{r} = d(\frac{1}{2} - \frac{1}{p}), r > 1.$$

and p and q are strict Strichartz pairs, i.e. numbers which satisfy

$$\frac{2}{q} + \frac{d}{p} = \frac{d}{2}. \tag{3.3}$$

and $2 < q \le \infty, 2 \le p \le \infty$. The left hand side of (3.2) controls

$$\|u\|_{L^q_t L^p_x} := \Big(\int \|u(t)\|^q_{L^p(\mathbb{R}^d)} dt \Big)^{1/q}$$

with the obvious modification if $q = \infty$ and we obtain

$$\|u\|_{L^q_t L^p_x} \le c\|f\|_{L^{q'}_t L^{p'}_x}$$

for all strict Strichartz pairs. Here $L^q_t L^p_x$ consists of all equivalence classes of measurable functions such that the integral expression for the norm is finite.

It is not hard to see that u measurable implies

$$t \to \|u(t, .)\|_{L^p}$$

is measurable, the expression for the norm actually defines a norm, and the space is closed and hence a Banach space. The duality of the Lebesgue spaces extends to duality of this mixed norm spaces: The map

$$L^{p',q'} \ni f \to (g \to \int f g dm^d dt) \in (L^{p,q})^*$$

is an isometry if $1 \le p, q \le \infty$ and surjective if $p, q < \infty$. Complex interpolation extends to the mixed norm spaces - this is quite evident from the definition.

We claim

Theorem 3.1. *The variation of constants formula defines a function u which satisfies*

$$i\partial_t u + \Delta u = f, \qquad u(0) = u_0$$

and let (q, p) be a strict Strichartz pair. Then

$$\|u\|_{C_b(\mathbb{R},L^2)} + \|u\|_{L^q L^p} \le c\Big(\|u(0)\|_{L^2} + \|f\|_{L^{q'} L^{p'}}\Big).$$

We will later improve this statement in several directions. Denote by T,

$$L^2 \ni v \to Tv \in C([0,\infty), L^2)$$

the operator which maps the initial data to the solution. Let (p,q) be Strichartz pairs. Then

$$\|T\|^2_{L(L^2,L^{q'}L^{p'})} = \|T^*\|^2_{L(L^{q,p},L^2)} = \|TT^*\|_{L(L^{q,p},L^{q',p'})}$$

and

$$TT^*f(t) = \int_0^\infty S(t+s)f(s)ds = \int_{-\infty}^0 S(t-s)f(-s)ds$$

and the bound follows as above.

3.2. **Strichartz estimates for the Airy equation.** This section follows Kenig, Ponce and Vega [14]. Scaling shows that the solution to the Airy equation satisfies

$$u(t,x) = \frac{1}{(t/3)^{1/3}} \int \mathrm{Ai}((x-y)/(t/3)^{\frac{1}{3}})u(0,y)dy$$

and we obtain the estimates

$$\|u(t)\|_{L^2} = \|u_0\|_{L^2}$$

$$\|u(t)\|_{L^\infty} \le ct^{-1/3}\|u_0\|_{L^1}$$

and

$$\||D|^{\frac{1}{2}}u(t)\|_{L^\infty} \le ct^{-\frac{1}{2}}\|u_0\|_{L^1}.$$

The Strichartz estimate is more complicated. Here we use complex interpolation to see for $2 < p \le \infty$

$$\|D^{\frac{1}{2}-\frac{1}{p}}S(t)v\|_{L^p} \le c|t|^{\frac{1}{p}-\frac{1}{2}}\|v\|_{L^{p'}} \tag{3.4}$$

where D^s is defined through the Fourier multiplier. The multiplication on the Fourier side commutes with the evolution, and hence this estimates is equivalent to

$$\|D^{\frac{1}{q}}S(t)v\|_{L^p(\mathbb{R})} \le c|t|^{-\frac{2}{q}}\|D^{-\frac{1}{q}}v\|_{L^{p'}}.$$

The Strichartz estimates take the form

Theorem 3.2. *The variation of constants formula defines a function u which satisfies*

$$\partial_t u + u_{xxx} = f, \qquad u(0) = u_0$$

and

$$\|u\|_{C_b(\mathbb{R},L^2)} + \||D|^{\frac{1}{q}}u\|_{L^qL^p} \le c\left(\|u(0)\|_{L^2} + \||D|^{-\frac{1}{q'}}f\|_{L^{q'}L^{p'}}\right)$$

for all Strichartz pairs (q,p).

Proof. It remains to prove (3.4).

We claim that it follows from

$$\left|\int |\xi|^{\frac{1}{2}+i\sigma}e^{i\xi^3+i\xi x}d\xi\right| \le C(1+|\sigma|) \tag{3.5}$$

uniformly in x - which has to be understood as oscillatory integral. We apply then complex interpolation with the family of operators

$$\widehat{T_\lambda u_0} = e^{\lambda^2}|D|^{\frac{\lambda}{2}}\widehat{S(t)u_0}$$

for which we easily see that

$$\|T_{i\sigma}u_0\|_{L^2} = e^{-\sigma^2}\|u_0\|_{L^2}$$

and

$$\|T_{i+\sigma}u_0\|_{L^\infty} \le ct^{-1/2}(1+|\sigma|)e^{-\sigma^2}\|u_0\|_{L^1}.$$

Now (3.4) follows by complex interpolation. We turn to (3.5).

There are three cases: $|x| \le 10$, $x \ge 10$ and $x \le -10$. The last one is the hardest since there are large critical points $\pm\xi_c = \sqrt{-x/3}$ in the phase, and we restrict to it. We split the integration into the intervals

$$(-\infty, -\xi_c - |x|^{-1/4}), (-\xi_c - |x|^{-1/4}, \xi_c + |x|^{-1/4}), (-\xi_c + |x|^{-1/4}, -1), (-1,1),$$

$$(1, \xi_c - |x|^{-1/4}), (\xi_c - |x|^{-1/4}, \xi_c + |x|^{-1/4}), (\xi_c + |x|^{-1/4}, \infty)$$

The argument is immediate for the second, the fourth and the sixth integral, which we estimate by $3\xi_c^{1/2}|x|^{-1/4}$. Now

$$\begin{aligned}\int_{-\infty}^{-\xi_c - |x|^{-1/4}} |\xi|^{\frac12 + i\sigma} e^{i\xi^3 + ix\xi} d\xi =& i\int_{-\infty}^{-\xi_c-|x|^{-1/4}} e^{i\xi^3+ix\xi}\frac{d}{d\xi}\frac{|\xi|^{\frac12+i\sigma}}{3\xi^2+x}d\xi \\ &+ \frac{(\xi_c+|x|^{-1/4})^{\frac12+i\sigma}}{3(\xi_c+|x|^{-1/4})^2+x}e^{-i(\xi_c+|x|^{-1/4})^3 - i(\xi_c+|x|^{-1/4})x}\end{aligned}$$

and the direct estimate as for stationary phase gives the result. The largest term (in terms of σ) occurs when the derivative falls on $|\xi|^{\frac12+i\sigma}$ - all the others are estimates as when $\sigma = 0$. We recall that

$$3(\xi_c + |x|^{-1/4})^2 + x \sim |x|^{\frac14}.$$

□

3.3. **The Kadomtsev-Petviashvili equation.** The symbol is $\xi^3 - \eta^2/\xi$, with gradient

$$\begin{pmatrix} 3\xi^2 + \eta^2/\xi^2 \\ -2\eta/\xi \end{pmatrix}$$

and Hessian matrix

$$\begin{pmatrix} 6\xi - 2\eta^2/\xi^3 & 2\eta/\xi^2 \\ 2\eta/\xi^2 & -2/\xi \end{pmatrix}$$

and Hessian determinant -12.

Lemma 3.3. *The following Strichartz estimate holds*

$$\|u\|_{L^\infty_t L^2_x} + \|u\|_{L^p_t L^q_x} \le c\left(\|u_0\|_{L^2} + \|f\|_{L^{p'}_t L^{q'}_x}\right).$$

The proof is the same (since the same dispersive estimate holds) as for the Schrödinger equation.

3.4. **The (half) wave equation and the Klein-Gordon equation.** Here we only state the result. The proof requires a sharpening of complex interpolation, replacing L^∞ by BMO. The estimates for the wave equation imply that

$$\| |D|^{-\frac{d+1}{2}} S(t)v\|_{BMO} \le ct^{-\frac{d-1}{2}} \|v\|_{L^1(\mathbb{R}^d)}$$

which implies

$$\| |D|^{-\frac{d+1}{2}(1-\frac{2}{p})} S(t)v\|_{L^p} \le ct^{-\frac{d-1}{2}(1-\frac{2}{p})} \|v\|_{L^{p'}}$$

where the half wave evolution operator $S(t)$ is defined by

$$S(t)v = \mathcal{F}^{-1}(e^{it|\xi|}\hat{v}).$$

As a consequence we obtain

Theorem 3.4. *Let $d \ge 2$. The variation of constants formula defines a function u which satisfies*

$$i\partial_t u + |D|u = f, \qquad u(0) = u_0$$

and

$$\|u\|_{C_b(\mathbb{R},L^2)} + \| |D|^{-\frac{d+1}{4}(1-\frac{2}{p})} u\|_{L^q L^p} \le c\|u(0)\|_{L^2} + \| |D|^{\frac{d+1}{4}(1-\frac{2}{p})} f\|_{L^{q'} L^{p'}}.$$

where q satisfies $2 < q < \infty$, $2 \le p \le \infty$ and

$$\frac{1}{q} + \frac{d-1}{p} = \frac{d-1}{2}.$$

3.5. **The endpoint Strichartz estimate.** We prove the endpoint Strichartz estimate for the Schrödinger equation

$$iu_t + \Delta u = f \qquad u(0) = u_0$$

for $d \ge 3$. The argument is due to Keel and Tao [13] and it applies to much more general situations.

Theorem 3.5. *The solution defined by the variation of constants formula satisfies*

$$\|u\|_{L^\infty_t L^2_x} + \|u\|_{L^2_t L^{\frac{2d}{d-2}}} \le c\left(\|u_0\|_{L^2} + \|f\|_{L^2_t L_x^{\frac{2d}{d+2}}} \right). \tag{3.6}$$

Before we prove the statement we need a robust estimate for integral operators.

Lemma 3.6 (Schur's lemma)**.** *Let μ and ν be measures,*

$$Tf(x) = \int K(x,y) f(y) d\mu(y)$$

where K satisfies

$$\sup_x \int |K(x,y)| d\mu(y) \le C_x, \quad \sup_y \int |K(x,y)) d\nu(y) \le C_y.$$

Then

$$\|Tf\|_{L^p(\nu)} \le C_x^{1-\frac{1}{p}} C_y^{\frac{1}{p}} \|f\|_{L^p(\mu)}.$$

Proof. By duality the claim is equivalent to

$$\left| \int f(x) g(y) K(x,y) d\mu(y) d\nu(x) \right| \le C_x^{1-\frac{1}{p}} C_y^{\frac{1}{p}} \|f\|_{L^{p'}(\mu)} \|g\|_{L^p(\mu)}.$$

This is obvious for $p = \infty$ and $p = 1$. Hence the operator satisfies the desired bounds on L^1 and L^∞. The claim follows by complex interpolation. $\square$

Proof. We denote by $S(t)$ the Schrödinger group. We first prove

$$\left| \int_{s<t} \langle S(-t)f(t), S(-s)g(s)\rangle \right| \le c\|f\|_{L^2_t L^{\frac{2d}{d-2}}} \|g\|_{L^2_t L^{\frac{2d}{d-2}}} \tag{3.7}$$

which implies by duality

$$\Big\| \int_{-\infty}^t S(t-s)f(s) \Big\|_{L^2 L^{\frac{2d}{d-2}}} \le c\|f\|_{L^2_t L^{\frac{2d}{d+2}}}$$

and, by the TT^* argument the full statement.

We define

$$T_j = \int_{t-2^{j+1}<s\le t-2^j} \langle S(-s)f(s), S(-t)g(t)\rangle dsdt$$

and claim

$$|T_j| \le C2^{-j\beta(p,\tilde p)}\|f\|_{L^2 L^{p'}} \|g\|_{L^2 L^{\tilde p'}} \tag{3.8}$$

for $j \in \mathbb{Z}$, p and $\tilde p$ in a neighborhood of $\frac{2d}{d+2}$ and

$$\beta(p,\tilde p) = \frac{d}{2} - 1 + \frac{d}{2p} + \frac{d}{2\tilde p}.$$

It vanishes for $p = \tilde p = \frac{2d}{d-2}$ as it should.

We set $\tilde t = t2^{-j}$, $\tilde s = s2^{-j}$, $\tilde x = 2^{-j/2}x$ and $\tilde y = 2^{-j/2}y$. This transformation of coordinates (which reflects the symmetry) reduces the estimate to the case $j = 0$.

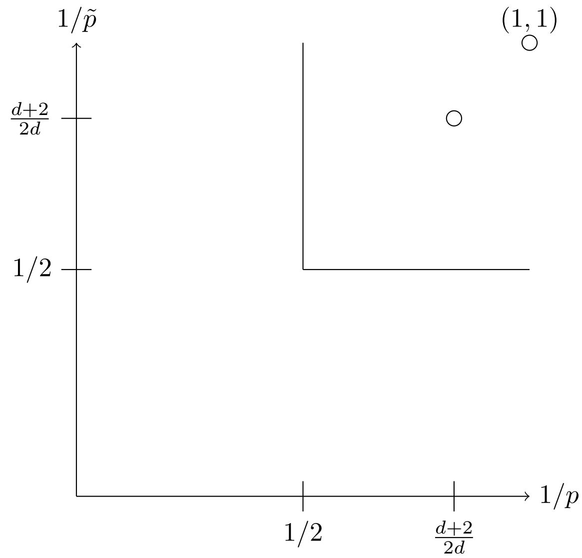
The estimate for $j = 0$

$$|T_0| \le C\|f\|_{L^2 L^{p'}} \|g\|_{L^2 L^{\tilde p'}} \tag{3.9}$$

holds for

(1) $p = \tilde p = 1$ by the dispersive estimate
(2) $\tilde p = 2$ and $\frac{2d}{d+2} < p' \le 2$
(3) $p = 2$ and $\frac{2d}{d+2} \le \tilde p' \le 2$

Then the estimate (3.8) follows by complex interpolation and duality. It is convenient to draw a diagram



Convex interpolation - this time for $L^{2,p'}$ spaces gives the convex envelope which contains the point $(\frac{d+2}{2d}, \frac{d+2}{2d})$ in its interior.

For the first case (which corresponds to $(1,1)$) observe that by the dispersive estimate, if $t-2 < s < -1$

$$|\langle S(t-s)g(s), f(t)| \le C\|f(t)\|_{L^1}\|g(s)\|_{L^1}$$

Let $h_f(t) = \|f(t)\|_{L^1}$ and $h_g(t) = \|g(t)\|_{L^1}$. Then

$$|T_0(f,g)| \le C\int\int K(t,s)h_g(s)ds h_f(t)dt$$

where $K(t-s) = 1$ if $t-2 < s < t-1$ and 0 otherwise. The first estimate follows by Schur's lemma.

For the second estimate (which corresponds to the horizontal line) we use nonendpoint Strichartz estimate and finally Hölder's inequality to bound

$$\begin{aligned}\left|\int_{s+1}^{s+2}\langle f(t), S(t-s)g(s)\rangle dt\right| \le & \|f\|_{L^{q'p'}([s+1,s+2]\times\mathbb{R}^d}\|S(t-s)g(s)\|_{L^{qp}} \\ \le & C\|f\|_{L^{2,p'}([s+1,s+2]\times\mathbb{R}^d}\|g(s)\|_{L^2}.\end{aligned}$$

where (q,p) is a strict Strichartz pair.

Thus

$$\left|\int_k^{k+1}\int_{t-2}^{t-1}\langle S(-t)f(t), S(-s)g(s)\rangle dsdt\right| \le c\|f\|_{L^{2,p'}([k,k+1]\times\mathbb{R}^d)}\|g\|_{L^{2,2}([k-2,k]\times\mathbb{R}^d)}.$$

The statement follows by summation with respect to k, and the Cauchy-Schwartz inequality with respect to k.

The third estimates follows by the same argument. This completes the estimate (3.8) for $(p,\tilde{p})$ close to $(\frac{2d}{d-2}, \frac{2d}{d-2})$.

To make use of the flexility we decompose $f = \sum f_k$, $g = \sum g_k$ such that

$$f_k(t,x) = c_k(t)\chi_{t,k}(x), g_k(t,x))d_k(t)\tilde{\chi}_{t,k}(x).$$

We define the decomposition as follows. Given $f:\mathbb{R}^d\to\mathbb{R}$ we define its distribution function

$$\lambda(s)=m^d\{x:|f(x)|>s\}.$$

It is monotonically decreasing and finite for $f\in L^p$. Let s_k be the infimum of all s so that $\lambda(s)<2^k$ - we allow $s=0$. We set $c_k=2^{k/p}s_k$ and

$$\chi_k(x)=c_k^{-1}\begin{cases} f & \text{if } s_k<|f|<s_{k+1}\\ 0 & \text{otherwise}\end{cases}$$

Then

$$f=\sum c_k\chi_k$$

and, for some $C>0$

$$C^{-1}\|f\|_{L^p}\le\|(c_k)\|_{l^p}\le C\|f\|_{L^p}$$

which can be seen by comparing to

$$\|f\|_{L^p}^p=p\int m^d(\{|f|>s\})s^{p-1}ds.$$

By definition

$$m^d(\text{ supp }\chi_k)\le 2^k \qquad |\chi_k|\le 2^{k/p}.$$

We apply this decomposition at every time t with $p=\frac{2d}{d+2}$. Then

$$f=\sum f_k$$

where at most one summand differs from 0.

We apply the first estimate (3.8):

$$\begin{aligned}|T_j(f_k,g_{k'})|\le&c2^{-\beta(p,\tilde p)}\|f_k\|_{L^{2,p'}}\|g_{k'}\|_{L^{2,\tilde p'}}\\ \le&c2^{-\left(\frac{d-2}{2}+\frac{d}{2p}+\frac{d}{2\tilde p}\right)j+k\left(\frac{1}{p'}+\frac{1}{p}-\frac{d+2}{d}\right)}\|f_k\|_{L^{2,\frac{2d}{d+2}}}\|g_{k'}\|_{L^{2,\frac{2d}{d+2}}}\end{aligned}$$

where the second inequality follows from

$$\|\chi_{t,k}\|_{L^p}\le c2^{k(\frac{1}{p}-\frac{2d}{d+2})}$$

We optimize p and $\tilde p$. Thus

$$|T_j(f_k,g_{k'})|\lesssim 2^{-\varepsilon(|k-jd/2|+|k'-jd/2|)}\|f_k\|_{L^2L^{\frac{2d}{d+2}}}\|g_{k'}\|_{L^2L^{\frac{2d}{d+2}}}$$

for some $\varepsilon>0$ and e sum with respect to j:

$$\begin{aligned}\sum_j|T_j|\le&C\sum_k\sum_{k'}(1+|k-k'|)2^{-\varepsilon|k-k'|}\|f_k\|_{L^2L^{\frac{2d}{d+2}}}\|g_k\|_{L^2L^{\frac{2}{d}d+2}}.\\ \le&C\left(\sum_k\|f_k\|^2_{L^2L^{\frac{2d}{d+2}}}\right)^{1/2}\left(\sum_k\|g_k\|^2_{L^2L^{\frac{2d}{d+2}}}\right)^{1/2}\end{aligned}$$

by Schur's lemma. By Minkowski's inequality

$$\begin{aligned}\sum_k \int \left(\int_{\mathbb{R}^d} |g_k|^{\frac{2d}{d+2}} dm^d\right)^{\frac{d+2}{d}} dt =& \int \sum_k \left(\int_{\mathbb{R}^d} |g_k|^{\frac{2d}{d+2}} dm^d\right)^{\frac{d+2}{d}} dt \\ \le& \int \left(\int_{\mathbb{R}^d} \sum_k |g_k|^{\frac{2d}{d+2}} dm^d\right)^{\frac{d+2}{2}} dt \\ =& \|g\|^2_{L^{2,\frac{2d}{d+2}}}\end{aligned}$$

and hence we obtain (3.7).

$\square$

3.6. **Small data solutions to the nonlinear Schrödinger equation.** Most of this section can be found in [4].

We study the initial value problem for initial data $u_0 \in L^2$ for

$$iu_t + \Delta u = \pm |u|^\sigma u \tag{3.10}$$

where $0 \le \gamma \le \frac{4}{d-2}$. The case of the plus sign is called defocusing and case of the minus sign is called focusing. At least formally

$$M = \int_{\mathbb{R}^d} |u|^2 dx$$

called mass, and

$$\int_{\mathbb{R}^d} iu\partial_i \bar{u} dx$$

called momentum

$$E = \int_{\mathbb{R}} \frac{1}{2}|\nabla u|^2 \pm \frac{1}{\sigma+2}|u|^{\sigma+2} dx$$

called energy are conserved. For most of this section there is no distinction between the focusing and the defocusing case.

The argument will rely on the Strichartz estimates with $p = q = \frac{2(d+2)}{d}$ and $p' = q' = \frac{2(d+2)}{d+4}$.

The sign of the coefficients is of almost no importance in this section, and we choose + to cover both signs, indicating differences whenever necessary. This section establishes basic schemes which will be used over and over again. Simultaneously it is a warm up the set up and the consequences of the key multilinear estimate. Lateron we will restrict ourselves often to giving the estimates of the nonlinearity, and stating the properties.

It provides also a play ground for stability estimates, qualitative properties, criticality and subcriticality.

3.7. **Initial data in L^2.** Our approach will be based on the Strichartz estimates of Theorem 3.1 with $p = q = \frac{(2(d+2)}{d}$.

$$\|v\|_{L^{\frac{2(d+2)}{d}}(\mathbb{R}\times\mathbb{R}^d)} + \|v\|_{C(\mathbb{R};L^2(\mathbb{R}^d))} \lesssim \|v(0)\|_{L^2(\mathbb{R}^d)} + \|i\partial_t v + \Delta v\|_{L^{\frac{2(d+2)}{d+4}}(\mathbb{R}\times\mathbb{R}^d)} \tag{3.11}$$

In order to prepare for variants and improvements we assume that there is a space X with

$$X \subset C(\mathbb{R}; L^2(\mathbb{R}^d)) \cap L^{\frac{2(d+2)}{d}}(\mathbb{R}\times\mathbb{R}^d) \tag{3.12}$$

and

$$\sup_t \|v(t)\|_{L^2} + \|v\|_{L^{\frac{2(d+2)}{d}}} \le c\|v\|_X$$

and

$$\|v\|_X \le c\left(\|v(0)\|_{L^2} + \|i\partial_t v + \Delta v\|_{L^{\frac{2(d+2)}{d+4}}(\mathbb{R}\times\mathbb{R}^d)}\right).$$

Clearly such a space exists: We could define X as the intersection in (3.12), and then the Strichartz estimates ensure that it has the desired properties. The choice of the function space is an important and nontrivial part of studying solutions to many different dispersive equations. Even though we do not need this flexibility here, and even though it complicates the notation a bit we prefer to do it here to indicate possible modifications later on.

In the sequel we denote by v the solution to the homogeneous equation

$$i\partial_t v + \Delta v = 0, \qquad v(0) = u_0$$

which we can write by the unitary Schrödinger group $S(t)$ as

$$v(t) = S(t)u_0.$$

To approach the question of existence and uniqueness we make the ansatz $u = v + w$ where v satisfies the linear Schrödinger equation with initial data u_0 , and w satisfies $w(0) = 0$ and

$$\begin{aligned} iw_t + \Delta w =& \chi_{(0,T)}(t)|v+w|^\sigma(v+w) \qquad \text{in } \mathbb{R}\times\mathbb{R}^d \\ w(0,x) =& 0 \qquad \text{in } \mathbb{R}^d \end{aligned} \tag{3.13}$$

where $T \in (0,\infty]$ will be chosen later. We will construct a unique w in X by a fixed point argument. It is obvious that $u = v + w$ is the unique solution up to time T. Then $u = v + w$ is the searched for solution on the time interval $(0,T)$.

We rewrite the problem as a fixed point problem: Given $\tilde w$ we write $w = J(\tilde w)$ where J maps $\tilde w$ to the function w which satisfies

$$iw_t + \Delta w = \chi_{(0,T)}(t)|v+\tilde w|^\sigma(v+\tilde w), \quad w(0) = 0. \tag{3.14}$$

Suppose first that $\frac{2(d+2)}{d+4}(1+\sigma) \ge 2$ and $\sigma \le \frac{4}{d}$. By Hölder's inquality

$$\|f\|^{1+\sigma}_{L^{(1+\sigma)\frac{2(d+2)}{d+4}}(\mathbb{R}^d)} \le \|f\|^{\frac{4-d\sigma}{2}}_{L^2(\mathbb{R}^d)}\|f\|^{\frac{d+2}{2}\sigma-1}_{L^{\frac{2(d+2)}{d}}(\mathbb{R}^d)}$$

Observe that the exponent of $\|f\|_{L^2}$ is non-negative if $\sigma < \frac{4}{d}$ and it vanishes if $\sigma = \frac{4}{d}$.

If $0 < \frac{2(d+2)}{d+4}(1+\sigma) \le 2$ we estimate again by Hölder's inequality

$$\|f\|^{1+\sigma}_{L^{(1+\sigma)2}(\mathbb{R}^d)} \le \|f\|^{1-\frac{d\sigma}{4}}_{L^2(\mathbb{R}^d)}\|f\|^{(1+\frac{d}{4})\sigma}_{L^{\frac{2(d+2)}{d}}(\mathbb{R}^d)}.$$

In the first case we obtain the space-time estimate

$$\|\chi_{(0,T)}|u|^{1+\sigma}\|_{L^{\frac{2(d+2)}{d+4}}} \le T^{1-\frac{d\sigma}{4}}\|u\|^{\frac{4-d\sigma}{2}}_{L^\infty L^2}\|u\|^{\frac{d+2}{2}\sigma-1}_{L^{\frac{2(d+2)}{d}}} \tag{3.15}$$

and in the second case

$$\|\chi_{(0,T)}|v|^{1+\sigma}\|_{L^1_tL^2_x(\mathbb{R}^d)} \le T^{1-\frac{d\sigma}{4}}\|u\|^{1-\frac{d\sigma}{4}}_{L^\infty L^2}\|u\|^{(1+\frac{d}{4})\sigma}_{L^{\frac{2(d+2)}{d}}([0,T]\times\mathbb{R}^d)}. \tag{3.16}$$

If $\sigma < \frac{4}{d}$ T carries a positive power and we call this situation L^2 subcritical. This power becomes zero if $\sigma = \frac{4}{d}$, which we call L^2 or mass critical.

In the both cases

$$\|J(\tilde{w})\|_X \le cT^{1-\frac{d\sigma}{4}}(\|\tilde{w}\|_X + \|v\|_X)^{1+\sigma}$$

which we complement by the similar estimate

$$\|J(w) - J(\tilde{w})\|_X \le cT^{1-\frac{d\sigma}{4}}(\|\tilde{w}\|_X + \|w\|_X + \|v\|_X)^{\sigma}\|w - \tilde{w}\|_X.$$

We set up the problem for an application of the contraction mapping principle Let $R = \|v\|_X$. If $\|\tilde{w}\|_X \le R$ then, for some $c > 0$,

$$\|w\|_X \le cT^{1-\frac{d\sigma}{4}}(2R)^{1+\sigma} \le R$$

where the last inequality holds provided

$$T \le (2c(2R)^{\sigma})^{-\frac{4}{4-d\sigma}} := T_0$$

which we assume in the sequel. Moreover, if w and $\tilde{w}$ have norm at most R then

$$\|J(w) - J(\tilde{w})\|_X \le cT^{1-\frac{d\sigma}{4}}R^{\sigma}\|w - \tilde{w}\|_X$$

We obtain a contraction after decreasing T if necessary.

The critical case requires slightly different arguments, and it yields different conclusions. This time we cannot gain a small power of T and the smallness must have a different source.

In the mass critical case we assume that $\|\chi_{(0,T)}v\|_{L^{\frac{2(d+2}{d}}L^{\frac{2(d+2)}{d}}} \le \varepsilon$ for some small ε.

This is true for all T by Lemma (3.11) if $\|u_0\|_{L^2}$ is sufficiently small. Moreover, for all initial data $u_0 \in L^2$ we have by dominated convergence

$$\|\chi_{(0,T)}v\|_{L^pL^q} \to 0 \qquad \text{as } T \to 0 \tag{3.17}$$

for all Strichartz pairs with $q < \infty$.

It is obvious from the argument above (where we replace

$$\|\chi_{(0,T)}v\|_X \quad \text{ by } \|\chi_{(0,T)}v\|_{L^{\frac{2(d+2)}{2}}}$$

for the mass critical case) that the iteration argument applies if ε is sufficiently small. We obtain local existence under the smallness assumption, and hence global existence provided the initial data are sufficiently small.

We collect the results in a theorem.

Theorem 3.7. *There exists $\varepsilon > 0$ such that the following is true. Suppose that $0 < \sigma \le \frac{4}{d}$, $u_0 \in L^2$ and*

$$T^{1-\frac{d\sigma}{4}}\|\chi_T v\|_X^{\sigma} < \varepsilon.$$

resp. $\sigma = \frac{4}{d}$ and

$$\|\chi_T v\|^{\sigma}_{L^{\frac{2(d+2)}{2d}}(\mathbb{R}\times\mathbb{R}^d)} < \varepsilon.$$

Then there is a unique solution in X up to time T which satisfies

$$\|u - v\|_X \lesssim T^{1-\frac{d\sigma}{4}}\|v\|_X^{1+\sigma} \tag{3.18}$$

resp, if $\sigma = \frac{4}{d}$,

$$\|u - v\|_X \lesssim T^{1-\frac{d\sigma}{4}}\|v\|^{1+\sigma}_{L^{\frac{2(d+2)}{d}}} \tag{3.19}$$

There is a unique global solution

$$u \in L^{\frac{2(d+2)}{d}}((-T,T)\times\mathbb{R}^d)\cap C((-T,T);L^2(\mathbb{R}^d))$$

for all T *if either* $0\le\sigma<\frac{d}{4}$, *or, if* $\|u_0\|_{L^2}\le\varepsilon$ *and* $\sigma=\frac{d}{4}$. *In the last case we have* (3.19) *with* $T=\infty$. *If* $0\le k<1+\sigma$ *then*

$$(u_0\to u)\in C^k(L^2(\mathbb{R}^d);X)$$

There is a stability estimate. Suppose that $\tilde u\in X$ *satisfies*

$$T^{1-\frac{d\sigma}{4}}\|\tilde u\|_X<\varepsilon$$

$$\|\tilde u-u_0\|_{L^2}+\|i\partial_t\tilde u+\Delta\tilde u-|\tilde u|^\sigma\tilde u\|_{L^{\frac{2(d+2)}{d+4}}}<\varepsilon.$$

Then there exists a unique solution up to time T *with*

$$\|u-\tilde u\|_X\le c\left(\|\tilde u-u_0\|_{L^2}+\|i\partial_t\tilde u+\Delta\tilde u-|\tilde u|^\sigma\tilde u\|_{L^{\frac{2(d+2)}{d+4}}}\right). \tag{3.20}$$

If $\sigma=\frac{4}{d}$ *it suffices to require*

$$\|\chi_{(0,T)}\tilde u\|_{L^{\frac{2(d+2)}{d}}}<\varepsilon$$

Proof. Local existence in the subcritical case has been shown above. The fixed point formulation leads to existence via the contraction mapping theorem on a time interval whose length depends only on $\|u_0\|_{L^2}$. We claim that the L^2 norm (mass) is conserved. Indeed, for sufficiently regular and decaying $\tilde u=v+\tilde w$ and $u=v+w$ we have

$$\frac12\|u(t)\|_{L^2}^2=\frac12\|u_0\|_{L^2}^2+\ \mathrm{real} i\int_{(0,t)\times\mathbb{R}^d}|\tilde u|^\sigma\tilde u\bar u\,dxdt$$

which remains true for general $\tilde u$ and initial data by an approximation argument. By then it also holds for the fixed point, for which the second term on the right hand side is the real part of something purely imaginary.

Thus we can extend the solution to a global solution in the subcritical case.

It follows from the construction by the contraction mapping principle that the solution depends Lipschitz continuously on the initial data.

The map

$$L^{\frac{2(d+2)}{d}}(\mathbb{R}\times\mathbb{R}^d)\ni w\to\chi_{(0,T)}|w|^\sigma w\in L^{\frac{2(d+2)}{d+4}}(\mathbb{R}\times\mathbb{R}^d)$$

is k times continuously differentiable for $k<1+\sigma$, and $\sigma\le\frac{4}{d}$.

Thus J is k times continuously Frechet differentiable. Moreover, by the very same estimates as for the contraction the derivative of J with respect to $\tilde w$ is invertible, and by the implicit function theorem from the initial data to the solution is k times continuously differentiable. Checking the norms implies the stability estimate.

□

We also have

$$\lim_{T\to\infty}\|\chi_{(T,\infty)}v\|_{L^{\frac{2(d+2)}{d}}(\mathbb{R}\times\mathbb{R}^d)}=0.$$

Suppose that $u\in X$ is a solution for $T=\infty$ and $\sigma=\frac{4}{d}$. One can deduce that the limit

$$\lim_{t\to\infty}S(-t)u(t)$$

exists in L^2. Let w_0 be this limit, and w the solution to the homogeneous equation with initial data w_0. Then the convergence statement can be formulated as

$$\lim_{t\to\infty} \|u(t)-w(t)\|_{L^2} = 0.$$

This is called scattering.

3.8. **Initial data in $\dot H^1$ for $d \geq 3$.** Consider

$$iu_t + \Delta u = \pm |u|^\sigma u \tag{3.21}$$

with initial data $u_0 \in \dot H^1$, by which we mean the space with the norm $\| \, |\nabla u_0| \, \|_{L^2}$. We want to use Strichartz spaces for the derivative and we define the function spaces X by

$$\|u\|_X := \sup_t \|\nabla u(t)\|_{L^2} + \|\nabla u\|_{L^{\frac{2(d+2)}{d}}}.$$

Then the Strichartz estimate 3.11 combined with Sobolev's estimate gives

$$\|u\|_X \leq c\left(\|\nabla u_0\|_{L^2} + \|\nabla f\|_{L^{\frac{2(d+2)}{d+4}}}\right)$$

for a solution u to the inhomogeneous linear problem.

Then, if $\sigma \leq \frac{4}{d-2}$, by Hölder's and Sobolev's inequality

$$\|\nabla |f|^\sigma f\|_{L^{\frac{2(d+2)}{d+4}}(\mathbb{R}^d)} \lesssim \|f\|^{\frac{4-(d-2)\sigma}{2}}_{L^{L^{\frac{2d}{d-2}}}(\mathbb{R}^d)} \|\nabla f\|^{-1+\frac{4-d}{2}\sigma}_{L^{\frac{2(d+2)}{d}}(\mathbb{R}^d)}$$

provided σ is not too small. For small σ we argue as for the case of L^2. We obtain in both cases

$$\|J(w)\|_X \lesssim T^{1-\frac{(d-2)\sigma}{4}} \left(\|v\|_X + \|w\|_X\right)^{1+\sigma}, \tag{3.22}$$

and, checking the same argument for differences,

$$\begin{aligned} \|J(w^2) - J(w^1)\|_{L^{\frac{2(d+2)}{d}}} &+ \|J(w^2)-J(w^1)\|_{L^\infty L^2} \\ &\lesssim T^{1-\frac{(d-2)\sigma}{4}} \left(\|v\|_X + \|w^1\|_X + \|w^2\|_X\right)^\sigma \\ &\quad \times \left(\|w^2-w^1\|_{L^{\frac{2(d+2)}{d}}} + \|w^2-w^1\|_{L^\infty L^2}\right) \end{aligned} \tag{3.23}$$

Theorem 3.8 (Local existence and uniqueness in energy space)**.** *Suppose that $0 < \sigma \leq \frac{4}{d-2}$. There exists $\varepsilon > 0$ such that the following is true. Let v be the solution to the homogeneous linear Schroedinger equation. Suppose that*

$$T^{1-\frac{(d-2)\sigma}{4}} \|v\|_X^\sigma \leq \varepsilon$$

Then there exists a unique solution $u = v + w$ with

$$\|\nabla w\|_{L^\infty L^2} + \|\nabla w\|_{L^{\frac{2(d+2)}{d}}} \lesssim T^{1-\frac{(d-2)\sigma}{4}} \|v\|_X^{1+\sigma}.$$

Again we may replace $\|v\|_X$ by $\|\chi_{0,T}\nabla v\|_{L^{\frac{2(d+2)}{d}}}$. In the defocusing case the solution is global if $\sigma < \frac{4}{d-2}$. In the energy critical case $\sigma = \frac{4}{d-2}$ there is global existence for small data, and local existence for all data in $\dot H^1$.

Proof. Again we characterize the solution as the fixed point of the same map as above, but now with respect to the norm X. By (3.22) we obtain a map of a closed ball in X to itself, but a contraction only in the metric of $L^{\frac{2(d+2)}{d}}$ in a ball in X- at least for large space dimensions and small σ. We change the space X slightly by replacing $C(\mathbb{R}; L^2)$ by $L^\infty(\mathbb{R}; L^2)$. We claim that sequences which are bounded in X and converge in $L^{\frac{2(d+2)}{d}}$ have a limit in X. There is a weak* converging subsequence in X, and the limits have to coincide.

It is not hard to complete the argument for initial data additionally in $L^2(\mathbb{R}^d)$: then $v \in L^{\frac{2(d+2)}{2}}$, and this remains true for the fixed point map. In general we define iteratively $v_{j+1} = J(v_j)$. We claim that there exists j so that

$$v_{j+1} - v_j \in L^{\frac{2(d+2)}{d}}(\mathbb{R} \times \mathbb{R}^d).$$

The contraction argument then completes the proof. This argument gives uniqueness in the set

$$v_j + X \cap L^{\frac{2(d+2)}{d}}(\mathbb{R} \times \mathbb{R}^d).$$

The proof of the claim is technical and omitted.

The remaining arguments are adaptations of similar arguments in Theorem 3.7. □

3.9. **Initial data in $H^1(\mathbb{R}^d)$.** In this case we combine the arguments. We obtain global well-posedness in the defocussing subcritical case $\sigma < \frac{d}{4}$, local existence in the subcritical and the critical case ($\sigma \le \frac{d}{4}$) and global existence in the critical case $\sigma = \frac{d}{4}$ and small initial data.

4. Functions of bounded p variation

The study of p variation of functions of one variable has a long history. Function of bounded p variation have been studied by Wiener in [31]. The generalization of the Riemann-Stieltjes integral to functions of bounded p variation against the deriative of a function of bounded q variation $1/p + 1/q > 1$ is due to Young [32]. Much later Lyons developed his theory of rough path [21] and [22], building on Young's ideas, but going much further.

In parallel D. Tataru realized that the spaces of bounded p variation, and their close relatives, the U^p spaces, allow a powerful sharping of Bourgain's technique of function spaces adapted to the dispersive equation at hand. These ideas were applied for the first time in the work of the author and D. Tataru in [16]. Since then there has been a number of questions in dispersive equations where these function spaces have been used. For example they play a crucial role in [17], but there they could probably be replaced by Bourgain's Fourier restriction spaces $X^{s,b}$. On the other hand, for well-posedness for the Kadomtsev-Petviashvili II in a critical function space (see [10]) the $X^{s,b}$ spaces seem to be insufficient. The theory of the spaces U^p and V^p and some of their basic properties like duality and logarithmic interpolation have been worked out for the first time in [10]. The development in stochastic differential equations and dispersive equations has been largely independent.

We will introduce and study functions from an interval (a, b) to $\mathbb{R}$, $\mathbb{R}^n$, a Hilbert space or a Banach space X, and spaces of such functions which are invariant under continuous monotone reparametrizations of the interval. For the most part of this section there are no more than the obvious modifications when considering Banach space valued functions. We allow $a = -\infty$ and $b = \infty$.

We call a function f ruled function if at every point (including the endpoints, which may be $\pm\infty$) left and right limits exist. The set of ruled functions is closed with respect to uniform convergence. We denote the Banach space of ruled functions equipped with the supremum norm by $\mathcal{R}$.

A step function is a function f for which there exists a partition so that f is constant on every interval (a,t_1), (t_i,t_{i+1}) and (t_n,b). We do not require that the value at a point coincides with the limit from either side. Step functions are dense in $\mathcal{R}$ (Aumann [1], Dieudonne [6]). We denote the set of step functions by $\mathcal{S}$.

Let $\mathcal{R}_{rc} \subset \mathcal{R}$ be the closed subspace of right continuous functions f with $\lim_{t\to a} f(t) = 0$. Similarly, if $A \subset \mathcal{R}$ we denote by A_{rc} the intersection with $\mathcal{R}_{rc}$.

Let X be a Banach space and X^* its dual. We consider functions with values in X resp X^* and we denote the corresponding spaces by $\mathcal{R}(X)$ reps. $\mathcal{S}(X)$.

There is a bilinear map B from $\mathcal{S}(X)_{rc} \times \mathcal{R}(X^*)$ to $\mathbb{R}$ resp. $\mathbb{C}$ defined by

$$B(u,v) = \sum_{i=1}^{n} v(t_i)(u(t_i) - u(t_{i-1})) \tag{4.1}$$

where $a = t_0 < t_1 < \cdots < t_n < b$ is the partition. In the sequel we will omit the space X and X^* from the notation unless there is some ambiguity. Similarly the formula above defines a bilinear on $\mathcal{R}(X^*) \times \mathcal{S}(X)$.

It will be convenient to extend every function on $[a,b)$ by zero to $[a,b]$, i.e. we will always set $f(b) = 0$, even if $a = -\infty$ or $b = \infty$. Similarly we extend every function by 0 to $\mathbb{R}$ whenever this is convenient.

Definition 4.1. *For $u \in \mathcal{R}$ and a partition*

$$\tau = (t_1, t_2 \dots t_n), \quad a < t_1 < t_2 < t_3 \cdots < t_n < b$$

we define (denoting the limit from the right by $f(t+)$)

$$u_\tau(t) = \begin{cases} u(t) & \text{if } t = t_j \qquad \text{for a } j \\ u(a+) & \text{if } a < t < t_1 \\ u(t_i+) & \text{if } t_i < t < t_{i+1} \\ u(t_n+) & \text{if } t_n < t \end{cases}$$

We observe that f_τ is a step function, and it is right continuous if f is right continuous.

Lemma 4.2. *Let $u \in \mathcal{R}_{rc}$ and $v \in \mathcal{R}$. Then*

$$B(u_\tau, v) = B(u_\tau, v_\tau) = B(u, v_\tau)$$

If $u, v \in \mathcal{S}_{rc}$ then, with t_i a partition containing all points of discontinuity of u and v,

$$B(u,v) + B(v,u) = \sum_t (v(t_i) - v(t_{i-1})(u(t_i) - u(t_{i-1})) + v(b)(u(b))$$

Proof. This follows immediately from the definitions. □

In particular, if there is no point where both u and w are discontinuous then the only term on the right hand side is $v(b)u(b)$.

4.1. **Functions of bounded p variation and the spaces U^p and V^p.** In the sequel $p \in [1,\infty]$. Unless explicitly stated otherwise we consider $p \in (1,\infty)$.

In later chapters we use U^p and V^p to study well-posedness questions for several dispersive PDEs, where we select a number of relevant and representative problems.

A partition τ of (a,b) is a strictly increasing finite sequence

$$a < t_1 < t_2 < \cdots < t_{n+1} < b$$

where we allow $b = \infty$ and $a = -\infty$.

Definition 4.3. *Let I be an interval, X a Banach space, $1 \le p < \infty$ and $f : I \to X$. We define*
$$\omega_p(v, I) := \sup_\tau \left(\sum_{i=1}^{n-1} \|v(t_{i+1}) - v(t_i)\|_X^p \right)^{1/p} \in [0, \infty].$$

There are obvious properties. The function $t \to \omega_p(v, [a, t))$ is monotonically increasing. The same is true if we consider closed or open intervals.

Lemma 4.4. *Suppose that $a < b < c$. Then*
$$\omega_p(v, [a, b)) \le \omega_p(v, [a, c)) \le 2^{1-1/p} \Big(\omega_p(v, [a, b]) + \omega_p(v, [b, c)) \Big).$$

Proof. Consider a partition τ. If b is a point of τ then the p-th power of the τ variation in the large interval is the sum of the p powers of the parts. If not we add the point b. This increases the right hand side of the second inequality, and it may decrease or increase the left hand side. The factor $2^{1-1/p}$ follows from
$$|a + b|^p \le 2^{p-1}(|a|^p + |b|^p).$$
□

The p variation can sometimes be explicitly estimated.

Lemma 4.5. *For bounded monotone functions we have*
$$\omega_p(v, [a, b)) = \sup v - \inf v.$$

We denote by $\dot C^s(I)$ the homogeneous Hölder norm:
$$\|f\|_{\dot C^s(I)} = \sup_{t \ne \tau} \frac{|u(t) - u(\tau)|}{|t - \tau|^s}.$$

Lemma 4.6. *We have*
$$\omega_p(v, (a, b)) \le \|v\|_{\dot C^{1/p}} (b - a)^{1/p}.$$
Suppose that
$$\omega_p(v, (a, b)) < \infty.$$
Then v has left and right limits at every point. The expression is invariant with respect to continuous monotone coordinate changes. Moreover
$$\omega_p(\lambda v, (a, b)) = |\lambda| \omega_p(v, (a, b)),$$
$$\omega_p(v + w, (a, b)) \le \omega_p(v, (a, b)) + \omega_p(w, (a, b)).$$

Proof. Let $t_0 < t_1 < \dots t_N$. Then
$$\sum_j \|v(t_{i+1}) - v(t_i)\|_X^p \le \sum_i (t_{i+1} - t_i) \|v\|_{\dot C^{1/p}}.$$
The other statement follow from a straightforward calculation. □

The p variation is continuous at points where v is continuous, provided the p variation is finite.

Lemma 4.7. *Suppose that $\omega_p(v, [a, b)) < \infty$ and v is continuous at $c \in [a, b)$. Then*
$$\lim_{t \to c} \omega_p(v, [a, t)) = \omega_p(v, [a, c]).$$

Proof. Suppose that

$$\lim_{t\to b,\, t>a} \omega(v,(a,t)) - \omega(v,(a,b)) = 2\delta > 0.$$

Then there is a sequence of points $c < t_1 < t_2 \dots t_n < b$ with

$$\sum \|v(t_{i+1}) - v(t_i)\|_X^p \geq \frac{\delta^p}{p\|v\|_{sup}^{p-1}}.$$

Similarly there is such a sequence in (c, t_1) and recursively we get an arbitrary large number of such sequences. Putting N of them together we see that

$$\omega_p(v,(c,b)) \geq Nc\delta$$

which would bound N. This is a contradiction. Similarly we argue for the limit from below. □

Definition 4.8. *Let X be a Banach space , $1 \leq p < \infty$ and $v : (a,b) \to X$. We define*

$$\|v\|_{V^p((a,b),X)} = \max\{\|v\|_{sup}, \omega_p(v,(a,b))\}.$$

Let $V^p = V^p((a,b)) = V^p(X) = V^p((a,b);X)$ be the set of all functions for which this expression is finite. We omit the interval and/or the Banach space in the notation when this seems appropriate.

The interval will usually be of minor importance. The following properties are immediate:

(1) $V^p(I)$ is closed with respect to this norm and hence $V^p(I)$ is a Banach subspace of $\mathcal{R}$. Moreover $V^p_{rc}(I)$ is a closed subspace.
(2) We set $V^\infty = \mathcal{R}$ with $\|.\|_{V^\infty} = \|.\|_{sup}$.
(3) If $1 \leq p \leq q \leq \infty$ then
$$\|v\|_{V^q} \leq \|v\|_{V^p}.$$
(4) Let X_i be Banach spaces, $T : X_1 \times X_2 \to X_3$ a bounded bilinear operator, $v \in V^p(X_1)$ and $w \in V^p(X_2)$. Then $T(v,w) \in V^p(X_3)$ and
$$\|T(v,w)\|_{V^p(X_3)} \leq 2\|T\|\|v\|_{V^p(X_1)}\|w\|_{V^p(X_2)}.$$
(5) We embed $V^p((a,b))$ into $V^p(\mathbb{R})$ by extending v by 0.
(6) The space V^1 has some additional structure: Every bounded monotone function is in V^1, and functions in V^1 can be written as the difference of two bounded monotone functions.

The space of bounded p variation is build on the sequence space l^p. We may also replace it by the weak space l^p_w, with

$$\|(a_j)\|_{l^p_w} = \sup_{\lambda>0} \lambda(\#\{j : |a_j| > \lambda\})^{\frac{1}{p}}.$$

This does not satisfy the triangle inequality, but if $p > 1$ there is an equivalent norm, which makes l^p_w a Banach space. We set $l^\infty_w = l^\infty$.

Definition 4.9. *Let $1 \leq p < \infty$. The weak V^p_w space consists of all functions such that*

$$\|v\|_{V^p_w} = \max\{\sup_{t_1<\dots<t_n} \|(v(t_{i+1}) - v(t_i))_{1\leq i\leq n-1}\|_{l^p_w}, \|v\|_{sup}\}$$

is finite.

By Tschebycheff's inequality

$$\|v\|_{V^p_w} \le \|v\|_{V^p}.$$

The spaces of bounded p variation are of considerable importance in probability and harmonic analysis. We shall see that V^p is the dual space of a space U^q, $1/p + 1/q = 1$, $1 < p < \infty$, with a duality pairing closely related to the Stieltjes integral, and its variant, the Young integral [32].

Definition 4.10. *A p-atom a is a step function in $\mathcal{S}_{rc}$,*

$$a(t) = \sum_{i=1}^n \phi_i \chi_{[t_i, t_{i+1})}(t)$$

where $\tau = (t_1 \dots t_n)$ is a partition, $t_{n+1} = b$, with $\sum |\phi_i|^p \le 1$. A p-atom a is called a strict p atom if

$$\max_i \|\phi_i\|_X (\#\tau)^{1/p} \le 1.$$

It is important that atoms are right continuous, zero in a neighborhood of a, but the limit as $t \to b$ may be different from 0.

Let a_j be a sequence of atoms and let λ_j be a summable sequence. Then

$$u = \sum \lambda_j a_j$$

is a U^p function. This is well defined since the right hand side converges in $\mathcal{R}$. We define U^p as the set of function having such a representation and give it the norm

$$\|u\|_{U^p} := \inf \Big\{ \sum |\lambda_j| : u = \sum \lambda_j a_j \Big\}.$$

The strict space U^p_{strict} is defined in the same fashion using strict p atoms.

We collect a number of elementary properties.

(1) If a is a p-atom then $\|a\|_{U^p} \le 1$. The norm of an atom may be less than 1. Determining the norm of an atom is a difficult task.
(2) Functions in U^p are continuous from the right. The limit as $t \to a$ vanishes.
(3) The expression $\|.\|_{U^p}$ defines a norm on U^p, and U^p is closed with respect to this norm. Moreover $U^p \subset \mathcal{R}_{rc}$ is a subspace with $\|.\|_{sup} \le \|.\|_{U^p}$.
(4) If $p < q$ then $U^p \subset U^q$ and
$$\|u\|_{U^q} \le \|u\|_{U^p}$$
(5) If $1 \le p < \infty$ then for all $u \in U^p$
$$\|u\|_{V^p} \le 2^{1/p} \|u\|_{U^p}$$
(6) Let Y be a Banach space, and let the linear operator $T : \mathcal{S}_{rc} \to Y$ satisfy
$$\|Ta\|_Y \le C \tag{4.2}$$
for every p atom. Then T has a unique extension to a bounded linear operator from U^p to Y which satisfies
$$\|Tf\|_Y \le C \|f\|_{U^p}. \tag{4.3}$$
(7) Let X_i be Banach spaces, $T : X_1 \times X_2 \to X_3$ a bounded bilinear operator, $v \in U^p(X_1)$ and $w \in U^p(X_2)$. Then $T(v, w) \in U^p(X_3)$ and
$$\|T(v, w)\|_{U^p(X_3)} \le 2\|T\| \|v\|_{U^p(X_1)} \|w\|_{U^p(X_2)}.$$
(8) We consider $U^p([a, b))$ in the same way as subspace of $U^p(\mathbb{R})$ as for V^p.

The following decomposition is crucial for most of the following. It is related to Young's generalization of the Stieltjes integral, and it deals with a crucial point in the theory. We denote the number of points in a partition τ by $\#\tau$.

Lemma 4.11. *There exists $\delta > 0$ such that for v right continuous with $\|v\|_{V^p_w} = \delta$ there are strict p atoms a_i with*

$$\|a_j(t)\|_{sup} \le 2^{1-j} \qquad \text{and} \qquad \#\tau_j \le 2^{jp}$$

such that in the sense of uniform convergence

$$v = \sum a_j.$$

Proof. We set $v_0 = v$, and we search for a recursive decomposition with

$$v_j = a_j + v_{j+1}$$

such that

$$\|v_j\|_{sup} \le 2^{-j}, \|a_j\|_{sup} \le 2^{-j}$$

and, with τ_j the partition related to a_j

$$\#\tau_j \le 2^{pj}.$$

Suppose we have constructed v_i for $i \le j$ and a_i for $i \le j-1$. We construct the a_j, which also defines v_{j+1}. We choose the unique partition τ so that

$$\sup_t \|v_j(t)\|_X < 2^{-1-j} \qquad \text{in } [a, t_1), \qquad \|v_j(t_1)\|_X \ge 2^{-1-j},$$

$$\|v_j(t) - v_j(t_i)\|_X < 2^{-1-j} \quad \text{in } t \in [t_i, t_{i+1})$$

and

$$\|v_j(t_{i+1}) - v_j(t_i)\|_X \ge 2^{-1-j}.$$

We define a_j as the step function adapted to the partition τ_j (recall Definition 4.1)

$$a_j = (v_j)_\tau$$

Then, by construction,

$$\|a_j\|_{sup} \le \|v_j\|_{sup} \le 2^{-j},$$

$$\|v_{j+1}\|_{sup} \le 2^{-1-j}$$

and since either $(t_j, t_{j+1}]$ contains no points of an earlier partition, in which case we estimate the sum of these differences using the V^p_w norm of v, or it does, and then we simply add the number of those terms, and iterate

$$\begin{aligned} \#\tau_j \le & 2^p \|v\|^p_{V^p_w} 2^{jp} + \sum_{i=0}^{j-1} \#\tau_i \\ \le & 2^p \|v\|^p_{V^p_w} \sum_{i=0}^{j} (j+1-i) 2^{ip} \\ \le & c_p \|v\|^p_{V^p_w} 2^{jp} \end{aligned} \tag{4.4}$$

We choose $\delta = c_p^{-1/p}$. □

There are a number of simple interesting and useful consequences.

Lemma 4.12. *Let $1 < p < q < \infty$. There exists $\kappa > 0$, depending only on p and q, such that for all $v \in V^p_{w,rc}$ and $M \geq 1$ there exist $u \in U^p_{strict}$ and $w \in U^q_{strict}$ with*

$$v = u + w$$

and

$$\frac{\kappa}{M}\|u\|_{u^p_{strict}} + e^M \|w\|_{U^q_{strict}} \leq \|v\|_{V^p_w}.$$

Observe that we may replace U^p_{strict} by U^p (since $U^p_{strict} \subset U^p$) and V^p_w by V^p (since $V^p \subset V^p_w$).

Proof. Multiplying v by $\delta/\|v\|_{V^p_{w,rc}}$ we may assume that $\|v\|_{V^p_w} = \delta$ as in Lemma 4.11 and setting $\tilde u = \sum_{j=1}^m a_j$ for some m to be chosen later we have

$$\|\tilde u\|_{U^p_{strict}} \leq m.$$

By construction $2^{j(1-p/q)} a_j$ is a strict q atom and hence, with $\tilde w = \sum_{j=m+1}^\infty a_j$,

$$\|\tilde w\|_{U^q_{strict}} \leq \sum_{j=m+1}^\infty \|a_j\|_{U^q_{strict}} \leq c_{p,q} 2^{(\frac{p}{q}-1)m}.$$

hence, with $u = \frac{\|v\|_{V^p_w}}{\delta}\tilde u$ and $w = \frac{\|v\|_{V^p_w}}{\delta}\tilde w$

$$u + w = v$$

and, with $\delta = -\ln 2(\frac{p}{q} - 1)$ there exists c depending only on p and w with

$$\frac{1}{m}\|u\|_{U^p} + e^{\delta m}\|v\|_{U^q} \leq c\|v\|_{V^p_{rc}}$$

We choose $m = (M + \ln 2c)/\delta$ and, for $M \geq \ln 2c$, $\kappa = \delta/2$ to obtain the claimed estimate. $\square$

We obtain the following embedding

Lemma 4.13. *Let $1 < p < q < \infty$. Then*

$$V^p_{rc} \subset V^p_{w,rc} \subset U^q_{strict} \subset U^q.$$

Proof. Apply Lemma 4.12 with $M = 1$. $\square$

4.2. **Duality and the Riemann-Stieltjes integral.** The Riemann-Stieltjes integral defines

$$\int f dg = \int f g_t dt$$

for $f \in \mathcal{R}$ and $g \in V^1$. If f or $g \in \mathcal{S}_{rc}$ then, with the obvious partition,

$$\int f g_t dt = \sum f(t_i)(g(t_i) - g(t_{i-1})). \tag{4.5}$$

This formula was the definition of the bilinear map B. We shall see that it uniquely defines an 'integral' for $f \in V^p$ and $g \in U^q$, for $1/p + 1/q = 1$, $q > 1$. Results become much cleaner when we use an equivalent norm in V^p,

$$\|v\|_{V^p} = \sup_{a<t_1\ldots t_n<b}\left(\sum_{j=1}^{n-1}|v(t_{j+1}) - v(t_j)|^p + |v(t_n)|^p\right)^{1/p} \tag{4.6}$$

which we do in the sequel. We also set $v(b) = 0$ and, for any partition, $t_{n+1} = b$.

Theorem 4.14. *The bilinear map B defines a unique continuous bilinear map*

$$B : U^q(X) \times V^p(X^*) \to \mathbb{R}$$

which satisfies (with $t_0 = a$ and $u(t_0) = 0$)

$$B(u,v) = \sum_{i=1}^{n} v(t_i)(u(t_i) - u(t_{i-1}))$$

for $v \in V^p$ and $u \in \mathcal{S}_{rc}$ with associated partition $(t_1, \dots t_n)$ and $v(t_i)(.)$ the evaluation of $v(t_i) \in X^*$ on the argument in X. It satisfies

$$|B(u,v)| \le \|u\|_{U^q(X)} \|v\|_{V^p(X^*)}. \tag{4.7}$$

The map

$$V^p(X^*) \ni v \to (u \to B(u,v)) \in (U^q(X))^*$$

is a surjective isometry if $1 \le q < \infty$. Moreover

$$\|v\|_{V^p(X^*)} = \sup_{u \in U^q(X), \|u\|_{U^q(X)}=1} B(u,v) = \sup_{a \text{ is a } q-atom} B(a,v). \tag{4.8}$$

The same statements up to constants are true if we replace U^p by U^p_{strict} and V^q by V^q_w.

Proof. Let $v \in V^p$. The expression

$$F_v(u) = \sum_{i=1}^{n} v(t_i)(u(t_i) - u(t_{i-1})) = -\sum_{i=1}^{n} (v(t_{i+1}) - v(t_i)) u(t_i)$$

is clearly defined for $v \in V^q$ and $u \in \mathcal{S}_{rc}$ with partition $\tau = (t_i)$. The product is an abuse of notation for the duality pairing between X and X^* which we suppress in the notation. The map is linear in v and u and satisfies for every atom (by Hölder's inequality, and using the right hand side of the equation for $F_v(u)$)

$$\begin{aligned} |F_v(a)| &\le \sum_{i=1}^{n} \|v(t_{i+1}) - v(t_i)\|_{X^*} \|a(t_i)\|_X \\ &\le \left(\sum_{i=1}^{n} \|v(t_{i+1}) - v(t_i)\|^p_{X^*} \right)^{1/p} \left(\sum_{i=1}^{n} \|a(t_i)\|^q_X \right)^{1/q} \end{aligned}$$

The first factor is bounded by $\|v\|_{V^p}$, and the second, by the definition of a q atom, by 1.

Existence of a unique extension to U^q follows from this estimate and (4.3). Linearity in v and estimate (4.7) are immediate consequences. Clearly B defines a map from V^p to the dual of U^q with norm at most 1. Let us prove that it defines an isometry and choose $v \in V^p$, $\varepsilon > 0$, and a partition $t_0 < t_1 < \dots < t_n$ with

$$\|v\|_{V^p} \le \left(\sum_{j=1}^{n} \|v(t_{j+1}) - v(t_j)\|^p_{X^*} \right)^{1/p} + \varepsilon.$$

Here we set again $t_{n+1} = b$ and $v(b) = 0$. We choose $x_i \in X$ of norm 1 with

$$(v(t_{i+1}) - v(t_i))(x_i) \ge (1 - \varepsilon) \| \|v(t_{i+1}) - v(t_i)\|_{X^*}$$

and

$$\phi_j := \mu \|v(t_{j+1}) - v(t_j)\|^{p-1}_{X^*} x_j$$

where $\mu = \|v\|_{V^p}^{1-p}$. Then

$$\sum_{j=1}^{n} \|\phi_j\|_x^{p'} \leq \mu^{-p} \sum_{j=1}^{n} \|v(t_{j+1}) - v(t_j)\|_{X^*}^p \leq 1.$$

Thus the partition and the ϕ_j define an atom a, and

$$\|v\|_{V^p} \geq B(a, v) - C\varepsilon.$$

The map is an isometry since ε is arbitrary. We turn to surjectivity. Let $F \in (U^q)^*$ and define the element $v(t) \in X^*$ by

$$v(t)(x) := F(x\chi_{[t,\infty)}) \qquad \text{for } x \in X.$$

Let a be an atom. Then

$$\begin{aligned} F(a) &= \sum_i F(\phi_i \chi_{[t_i,b)}) - F(\phi_i \chi_{[t_{i+1},b)}) = -\sum \phi_i (v(t_{i+1}) - v(t_i)) \\ &= \sum v(t_i)(a(t_i) - a(t_{i-1})) = B(a, v) \end{aligned}$$

By the previous estimate

$$\|v\|_{V^p} \leq \|F\|_{(U^q)^*}.$$

Hence both sides coincide on U^q. The remaining claims are simple consequences. $\square$

The previous results show that $U^p \subset V^p_{rc}$, and both spaces are very close. They are, however, not equal. The following example goes back to Young [32] with the same intention, but in a slightly different context.

Lemma 4.15. *Let ϕ be a smooth function with compact support, $1 < q < \infty$. Then*

$$u_q(t) = \phi(x) \sum_{j=1}^{\infty} 2^{-j/q} \cos(2^j t) \in V^q_{rc}$$

but not in U^q.

Proof. Let p be the Hölder dual exponent of q and

$$v_p^N(t) = \phi \sum_{j=1}^{N} 2^{-j/p} \sin(2^j t).$$

where we allow $N = \infty$. Then, with $M = [\ln_2(|t - s|)]$, $[\,]$ the Gauss bracket,

$$\begin{aligned} |v_p^N(t) - v_p^N(s)| \leq& \sum_{j=1}^{M} 2^{-j/p} |\phi(t) \sin(2^j t) - \phi(s) \sin(2^j s)| + c_1 \sum_{j=M+1}^{N} 2^{-j/p} \\ \leq& c_2 \left(\sum_{j=1}^{M} 2^{-j/p+j} |t - s| + 2^{-j/M} \right) \\ \leq& c_3 \left(2^{-M/p+M} |t - s| + 2^{-j/M} \right) \\ \leq& c_4 |t - s|^{\frac{1}{p}} \end{aligned}$$

and hence, by Lemma 4.6

$$\sup_N \|v_p^N\|_{V^p} < \infty,$$

and similarly $u_q \in V^q_{rc}$. Now, assuming that $u_q \in U^q$, with $1/p+1/q=1$, we claim

$$\|u_q\|_{U^p}\|v_p^N\|_{V^q} \geq \left|\int (u_q^\infty)' v_p^N dx\right| = N/2\int \phi^2 dx + O(1) \tag{4.9}$$

which is unbounded, hence a contradiction and $V^q_{rc} \ni u_q^\infty \notin U^q$. It remains to verify (4.9). The first inequality is a consequence of the duality theorem. We expand both factors in the integral and claim for $j \neq l$ by stationary phase

$$\left|\int \phi(t) 2^{-j/p-l/q}\cos(2^j t)(\phi(t)\sin(2^l t))' dt\right| \leq c_M 2^{-j}|2^j - 2^l|^{-M}$$

for every $M \in \mathbb{N}$. Thus

$$\sum_{j\neq l, l\leq N}\left|\int\int \phi(t) 2^{-j/p-l/q}\cos(2^j t)(\phi(t)\sin(2^l t))' dt\right| \leq c\sum_{j=1}^\infty 2^{-j}\sum_{l=1,l\neq j}^N 2^{-l}$$

which is bounded independent of N. Next

$$\left|\int\int \phi(t)2^{-j/p-j/q}\sin(2^j t)\cos(2^j t)\phi'(t))dt\right| \leq c_1 2^{-j}$$

and

$$\begin{aligned}\left|\int\int \phi^2(t)2^{-j/p-j/q+j}\cos^2(2^j t)dt\right| =& \left|\int\int \phi^2(t)\frac{1}{2}(1+\cos(2^{j+1}t))dt\right| \\ =& \frac{1}{2}\int \phi^2(t)dt + c_2^{-j}.\end{aligned}$$

We expand (4.9). Only the diagonal terms contribute. This completes the proof. □

4.3. **Step functions are dense.**

Lemma 4.16. *For all $v \in V^p$ and all partitions τ we have (recall Definition 4.1)*

$$\|v_\tau\|_{V^p} \leq \|v\|_{V^p}. \tag{4.10}$$

and for all $u \in U^p$

$$\|u_\tau\|_{U^p(I)} \leq \|u\|_{U^p(I)}. \tag{4.11}$$

For $v \in V^p$ and $\varepsilon > 0$ there is a partition τ so that

$$\|v - v_\tau\|_{V^p} < \varepsilon. \tag{4.12}$$

Given $u \in U^p$ and $\varepsilon > 0$ there exists τ with

$$\|u - u_\tau\|_{U^p} < \varepsilon. \tag{4.13}$$

In particular the step functions $\mathcal{S}$ are dense in V^p and $\mathcal{S}_{rc}$ is dense in U^p.

Proof. When we take the supremum over partitions for v_τ we may restrict to subsets of τ and the first statement becomes obvious. For U^p it suffices to check p atoms a,

$$\|a_\tau\|_{U^p} \leq 1.$$

Density of step functions in U^p follows from the atomic definition of the space: Let $u \in U^p$ and $\varepsilon > 0$. By definition there exists a finite sum of atoms (which is a right continuous step function u_{step}) such that

$$\|u - u_{step}\|_{U^p} < \varepsilon/2$$

Let τ be the partition associated to u_{step}. Then

$$\begin{aligned}\|u-u_\tau\|_{U^p} \le & \|u_{step}-u_\tau\|_{U^p} + \|u-u_{step}\|_{U^p}\\ < & \|(u_{step}-u)_\tau\|_{U^p} + \varepsilon/2\\ < & \varepsilon.\end{aligned}$$

which is the claim for U^p. Let $\tilde V^p$ be the closure of the step functions in V^p. Suppose there exists $v \in V^p$ with distance > 1 to $\tilde V^p$, and $\|v\|_{V^p} < 1+\varepsilon$. Such a function exists when $\tilde V^p$ is not V^p. Let $D \subset U^q$ be the subset such $B(u,v) = 0$ whenever $u \in D$ and $v \in \tilde V^p$. Since the dual space of D is naturally given by $D^* = V^p/\tilde V^p$, and since v defines an element in D^* of norm > 1 there exists $u \in D$ with $B(u,v) = 1$, and a partition τ so that $\|u - u_\tau\|_{U^p} < \varepsilon$. However

$$0 = B(u, v_\tau) = B(u_\tau, v) = B(u,v) + B(u_\tau - u, v) \ge 1 - \varepsilon(1+\varepsilon)$$

which is a contradiction if $\varepsilon < \frac12$. Hence the step functions are dense in V^p and, given $v \in V^p$ and $\varepsilon > 0$ there is a step function v_{step} with $\|v - v_{step}\|_{V^p} < \varepsilon$ and partition τ. Then

$$\begin{aligned}\|v-v_\tau\|_{V^p} \le & \|v_{step}-v_\tau\|_{V^p} + \|v-v_{step}\|_{V^p}\\ < & \|(v_{step}-v)_\tau\|_{V^p} + \varepsilon/2\\ < & \varepsilon.\end{aligned}$$

which is the density assertion. □

4.4. **Convolution and regularization.** Convolution by an L^1 function defines a bounded operator on U^p and V^p. Ruled functions are in L^∞ and hence the product of a function in U^p or V^p with an L^1 function can be integrated. In particular the convolution of a ruled function and an L^1 function is well defined.

Lemma 4.17. *Let $a = -\infty$ and $b = \infty$, $v \in V^p$ and $\phi \in L^1$. Then*

$$\|v * \phi\|_{V^p(X)} \le \|\phi\|_{L^1} \|v\|_{V^p(X)}$$

and

$$\|u * \phi\|_{U^p(X)} \le \|\phi\|_{L^1} \|u\|_{U^p(X)}.$$

Proof. Let τ be a partition. It suffices to consider ϕ non negative and with integral 1. Then, by convexity and Jensen's inequality

$$\sum |\phi * v(t_{i+1}) - \phi * v(t_i)|^p \le \int |\phi(h)| \sum_i |v(t_{i+1}+h) - v(t_i+h)|^p dh \le \|v\|_{V^p}$$

The statement for U^p follows by duality: We have

$$B(\phi * a, v) = B(a, \tilde\phi * v)$$

with $\tilde\phi(t) = \phi(-t)$. □

The first part of the next result it due to Hardy and Littlewood [11]. The Besov spaces of the lemma will be explained in the proof. We include third statement for completeness, but it will not be used later on.

Lemma 4.18. *Let $I = \mathbb{R}$, $h > 0$ and $v \in V^p$. Then*

$$\|v(.+h) - v(.)\|_{L^p} \le (2h)^{1/p} \|v\|_{V^p}. \tag{4.14}$$

In particular, if $1 < p < \infty$,

$$\|v\|_{\dot B^{1/p,p}_\infty} \le c\|v\|_{V^p}$$

and

$$\|u\|_{U^p} \le c\|u\|_{\dot B^1_{p,1}}$$

Proof. Let $I_j = [jh,(j+1)h]$ where

$$|v(t+h)-v(t)| \le \max\{\sup_{[jh,(j+1)h]} v - \inf_{[(j+1)h,(j+2)h]} v, \sup_{[(j+1)h,(j+2)h]} v - \inf_{[jh,(j+1)h]} v\}.$$

For $\varepsilon > 0$ there exist two points $t_{j,0} \in I_j$ and $t_{j,1} \in I_{j+1}$ with

$$\sup_{t\in I_j} |v(t+h)-v(t)| \le (1-\varepsilon)|v(t_{j+1})-v(t_j)|.$$

For simplicity we assume that v is continuous, in which case we may choose $\varepsilon = 0$, which is the only use we will make of the continuity assumption. Hence

$$\begin{aligned}\int |v(t+h)-v(t)|^p dx \le & h\Big(\sum_i |v(t_{2i+1,1})-v(t_{2i+1,0})|^p + \sum_i |v(t_{2i,1})-v(t_{2i,0})|^p\Big)\\ \le & 2h\|v\|^p_{V^p}.\end{aligned}$$

All partial sums on the right hand side are bounded by $2h\|v\|^p_{V^p}$ and hence the same is true for the sum. There are many equivalent norms on the homogeneous Besov space, one of them being

$$\|v\|_{\dot B^{1/p}_{p,\infty}} = \sup_{h>0} h^{-1/p}\|v(.+h)-v\|_{L^p}$$

and the bound follows from the estimate for the difference. The last statement follows by duality: The bilinear map

$$\dot B^{\frac1p}_{p,\infty} \times \dot B^{1-\frac1p}_{\frac{p}{p-1},1} \ni (f,g) \to \int f dg$$

defines an isomorphism $\dot B^{\frac1p}_{p,\infty} \to (\dot B^{1-\frac1p}_{\frac{p}{p-1},1})^*$. Here for $0 < s < 1$ and $1 \le q < \infty$

$$\|v\|_{\dot B^s_{p,q}} = \left(\int_0^\infty (h^{-1}\|v(.+h)-v\|_{L^p})^q \frac{dh}{h}\right)^{1/q}.$$

See Triebel [30] for the theory of these spaces. □

Let $\phi \in C_0^\infty$ with $\int \phi = 0$. Then it is an immediate consequence that

$$\begin{aligned}\|v*\phi\|_{L^p} =& \|\int (v(t+h)-v(t))\phi(h)dh\|_{L^p}\\ \le& \sup_h h^{-1/p}\|v(t+h)-v(t)\|_{L^p}\int h^{1/p}|\phi(h)|dh\\ \le& c\|v\|_{V^p}\end{aligned} \tag{4.15}$$

and, by duality, for $\phi \in C_0^\infty$,

$$\begin{aligned} \|u*\phi\|_{U^p} \le & \sup_{\|v\|_{V^q}\le 1} B(\phi*u,v) \\ =& \sup_{\|v\|_{V^q}\le 1} \int \phi' * uvdt \\ =& \sup_{\|v\|_{V^q}\le 1} \int u\tilde{\phi}' vdt \\ \le & \sup_{\|v\|_{V^q}\le 1} \|u\|_{L^p} \|\phi' * v\|_{L^q} \\ \le & C\|u\|_{L^p} \end{aligned} \tag{4.16}$$

Clearly $C_0^\infty \subset V^1_{rc}$. Let $\tilde V^p \subset V^p$ be the closed subspace of functions with $f(t) = \frac12(\lim_{h\to 0}(f(t+h)+f(t-h)))$. We consider functions on $\mathbb{R}$. If $v \in V^p$ is continuous then

$$B(\phi_h * a, v) \to B(a,v) \text{ as } h \to 0$$

for all atoms a. Here $\phi \in L^1$ with $\int \phi dx = 1$ and $\phi_h(x) = h^{-1}\phi(x/h)$. If moreover ϕ is symmetric then

$$\phi_h * v \to v$$

pointwise for all $v \in \tilde V^p$ and $B(\phi_h * u, v) = B(u, \phi_h * v)$ for all $u \in U^q$ and $v \in V^p$.

Lemma 4.19. *We have*

$$B(\phi_h * u, v) \to B(u,v)$$

for $u \in U^p(\mathbb{R})$ and $v \in V^q \cap C$ and

$$\phi_h * v \to v$$

in the weak * topology for $v \in \tilde V^p(\mathbb{R})$ for $1 \le p < \infty$.

Proof. Only the last statement needs a proof. By definition and the pointwise convergence $B(u, \phi_h * v) \to B(u,v)$ for all $u \in \mathcal{R}_{rc}$. This implies weak star convergence. □

4.5. **More duality.** The space $U^q \cap C(X)$ is a closed subspace of U^q.

Lemma 4.20. *The bilinear map B defines a surjective isometry*

$$\tilde V^p(X^*)_{rc} \to (U^q \cap C(X))^*, \frac1p + \frac1q = 1, 1 < p,q < \infty.$$

Proof. The kernel of the duality map composed with the inclusion $(U^p \cap C) \subset U^p$ consists exactly of those elements of V^q which are nonzero at most at countably many points. We claim that the duality map is an isometry. Let $v \in \tilde V^p$, and let a be an atom so that

$$\|v\|_{V^p} \le (1+\varepsilon) B(a,v)$$

If ϕ_h is a symmetric mollifier then, if h is sufficiently small

$$B(a, \phi_h * v) = B(\phi_h * a, v)$$

which shows that the duality map is an isometry.

It remains to prove surjectivity. Let $L : U^p \cap C(X) \to \mathbb{R}$ be linear and continuous. By the theorem of Hahn-Banach there is a extension with the same norm to U^p, and by duality there is $v \in V^q$ with $\|v\|_{V^q} = \|L\|$ and $L(u) = B(u,v)$ for all $u \in U^p$. Changing v at a countable set does not change the image in $(U^p \cap C(X))^*$, hence we may choose $v \in V^p_{rc}$. □

In the sequel we identify $u(a)$ resp. $u(b)$ with the limit from the right resp. the left.

Lemma 4.21. *Let $u \in U^q$ and $v \in U^p$, $1/p + 1/q = 1$ and let $(t_j)_{j\to 1}$ be the points where both v and u have jumps, and denote the size of the jumps by $\Delta u(t_j)$. Then*

$$B(u,v) + B(v,u) = \sum_j \Delta u(t_j)\Delta v(t_j) + u(b)v(b) \tag{4.17}$$

Proof. The right hand side of (4.17) is continuous with respect to $u \in V^q$ and $v \in V^p$, with the jump understood as the difference between the limit from the right and the left - the sum over the jumps to the power p is bounded by the V^p norm. The left hand side is continuous with respect to $u \in U^q$ and $v \in V^p$, and it suffices to verify the formula for $u, v \in \mathcal{S}_{rc}$ with joint partition (where we add $t_0 = a$) $a = t_0 < t_1 \dots t_N < b$. Then the statement follows from Lemma 4.2. $\square$

Lemma 4.22. *Test functions C_0^∞ are weak* dense in V^p.*

Proof. Step functions are dense in V^p, and it suffices to verify that step functions can be approximated by C_0^∞ functions in the weak$*$ sense. Moreover it suffices to consider test functions with a partition consisting of a single point, which we choose to be 0. Hence we reduce the problem to a proof for three functions. We fix $\phi \in C_0^\infty(\mathbb{R})$, identically 1 in $[-1,1]$, and $\eta \in C^\infty(\mathbb{R})$ supported in $(0,\infty)$ and identically 1 for $t \geq 1$. Then for $u \in \mathcal{S}_{rc}$ checking the definition shows

$$B(u, \phi(t/j)) \to B(u,1)$$

and with $v(t) = 0$ for $t \neq 0$ and $v(0) = 1$

$$B(u, \phi(jt) \to B(u,v)$$

and, with $v(t) = 0$ for $t \leq 0$ and 1 for $t > 0$

$$B(u, \phi(t/j)\eta(jt)) \to B(u,v)$$

with $j \to \infty$. $\square$

We define

$$V_C^q = \{v \in V^q \cap C, v(b) = 0\}. \tag{4.18}$$

Lemma 4.23. *The map*

$$U^p(X^*) \to (V_C^q(X))^*,$$

$$u \to (v \to B(u,v))$$

is a surjective isometry.

Proof. By the duality estimates the duality map is defined, and it is an isometry since the space V_C^q is weak star dense in V^q. Let $L : V_C^q \to \mathbb{R}$ by linear and continuous. By Hahn Banach L can be extended to continuous linear form on $\tilde{L} \in (V^q)^*$. Since $U^q \subset V_{rc}^q$ by an abuse of notation $L \in (U^q)^*$ and there exists $\tilde{u} \in V^p$ such that

$$B(w, -\tilde{u}) = \tilde{L}(w)$$

for all $w \in U^q$. We define (with $t\pm$ the limit from the left resp. the right)

$$u(t) = \tilde{u}(t+) - \tilde{u}(a).$$

Then $u \in \cap_{\tilde p > p} U^{\tilde p}$ and by Lemma 4.21 below, for all $v \in V^p_C$,

$$\begin{aligned} L(v) =& L(v - v(a)) + v(a)L(1) \\ =& B(v - v(a), \tilde u) + v(a) \lim_{t \to a} L(\chi_{(t,b)}) \\ =& B(v - v(a), u) - (v(b-) - v(a))\tilde u(a) + v(a)\tilde u(a) \\ =& B(u, v - v(a)) + (\tilde u(b) - \tilde u(a))v(a) \\ =& B(u, v) \end{aligned}$$

where we used that $v(b) = 0$ and that v is continuous.

For every partition we have $u_\tau \in U^p$, with

$$\|u_\tau\|_{U^p} \le \sup_{v \in V^p_C, \|v\|_{V^p} \le 1} B(v, u_\tau) = \sup_{\|v_\tau\|_{V^p} = 1} L(v_\tau)$$

Since $u \in V^p_{rc}$ there is a sequence of partitions τ_i so that $u_{\tau_i} \to u \in V^p$ and hence the sequence converges uniformly. Thus for every step function v

$$B(u_{\tau_i}, v) \to B(u, v).$$

Since step functions are dense in V^q even

$$B(u_{\tau_i}, v) = B(u, v_{\tau_i}) \to B(u, v)$$

For all $v \in V^q$. Let U^{p**} be the bidual space of U^p , which we consider as isometric closed subspace of X^{**}. By an abuse of notation we consider u as element of U^{p**}. Then

$$B(u_{\tau_i}, v) \to u(v)$$

for all $v \in V^q$ and the distance between u and U^p in U^{p**} is zero, and hence $u \in U^p$. □

Corollary 4.24. *We have*

$$\|u\|_{U^p(X)} = \sup\{B(u, v) : v \in C^\infty_0(X), \|v\|_{V^q(X^*)} = 1\}.$$

and

$$\|v\|_{V^p_{rc}(X)} = \sup\{B(u, v) : u \in C^\infty_0, \|u\|_{U^q(X^*)}\} = 1\}.$$

Proof. Clearly C^∞_0 is weak dense in $V^p(X^*)$. This implies the first statement. Given $\varepsilon > 0$ there exists a q atom in $U^q(X^*)$ with

$$B(a, v) \ge \|v\|_{V^p} - \varepsilon$$

Since

$$B(x\chi_{[t,b)}, v) \to 0$$

as $t \to a$ we may assume that $a(b) = 0$. A standard regularization implies the full statement. □

4.6. **Consequences of Minkowski's inequality.** For a Banach space Y we denote by $L^p(Y)$ the weakly measurable maps with values in Y; for which the norm is p integrable.

Lemma 4.25. *We have for* $1 < p \le q < \infty$

$$\|u\|_{L^q_x(U^p)} \le \|u\|_{U^p(L^q_x)} \tag{4.19}$$

and

$$\|v\|_{V^p(L^q_x)} \le \|v\|_{L^q_x(V^p)}. \tag{4.20}$$

Proof. It suffices to verify the first inequality for a p atom

$$a(t,x) = \sum \chi_{[t_i,t_{i+1})}(t)\Phi_i(x)$$

with values in L^q. This is a function of x and t. Then $t \to a_x(t)$ is a step function. Let

$$f(x) = \left(\sum_i |\Phi_i(x)|^p\right)^{1/p}$$

Then

$$\begin{aligned}
\|a\|_{L^q_x(U^p)} &= \left(\int f(x)^q dx\right)^{1/q} \\
&= \left(\int \left(\sum_j |\Phi_j(x)|^p\right)^{q/p}\right)^{1/q} \\
&\le \left(\sum_j \|\Phi_j\|^p_{L^q}\right)^{1/p} \\
&\le 1
\end{aligned}$$

where we use Minkowski's inequality for the first inequality. The argument for the V^p space is similar. □

The argument works the same way if we consider Banach space valued functions in U^pL^q etc.

4.7. **The bilinear form as integral.** Here we consider scalar valued functions.

Definition 4.26. *Let $v \in V^p(a,c)$ and $u \in U^q(a,c)$. We define for $a \le s < t \le b$*

$$\int_s^t vdu := B_{(s,t)}(u - u(s), v) + (u(t) - u(t-))v(t) \tag{4.21}$$

and

$$\begin{aligned}
\int_s^t udv := -\int_s^t vdu + \sum_j (u(t_j) - u(t_j-))(v(t_j) - v(t_j-)) \\
+ u(t-)v(t-) - u(s)v(s+) + u(t)(v(t+) - v(t-)) + v(t)(u(t) - u(t-))
\end{aligned} \tag{4.22}$$

with the sum over all joined jumps in (s,t).

The second definition is partly motivated by

1. The integration by parts formula (4.17). It should reduce to integration by parts if $v \in U^q$, and if there are no jumps at t
2. The desire to have a certain symmetry with time reversion if v is continuous the left and u is continuous from the right.
3. We want the integral to be additive in the interval.

Lemma 4.27. *For $u \in U^q$ and $v \in V^p$, $1/p + 1/q = 1$ we have*

$$\int_a^c vdu = \int_a^b vdu + \int_b^c vdu$$

and

$$\int_a^c u dv = \int_a^b u dv + \int_b^c u dv.$$

With the obvious notation,

$$\Big\| \int_a^t u dv \Big\|_{V^p} \le \|u\|_{U^q} \|v\|_{V^p} \tag{4.23}$$

and

$$\Big\| \int_a^t v du \Big\|_{U^q} \le \|u\|_{U^q} \|v\|_{V^p}. \tag{4.24}$$

Proof. It suffices to check the first formula for atoms u. Suppose that $t_j < b \le t_{j+1}$. On both sides we have a sum over

$$v(t_{j+1})(u(t_{j+1}) - u(t_j)).$$

For the second formula we see from the definition

$$\int_a^c u dv = \int_a^b u dv + \int_b^c u dv$$

where we have to check the contribution at $t = b$.

Formally, for smooth functions

$$\begin{aligned} B(\int_a^t v du, w) = & \int_a^b w(t) v(t) u'(t) dt \\ =& B(u, vw) \\ \le& \|vw\|_{V^q} \|u\|_{U^p} \\ \le& 2|v\|_{V^q} \|w\|_{V^q} \|u\|_{U^p} \end{aligned} \tag{4.25}$$

which formally implies (4.24).

For a rigorous proof we verify the formula in the case when u is a atom, and v and w are step function with a common partition all functions. Then $\int_a^t v du$ is a right continuous step function and

$$\sum_j (v(t_j)(u(t_j) - u(t_{j-1}))w(t_j) = \sum_j [v(t_{j+1})w(t_{j+1}) - v(t_j)w(t_j)]u(t_j)$$

where we neglect the boundary terms. We apply Hölder's inequality to bound the expression by

$$\Big(\sum |v(t_{j+1})w(t_{j+1}) - v(t_j)w(t_j)|^q \Big)^{1/q} \Big(\sum |u(t_j)|^p \Big)^{1/p}.$$

Again formally for smooth functions

$$\begin{aligned} B(w, \int_a^t u dv) = & - \int_a^b v w u' dt + (w(b) - w(a)) \int_a^b u v' dt \\ = & \int_a^b v(uw)' dt - (w(b) - w(a)) \int_a^b v u' dt \\ & - u(b)v(b)w(b) + u(a)v(a)w(a) \\ & + (w(b) - w(a))(u(b)v(b) - u(a)v(a)) \\ = & B(uw, v) - (w(b) - w(a)) B(v, u) \end{aligned} \tag{4.26}$$

if $u(a) = w(a) = 0$. This implies formally (4.24). For a rigorous proof we apply integration by parts several times. First

$$\begin{aligned}\int_{t-}^{t+} udv =&(u(t) - u(t-))(v(t) - v(t-)) + u(t)v(t+) - u(t-)v(t-) \\ &- v(t)(u(t) - u(t-)) \\ =&u(t)(v(t+) - v(t-))\end{aligned}$$

and

$$\int_t^{t+} udv = u(t)(v(t+) - v(t))$$

and hence l^p sum over the jumps is bounded. Thus the bound reduces to the bound for

$$B(w, \int_a^t vdu)$$

and by the same token to

$$B(\int_a^t vdu, w)$$

which we have proven above. □

Sometimes it is convenient to have a notation for spaces of derivatives of functions in U^p resp. V^p.

Definition 4.28. *We define dU^p as the space of all distributions f for which there exists an antiderivative in U^p, equipped with the norm in U^p. Similarly, let dV^p be the space of all distributions which have an antiderivative in $\tilde{V}^p_{rc}$, equipped with the obvious norm.*

4.8. **Differential equations with rough paths.** This type of study was initiated by Lyons [21]. We will only scratch on the surface. We observe that the duality mapping extends the Young integral.

We consider the differential equation

$$\dot{y} = F(y, x)\dot{x}, \qquad y(0) = y_0$$

where $x \in U^2$ and F is a bounded Lipschitz function continuously Frechet differentiable with respect to y, and d_yF is uniformly Lipschitz continuous. We denote by an abuse of notation the bound for F by $\|F\|_{sup}$, the Lipschitz bound with respect to y by $\|D_Y F\|_{sup}$, and the homogeneous Hölder bound with respect to y by $\|F\|_{C^s(Y)}$.

Suppose that y is a solution, i.e

$$y(t) = y(a) + \int_a^t F(y, x)dx$$

Then, by (4.24)

$$\begin{aligned}\|y(t) - y(a)\|_{U^2} \leq&\|F(y, x)\|_{V^2}\|x\|_{U^2} \\ \leq&(\|F\|_{sup} + \|D_yF\|_{sup}\|y\|_{V^2}) + \|D_xF\|_{sup}\|x\|_{V^2})\|x\|_{U^2}\end{aligned} \tag{4.27}$$

It is trivial that there is a unique solution if x is a step function in $\mathcal{S}_{rc}$ - for that we consider a finite number of differences. We shall construct a solution to the initial value problem for $\|x\|_{U^p}$ small. This implies existence of a unique solution since we may first approximate x by a step function, and then solve the differential equation on each of the intervals of the step function.

We want to construct a solution as fixed point of

$$y(t) = y_0 + \int_0^t F(y(s), x(s))\dot{x} ds.$$

We claim that there is a unique solution y with $y - y(a) \in U^2$ provided

$$\|x\|_{U^2} < \varepsilon$$

with ε sufficiently small. Let

$$y(t) = y_0 + \int_0^t F(\tilde{y}(s), x(s))\dot{x} ds.$$

Now, by (4.27),

$$\|y - y(a)\|_{U^2} \le (\|F\|_{sup} + \|D_y F\|_{sup}\|\tilde{y}\|_{V^2} + \|D_x F\|_{sup}\|x\|_{V^2})\|x\|_{U^2}$$

and we obtain a uniform bound R on the iteration provided $\|D_y F\|_{sup}\|x\|_{U^2} \le \frac{1}{2}$. If $\tilde{y}_1, \tilde{y}_2 \in U^2$ and y_i is defined by the Young integral above we get- we consider scalar valued functions to simplify the notation -

$$\begin{aligned} \|y_2 - y_1\|_{U^2} \le & 2\|F(\tilde{y}_2, x) - F(\tilde{y}_1, x)\|_{V^2}\|x\|_{U^2} \\ \le & \Big(\|D_y F\|_{sup}\|\tilde{y}_2 - \tilde{y}_1\|_{V^2} + \|D^2_{yy} F\|_{sup}\|\tilde{y}_2 - \tilde{y}_1\|_{sup}\|\tilde{y}_2 - \tilde{y}_1\|_{V^2} \\ & + \|D^2_{yx} F\|_{sup}\|\tilde{y}_1 - \tilde{y}_1\|_{sup}\|x\|_{U^2}\Big)\|x\|_{U^2} \end{aligned}$$

We easily construct a unique solution by a standard contraction argument provided

$$\left(\|D_y F\|_{\sup} + \|D^2_{yy} F\|_{sup}\|R + \|D^2_{xy}\|x\|_{U^2}\right)\|x\|_{U^2} < \frac{1}{2}.$$

where R is the uniform bound from above.

The modifications for U^p, $p < 2$ are as follows. The differentiability requirements on F are weaker: Let $1 < p < 2$ and $\frac{1}{p} + \frac{1}{q} = 1$. The apriori estimate requires few changes and we concentrate on the contraction, for which we consider

$$\begin{aligned} \|F(\tilde{y}_2, x) - F(\tilde{y}_1, x)\|_{V^q} \le & \|D_y F\|_{sup}\|\tilde{y}_2 - \tilde{y}_1\|_{V^q} + \|D_y F\|_{C^{p/q}}(\|y_2 - \tilde{y}_1\|^{p/q}_{V^p} \\ & + \|x\|^{p/q}_{V^p})\|\tilde{y}_2 - \tilde{y}_1\|_{sup}. \end{aligned}$$

We recall that $p - 1 = p/q$. We obtain the contraction as above.

Theorem 4.29. *Let $1 < p \le 2$, $F : X \times Y \to Y$ be bounded, uniformly Lipschitz continuous, Frechet differentiable with respect to X and Y, and dF is Hölder continuous with respect to y with Hölder exponent $p - 1$. We study*

$$dy = F(x, y)dx, y(a) = y_0$$

Then there exists a unique solution $y \in U^p(Y)$ if $x \in U^p$ if $1 \le p \le 2$ and $y \in V^p$ if $x \in V^p$ and dF is Hölder continuous with exponent $s > p - 1$.

4.9. **The Brownian motion.** The Brownian motion is almost surely in V^p for $p > 2$. We denote by $B_t(\omega)$ the path of the Brownian motion as a function of t and the element of the probability space ω. If the Brownian motion would be in U^2 with positive probability we could solve stochastic differential equations in a pointwise sense. The 2-variation however is almost certainly infinite.

The regularity of the Brownian motion is characterized by the following fairly sharp result of Taylor [29] , see also [7].

Theorem 4.30. *Let*

$$\psi_{2,1}(h) = \begin{cases} h^2 & \text{for } h \ge e^{-e} \\ \frac{h^2}{\ln\ln(1/h)} & \text{if } h < e^{-e} \end{cases}$$

There exists $\eta > 0$ so that

$$\mathbb{E}(\exp(\frac{\eta}{T}\|B\|^2_{\psi_{2,1};[0,T]}) < \infty$$

where

$$\|B\|_{\psi_{2,1};[0,T]} = \inf\{M > 0 : \sup_\tau \sum \psi_{2,1}(|B_{t_{i+1}} - B_{t_i}|/M) \le 1\}.$$

Moreover, if

$$\frac{h^2}{\psi(h)\ln\ln(1/h)} \to 0 \text{ as } h \to 0$$

then

$$\sup_{\tau_T} \sum \psi(|B_{t_{i+1}} - B_{t_i}|) = \infty.$$

See Theorem 13.15 and Theorem 13.69 in [7]. This result deviates from the V^p spaces by an iterated logarithm.

Let (Ω, μ) be a probability space with a filtration μ_t, $t \in \mathbb{R}$, $f \in L^p$ and $f_t = \mathbb{E}(f, \mu_t)$. Then

$$\|f_t\|_{L^p(\Omega, V^2_w)} \le c_p \|f\|_{L^p} \tag{4.28}$$

is a consequence of Doob's oscillation lemma for martingals [23], see also Bourgain's proof of p-variation estimate [2]. A weaker version is due to Lepingle [19].

For the Brownian motion B_t we obtain

Theorem 4.31.

$$\|B_t\|_{L^p(\Omega, V^2_w([0,1)))} \le c_p.$$

This has been a motivation to introduce V^p_w.

4.10. **Adapted function spaces.** Given distribution T want to construct an element in U^p or V^p which has T as derivative. This is the done in the next lemma. Again $\frac{1}{p} + \frac{1}{q} = 1$.

Lemma 4.32. *Suppose that T is a distribution supported in $[0, \infty)$ so that*

$$\sup\{T(\phi) : \phi \in C_0^\infty, \|v\|_{U^q} \le 1\} = C_1 < \infty$$

then there exists a unique $v \in V^p_{rc}$ with

$$T(\phi) = B(v, \phi),$$

$$C_1 \le \|v\|_{V^p} \le 2C_1$$

and $v_t = T$ in the sense of distributions. Suppose that T is a distribution supported in $[0,\infty)$ so that

$$\sup\{T(\phi) : \phi \in C_0^\infty, \|v\|_{V^q} \le 1\} = C_1 < \infty$$

then there exists a unique $u \in U^p$ with

$$T(\phi) = B(u,\phi),$$

$$\|u\|_{U^p} = C_1$$

and $u_t = T$ in the sense of distributions.

Proof. There exists a unique distribution V supported in $[0,\infty)$ with $\partial_t V = T$ which is defined as follows. We fix a function $\eta \in C^\infty$ supported in $[-2,\infty)$ and identically 1 in $[-1,\infty)$. Then

$$V(\phi) := T(\eta \int_t^\infty \phi)$$

which does not depend on the choice of η. Then

$$V(\partial_t \phi) = T(\eta\phi) = T(\phi)$$

by definition. The difference of two such distributions has zero derivative, hence it is constant, and by the assumption on the support it is unique.

Next we choose a function $\psi \in C_0^\infty(\mathbb{R})$ supported in $(-1,1)$ with $\int \psi dx = 1$ and define for $h > 0$ and $s \in \mathbb{R}$

$$\phi(t) = \eta(t) h^{-1} \int_t^\infty \psi((t-s)/h) dt.$$

Then by the support property,

$$V(h^{-1}\psi((t-s)/h)) = -V(\partial_t \phi) = T(\phi)$$

and, since, for suitably choosen ψ

$$\|\phi\|_{U^q} \le 1$$

and hence

$$|V(h^{-1}\psi((t-s)/h)| \le C,$$

which implies

$$\sup_t |V * h^{-1}\psi(./h)| \le C$$

and thus there exists a bounded and measurable function v with

$$V(\phi) = \int v\phi dt$$

and moreover v is supported in $[t_0,\infty)$. At Lebesgue points

$$|V(h^{-1}\psi((t-s)/h))| = h^{-1} \int v(t)\psi((t-s)/h) dt \to v(s)$$

as $h \to 0$. Similarly, if τ is partition for which all points are Lebesgue points, and arguing as for duality we see that

$$\Big(\sum |v(t_j) - v(t_{j-1})|^p\Big)^{\frac{1}{p}} \le C$$

In particular left and right limits at $t \in \mathbb{R}$ exist if we restrict the approach to Lebesgue points. Hence we may assume that v is a right continuous ruled function, supported in $[t_0, \infty)$. But then the very same argument shows (since we have to include the supremum in the norm) that

$$\|v\|_{V^p_{rc}} \le 2C.$$

By construction the weak derivative of v is T. We conclude that T defines an element of $(U^q)^*$ which is represented by same function which has to coincide with v. This completes the argument in this case.

In the second part we construct the function u as above. Then

$$T(\phi) = -\int u\partial_t \phi = -B(\phi, u) = B(u, \phi).$$

In particular, for every partition, since C_0^∞ is weak star dense,

$$\|u_\tau\|_{U^p} \le C$$

We conclude as for the duality that

$$\|u\|_{U^p} \le C.$$

□

we observe that there are not more than obvious changes if we consider Hilbert spaces valued functions, and if we replace the product by the inner product.

We briefly survey constructions going back to Bourgain, which have become standard. The following situation will be of particular interest. Let $t \to S(t)$ be a continuous unitary group on a Hilbert space H. We define U^p_S and V^p_S by

$$\|v\|_{V^p_S(H)} = \|S(-t)v(t)\|_{V^p(H)},$$

or, to put it differently, we say that $u \in V^p_S$ if and only if $S(-t)v \in V^p$. Similarly we define U^p_S. Alternatively we could define U^p_S by U^p_S atoms. Such an atom is given by a partition $t_1 < t_2 \dot{t}_n$ and n elements $\phi_j \in H$, with $\sum \|\phi_j\|^p \le 1$, and $a(t) = 0$ if $t < t_1$, and $a(t) = S(t - t_j)\phi_j$ if $t_j \le t < t_{j+1}$, with the obvious modification if $t \ge t_n$.

By Stone's theorem unitary groups are in one-one correspondence with self-adjoint operators, in the sense that

$$i\partial_t u = AU$$

with a self adjoint operator defines unitary group $S(t)$ and vice versa. At least formally

$$i\partial_t(S(-t)u(t)) = S(-t)(i\partial_t u - Au)$$

and hence the duality assertion is

$$\|u\|_{U^q_S} = \sup_{\|v\|_{V^p_S} \le 1} B(S(-t)u(t), S(-t)v(t)).$$

Now suppose that - again formally -

$$i\partial_t u + Au = f$$

then, if we choose by Duhamel's formula the solution

$$u(t) = \int_{-\infty}^t S(t-s)f(s)ds.$$

A related construction goes back to Bourgain. He defines

$$\|u\|_{X^{0,b}_S} = \|S(-t)u(t)\|_{H^b L^2} \tag{4.29}$$

where the Sobolev space H^b is defined by the Fourier transform,

$$\|f\|_{H^b} = \|(1+|\tau|^2)^{b/2}\hat{f}\|_{L^2}$$

Clearly

$$X_S^{0,b} \subset X_S^{0,b'}$$

whenever $b \geq b'$. We may use a Besov refinement of the right hand side of (4.29), i.e.

$$\|u\|_{\dot{X}^{s,b,q}} = \left(\sum_{N\in 2^{\mathbb{Z}}} N^{sq}\|u_N\|^q_{H^b(L^2)}\right)^{1/q}$$

where we choose a disjoint partition $A_N = \{(\tau,\xi) : 2^N \leq |\tau+\phi(\xi)| \leq 2^{1+N}\}$ and define u_N by the Fourier multiplication by the characteristic function of A_N.

Then

$$\dot{X}_S^{0,\frac12,1} \subset U_S^2 \subset V^2_{S,rc} \subset \dot{X}^{0,\frac12,\infty}$$

follows from Lemma 4.18.

There is an obvious generalization to the case of time dependent operators $A(t)$. Definitions are simple, but this often leads to technical questions.

Now

$$\mathcal{F}_{t,x}(S(-t)u)(\tau,\xi) = \mathcal{F}_t e^{-it\phi(\xi)}\hat{u}(t,\xi) = \mathcal{F}_{t,x}u(\tau - t\phi(\xi),\xi)$$

and hence by the formula of Plancherel and a translation in τ variable

$$\|u\|_{X^{0,b}} = \|(1+\tau^2)^{b/2}\mathcal{F}_{t,x}(u)(\tau - t\phi(\xi),\xi)\|_{L^2} = \|(1+(\tau+\phi(\xi))^2)^{b/2}\mathcal{F}_{t,x}(u)\|_{L^2}.$$

4.10.1. *Strichartz estimates.* We want to use this construction for dispersive equations. There A is often defined by a Fourier multiplier, most often even by a partial differential operator with constant coefficients.

We consider the Schrödinger equation

$$i\partial_t u + \Delta u = 0 \qquad \text{in } [0,\infty)$$

$$u(0) = u_0 \qquad \text{on } \mathbb{R}^d$$

Let $u(t) = 0$ for $t<0$ and the solution otherwise. Then

$$\|u\|_{U^1_S} = \|u_0\|_{L^2(\mathbb{R}^d)}.$$

One of the Strichartz estimates states

$$\|u\|_{L^p_t L^q_x} \leq \|u_0\|_{L^2} \tag{4.30}$$

whenever

$$\frac{2}{p} + \frac{d}{q} = \frac{d}{2}, \qquad 2 \leq p,q, \quad (p,q,d) \neq (2,\infty,2).$$

We claim that this implies

$$\|u\|_{L^p_t L^q_x} \leq c\|u\|_{U^p}.$$

It suffices to verify this if $S(-t)u$ is an atom with partition $(t_1, t_2 \dots t_n)$. Then, with $t_{n+1} = \infty$, by the Strichartz estimate

$$\|u\|_{L^p_t(t_j,t_{j+1});L^q_x)} \leq c\|u(t_j)\|_{L^2}.$$

We raise this to the pth power, and add over j. Then

$$\|u\|_{L^pL^q} \leq c\left(\sum \|u(t_j)\|^p_{L^2}\right)^{1/p} \leq c$$

since $S(-t)u$ is a p atom.

Consider $v(t) = \int_{-\infty}^t S(t-s)f(s)ds$ and let $\tau = (t_j)$ be a partition. Then

$$v(t_j) - S(t_j - t_{j-1})v(t_{j-1}) = \int_{t_{j-1}}^{t_j} S(t_j - t)f(t)dt$$

and by the Strichartz estimate

$$\|S(-t_j)v(t_j) - S(-t_{j-1})v(t_{j-1})\|_{L^2} \le c\|f\|_{L^{p'}_t L^{q'}_x}$$

and

$$t \to S(-t)v(t)$$

is continuous.

We take the power p' and sum over j to reach the conclusion

$$\|v\|_{V^{p'}_S} \le c\|f\|_{L^{p'}L^{q'}}$$

This implies the dual estimate to (4.30). If $p > 2$ we may combine the estimates with an embedding to obtain the full Strichartz estimate. In particular we arrive at the non symmetric improvement for the Strichartz estimate

$$\|u\|_{L^\infty(L^2)} + \|u\|_{L^{q_0,p_0}} \le c\left(\|u_0\|_{L^2} + \|f\|_{L^{q_1' p_1'}}\right)$$

if both (q_1, p_1) and (q_0, p_0) are Strichartz pairs, but not necessarily the same ones.

We prove this estimate over the interval $(0,\infty)$ and extend u by 0 to negative t. Then

$$\|u\|_{L^\infty(L^2)} + \|u\|_{L^{p_0,q_0}} \le c\|u\|_{U^{p_0}} \le c\|u\|_{V^{p_1'}} \le c\|u_0\|_{L^2} + \|f\|_{L^{p_1',q_1'}}.$$

Lemma 4.33. *The following estimates hold for Strichartz pairs*

$$\|u\|_{L^{p,q}} \le c\|u\|_{U^p}$$

and

$$\left\|S(t)u_0 + \int_0^t S(t-s)f(s)ds\right\|_{V^{p'}} \le c(\|u_0\|_{L^2} + \|f\|_{L^{p',q'}}).$$

4.10.2. *Estimates by duality.* We return to duality questions and calculate formally

$$\begin{aligned}
\|u\|_{U^q_S} &= \sup_{\|v\|_{V^p_S}\le 1} |B(S(-t)u(t), S(-t)v(t))| \\
&= \sup_{\|v\|_{V^p_S}\le 1} \left|\int_{\mathbb{R}} \langle \partial_t S(-t)u(t), S(-t)v(t)\rangle dt\right| \\
&= \sup_{\|v\|_{V^p_S}\le 1} |-i\langle S(-t)(i\partial_t u - Au), S(-t)v\rangle dt| \\
&= \sup_{\|v\|_{V^p_S}\le 1} \int_{\mathbb{R}} \langle f, v\rangle dt
\end{aligned} \tag{4.31}$$

with a similar statement for V^p_S. This observation will be crucial for nonlinear dispersive equations.

Lemma 4.34. *Let $\phi \in C^\infty(\mathbb{R}^d)$ be a real polynomial and let S be the unitary group defined by the Fourier multiplier $e^{it\phi(\xi)}$. Let $1 < p, q < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Let T be a tempered distribution in $(a,b) \times \mathbb{R}^d$ which satisfies*

$$\sup\{|T(\bar u)| : u \in C_0^\infty((a,b)\times\mathbb{R}^d), \|u\|_{U^p_S} \le 1\} = C_1 < \infty$$

Then there is a unique $v \in V^q_{S,rc}(a,b)$ *with*

$$T(u) = \int v\overline{iu_t + \phi(D)u}dxdt$$

and $\|v\|_{V^q} = C_1$. *Let* T *be a distribution in space time which satisfies*

$$\sup\{|T(\bar v)| : v \in C_0^\infty(\mathbb{R}\times\mathbb{R}^d), \|v\|_{V^p_S}\} = C_2 < \infty$$

Then there is a unique $u \in U^q_S$ *with*

$$T(\bar v) = \int u\overline{iv_t + \phi(D)v}dxdt$$

and $\|u\|_{U^q} = C_2$.

Proof. Fourier multiplication commutes with the evolution. We convolve T with the inverse Fourier transform of a non-negative function with compact support. To this we apply Lemma 4.22. But this gives the full statement. □

The theorem implies existence of a weak solution to

$$i\partial_t u + \phi(D)u = f, \quad u(a) = 0,$$

together with an estimate for u.

4.10.3. *High modulation estimates.* We denote by $f(D)$ the Fourier multiplier defined by a function f. Let

$$f = 1 - \chi(\tau/\Lambda)$$

where τ is the Fourier variable corresponding to t and χ is an approximate characteristic function, i.e. χ is supported on a ball of radius 2, and identically 1 on a ball of radius 1.

Lemma 4.35. *The following estimate holds.*

$$\|f(D)v\|_{L^2} \le c\Lambda^{-1/2}\|v\|_{V^2}$$

Suppose the group $S(t)$ *is defined by the Fourier multiplier* $e^{it\phi(\xi)}$ *then, with*

$$f(D) = 1 - \chi(\tau + \phi(\xi))$$

$$\|f(D)u\|_{L^2} \le c\Lambda^{-1/2}\|v\|_{V^2_S}$$

Proof. We have

$$\mathcal{F}_t(e^{-it\phi(\xi)}\hat u(t,\xi)) = \mathcal{F}_{x,t}u(\tau - \phi(\xi),\xi)$$

and the second claim follows from the first one. Let

$$g = \mathcal{F}^{-1}\chi(\xi/\Lambda).$$

Then

$$g(t) = \Lambda^{-1}(\mathcal{F}^{-1}\chi)(\Lambda\xi)$$

and

$$\begin{aligned}\Big\|\int (v(t+h) - v(t))g(h)dh\Big\|_{L^2}& \\ \le & \sup_h |h|^{-1/2}\|v(t+h)-v(t)\|_{L^2}\int |h|^{1/2}\Lambda^{-1/2}|\mathcal{F}^{-1}\chi(h\Lambda)|dh \\ \le & c\|u\|_{V^2}\Lambda^{-1/2}\int |h|^{1/2}|\mathcal{F}^{-1}\chi|dh.\end{aligned}$$

□

5. Convolution of measures on hyper surfaces, bilinear estimates and local smoothing

The contents of this section developed in discussions with S. Herr, T. Schottdorf and J. Li. The bilinear estimates for the Kadomtsev-Petviashvili equation have been influenced by the careful work of M. Hadac. Bilinear estimates are standard tools in dispersive equations. Here we attempt to streamline arguments and sharpen the results. In particular the bilinear estimates for the KP-II seem to be new.

The transformation formula for a diffeomorphism $\phi : U \to V$ $U, V \subset \mathbb{R}^d$, states

$$\int_V f dm^d = \int_U f \circ \phi |\det D\phi| dm^d.$$

Its relative, the area formula for $n \ge d$,

$$\phi : U \to S \subset \mathbb{R}^n,$$

ϕ continuously differentiable and injective reads as

$$\int_S f d\mathcal{H}^d = \int_U f \circ \phi (\det D\phi^T D\phi)^{1/2} dm^d.$$

where $\mathcal{H}^d$ denotes the Hausdorff measure. The coarea formula deals with the opposite situation $d \ge n$ and

$$\phi : U \to V \subset \mathbb{R}^n$$

surjective. It states for $f : U \to \mathbb{R}$ measurable

$$\int_V \int_{\phi^{-1}(y)} f d\mathcal{H}^{d-n} dm^n(y) = \int_U f \det(D\phi D\phi^T)^{1/2} dm^d.$$

Often it is useful to write it in the form

$$\int_V \int_{\phi^{-1}(y)} \det(D\phi(x) D\phi^T(x))^{-1/2} f(x) d\mathcal{H}^{d-n}(x) dm^n(y) = \int_U f dm^d. \tag{5.1}$$

The Fourier transform transforms a product into a convolution, and vice verse. Let Σ_1 and Σ_2 be two $d-1$ dimensional hyper surfaces in $\mathbb{R}^d$ such that for all $x_i \in \Sigma_i$ the tangent spaces of Σ_i at x_i are transversal, for $i = 1, 2$.

Let Σ_1 and Σ_2 be non degenerate level sets of functions ϕ_1 and ϕ_2. Let h be a continuous function. Then, by the coarea formula

$$\int_{\mathbb{R}^d} f(x) h \circ \phi_1(x) dm^d(x) = \int_{\mathbb{R}} h(s) \int_{\phi_1^{-1}(s)} f(x) |\nabla \phi_1|^{-1}(x) d\mathcal{H}^{d-1}(x) ds.$$

This motivates the notation

$$\delta_\phi = |\nabla \phi|^{-1} d\mathcal{H}^{d-1}\Big|_{\phi=0}.$$

We study the convolution of two measures supported on the hyper surfaces Σ_1 and Σ_2.

Theorem 5.1. *Let $\Sigma_i \subset \mathbb{R}^d$ hyper surfaces and ϕ_i as above, and f_i square integrable functions on Σ_i with respect to δ_{ϕ_i}. Then*

$$\| f_1 \delta_{\phi_1} * f_2 \delta_{\phi_2} \|_{L^2(\mathbb{R}^d)} \le L \| f_1 |\nabla \phi_1|^{-1/2} \|_{L^2(\Sigma_1)} \| f_2 |\nabla \phi_2|^{-1/2} \|_{L^2(\Sigma_2)}$$

where with $\Sigma(x, y) = \{y + \Gamma_1\} \cap \{x + \Gamma_2\}$

$$L = \sup_{x \in \Sigma_1, y \in \Sigma_2} L(x, y),$$

and where $L(x,y)$ is the square root of

$$\int\limits_{\Sigma(x,y)} \left[|\nabla\phi_1(z-y)|^2|\nabla\phi_2(z-x)|^2 - \langle\nabla\phi_1(z-x),\nabla\phi_2(z-y)\rangle^2\right]^{-1/2} d\mathcal{H}^{d-2}.$$

Proof. Let f_i be measurable functions in a neighborhood of Σ_i, let h be continuous and non negative, and $g_i = h\circ\phi_i$. Then, by Cauchy Schwartz and Fubini,

$$\begin{aligned}
&\|f_1g_1 * f_2g_2\|_{L^2}^2 \\
&\quad= \int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d} f_1(x)g_1^{\frac12}(x)g_2^{\frac12}(z-x)f_2(z-x)g_1^{\frac12}(x)g_2^{\frac12}(z-x)dm^d(x)\right)^2 dm^d(z)\\
&\quad\le \int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d}|f_1(x)|^2g_1(x)g_2(z-x)dm^d(x)\int\limits_{\mathbb{R}^d}|f_2(y)|^2g_2(y)g_1(z-y)dm^d(y)dm^d(z)\\
&\quad= \int_{\mathbb{R}^{2d}}|f_1(x)|^2g_1(x)|f_2(y)|^2g_2(y)\int g_2(z-x)g_1(z-y)dm^d(z)dm^{2d}(x,y).
\end{aligned}$$

By the coarea formula

$$\int g_2(z-x)g_1(z-y)dm^d = \int_{\mathbb{R}^2} h(s)h(t)I(s,t)dsdt$$

where, with

$$\Sigma_{s,t} = \{z : \phi_1(y+z) = s, \phi_2(x+z) = t\}$$

and

$$\rho(s,t,z) = \left||\nabla\phi_1(z-y)|^2|\nabla\phi_2(z-x)|^2 - (\nabla\phi_1(z-y)\cdot\nabla\phi_2(z-x))^2\right|^{-1/2}$$

$$I(s,t) = \int_{\Sigma_{s,t}} \rho(s,t,z)d\mathcal{H}^{d-2}(z).$$

Here we suppress the dependence on x and y, but we set

$$\gamma(x,y) = I(0,0).$$

Again by the coarea formula

$$\int_{\mathbb{R}^d}|f_1(x)|^2g_1(x)dm^d(x) = \int_{\mathbb{R}} h(s)\int_{\phi_1^{-1}(s)}|f_1(x)|^2\,|\nabla\phi_1(x)|^{-1}d\mathcal{H}^{d-1}(x)ds.$$

There is a similar formula for the second integral. We assume that f_i is continuous and choose a Dirac sequence for h to obtain the estimate. The statement for measurable functions on the surfaces follows by a standard approximation argument. □

Using the coarea formula we obtain a more explicit formula for the convolution:

$$\begin{aligned}
f_1h\circ\phi_1 * f_2h\circ\phi_2(z) &= \int (f_1h\circ\phi_1)(z-y)(f_2h\circ\phi_2)(y)dm^d(y)\\
&= \int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}} h(s)h(t)\int_{\Sigma(s,t)} f_1(z-y)f_2(y)\rho(s,t,z)d\mathcal{H}^{d-2}(y)dsdt.
\end{aligned}$$

hence
(5.2)

$$f_1\delta_{\phi_1} * f_2\delta_{\phi_2}(x) = \int_{\Gamma_1\cap(x-\Gamma_2)} \big||\nabla\phi_1(y)|^2|\nabla\phi_2(x-y)|^2 - (\nabla\phi_1(y)\cdot\nabla\phi_2(x-y))^2\big|^{-1/2} d\mathcal{H}^{d-2}(y).$$

There is a trivial and useful improvement of the convolution estimate of Theorem 5.1

$$\Big\| \int_{\Gamma_1\cap(z-\Gamma_2)} \gamma^{-1/2}(x,y)\rho(0,0,z)f_1(x)f_2(y)d\mathcal{H}^{d-2}\Big\|_{L^2} \le \|f_1\|_{L^2(\Sigma_1,\delta_{\phi_1})}\|f_2\|_{L^2(\Sigma_2,\delta_{\phi_2})}. \tag{5.3}$$

It follows from the same brief as Theorem 5.1. Here $L^2(\Sigma_i,\delta_{\phi_i})$ denotes the space of square integrable functions on the hyper surface with respect to the measure δ_{ϕ_i}.

We use the convolution estimate to bound products of solutions to dispersive equations. Consider

$$iu_t - \psi(D)u = 0$$

where the operator $\psi(D)$ is defined as the multiplication of the Fourier transform by the real function ψ. The characteristic surface Σ is defined as the surface in $\mathbb{R}^{d+1}$ defined by the zero level set of the function

$$\phi(\tau,\xi) = \tau - \psi(\xi).$$

Let u be the solution with initial data u_0. Then

$$\mathcal{F}_x u(t,\xi) = e^{it\psi(\xi)}\mathcal{F}_x u_0(\xi)$$

and, for any Schwartz function $f\in\mathcal{S}(\mathbb{R}^{d+1})$ with Fourier transform g, by Plancherel

$$\begin{aligned}\int_{\mathbb{R}\times\mathbb{R}^d} u\bar{f}dm^{d+1}(t,x) =& \int_{\mathbb{R}}\int_{\mathbb{R}^d} \mathcal{F}_x u(t,\xi)\overline{\mathcal{F}_x f(t,\xi)}dm^d(\xi)dt\\ =& \int_{\mathbb{R}}\int_{\mathbb{R}^d} e^{it\psi(\xi)}\hat{u}_0(\xi)\overline{\mathcal{F}_x f(t,\xi)}dm^d(\xi)dt\\ =& \int_{\mathbb{R}^d}\hat{u}_0(\xi)\int\overline{e^{-it\psi(\xi)}\mathcal{F}_x f(t,\xi)}dtdm^d(\xi)\\ =&\sqrt{2\pi}\int_{\mathbb{R}^d}\hat{u}_0(\xi)\overline{g(\psi(\xi),\xi)}d\xi\\ =&\sqrt{2\pi}\int_{\tau=\psi(\xi)}|\nabla_{\tau,\xi}\phi(\tau,\xi)|^{-1}u_0(\xi)\bar{g}(\tau,\xi)d\mathcal{H}^d(\tau,\xi)\\ =:&\sqrt{2\pi}\int\bar{g}(\tau,\xi)\hat{u}_0(\xi)\delta_\phi.\end{aligned}$$

This calculation implies the following lemma.

Lemma 5.2. *Let $\mathcal{F}_x u(t,x) = e^{it\psi}\mathcal{F}_x u_0$. Then the space time Fourier transform of u is the the measure $\sqrt{2\pi}\hat{u}_0\delta_\phi$.*

Let ψ_1 and ψ_2 be real smooth functions and, as above,

$$\phi_1(\tau,\xi) = \tau-\psi_1(\xi) \quad \text{resp. } \phi_2(\tau,\xi) = \tau-\psi_1(\xi).$$

The product uv of two solution of the linear equations

$$iu_t - \psi_1(D)u = 0, \qquad iv_t - \psi_2(D)v = 0$$

is the convolutions of the Fourier transforms in Lemma 5.2, which in turn can be estimated by Theorem 5.1. We identify the terms occurring in Theorem 5.1.

The set of integration for given ξ_j is

$$M = \{(\tau, \xi) : \tau = \psi_2(\xi_2) + \psi_1(\xi - \xi_2) = \psi_1(\xi_1) + \psi_2(\xi - \xi_1)\}.$$

The most important case will be $\psi_i = \psi$. We express the integrand in terms of $\nabla\psi_i$ using

$$\begin{aligned} &|\nabla_{\tau,\xi}\phi_1|^2|\nabla_{\tau,\xi}\phi_2|^2 - (\nabla_{\tau,\xi}\phi_1 \cdot \nabla_{\tau,\xi}\phi_2)^2 \\ &\qquad = |\nabla\psi_1 - \nabla\psi_2|^2 + |\nabla\psi_1|^2|\nabla\psi_2|^2 - (\nabla\psi_1 \cdot \nabla\psi_2)^2. \end{aligned} \tag{5.4}$$

The first term is the square of the distance of the gradients, and the second is the square of product of length multiplied by $\sin^2$ of the angle between them. Here we did suppress the arguments. With them the integrand reads as

$$\begin{aligned} \Big[&|\nabla\psi_1(\xi - \xi_2) - \nabla\psi_2(\xi - \xi_1)|^2 + |\nabla\psi_1(\xi - \xi_2)|^2|\nabla\psi_2(\xi - \xi_1)|^2 \\ &\qquad - (\nabla\psi_1(\xi - \xi_2) \cdot \nabla\psi_2(\xi - \xi_1))^2\Big]^{-\frac{1}{2}}. \end{aligned} \tag{5.5}$$

The proof of bilinear estimates reduces to bounding the integral over this expression over M.

We first consider one space dimension where the second term of (5.4) vanishes. The set $x + \Sigma_1 \cap y + \Sigma_2$ consists generically of a discrete set of points and we obtain a sum of $|\psi_1'(z - x) - \psi_2'(z - y)|^{-1}$ over the points of the intersection. Often the intersection consists of one point as for the Schrödinger equation or up to two points as for the Airy equation. We consider the more general case of $\psi(\xi) = \xi^N$ for an even integer N. Then the equation

$$\xi_1^N + (\xi - \xi_1)^N = \xi_2^N + (\xi - \xi_2)^N$$

has the obvious and unique solution $\xi = \xi_2 + \xi_1$ unless $\xi_1 = \xi_2$. If N is odd there are the exactly two solutions $\xi = \xi_1 + \xi_2$ and $\xi = 0$, unless $\xi_2 = \xi_1$.

At these points

$$|\psi'(\xi - \xi_1) - \psi'(\xi - \xi_2)| = |\psi'(\xi_1) - \psi'(\xi_2)|$$

and we obtain from inequality (5.3):

Theorem 5.3. *With the notation introduced above*

$$\begin{aligned} \Big\| \int |N[(\xi - \eta)^{N-1} - \eta^{N-1}]|^{1/2} e^{it(\xi-\eta)^N + it\eta^N} \hat{u}_0(\xi - \eta)\hat{u}_1(\eta) d\eta \Big\|_{L^2(\mathbb{R}_t \times \mathbb{R}_\xi)} \\ \le 2\pi \|u_0\|_{L^2(\mathbb{R})}\|u_1\|_{L^2(\mathbb{R})} \end{aligned} \tag{5.6}$$

if N is even and if N is odd

$$\begin{aligned} \Big\| \int |N[(\xi - \eta)^{\frac{N-1}{2}} - \eta^{\frac{N-1}{2}}]|^{1/2} e^{it(\xi-\eta)^N + it\eta^N} \hat{u}_0(\xi - \eta)\hat{u}_1(\eta) d\eta \Big\|_{L^2(\mathbb{R}_t \times \mathbb{R}_\xi)} \\ \le \sqrt{2}\, 2\pi \|u_0\|_{L^2(\mathbb{R})}\|u_1\|_{L^2(\mathbb{R})}. \end{aligned} \tag{5.7}$$

We will use this estimate often via the following corollary. Given $\lambda \in (0, \infty)$ we define

$$u_{>\lambda} = \mathcal{F}^{-1}(\chi_{|\xi|>\lambda}\hat{u})$$

and similarly $u_{<\lambda}$.

Corollary 5.4. *Let $0 < \mu < \lambda$ and $u(t,x) = S(t)u_0(x)$, $v(t,x) = S(t)v_0(x)$ where S is the unitary group defined by $\phi = \xi^N$. Then*

$$\|u_{<\mu} v_{>\lambda}\|_{L^2(\mathbb{R}^2)} \le \frac{4\pi}{\left[N|\lambda^{N-1} - \mu^{N-1}|\right]^{\frac{1}{2}}} \|u_0\|_{L^2(\mathbb{R})} \|v_0\|_{L^2(\mathbb{R})}$$

and

$$\|(u_{>\lambda} v_{>\lambda})_{>\mu}\|_{L^2} \le \frac{4\pi}{\left[N|\lambda^{N-1} - (\lambda-\mu)^{N-1}|\right]^{\frac{1}{2}}} \|u_0\|_{L^2(\mathbb{R})} \|v_0\|_{L^2(\mathbb{R})}.$$

There is an interesting special case of the bilinear estimate: Local smoothing corresponds to $\Sigma_1 = \{(\xi^N, \xi)\}$ and Σ_0 is given by $\tau = 0$.

Theorem 5.5. *Let $\psi(\xi) = \xi^N$ be as above. Then*

$$\| |ND^{N-1}|^{1/2} S(t) u_0 \|_{L^\infty_x L^2_t} \le 4\pi^2 \|u_0\|_{L^2(\mathbb{R})}.$$

if N is odd and if N is even

$$\| |ND^{N-1}|^{1/2} S(t) u_0 \|_{L^\infty_x L^2_t} \le \sqrt{2}\, 4\pi^2 \|u\|_{L^2(\mathbb{R})}.$$

Proof. We apply the convolution estimate with $\psi_1(\xi) = \xi^N$ and $\psi_0 = 0$. The set M is given by

$$\tau = (\xi - \xi_0)^N = \xi_1^N$$

which has the unique solution $\xi = \xi_1 - \xi_0$ if N is odd, and $\xi = \xi_0 \pm \xi_1$ if N is even and the integrand is

$$|\psi'(\xi - \xi_0)|^{-1} = N|\xi_1|^{1-N}.$$

Thus, if N is odd

$$\sqrt{N} \int |(|D|^{\frac{N-1}{2}} S(t) u_0) v(x)|^2 dx dt \le 2\pi \|u_0\|^2_{L^2(\mathbb{R})} \|v\|^2_{L^2(\mathbb{R})}$$

and we choose v so that $|v|^2$ is a Dirac sequence. There are not more than obvious adaptations if N is even. $\square$

In particular, if u satisfies the Airy equation then

$$\|\partial_x u\|_{L^\infty_x L^2_t(\mathbb{R})} \le 2\pi \|u_0\|_{L^2} \tag{5.8}$$

and u has locally square integrable derivatives for almost all t.

We continue with a case by case study of several linear dispersive equations in several space dimensions. The first is the Schrödinger equation in higher space dimension. Here the characteristic set Σ is a standard parabola. The set

$$\{(\tau_1, \xi_1) + \Sigma\} \cap (\tau_2, \xi_2) + \Sigma\}$$

is the intersection of two paraboloids, and hence a paraboloid of dimension $d-1$. It is given by the equations

$$\tau = |\xi_1|^2 + |\xi - \xi_1|^2 = |\xi_2|^2 + |\xi - \xi_2|^2.$$

The first equality determines τ, which is of minor importance, and the second is equivalent to

$$\langle \xi, \xi_2 - \xi_1 \rangle = |\xi_2|^2 - |\xi_1|^2$$

resp.

$$\langle \xi - (\xi_2 + \xi_1), \xi_2 - \xi_1 \rangle = 0 \tag{5.9}$$

which is a hyper plane with normal $\xi_2 - \xi_1$, if $\xi_2 \neq \xi_1$. We restrict to this non degenerate situation. This suffices for the estimate.

Let w be the closest point of the hyper plane defined by (5.9) to ξ_1 resp. ξ_2. With this notation the intersection is given by

$$\{(\tau, w+v) : \tau = \xi_1^2 + |w-\xi_1|^2 + |v|^2 = \xi_2^2 + |w-\xi_2|^2 + |v|^2, \langle v, \xi_2 - \xi_1 \rangle = 0\} \tag{5.10}$$

If we integrate with respect to v we obtain by the coarea formula an integral

$$\int \dots \sqrt{1+4|v|^2} dv$$

At $\xi = w + v$

$$\nabla |\xi - \xi_1|^2 = 2v + 2(w - \xi_1)$$

and similarly

$$\nabla |\xi - \xi_2|^2 = 2v + 2(w - \xi_2)$$

Thus the square of the difference is given by

$$4|\xi_2 - \xi_1|^2$$

and

$$(|v|^2 + |w-\xi_1|^2)(|v|^2 + |w-\xi_2|^2) - (|v|^2 + (w-\xi_1)(w-\xi_2))^2 = |v|^2 |\xi_2 - \xi_1|^2$$

and the integrand is

$$(|\xi_2 - \xi_1| \sqrt{4 + 4|v|^2})^{-1}.$$

We will choose Σ_1 to be the part of the parabola above $|\xi| \geq \lambda$ and Σ_2 the part of the parabola above the ball of radius μ.

Lemma 5.6 (Schrödinger, d dimensions)**.** *Let $d \geq 2$, $u(t,x) = S(t)u_0$, $v(t) = S(t)v_0$ where S denotes the Schrödinger group. Let $\mu \leq \frac{1}{2}\lambda$. Then*

$$\|u_{>\lambda} v_{<\mu}\|_{L^2} \leq c_d \mu^{\frac{d-1}{2}} \lambda^{-1/2} \|u_\lambda(0)\|_{L^2} \|v_\mu(0)\|_{L^2}$$

and

$$\|(uv)_{<\mu}\|_{L^2(\mathbb{R}^2)} \leq c_d \mu^{\frac{d-2}{2}} \|u_\lambda(0)\|_{L^2(\mathbb{R}^d)} \|v_\lambda(0)\|_{L^2(\mathbb{R}^d)}.$$

Proof. In the first case $|\xi_2 - \xi_1| \geq \lambda/2$, and we integrate over a ball of radius μ. The factor from the area formula cancels the one from the integrand, hence the first estimate. It is not difficult to determine the constant c_d.

The second estimate could probably be proven with the arguments here. We derive it from the Strichartz estimate

$$\|u\|_{L^4_t L^{\frac{2d}{d-1}}_x} \leq c \|u_0\|_{L^2(\mathbb{R}^d)}.$$

We combine it with Bernstein's inequality for $p \leq q$

$$\|v_{<\mu}\|_{L^q} \leq c \mu^{\frac{d}{p} - \frac{d}{q}} \|v_{<\mu}\|_{L^p}.$$

With a smooth truncation (instead of the Fourier multiplication by a characteristic function) we obtain for fixed t

$$\|(uv)_{<\mu}(t)\|_{L^2(\mathbb{R}^d)} \leq c\mu^{\frac{d-2}{2}} \|uv\|_{L^{\frac{d}{d-1}}} \leq c\mu^{\frac{d-2}{2}} \|u\|_{L^{\frac{2d}{d-1}}(\mathbb{R}^d)} \|v\|_{L^{\frac{2d}{d-1}}(\mathbb{R}^d)}$$

and we complete the argument by taking the L^2 norm with respect to t. $\square$

The case of the Kadomtsev-Petviashvili-II equation is considerably more intricate. We study

$$u_t + u_{xxx} + \partial_x^{-1} u_{yy} = 0.$$

The symbol resp. Fourier multiplier is

$$\psi(\xi,\eta) = \xi^3 - \eta^2/\xi.$$

Here the formal notation ∂_x^{-1} has to be understood as Fourier multiplier. Here is it useful to first apply Fubinis theorem for the integration over M, or more precisely in its derivation, and to integrate first with respect to ξ.

For fixed ξ the intersection consists of at most to points η and the considerations in one space dimensions show that the integrand for the integration with respect to ξ is the following to the power $^{-1/2}$:

$$\begin{aligned}\big|\partial_\eta[(\xi-\xi_1)^3 - (\eta-\eta_1)^2/(\xi-\xi_1) - (\xi-\xi_2)^3 - (\eta-\eta_2)^2/(\xi-\xi_2)]\big| \\ = 2\left|\frac{\eta-\eta_1}{\xi-\xi_1} - \frac{\eta-\eta_2}{\xi-\xi_2}\right|.\end{aligned} \tag{5.11}$$

The curve of integration is described by the equations

$$\tau = \xi_1^3 - \frac{\eta_1^2}{\xi_1} + \tau - \xi_1^3 + (\xi-\xi_1)^3 - \frac{(\eta-\eta_1)^2}{\xi-\xi_1} = \xi_2^3 - \frac{\eta_2^2}{\xi_2} + (\xi-\xi_2)^3 - \frac{(\eta-\eta_2)^2}{\xi-\xi_2}.$$

We reorganize the second identity to

$$\begin{aligned}&\xi_1^3 - \frac{\eta_1^2}{\xi_1} + \left[(\xi_2-\xi_1)^3 - \frac{(\eta_2-\eta_1)^2}{\xi_2-\xi_1}\right] - \xi_2^3 + \frac{\eta_2^2}{\xi_2} = \\ &\quad\left[(\xi-\xi_2)^3 - \frac{(\eta-\eta_2)^2}{\xi-\xi_2}\right] + \left[(\xi_2-\xi_1)^3 - \frac{(\eta_2-\eta_1)^2}{\xi_2-\xi_1}\right] - \left[(\xi-\xi_1)^3 - \frac{(\eta-\eta_1)^2}{\xi-\xi_1}\right]\end{aligned}$$

and use the algebraic resonance relation

$$(\xi_1+\xi_2)^3 - \frac{(\eta_1+\eta_2)^2}{\xi_1+\xi_2} - (\xi_1^3 - \eta_1^2/\xi_1) - (\xi_2^3 - \eta_2^2/\xi_2) = \xi_1\xi_2(\xi_1+\xi_2)\left[3 + \frac{\left|\frac{\eta_1}{\xi_1} - \frac{\eta_2}{\xi_2}\right|^2}{|\xi_1+\xi_2|^2}\right] \tag{5.12}$$

to arrive at

$$\begin{aligned}\omega :=& \xi_1\xi_2(\xi_1-\xi_2)\left(3 + \frac{\left|\frac{\eta_1}{\xi_1} - \frac{\eta_2}{\xi_2}\right|^2}{(\xi_2-\xi_1)^2}\right) \\ =& (\xi-\xi_2)(\xi-\xi_1)(\xi_1-\xi_2)\left(3 + \frac{\left|\frac{\eta-\eta_1}{\xi-\xi_1} - \frac{\eta-\eta_2}{\xi-\xi_2}\right|^2}{|\xi_2-\xi_1|^2}\right)\end{aligned} \tag{5.13}$$

Here we used the elementary identities which express a high degree of symmetry

$$\frac{\left|\frac{\eta_1}{\xi_1} - \frac{\eta_2}{\xi_2}\right|^2}{|\xi_1+\xi_2|^2} = \frac{|\xi_1\eta_2 - \xi_2\eta_1|^2}{(\xi_1\xi_2(\xi_1+\xi_2))^2} = \frac{|(\xi_1+\xi_2)\eta_2 - \xi_2(\eta_1+\eta_2)|^2}{(\xi_1\xi_2(\xi_1+\xi_2))^2} = \frac{\left|\frac{\eta_1+\eta_2}{\xi_1+\xi_2} - \frac{\eta_2}{\xi_2}\right|^2}{|\xi_1|^2}.$$

The left hand side of (5.13) is the called modulation of the input. Assuming neither $\xi_1 = 0$ nor $\xi_2 = 0$ nor $\xi_1 = \xi_2$ there is only a solution if $\xi_1\xi_2$ as the same sign as $(\xi-\xi_2)(\xi-\xi_1)$. Below we neglect the question whether there is a solution and we rewrite the identity as

$$\left|\frac{\eta-\eta_1}{\xi-\xi_1}-\frac{\eta-\eta_2}{\xi-\xi_2}\right|^2=\frac{\xi_2-\xi_1}{(\xi-\xi_1)(\xi-\xi_2)}(\omega-3(\xi-\xi_1)(\xi-\xi_2)(\xi_2-\xi_1))=f(\xi) \tag{5.14}$$

which is useful to determine η as a function of ξ. The left hand side coincides with (5.11) and allows us to determine the integrand as a function of ξ.

Algebraic manipulation allow a fairly explicite determination of the solutions to the polynomial equation (5.13) is a polynomial equation. To shorten the notation we write $\tilde\xi=\xi_2-\xi_1$ in the sequel. Then

$$0=3\tilde\xi^2(\xi-\xi_1)^2(\xi-\xi_2)^2+\big((\eta-\eta_1)(\xi-\xi_2)-(\eta-\eta_2)(\xi-\xi_1)\big)^2+\omega\tilde\xi(\xi-\xi_1)(\xi-\xi_2) \tag{5.15}$$

which we rewrite using in terms of

$$\hat\xi=\xi-\frac12(\xi_1+\xi_2)$$

and

$$\begin{aligned}\hat\eta=&(\eta-\eta_1)(\xi-\xi_2)-(\eta-\eta_2)(\xi-\xi_1)\\ =&\eta(\xi_1-\xi_2)+\xi(\eta_2-\eta_1)+\eta_1\xi_2-\eta_2\xi_1.\end{aligned}$$

We observe that

$$f(\xi)=\frac{\hat\eta^2}{\hat\xi^2-\frac14\tilde\xi^2} \tag{5.16}$$

since

$$(\xi-\xi_1)(\xi-\xi_2)=\hat\xi^2-\left(\frac{\xi_1-\xi_2}{2}\right)^2$$

we obtain

$$3\tilde\xi^2\Big(\hat\xi^2-\frac14\tilde\xi^2\Big)^2+\omega\tilde\xi\big(\hat\xi^2-\frac14\tilde\xi^2\big)+\hat\eta^2=0. \tag{5.17}$$

We arrive at

$$\Big[\sqrt3\tilde\xi(\hat\xi^2-\frac14\tilde\xi^2)+\frac{\omega}{2\sqrt3}\Big]^2+\hat\eta^2=\frac{\omega^2}{12}. \tag{5.18}$$

It remains to partly undo and interpret the formulas and transformations. For simplicity we assume $\xi_1<\xi_2$. All solutions of the polynomial equation satisfy

$$\left|\sqrt3\tilde\xi(\hat\xi^2-\frac14\tilde\xi^2)+\frac{\omega}{2\sqrt3}\right|\le\frac{|\omega|}{2\sqrt3}$$

resp.

$$\frac16(\omega-|\omega|)\le(\xi_2-\xi_1)(\xi-\xi_1)(\xi-\xi_2)\le\frac16(\omega+|\omega|) \tag{5.19}$$

which we could have read from (5.13). Clearly

$$|\xi_1-\xi_2|(\xi_1-\xi)(\xi-\xi_2)\le\frac14|\xi_1-\xi_2|^3$$

with equality if $\rho=0$ resp. $\xi=\frac{\xi_1+\xi_2}{2}$. This set always contains the points $\xi=\xi_1$, $\eta=\eta_1$ and $\xi=\xi_2$, $\eta=\eta_2$. We list the geometric cases. For simplicity we assume that $\xi_1\le\xi_2$.

(1) If $\xi_1\xi_2 > 0$ then (5.13) and (5.19) describes two intervals. The set is a union of two topological circles contained in $\{\xi \le \xi_1\} \cup \{\xi \ge \xi_2\}$. The size of the circles is given by ω.
(2) If $\xi_1\xi_2 < 0$ and
$$|\omega| < \frac{4}{3}|\xi_1 - \xi_2|^3$$
then there are again two topological circles, but this time contained in $\{\xi_1 \le \xi < \frac{\xi_2+\xi_1}{2}\}$ and $\{\frac{\xi_2-\xi_1}{2} < \xi \le \xi_2\}$.
(3) If $\xi_1\xi_2 > 0$ and
$$|\omega| = \frac{4}{3}|\xi_1 - \xi_2|^3$$
then the intersection is a topological 8 contained in $\xi_1 \le \xi \le \xi_2$. The center of the figure eight is at $\xi = \frac{\xi_1+\xi_2}{2}$ and $\eta = \frac{\eta_1+\eta_2}{2}$.
(4) If $|\omega| > \frac{4}{3}|\xi_1 - \xi_2|^3$ then the intersection is a topological sphere in $\xi_1 < \xi < \xi_2$. In this case
$$f(\xi) \sim \frac{\xi_2 - \xi_1}{(\xi-\xi_1)(\xi-\xi_2)}\omega$$

The set expressed with respect to $\hat\eta$ and ξ is always symmetric with respect to the reflection at $\frac{\xi_1+\xi_2}{2}$ and $\hat\eta = 0$. We choose various subsets of the characteristic surface. Let $\mu \le \lambda$, $\Sigma_1 = \Sigma \cap \{\mu/2 \le |\xi| \le \mu\}$ and $\Sigma_2 = \Sigma \cap \{\lambda \le |\xi|\}$.

If $\mu \le \lambda/10$ then we obtain only the parts of the curves with $|\xi_2 - \xi| \sim \mu$ and $\eta \sim \eta_2$. In particular we stay away from $\xi = \frac{\xi_1+\xi_2}{2}$. If $\left|\frac{\eta_1}{\xi_1} - \frac{\eta_2}{\xi_2}\right| \ge 5\lambda$ then $|f| \ge \frac{\omega}{\mu}$, the ξ integral is over an interval of length μ and

$$\int_I |f|^{-\frac12} d\xi \sim \frac{\mu^{\frac32}}{\sqrt{\omega}}. \tag{5.20}$$

If $\mu \sim \lambda$ and $\left|\frac{\eta_1}{\xi_1} - \frac{\eta_2}{\xi_2}\right| \le 5\lambda$ we apply the L^4 Strichartz estimate. In the opposite case we argue as above.

Let
$$A_{\mu,\Lambda,k} = \Big\{(\xi,\eta) : \mu \le |\xi| \le 2\mu, k\mu - \frac{\Lambda}{\mu} \le \frac{\eta}{\xi} < k\mu + \frac{\Lambda}{\mu}\Big\}$$
We use the Strichartz estimate for $\mu \sim \lambda$.

Theorem 5.7. *The following estimate holds with suggestive notation and, if $\mu \le \lambda$,*
(5.21)
$$\left\| \int \Big|\Big(3 + \frac{\big|\frac{\eta_1}{\xi_1} - \frac{\eta_2}{\xi_2}\big|^2}{|\xi_1-\xi_2|^2}\Big)\Big|^{1/4} \hat u_{<\mu}(t,\xi_1)\hat v_{>\lambda}(t,\xi_2) \right\|_{L^2} \le c\Big(\frac{\mu}{\lambda}\Big)^{\frac12} \|v_\mu(0)\|_{L^2}\|u_\lambda(0)\|_{L^2}$$
where the inner integral is a two dimensional integral with respect to ξ_1 and η_1, and $\xi_2 = \xi - \xi_1$ resp $\eta_2 = \eta - \eta_1$. Similarly
(5.22)
$$\left\| \int \Big|\Big(3 + \frac{\big|\frac{\eta_1}{\xi_1} - \frac{\eta_2}{\xi_2}\big|^2}{|\xi_1-\xi_2|^2}\Big)\Big|^{1/2} \hat u_{A_{\mu,\Lambda,k}}(t,\xi_1)\hat v_{>\lambda}(t,\xi_2) \right\|_{L^2} \le c\frac{\sqrt{\Lambda}}{\lambda} \|v_\mu(0)\|_{L^2}\|u_{>\lambda}(0)\|_{L^2}.$$

Proof. The first estimate follows from the previous estimates.

Only the second estimate remains to be shown. We prove the estimate first for $k = 0$. The curve described by (5.18) lies on one side of ξ_1 resp. ξ_2, and hence it is vertical there. Assuming $\eta_1 = 0$ (related to $k = 0$) we expand equation (5.18) to

$$\begin{aligned}3(\xi_2-\xi_1)^2(\xi-\xi_1)^2(\xi-\xi_2)^2 + \omega(\xi_2-\xi_1)(\xi-\xi_1)(\xi-\xi_2)\\ + \eta^2(\xi_2-\xi_1)^2 - 2\eta\eta_2(\xi-\xi_1)(\xi-\xi_2) + \eta_2^2(\xi-\xi_1)^2\end{aligned}$$

We consider the situation where $\Lambda \le \mu^{\frac12}\omega^{\frac12}$ - in the complementary case estimate (5.21) is stronger.

The dominant terms are the second and the third term and hence in that range

$$|\xi - \xi_1| \le C\frac{\eta^2}{\omega}. \tag{5.23}$$

This bounds the interval of integration in (5.20) and implies the estimates.

The bound (5.23) follows from our discussion above - which controls the global geometry - and a continuity argument from $\xi = \xi_1$ and $\eta = \eta_1$:

$$3(\xi_2-\xi_1)^2(\xi-\xi_1)^2(\xi-\xi_2)^2 = \left(\frac{(\xi_2-\xi_1)(\xi-\xi_1)(\xi-\xi_2)}{\omega}\right)\omega(\xi_2-\xi_1)(\xi-\xi_1)(\xi-\xi_2)$$

where the bracket is small compared to the next term provided $|\xi - \xi_1| << \mu|\omega|$. Similarly

$$\eta_2^2(\xi-\xi_1)^2 = \left(\frac{\eta_2^2(\xi-\xi_1)}{\omega(\xi_2-\xi_1)(\xi-\xi_2)}\right)\omega(\xi_2-\xi_1)(\xi-\xi_1)(\xi-\xi_2)$$

is small by a continuity argument. The restriction $k = 0$ resp. $\eta_1 = 0$ is possible due to the Galilean symmetry,

$$(t, x, y) \to (t, x - c^2t - cy, y + 2ct)$$

which is a symmetry of the linear and nonlinear $KP-II$ equation, and it respects the bilinear estimate. On the Fourier side this corresponds to

$$(\tau, \eta, \xi) \to (\tau - 2c\eta - c^2\xi, \eta + c\xi, \xi).$$

If we neglect τ then the lines through the origin in the (ξ, η) are mapped to such lines, and the lines $\xi = d$ are preserved.

The center of the figure 8 never plays a role unless $\mu \sim \lambda$ and $|\frac{\eta_1}{\xi_1} - \frac{\eta_2}{\xi_2}| \le \lambda$ but then its contribution is not hard to control.

$\square$

We conclude this section by explaining the relation to U^2 spaces. Let as above u_A resp. u_B be the projection on the Fourier side to sets A resp. B.

Theorem 5.8. *Suppose that*

$$\|S(t)u_{0,A}S(t)v_{0,B}\|_{L^2} \le c_{A,B}\|u_{0,A}\|_{L^2}\|u_{0,B}\|_{L^2}$$

Then we have we the same constant

$$\|u_Av_B\|_{L^2} \le c_{A,B}\|u_A\|_{U^2_S}\|u_B\|_{U^2_S}.$$

Proof. As for the Strichartz estimates the assertion reduces to the assuption and a summation for 2 atoms. We first write the second term as a sum of atoms to obtain the statement of the first factor is a an atom, and the second factor is in U^2, and then we expand the first factor to obtain the full statement. $\square$

6. Well-posedness for nonlinear dispersive equations

In this section we will study local and global well-posedness for a number of different equations where the techniques developed so far are relevant. The first example describes the interaction of three waves of different velocities. It is elementary and displays the role of adapted function spaces on an elementary level. The limitations of our current understanding become obvious as well: The result should remain true under small perturbations of the system, but I have no idea how to approach perturbed equations.

Next we turn to generalized KdV equations and establish global well-posedness and scattering in a large scale invariant Besov space for the quartic and the quintic equation, and local existence for modified KdV and KdV in the spaces $B^{\frac14}_{2,\infty}$ and $B^{-\frac34}_{2,\infty}$ using the U^2-V^2 spaces, bilinear estimates, Strichartz estimates and, for KdV, modulation arguments. This is basically well known, but for KdV and mKdV slightly stronger than available results in the literature. Going from $H^{-\frac34}$ to $B^{-\frac34}_{2,\infty}$ for the initial data for Korteweg de-Vries requires a new technique, which also allows to treat low frequencies similarly to high frequencies.

Next we turn to higher dimensional non resonant derivative Schrödinger equations, following the dissertation of T. Schottdorf, and conclude with a discussion of the two dimensional Kadomtsev-Petviashvili II equation.

6.1. **Adapted function spaces approach for a model problem.**

To motivate the relevance of adapted function spaces we begin with a self contained study of a simple toy problem, where a nonstandard choice of adapted function spaces leads to global well-posedness for small data in L^2, and where I know of no other technique which allows to prove this result. Consider the three wave interaction

$$\begin{aligned} u_t + u_x &= vw \\ v_t + v_y &= uw \\ w_t &= -2uv. \end{aligned} \tag{6.1}$$

It is easy to solve the linear equation for given initial data. We define the evolution operator

$$S(t)[u_0, v_0, w_0](x, y) = [u_0(x-t, y), v_0(x, y-t), w(x, y)]$$

and the operator adapted function space

$$\begin{aligned} \|[u,v,w]\|_X = \max\big\{ &\| \sup_t |u(t, x+t, y)| \|_{L^2(\mathbb{R}^2)}, \| \sup_t |v(t, x, y+t)| \|_{L^2(\mathbb{R}^2)}, \\ &\| \sup_t |w(t,x,y)| \|_{L^2(\mathbb{R}^2)} \big\} \end{aligned}$$

or, written differently with an equivalent norm,

$$\|[u,v,w]\|_X \sim \| \sup_t S(-t)[u(t,x,y), v(t,x,y), w(t,x,y)] \|_{L^2(\mathbb{R}^2}.$$

Theorem 6.1. *There exists $\varepsilon > 0$ so that, if*

$$\max\{\|u_0\|_{L^2}, \|v_0\|_{L^2}, \|w_0\|_{L^2}\} \le \frac{1}{4}$$

there exists a unique global solution $[u,v,w] \in X$ which satisfies

$$\|[u,v,w] - S(t)[u_0,v_0,w_0]\|_X \le 2\max\{\|u_0\|_{L^2}, \|v_0\|_{L^2}, \|w_0\|_{L^2}\}^2.$$

Proof. The assertion follows by an easy duality argument from the trilinear estimate (6.2)

$$\left|\int uvw\,dx\,dy\,dt\right| \le \|\sup_t |u(t,x+t,y)|\|_{L^2}\|\sup_t |v(t,x+y+t)|\|_{L^2}\|\sup_t |w(t,x,y)|\|_{L^2}$$

To prove this estimate we denote

$$\tilde u(x,y) = \sup_t |u(t,x+t,y)|, \quad \tilde v = \sup_t |v(t,x+y+t)|, \quad \tilde w(x,y) = \sup_t |w(t,x,y)|.$$

Then

$$\int |uvw| dxdydt \le \int \tilde u(x-t,y)\tilde v(x,y-t)\tilde w(x,y)dtdxdy \le \|\tilde u\|_{L^2}\|\tilde v\|_{L^2}\|\tilde w\|_{L^2}$$

by a multiple application of the Cauchy Schwartz inequality.

It is not difficult to set up an iteration argument to construct a global solution for small data, which depends analytically on the initial data. □

6.2. **The (generalized) KdV equation.** For integers $p \ge 1$ we consider the initial value problems

$$u_t + u_{xxx} + (u^p u)_x = 0 \tag{6.3}$$

$$u(0) = u_0 \tag{6.4}$$

- the case $p = 1$ is the Korteweg-de-Vries equation, and $p = 2$ the modified Korteweg-de-Vries equation, and

$$u_t + u_{xxx} + (|u|^p u)_x = 0 \tag{6.5}$$

$$u(0) = u_0 \tag{6.6}$$

for positive real p.

Both equations have soliton solutions

$$u(x,t) = c^{\frac{1}{p}} Q(c^{1/2}(x-ct))$$

with

$$Q_p = \left(\frac{p+1}{2}\right)^{2/p} \cosh^{2/p}(\frac{2}{p}x).$$

The equation is invariant with respect to scaling: $\lambda^{2/p}u(\lambda x, \lambda^3 t)$ is a solution if u satisfies the equation. The mass $\int u^2 dx$ and energy $\int \frac{1}{2}u_x^2 - \frac{1}{p+2}u^{p+2}$ are conserved. The energy however is not bounded from below.

The space $\dot H^{\frac{1}{2}-\frac{2}{p}}$ (with norm $\|u\|_{\dot H^s} = \||\xi|^s \hat u\|_{L^2}$) is invariant with respect to scaling and it is not hard to see that the generalized KdV equation is globally well posed in H^1 if $p < 4$. For $p \ge 4$ one expects blow-up. This has been proven in series of seminal papers by Martel, Merle and Martel, Merle and Raphael.

Using the Fourier transform we see that

$$v_t + v_{xxx} = 0 \qquad v(0,x) = v_0(x)$$

defines a unitary group on L^2. We denote

$$S(t)v_0 = v(t)$$

for $t \ge 0$ and $v(t) = 0$ otherwise and define the adapted function spaces by

$$\|u\|_{U^p_{KdV}} = \|S(-t)u(t)\|_{U^p}, \qquad \|u\|_{V^p_{KdV}} = \|S(-t)u(t)\|_{V^p}.$$

The Strichartz estimates are

$$\|u\|_{L^p_t L^q_x} \le c\||D|^{-1/p} u_0\|_{L^2} \tag{6.7}$$

for

$$\frac{2}{p} + \frac{1}{q} = \frac{1}{2}.$$

We have seen they imply the embedding estimates

$$\||D|^{1/p} u\|_{L^p_t L^q_x} \le c\|u\|_{U^p_{KdV}} \tag{6.8}$$

in the same range.

For $\lambda > 0$ we denote

$$u_\lambda = \chi_{\lambda \le |\xi| \le 1.01\lambda}(D) u$$

the projection of the Fourier transform. Then the Strichartz embedding applied to $g(D)u$, $g(\xi) = |\xi|^{-\frac{1}{p}}$ gives

$$\|u_\lambda\|_{L^p_t L^q_x} \le c\lambda^{-1/p}\|u\|_{U^p_{KdV}} \tag{6.9}$$

- checking atoms one sees that Fourier multipliers act nicely on U^p and V^p.

The bilinear estimates for $\mu \le \frac{9}{10}\lambda$

$$\|S(t)u_{0,\lambda} S(t) v_{0,\mu}\|_{L^2} \le c\lambda^{-1}\|u_{0,\lambda}\|_{L^2}\|v_{0,\mu}\|_{L^2}$$

are a direct consequence of the bilinear estimate of the last section. Hence

$$\|u_\lambda v_\mu\|_{L^2} \le c\lambda^{-1}\|u_\lambda\|_{U^2_{KdV}}\|v_\mu\|_{U^2_{KdV}}. \tag{6.10}$$

After these preparations we turn to the cases $p = 4$ and $p = 3$. There is a number of aspects which are the same for both cases, and also for many other equations. We discuss them in detail for the case $p = 4$ and only sketch them at later on.

We begin with the L^2 critical case

$$u_t + u_{xxx} + u_x^5 = 0. \tag{6.11}$$

and choose the norm

$$\|u_0\|_{\dot B^0_{2,\infty}} = \sup_{\lambda \in 1.01^{\mathbb{Z}}} \|u_{0,\lambda}\|_{L^2(\mathbb{R})}$$

for the initial data, and, with $I = [0, T)$, $T \in (0, \infty]$,

$$\|u\|_X = \sup_{\lambda \in 1.01^{\mathbb{Z}}} \|u_\lambda\|_{V^2_{KdV}(I)}.$$

We will usually suppress I in the notation.

Theorem 6.2. *There exists $\varepsilon > 0$ such that if*

$$\|u_0\|_{\dot B^0_{2,\infty}} < \varepsilon$$

there is a unique global weak solution u in X with

$$\|u - S(t)u_0\|_X \le c\|u_0\|^5_{\dot B^0_{2,\infty}}.$$

We need Bernstein's inequality for the proof. For $q \ge p$

$$\|u_\lambda\|_{L^q(\mathbb{R})} \le \lambda^{\frac{1}{p} - \frac{1}{q}}\|u_\lambda\|_{L^p(\mathbb{R})}. \tag{6.12}$$

Bernstein's inequality is easy to prove. Scaling reduces to question to $\lambda = 1$. So we consider u with Fourier transform supported in $[-2, 2]$. We choose a Schwartz function η with $\hat\eta(\xi) = 1$ for $|\xi| \le 2$. Then

$$\eta * u_1 = u_1$$

and Young's inequality gives the bound.

Proof of Theorem 6.2. We claim that the assertion follows from the estimate

$$\int u_1 u_2 u_3 u_4 u_5 v dxdt \leq c \prod_{i=1}^{5} \|u_i\|_X \|v\|_{V^2_{KdV}}. \tag{6.13}$$

Suppose that this estimate is true. We search a solution $u = S(t)u_0 + w$ where

$$w_t + w_{xxx} + (S(t)u_0 + w)^4_x = 0$$

with initial values $w(0) = 0$, which we formulate as fixed point problem of the map $w \to \tilde{w}$ where

$$\tilde{w}_t + \tilde{w}_{xxx} = -(S(t)u_0 + w)^5_x.$$

This equation has to be understood as follows: $\tilde{w}_\lambda$ satisfies

$$\tilde{w}_{\lambda,t} + \tilde{w}_{\lambda,xxx} = (-(S(t)u_0 + w)^5_x)_\lambda$$

in the sense of Lemma 4.34 with $a = 0$ and $b = \infty$. The derivative can be replace by the multiplication by λ after the frequency localization.

By Lemma 4.34 there exists a unique such $\tilde{w}_\lambda \in U^2_{KdV}$ with

$$\|\tilde{w}_\lambda\|_{U^2_{KdV}} \leq c\|S(t)u_0 + w\|^5_X$$

and, for the difference for two different data

$$\|\tilde{w}^2_\lambda - \tilde{w}^1_\lambda\|_{U^2_{KdV}} \leq c\big(\|S(t)u_0 + w^1\|_X + \|S(t)u_0 + w^2\|_X\big)^4 \|w^2 - w^1\|_X.$$

We take the supremum with respect to λ and arrive at, denoting the map from w to $\tilde{w}$ by J,

$$\|J(w)\|_X \leq c(\|w\|_X + \|u_0\|_{\dot{B}^0_{2,\infty}})^5$$

$$\|J(w^2) - J(w^1)\|_X \leq c(\|w^2\|_X + \|w^1\|_X + \|u_0\|_{\dot{B}^0_{2,\infty}})^4 \|w^2 - w^1\|_X.$$

Thus J maps a ball of radius R to a ball of radius

$$c(R + \|u_0\|_{\dot{B}^0_{2,\infty}})^4 < R$$

provided

$$\max\{R^3, \|u_0\|^3_{\dot{B}^0_{2,\infty}}\} < \frac{1}{16c}.$$

Then

$$\|J(w^2) - J(w^1)\|_X \leq \frac{1}{2}\|w^2 - w^1\|_X$$

provided $\|w^j\|_X \leq R$, $\|u_0\| < \frac{c^{1/3}}{10}$ and $R < \frac{1}{10c^{1/3}}$. We choose $R = \delta = \frac{1}{10c^{1/3}}$. Then J defines a contraction on the closed ball of radius R in X. The contraction mapping theorem implies existence of a unique fixed point, which by Lemma 4.34 is the unique weak solution in X. The map J is a polynomial, and hence analytic. The map J is a contraction, and this implies that its derivative is invertible. Now the analytic implicit function theorem in Banach spaces implies an analytic dependence on the initial data.

These arguments make little difference between most dispersive equations, some wave equations, parabolic equations and even ordinary differential equations.

It remains to prove (6.13). We expand the terms and claim

$$\int \prod_{i=1}^{6} u_{i,\lambda_i} dxdt \leq c\lambda_6^{-1+\frac{1}{10}} \lambda_1^{\frac{1}{2}-\frac{1}{10}} (\lambda_3\lambda_4\lambda_5)^{-\frac{1}{6}} \prod \|u_{i,\lambda_i}\|_{V^2_{KdV}} \tag{6.14}$$

for $\lambda_1 \le \lambda_2 \le \lambda_3 \le \lambda_4 \le \lambda_5 \le \lambda_6$.

Let us check that this gives the summation. We break the sum up depending on the relative size of λ compared to λ_i. We begin with the case $\lambda = \lambda_6$. Then necessarily $\lambda_6 \sim \lambda_5$ otherwise the frequencies cannot add up to 0, and it remains to sum - taking account that the derivative contributes a factor λ -

$$\sum_{\lambda_1 \le \lambda_2 \le \lambda_3 \le \lambda_4 \le \lambda} \lambda_1^{\frac12 - \frac1{10}} \lambda_3^{-\frac16} \lambda_4^{-\frac16} \lambda^{-\frac16 + \frac1{10}}$$

and to verify that this is uniformly bound. This is done by summing first over λ_1, then λ_2, λ_3 and λ_4.

Next consider $\lambda = \lambda_4$, which leads to the sum

$$\sum_{\lambda_1 \le \lambda_2 \le \lambda_3 \le \lambda \le \lambda_6} \lambda_1^{\frac12 - \frac1{10}} \lambda_3^{-\frac16} \lambda^{\frac56} \lambda_6^{-\frac76 + \frac1{10}}.$$

We obtain a uniform bound by first summing with respect to λ_1, then λ_2, λ_3 and $\lambda_5 \sim \lambda_6$.

If $\lambda = \lambda_3$ we are led to

$$\sum_{\lambda_1 \le \lambda_2 \le \lambda \le \lambda_4 \le \lambda_6} \lambda_1^{\frac12 - \frac1{10}} \lambda^{\frac56} \lambda_4^{-\frac16} \lambda_6^{-\frac76 + \frac1{10}},$$

if $\lambda = \lambda_2$ we get

$$\sum_{\lambda_1 \le \lambda \le \lambda_3 \le \lambda_4 \le \lambda_6} \lambda_1^{\frac12 - \frac1{10}} \lambda \lambda_3^{-\frac16} \lambda_4^{-\frac16} \lambda_6^{-\frac76 + \frac1{10}},$$

and finally, if $\lambda = \lambda_1$,

$$\sum_{\lambda \le \lambda_2 \le \lambda_3 \le \lambda_4 \le \lambda_6} \lambda_1^{\frac32 - \frac1{10}} \lambda_3^{-\frac16} \lambda_4^{-\frac16} \lambda_6^{-\frac76 + \frac1{10}}.$$

None of the summation poses difficulties. We observe that $\lambda \sim \lambda_6$ has been the most difficult, and in later proofs we often name the most difficult case, and neglect the others. This has to be done with care.

We turn to the proof of (6.14). The Strichartz estimate gives

$$\int \prod_{j=1}^{6} u_{j,\lambda_j} dxdt \le \prod \lambda_j^{-1/6} \|u_{j,\lambda_j}\|_{U^6_{KdV}}.$$

The product $\prod_{j=1}^6 \lambda_j^{1/6}$ compensates for the derivative if the output frequency is λ_1, which is in particular the case if all frequencies are of the same size.

Now suppose that λ_1 is much smaller than λ. Then the integral vanishes unless

$$\lambda_6 - \lambda_2 \ge \frac15 \lambda_6$$

since otherwise no frequencies in the Fourier supports can add up to zero. We assume that this inequality holds and estimate using Bernstein's inequality on the first factor

$$\begin{aligned}\int \prod_{j=1}^{6} u_{j,\lambda_j} dxdt \le & \|u_{2,\lambda_2} u_{6,\lambda_6}\|_{L^2} \|u_1\|_{L^\infty} \prod_{j=3}^{5} \|u_{j,\lambda_j}\|_{L^6} \\ & \le \lambda_1^{1/2} (\lambda_3\lambda_4\lambda_5)^{-1/6} \lambda^{-1} \|u_{2,\lambda_2}\|_{U^2_{KdV}} \|u_{6,\lambda_6}\|_{U^2_{KdV}} \|u_{1,\lambda_1}\|_{V^\infty_{KdV}} \prod_{j=3}^{5} \|u_{j,\lambda_j}\|_{U^6_{KdV}}.\end{aligned}$$

We recall the embedding $U^p \subset V^2_{rc}$ if $p > 2$. This is almost good enough, upon replacing U^2 by V^2. Let $\mu \le \frac{9}{10}\lambda$. Then

$$\begin{aligned}\|S(t)u_{0,\mu}S(t)v_{0,\lambda}\|_{\frac{25}{12}} \le& c\|S(t)u_{0,\mu}S(t)v_{0,\lambda}\|_{L^2}^{\frac{21}{25}}\|S(t)u_{0,\mu}\|_{L^6}^{\frac{4}{25}}\|S(t)v_{0,\lambda}\|_{L^6}^{\frac{3}{25}}\\ \le& c\lambda^{-\frac{21}{25}}\lambda^{-\frac{2}{75}}\mu^{-\frac{2}{75}}\|u_{0,\mu}\|_{L^2}\|v_{0,\lambda}\|_{L^2}\end{aligned}$$

and hence, if $\mu \le \frac{9}{10}\lambda$

$$\|u_\mu v_\lambda\|_{L^{\frac{25}{12}}} \le c\lambda^{-\frac{61}{75}}\mu^{-\frac{2}{75}}\|u_\mu\|_{V^2_{KdV}}\|v_\lambda\|_{V^2_{KdV}} \tag{6.15}$$

and hence

$$\begin{aligned}\int\prod_{j=1}^6 u_{j,\lambda_j}dxdt \le& \|u_{2,\lambda_2}u_{6,\lambda_6}\|_{L^{\frac{25}{12}}}\|u_1\|_{L^{50}}\prod_{j=3}^5\|u_{j,\lambda_j}\|_{L^6}\\ \le& \lambda_1^{\frac{22}{50}}\|u_1\|_{L^{50}_tL^{\frac{50}{23}}x}\lambda_6^{-\frac{61}{75}}\lambda_2^{-\frac{2}{75}}(\lambda_3\lambda_4\lambda_5)^{-1/6}\prod_{i=2}^6\|u_{i,\lambda_i}\|_{V^2_{KdV}}\\ \le& \lambda_1^{\frac{21}{50}}\lambda_2^{-\frac{2}{75}}\lambda_6^{-\frac{61}{75}}(\lambda_3\lambda_4\lambda_5)^{-1/6}\prod_{i=1}^6\|u_{i,\lambda_i}\|_{V^2_{KdV}}\end{aligned}$$

This is slightly stronger than the claimed estimate. It completes the proof. $\square$

A variant yields local existence. There are two key observations. First we may expand

$$\prod(S(t)u_0+w)_{\lambda_j} = \prod(S(t)u_0)_{\lambda_j} + .. + \prod w_{\lambda_j}$$

there is one term without w, a term linear in w, and higher order terms in w. If w is small than the higher order terms are even smaller. So we need some smallness of the first and the second term. We do not want to assume that the initial data are small, but we are willing to choose a small time.

Theorem 6.3. *There exist $\delta > 0$ such that, if $R > 0$*

$$\|u_0\|_{\dot B^0_{2,\infty}} \le R$$

and with $v = S(t)u_0$

$$(1+R^3)\sup_\lambda \lambda^{-\frac{1}{6}}\|v_\lambda\|_{L^6([0,T]\times\mathbb{R})} \le \delta \tag{6.16}$$

then there is a unique solution u to

$$u_t + u_{xxx} + \partial_x(\chi_{[0,T]}(t)u^5) = 0$$

with initial data u_0 which satisfies

$$\|u - S(t)u_0\|_X \le cR^3\sup_\lambda \lambda^{-\frac{1}{6}}\|v_\lambda\|^2_{L^6([0,T]\times\mathbb{R})}$$

and which depends analytically on the initial data.

Proof. By the discussion above it suffices to consider integrals

$$\int_0^T\int_{\mathbb{R}}(S(t)u_0)^5 v dxdt.$$

and

$$\int_0^T \int_{\mathbb{R}} (S(t)u_0)^4 wv dx dt.$$

We observe that here we may always estimate two $S(t)u_0$ factors in L^6. Thus

$$\|w\|_X \le cR^3\delta^2$$

which is small provided δ is sufficiently small. The rest of the proof works with virtually no change in the argument. $\square$

Assumptions and statement of Theorem 6.3 are uniform with respect to T. Here $T=\infty$ is allowed even for large initial data. In that case the solution is in U^2_{KdV} and hence

$$w_\lambda = \lim S(-t)u_\lambda(t)$$

exists - since all one sided limits exists. Equivalently

$$u_\lambda(t) - S(t)w_\lambda \to 0$$

in L^2 and the solution to the nonlinear equation is for large t close to a solution to the linear equation. This is called scattering.

Suppose that

$$\lim_{\lambda\to\infty} \|u_{0,\lambda}\|_{L^2} = 0. \tag{6.17}$$

Since by dominated convergence

$$\lim_{T\to 0} \lambda^{-1/6}\|v_\lambda\|_{L^6([0,T]\times\mathbb{R})} = 0$$

whenever $v_\lambda \in L^6$ there exists T such that

$$\sup_{\lambda\ge 1} \lambda^{-1/6}\|v_\lambda\|_{L^6([0,T]\times\mathbb{R})} < \delta.$$

Trivially

$$\|v_\lambda\|_{L^6([0,T];L^2)} \le cT^{1/6}\|u_{0,\lambda}\|_{L^2}$$

and, together with Bernstein's inequality

$$\|v_\lambda\|_{L^6([0,T]\times\mathbb{R})} \le \lambda^{\frac12}T^{\frac16}(\lambda^{-\frac16}\|u_{0,\lambda}\|_{L^2},$$

which is much stronger than needed to ensure the smallness assumption (6.16) for sufficiently small time. As a consequence we obtain existence of unique local solutions provided (6.17) is satisfied.

Since there are solitons in general solutions are not in L^6 of space-time. Solitons clearly do not scatter. This version of well-posedness has been proven by Strunk [26]. The result in L^2 is due to Kenig, Ponce and Vega.

We turn to

$$u_t + u_{xxx} + u_x^4 = 0. \tag{6.18}$$

Here $\dot H^{-1/6}$ is the critical Sobolev space. We choose a slightly larger space

$$\|u\|_X = \sup_{\lambda\in 1.01^{\mathbb{Z}}} \lambda^{-1/6}\|u_\lambda\|_{V^2_{KdV}(0,\infty)}$$

for the solution and

$$\|u_0\|_{\dot B^{-1/6}_{2,\infty}} = \sup_{\lambda\in 1.01^{\mathbb{Z}}} \lambda^{-1/6}\|u_{0,\lambda}\|_{L^2}.$$

Then

$$\sup_\lambda \lambda^{-1/6}\|S(t)u_{0,\lambda}\|_{V^2_{KdV}} \sim \sup_\lambda \lambda^{-1/6}\|u_{0,\lambda}\|_{L^2}$$

Theorem 6.4. *There exists $\delta > 0$ such that for all u_0 with*

$$\|u_0\|_{\dot{B}^{-\frac{1}{6}}_{2,\infty}} < \delta.$$

there is a unique global solution u which satisfies

$$\|u - S(t)u_0\|_X \leq c\|u_0\|^4_{\dot{B}^{-1/6}_{2,\infty}}$$

which depends analytically on the initial data.

Proof. We claim

$$\left|\int u_1u_2u_3u_4v_\lambda dxdt\right| \leq \lambda^{-\frac{5}{6}} \prod \|u_i\|_X\|v_\lambda\|_{V^2}. \tag{6.19}$$

The theorem follows from this estimate in the same fashion as for $p = 4$. As there (6.19) follows from

$$\left|\int u_1u_2u_3u_4v_\lambda dxdt\right| \leq \lambda^{-\frac{5}{6}} \prod \|u_i\|_X\|v_\lambda\|_{V^2}. \tag{6.20}$$

To prove it we expand the left hand side into a dyadic sum and we try to bound

$$I = \left|\int \prod_{i=1}^5 u_{i,\lambda_i} dxdt\right|$$

where (by symmetry) $\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \lambda_4 \leq \lambda_5$. We claim

$$\left|\int \prod_{i=1}^5 u_{i,\lambda_i} dxdt\right| \leq c_\varepsilon\lambda_5^{-1}(\lambda_2\lambda_3\lambda_4)^{-1/6}(\lambda_5/\lambda_1)^\varepsilon \prod \|u_{i,\lambda_i}\|_{V^2}. \tag{6.21}$$

We assume that (6.21) holds. The integral with respect to x vanishes unless there are frequencies in the support of the Fourier transform which add up to zero. Since, if $|\xi_1| \leq |\xi_2| \leq \dots |\xi_5|$ the frequencies can only add up to zero, $\xi_1+\xi_2+\xi_3+\xi_4+\xi_5 = 0$ if $|\xi_5| - |\xi_1| \geq \frac{1}{10}|\xi_5|$, which we restrict to in the sequel. We observe that we may restrict to $\lambda_4 \geq \lambda_5/8$ - otherwise the integral vanishes. The summations is done as for $p = 4$.

It remains to prove (6.21). We recall have seen that we may assume that $\lambda_1 \leq 4\lambda_5/5$ and $\lambda_4 \geq \lambda_5/8$ The first attempt is

$$\begin{aligned} I \leq &\|u_{1,\lambda_1}u_{5,\lambda_5}\|_{L^2} \prod_{j=2}^4 \|u_{j,\lambda_j}\|_{L^6} \\ \leq &(\lambda_2\lambda_3\lambda_4)^{-1/6}\lambda_5^{-1}\|u_{1,\lambda_1}\|_{U^2_{KdV}}\|u_{5,\lambda_5}\|_{U^2_{KdV}} \prod_{j=2}^4 \|u_{j,\lambda_j}\|_{U^6_{KdV}} \end{aligned} \tag{6.22}$$

where we used Hölder's inequality for the first inequality, the bilinear estimate for the first factor, and the L^6 Strichartz embedding for the remaining factors. This is almost what we need - we still have to replace the norm U^2_{KdV} by V^2_{KdV}.

The Strichartz estimates imply

$$\|S(t)u_{0,\lambda}S(t)u_{0,\mu}\|_{L^3} \leq c(\lambda\mu)^{-1/6}\|u_{0,\mu}\|_{L^2}\|u_{0,\lambda}\|_{L^2}$$

and the bilinear estimate is - for $\mu \le \lambda/1.03$

$$\|S(t)u_{0,\lambda}S(t)u_{0,\mu}\|_{L^2} \le c\lambda^{-1}\|u_{0,\mu}\|_{L^2}\|u_{0,\lambda}\|_{L^2}.$$

Thus, for $2 \le p \le 3$

$$\|S(t)u_{0,\lambda}S(t)u_{0,\mu}\|_{L^p} \le c\lambda^{-6(\frac{1}{p}-\frac{1}{3})}(\lambda\mu)^{-(\frac{1}{2}-\frac{1}{p})}\|u_{0,\mu}\|_{L^2}\|u_{0,\lambda}\|_{L^2}$$

and hence, by Hölder's inequality

$$\|u_\lambda u_\mu\|_{L^p} \le c\lambda^{2-\frac{5}{p}}\mu^{\frac{1}{p}-\frac{1}{2}}\|u_\mu\|_{U^p_{KdV}}\|u_\lambda\|_{U^p_{KdV}}$$

With this argument we may replace the U^2 by V^2 norms - but now the remaining terms are not square integrable anymore. We use this modified bilinear estimate twice if there are two pairs of λ_i with quotient at least $\ge 1.01^2$. Oversimplifying slightly this leaves us with $\lambda_2 = \lambda_3 \cdots = \lambda_5$ and $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4$ and $\lambda_5 \sim 3\lambda_1$. The second case is easier, and we focus on the first. We again turn our attention to

$$\xi_1 + \xi_2 + \xi_3 + \xi_4 + \xi_5 = 0$$

assuming $|\xi_1| \le |\xi_2| \le |\xi_3| \le |\xi_4| \le |\xi_5|$. We have already seen that $|\xi_1| \le 0.9|\xi_5|$. We compose the set $\{\xi : \lambda_j \le |\xi| < 1.01\lambda_j\}$ for $2 \le j \le 5$ into symmetric unions of intervals of length $\lambda_1/100$. We label this intervals by μ_{ij} with $2 \le i \le 5$ and $j \le \lambda_5/\lambda_1$ and expand the sums in

$$\left|\int u_{1,\lambda_1}u_{2,\lambda_5}u_{3,\lambda_5}u_{4,\lambda_5}u_{5,\lambda_5}dxdt\right| = \sum_{90 \le |\sum_{j=2}^5 \mu_j| \le 110} \int u_{1,\lambda_1}u_{2,\mu_2}u_{3,\mu_3}u_{4,\mu_4}u_{5,\mu_5}dxdt$$

there are at most $\sim (\lambda_5/\lambda_1)^4$ terms. We fix μ_j and assume that they are ordered. Then $\mu_5 - \mu_2 \ge 2$ and we estimate

$$\int u_{1,\lambda_1}u_{2,\mu_2}u_{3,\mu_3}u_{4,\mu_4}u_{5,\mu_5}dxdt \le \|u_{\lambda_1}u_{4,\mu_4}\|_{L^p}\|u_{\mu_2}u_{\mu_5}\|_{L^q}\|u_{\lambda_3}\|_{L^6}$$

and hence (changing indices if necessary, or summing over similar terms)

(6.23)
$$\left|\int u_{1,\lambda_1}u_{2,\mu_2}u_{3,\mu_3}u_{4,\mu_4}u_{5,\mu_5}dxdt\right| \le c\lambda_5^{-1}(\lambda_2\lambda_3\lambda_4)^{-1/6}(\lambda_5/\lambda_1)\prod\|u_{i,\lambda_i}\|_{U^p}$$

since p is the smallest exponent. This is almost good - but $(\lambda_5/\lambda_1)^5$ is too big.

We recall Lemma 4.12 which allows us to write for given M

$$u = v + w$$

with

$$\frac{\kappa}{M}\|w\|_{U^2_{KdV}} + e^M\|v\|_{U^p_{KdV}} \le \|u\|_{V^2_{KdV}}.$$

We expand all the u_i. This yields by (6.22)

$$\left|\int v_{1,\lambda_1}v_{2,\mu_2}v_{3,\mu_3}v_{4,\mu_4}v_{5,\mu_5}dxdt\right| \le cM^5\lambda_5^{-1}(\lambda_2\lambda_3\lambda_4)^{-\frac{1}{6}}\prod\|u_{i,\lambda_i}\|_{V^2_{KdV}}$$

and

$$\begin{aligned}
&\left|\int w_{1,\lambda_1} v_{2,\lambda_2} v_{3,\lambda_3} v_{4,\lambda_4} v_{5,\lambda_5} dxdt\right| \\
&\qquad \le c\lambda_5^{-1}(\lambda_2\lambda_3\lambda_4)^{-1/6}(\lambda_5/\lambda_1)^5 \|w_{1,\lambda_1}\|_{u^p_{KdV}} \prod_{i=2}^5 \|v_{i,\lambda_i}\|_{U^p_{KdV}} \\
&\qquad \le e^{-M}\lambda_5^{-1}(\lambda_2\lambda_3\lambda_4)^{-1/6}(\lambda_5/\lambda_1)^5 \prod \|u_{i,\lambda_i}\|_{V^2_{KdV}}
\end{aligned}$$

Similarly we estimate all the other terms in the expansion. Then

$$\begin{aligned}
\left|\int u_{1,\lambda_1} u_{2,\mu_2} u_{3,\mu_3} u_{4,\mu_4} u_{5,\mu_5} dxdt\right| \le & c(M^5 + e^{-M}(\lambda_5/\lambda_1)^5)\lambda_5^{-1}(\lambda_2\lambda_3\lambda_4)^{-1/6} \\
& \times \prod \|u_{i,\lambda_i}\|_{V^2_{KdV}}. \\
\le & \, c\ln(1+(\lambda_5/\lambda_1))^5 \lambda_5^{-1}(\lambda_2\lambda_3\lambda_4)^{-1/6} \prod \|u_{i,\lambda_i}\|_{V^2_{KdV}}.
\end{aligned}$$

if we choose $M = 5\ln(\lambda_5/\lambda_1)$. This completes the proof of estimate (6.19), and hence the proof of the theorem. □

Again there are similar refinements as for the critical gKdV equation. Well-posedness in a slightly smaller spaces has been proven by Grünrock [9] and Tao [28] based on a modification of the Fourier restriction spaces of Bourgain at the critical level.

Statement and proof are based on [18], where it was one step to prove stability of the soliton in $\dot{B}^{-1/6}_{2,\infty}$, and scattering, which is probably the first stability result of solitons for gKdV which is not based on Weinstein's convexity argument.

Next we turn to the modified KdV equation

$$u_t + u_{xxx} + u_x^3 = 0. \tag{6.24}$$

The space $\dot{H}^{-1/2}$ is scaling invariant, but we are not able the reach the critical space. Instead we construct global in time solutions to

$$u_t + u_{xxx} + \partial_x(\chi_{[0,T]}u^3) = 0$$

for given initial data u_0 and $T > 0$. We aim for a scale invariant formulation. Given $T > 0$ we define the equivalent norm on $B^{\frac14}2,\infty$,

$$\|u_0\|_E = \max\{T^{\frac16}\|u_{0,<T^{-\frac13}}\|_{L^2}, \sup_{\lambda \ge T^{-\frac13}} (\lambda T)^{\frac14}\|u_{0,\lambda}\|_{L^2}$$

and

$$\|u\|_E = \max\{T^{\frac16}\|u_{<T^{-\frac13}}\|_{V^2_{KdV}}, \sup_{\lambda \ge T^{-\frac13}} (\lambda T)^{\frac14}\|u_\lambda\|_{V^2_{KdV}}$$

Well-posedness by different arguments has been shown by [15] in a slightly smaller space of initial data.

Theorem 6.5. *There exists $\varepsilon > 0$ such that for $u_0 \in B^{\frac14}_{2,\infty}$ with*

$$\|u_0\|_E \le \varepsilon$$

there is a unique weak solution $u \in X$ with

$$\|u - S(t)u_0\|_X \le c\|u_0\|_E^3.$$

Proof. We want to construct a fixed point of

$$v = \int_0^t S(t-s)\chi_{[0,T]}(s)\partial_x (w+v)^3 ds$$

The key estimate (for small data) is

$$\lambda^{\frac14} \left| \int_{\mathbb{R}\times R} \chi_{[0,T]} u_1 u_2 u_3 \partial_x v_\lambda dxdt \right| \le c \prod_{j=1}^3 \|u_j\|_X \|v_\lambda\|_{V^2}. \tag{6.25}$$

The theorem follows from it by repeating the arguments for the L^2 critical case.

To prove (6.25) we expand the left hand side into a dyadic sum. The pieces are estimated by

$$\left| \int_0^T \int_{\mathbb{R}} \prod_{j=1}^4 u_{i,\lambda_i} dxdt \right| \le cT^{\frac12} \prod_{j=1}^4 \lambda_j^{-1/8} \|u_{j,\lambda_j}\|_{U^8_{KdV}} \tag{6.26}$$

if $\lambda_1 \ge 2$ using the Strichartz embedding

$$\|u_{j,\lambda_j}\|_{L^8_t L^4_x} \le c\lambda_j^{-1/8} \|u_{j,\lambda_j}\|_{U^8_{KdV}}.$$

This is good enough if $\lambda_1 \sim \lambda_4$ and $\lambda_1 \ge 2$. If $\mu \le \lambda/4$ there is the bilinear estimate

$$\begin{aligned} \|S(t)u_{0,\mu} S(t) v_{0,\lambda}\|_{L_t^{\frac83} L^2_x} \le & \|S(t)u_{0,\mu} S(t) v_{0,\lambda}\|^{\frac12}_{L^4_t L^2_x} \|S(t)u_{0,\mu} S(t) v_{0,\lambda}\|^{\frac12}_{L^2_t L^2_x} \\ \le & c\lambda^{-\frac98} \mu^{-\frac{1}{16}} \|u_{0,\mu}\|_{L^2} \|v_{0,\lambda}\|_{L^2} \end{aligned}$$

and hence if $\lambda_1 \le \lambda_3/4$ and $\lambda_2 \le \lambda_4/4$

$$\begin{aligned} \left| \int_0^T \int \prod_{j=1}^4 u_{i,\lambda_i} dxdt \right| \le & cT^{\frac14} \|u_{1,\lambda_1} u_{3,\lambda_3}\|_{L^{\frac83,2}} \|u_{2,\lambda_2} u_{4,\lambda_4}\|_{L^{\frac83,2}} \\ \le & cT^{\frac14} \lambda_4^{-\frac98} \lambda_1^{-\frac{1}{16}} \lambda_2^{-\frac{1}{16}} \prod_j \|u_{j,\lambda_j}\|_{V^2_{KdV}}. \end{aligned} \tag{6.27}$$

If $\lambda_4 \le 2T^{\frac13}$ we estimate

$$\begin{aligned} \int_0^T \int_{\mathbb{R}} \prod_{i=1}^4 u_{i,\lambda_i} dxdt \le & T\|u_{1,\lambda_1}\|_{L^\infty} \|u_{2,\lambda_2}\|_{L^\infty} \|u_{3,\lambda_3}\|_{L^\infty L^2} \|u_{4,\lambda_4}\|_{L^\infty L^2} \\ \le & c\prod \|u_{i,\lambda_i}\|_{V^\infty_{KdV}}. \end{aligned}$$

Checking the support we see that the integral vanishes unless either $\lambda_1 \ge \lambda_4/16$ or $\lambda_1 \le \lambda_3/4$ and $\lambda_2 \le \lambda_4/4$ or $\lambda \le 16$.

We turn to the summation.

(1) $\lambda > \lambda_4/16$, $\lambda_4 \ge 16T^{-\frac13}$ The sum can be bounded using (6.26) for $\lambda_1 \ge \lambda_4/16$ and (6.27) for $\lambda_1 \le \lambda_4/16$ and $\lambda_4 \ge 16$ where the sum takes the form

$$\begin{aligned} \Big(\sum_{1\le\lambda_1\le\lambda_2\le\lambda_4/4} & (T^{\frac13}\lambda_1)^{-\frac{5}{16}} (T^{\frac13}\lambda_2)^{-\frac{5}{16}} (T^{\frac13}\lambda_4)^{-\frac18} \\ & \times (T\lambda_1)^{\frac14} \|u_{1,\lambda_1}\|_{V^2_{KdV}} (T\lambda_2)^{\frac14}) \|u_{2,\lambda_2}\|_{V^2_{KdV}} \Big) \|u_{4,\lambda_4}\|_{V^2_{KdV}} \|v_\lambda\|_{V^2_{KdV}}. \end{aligned}$$

The bound is obvious.

(2) $\max\{T^{-\frac13}, \lambda\} \le \lambda_4/16$. Here we use (6.27). The uniform bound for the sum is immediate.

(3) $\lambda_4 \le 16T^{-1/3}$. Now the estimate follows from the last estimate.

The proof is complete. □

The proof could easily be simplified by first rescaling to $T=1$. The advantage of the current proof is that it makes the behavior of all terms with respect to scaling transparent.

Finally we study the Korteweg de-Vries equation

$$u_t + u_{xxx} + u_x^2 = 0.$$

The well-posedness result in $H^{-\frac{3}{4}}$ is due to Christ, Colliander and Tao [5] who also prove that below $-\frac{3}{4}$ some sort of ill-posedness must occur. Despite this there are uniform global apriori estimates in H^{-1}, see [3]. Uniqueness between $-\frac{3}{4}$ and -1 is entirely open.

We search a solution u to

$$u_t + u_{xxx} + \partial_x(\chi[0,1](t)u^2) = 0$$

with the given initial data. We again make the ansatz

$$u = v + w$$

where $v = S(t)u_0$ and

$$w_t + w_{xxx} + \partial_x(\chi(t)(v+w)^2) = 0.$$

The identity

$$(\xi_1+\xi_2)^3 - \xi_1^3 - \xi_2^3 = 3\xi_1\xi_2(\xi_1+\xi_2)$$

describes the vertical distance of the sum of two points (τ_j, ξ_j) from the characteristic set. We will make use of this property through 'high modulation' L^2 estimates. For this purpose we fix a smooth function ϕ supported in $[-2,2]$, identically 1 in $[-1,1]$ and define $u^\Lambda(t)$ by the Fourier multiplier $1-\phi(\tau/\Lambda)$. The Fourier multiplier $\phi(\tau/\Lambda)$ defines a convolution. Let ψ be the inverse Fourier transform. Then up to a power of $\sqrt{2\pi}$,

$$(1-\phi(\tau/\Lambda))u = u - \Lambda\psi(\Lambda t)u$$

and hence

$$\|u^\Lambda\|_{U^p} \le c\|u\|_{U^p}$$
$$\|u^\Lambda\|_{V^p} \le c\|u\|_{V^p}$$

Moreover, for $\Lambda = 1$

$$u^1(t) = u(t) - \int \psi(t-s)u(s)ds = \int (u(t)-u(s))\psi(t-s)ds$$

and hence

$$\|u^1\|_{L^2} \le c\|u\|_{V^2}.$$

Rescaling gives

$$\|u^\Lambda\|_{L^2} \le c\Lambda^{-1/2}\|u\|_{V^2}$$

since the right hand side is invariant with respect to rescaling.

We consider solutions in a space defined by

$$\|u\|_X = \Big(\|u_{<0}\|_{X_0}^2 + \sum_{\lambda\in 2^{\mathbb{N}}} \lambda^{-\frac{3}{2}}\|u_\lambda\|_{U^2_{KdV}}^2\Big)^{\frac{1}{2}}.$$

Theorem 6.6. *There exists $\delta > 0$ such that for all initial data*

$$\|u_0\|_{H^{-\frac{3}{4}}} < \delta$$

there is a unique function $u \in X$ with

$$\|\chi_T u - S(t)u\|_X \le c\|u\|^2_{B^{-\frac{3}{4}}_{2,\infty}}$$

which satisfies the equation up to time 1*. It depends analytically on the initial data.*

Proof. We define the sets

$$A(0) = \{(\tau,\xi) | |\xi| \le 1, |\tau - \xi^3| \le 1\}$$
$$A(\lambda) = \{(\tau,\xi) | \lambda \le |\xi| \le 2\lambda, |\tau - \xi^3| \le \lambda^3\}$$
$$B(\lambda) = \{(\tau,\xi) | |\xi| \le \lambda, 1 \le |\tau - |\xi|^3| \le |\xi|\lambda^2\}.$$

Then

$$\begin{aligned}\Big\| |D_x|^{\frac{1}{2}} \int S(t-s)\rho(s)\partial_x(u_{A(\lambda)}u_{A(\lambda)})dt_{B(\lambda)}\Big\|_{L^2} \le & \lambda^{-2}\||D_x|^{\frac{1}{2}} u_{A(\lambda)}u_{A(\lambda)}\|_{L^2} \\ \le & \lambda^{-\frac{5}{2}}\|u_{A(\lambda)}\|^2_{U^2_{KdV}}\end{aligned}$$

which is scale invariant. Alternatively we may estimate

$$\begin{aligned}\Big\| |D_x|^{-\frac{1}{2}} & \int S(t-s)\rho(s)\partial_x(u_{A(\lambda)}u_{A(\lambda)})dt_{\mu,B(\lambda)}\Big\|_{L^2} \\ & \le \lambda^{-1}\Big\|\int S(t-s)\rho(s)u_{A(\lambda)}u_{A(\lambda)}ds_{\mu,B(\lambda)}\Big\|_{V^2_{KdV}} \\ & \le \lambda^{-2}\|u_{A(\lambda)}\|_{U^2_{KdV}}\|u_{A(\lambda)}\|_{U^2_{KdV}}\end{aligned}$$

Observe that the two terms are of the same size for $\mu = \lambda^{-1/2}$.

More precisely the L^2 norm is of unit size.

$$\Big\|\int S(t-s)\rho(s)\partial_x(u_{A(\lambda)}u_{A(\lambda)})_{\lambda^{-1/2},B(\lambda)}\Big\|_{L^2} \le c\lambda^{-\frac{3}{4}}(\lambda^{-\frac{3}{4}}\|u_{A(\lambda)}\|_{U^2_{KdV}})^2$$

There is nothing to loose, and hence we need to control $u_{A(\lambda)}$ in U^2_{KdV}. Similarly

$$\Big\|\int S(t-s)\rho(s)\partial_x(u_{A(\lambda)}u_{B(\lambda)})_{A(\lambda)}ds\Big\|_{V^2_{KdV}} \le c\lambda^{\frac{1}{2}}\|u_{A(\lambda)}\|_{U^2_{KdV}}\||D_x|^{-\frac{1}{2}}u_{B(\lambda)}\|_{L^2}$$

and

$$\Big\|\int S(t-s)\rho(s)\partial_x(u_{A(\lambda)}u_{\mu,B(\lambda)})_{A(\lambda)}ds\Big\|_{U^2_{KdV}} \le c\|u_{A(\lambda)}\|_{U^2_{KdV}}\||D_x|^{\frac{1}{2}}u_{\mu,B(\lambda)}\|_{L^2}.$$

This is the only place which does not allow us to go beyond $B^{-\frac{3}{4}}_{2,\infty}$.

We only consider the most important term. The remaining estimates are tedious, and we do not work them out.

□

The interest in this setup is twofold: It shows how to go beyond $H^{-\frac{3}{4}}$. Then X is not a subset of $L^\infty(\mathbb{R}; B^{-\frac{3}{4}}2,\infty)$ and one has to use energy estimates to see that the solution is bounded and weakly continuous as a map to $B^{-\frac{3}{4}}2,\infty$. This difficulty is related to the classical ill-posedness results: The flow map does not extend to a differentiable map from the initial data to $u(t) \in \mathcal{S}$ below $-\frac{3}{4}$.

6.3. **The derivative nonlinear Schrödinger equation.** We consider

$$iu_t + \Delta u = \bar u \partial_1 \bar u. \tag{6.28}$$

This equation has no significance from applications as far as I know. The choice of the non-linearity is crucial. If u satisfies (6.28) then the same is true for

$$\lambda u(\lambda^2 t, \lambda x)$$

and critical space is $\dot H^{\frac{d-2}{2}}$.

The Strichartz with $\frac{2}{4} + \frac{d}{p} = \frac{d}{2}$ and Bernstein, gives for $d \ge 2$

$$\|u_\lambda\|_{L^4(\mathbb{R}\times\mathbb{R}^d)} \le \lambda^{\frac{d-2}{4}} \|u_{i,\lambda_i}\|_{L^{4,p}(\mathbb{R}^d)} \le \lambda^{\frac{d-2}{4}} \|u_{i,\lambda_i}\|_{U^4}. \tag{6.29}$$

The bilinear estimates are

$$\|u_\lambda v_\mu\|_{L^2} \le c\mu^{\frac{d-1}{2}} \lambda^{-1/2} \|u_\lambda\|_{U^2_{i\Delta}} \|v_\mu\|_{U^2_{i\Delta}} \tag{6.30}$$

and

$$\|(u_\lambda v_\lambda)_\mu\|_{L^2} \le c\mu^{\frac{d-2}{2}} \|u_\lambda\|_{U^2_{i\Delta}} \|v_\lambda\|_{U^2_{i\Delta}}.$$

if $\mu < \lambda/4$. We may improve the second estimate by Bernstein and Strichartz (using a smooth Fourier projection for $_\mu$)

$$\begin{aligned} \|(u_\lambda v_\lambda)_\mu\|_{L^2} \le & c\mu^{\frac{d-2}{2}} \|u_\lambda v_\lambda\|_{L^2_t L^{\frac{p}{2}}_x} \\ \le & c\mu^{\frac{d-2}{2}} \|u_\lambda\|_{L^{4,p}} \|v_\mu\|_{L^{4,p}} \\ \le & c\mu^{\frac{d-2}{2}} \|u_\lambda\|_{U^4_{i\Delta}} \|v_\lambda\|_{U^4_{i\Delta}}. \end{aligned} \tag{6.31}$$

This time we need the complex inner product. The modulation relation is

$$\xi_1^2 + \xi_2^2 + (-\xi_1 - \xi_2)^2 \ge \xi_1^2 + \xi_2^2$$

which is a particularly pleasant situation.

The dyadic estimates become for $\lambda_1 << \lambda_2 \sim \lambda_3$

$$\left| \int u^h_{\lambda_1} u_{\lambda_2} u_{\lambda_3} dx\, dt \right| \le c\lambda_3^{-1} \lambda_1^{\frac{d-2}{2}} \|u_{1,\lambda_1}\|_{V^2_{i\Delta}} \|u_{2,\lambda_2}\|_{V^2_{i\Delta}} \|u_{3,\lambda_3}\|_{V^2_{i\Delta}} \tag{6.32}$$

and

$$\left| \int u_{\lambda_1} u^h_{\lambda_2} u_{\lambda_3} dx\, dt \right| \le c\lambda_3^{-\frac{3}{2}} \lambda_1^{\frac{d-1}{2}} \|u_{1,\lambda_1}\|_{U^2_{i\Delta}} \|u_{2,\lambda_2}\|_{V^2_{i\Delta}} \|u_{3,\lambda_3}\|_{U^2_{i\Delta}} \tag{6.33}$$

and hence

$$\left| \int \prod u_{1,\lambda_1} u^h_{2,\lambda_2} u_{3,\lambda_3} dx dt \right| \le c\lambda_3^{-\frac{3}{2}} \lambda_1^{\frac{d-1}{2}} (\lambda_3/\lambda_1)^\varepsilon \prod_{i=1}^3 \|u_{i,\lambda_i}\|_{V^2_{i\delta}}$$

Theorem 6.7. *Let $d = 2$. There exists $\varepsilon > 0$ so that if*

$$\|u_0\|_{L^2} < \varepsilon$$

then there is a unique solution to

$$iu_t + \Delta u = \bar u \partial_{x_1} \bar u$$

with

$$\|u\|_X := \left(\sum_{\lambda \in 2^{\mathbb{Z}}} \|u_\lambda\|^2_{U^2_{KdV}} \right)^{1/2} \le c\|u_0\|_{L^2}.$$

If $d \geq 3$ there exists $\varepsilon > 0$ so that if

$$\|u_0\|_{\dot{B}^{\frac{d-2}{2}}_{2,1}} = \sum_\lambda \lambda^{\frac{d+2}{2}} \|u_{0,\lambda}\|_{L^2} < \varepsilon$$

then there is a unique weak solution with

$$\|u\|_X := \sum_\lambda \lambda^{\frac{d-2}{2}} \|u_\lambda\|_{U^2} \leq c\|u_0\|_{\dot{B}^{\frac{d-2}{2}}_{2,1}}$$

Proof. The key estimates are again

$$\left| \int_{\mathbb{R}\times\mathbb{R}^d} (\partial_{x_1}\bar{u}_1)\bar{u}_2\bar{v}dxdt \right| \leq \|u_1\|_X \|u_2\|_X \left(\sum_\lambda \|v_\lambda\|^2_{V^2_{KdV}} \right)^{1/2}$$

resp.

$$\left| \int_{\mathbb{R}\times\mathbb{R}^d} (\partial_{x_1}\bar{u}_1)\bar{u}_2\bar{v}dxdt \right| \leq \|u_1\|_X \|u_2\|_X \sup_\lambda \lambda^{-\frac{d-2}{2}} \|v_\lambda\|_{V^2_{KdV}}$$

if $d \geq 3$. We abuse the notation and set $\lambda_2 = \lambda_3 = \lambda$ and compute for $d = 2$

$$\begin{aligned} \sum_{\mu<\lambda} \lambda \left| \int \bar{u}^h_\mu \bar{u}_{2,\lambda} \bar{v}_\lambda dxdt \right| \leq & \sum_{\mu\leq\lambda} \lambda \|u^h_\mu\|_{L^2} \|(u_{2,\lambda}v_\lambda)_\mu\|_{L^2} \\ \leq & \left(\sum_{\mu\leq\lambda} \|u_{1,\mu}\|_{V^2_{i\Delta}})^2 \right)^{1/2} \|u_\lambda v_\lambda\|_{L^2(\mathbb{R}^2)} \\ \leq & \|u_1\|_X \|u_\lambda\|_{U^4_{i\Delta}} \|v_\lambda\|_{U^4_{i\Delta}} \end{aligned}$$

The factor λ^{-1} compensates for the derivative. The summation with respect to λ is trivial. The estimate is easier if the high modulation falls on other terms.

$$\begin{aligned} \sum_{\mu<\lambda} \lambda \Big| \int \bar{u}_\mu \bar{u}^h_{2,\lambda} \bar{v}_\lambda dxdt \Big| \leq & \sum_{\mu\leq\lambda} \lambda \|u^h_{2,\lambda}\|_{L^2} \|u_{1,\mu}v_\lambda\|_{L^2} \\ \leq & \mu^{1/2} \|u_{1,\mu}\|_{V^2_{i\Delta}} \lambda^{-1/2} \|u_{1,\mu}\|_{U^2_{i\Delta}} \|v_\lambda\|_{U^2_{i\Delta}} \end{aligned}$$

By logarithmic interpolation

$$\begin{aligned} \sum_{\mu<\lambda} \lambda \left| \int \bar{u}_\mu \bar{u}^h_{2,\lambda} \bar{v}_\lambda dxdt \right| \leq & \sum_{\mu\leq\lambda} \lambda \|u^h_{2,\lambda}\|_{L^2} \|u_{1,\mu}v_\lambda\|_{L^2} \\ \leq & \sum_{\mu\leq\lambda} (\mu/\lambda)^{\frac{1}{2}-\varepsilon} \|u_{1,\mu}\|_{V^2_{i\Delta}} \lambda^{-1/2} \|u_{1,\mu}\|_{V^2_{i\Delta}} \|v_\lambda\|_{V^2_{i\Delta}} \end{aligned}$$

and the summation is straight forward.

The modification for $d \geq 3$ is simply: We give up orthogonality and sum for the first estimate

$$\begin{aligned} \sum_{\mu<\lambda} \lambda \left| \int \bar{u}^h_\mu \bar{u}_{2,\lambda} \bar{v}_\lambda dxdt \right| \leq & \sum_{\mu\leq\lambda} \lambda \|u^h_\mu\|_{L^2} \|(u_{2,\lambda}v_\lambda)_\mu\|_{L^2} \\ \leq & \sum_{\mu\leq\lambda} \mu^{\frac{d-2}{2}} \|u_{1,\mu}\|_{V^2_{i\Delta}} \|u_\lambda\|_{V^2_{i\Delta}} \|v_\lambda\|_{V^2_{i\Delta}} \end{aligned}$$

For the second estimate we put in powers of μ resp. λ.

□

6.4. **The Kadomtsev-Petviashvili II equation.** The Kadomtsev-Petviashvili-II (KP-II) equation

$$\begin{aligned} \partial_x(\partial_t u + \partial_x^3 u + u\partial_x u) + \partial_y^2 u = 0 \quad & \text{in } (0,\infty)\times\mathbb{R}^2 \\ u(0,x,y) = u_0(x,y) \quad & (x,y)\in\mathbb{R}^2 \end{aligned} \tag{6.34}$$

has been introduced by B.B. Kadomtsev and V.I. Petviashvili [12] to describe weakly transverse water waves in the long wave regime with small surface tension. It generalizes the Korteweg - de Vries equation, which is spatially one dimensional and thus neglects transversal effects. The KP-II equation has a remarkably rich structure.

Here we describe a setup leading to global well-posedness and scattering for small data. The Hilbert space will be denoted by $\dot H^{-1/2,0}$ which is defined by through the norm

$$\|u_0\|_{\dot H^{-1/2}} = \||\xi|^{-1/2}\hat u_0\|_{L^2}.$$

where ξ is the Fourier multiplier with respect to x. The Fourier multiplier $|\xi|^{-1/2}$ defines an isomorphism from L^2 to $\dot H^{-1/2}$.

For $\lambda > 0$ we write define the projection to the $1 \le |\xi|/\lambda < 2$

$$\mathcal{F}(u_\lambda) = \chi_{\lambda\le|\xi|\le 2\lambda}\mathcal{F}u$$

where $\mathcal{F}$ denotes the Fourier transform. Usually we choose $\lambda \in 2^{\mathbb{Z}}$, the set of integer powers of 2.

Let $u(t) = S(t)u_0$. The Strichartz estimate is

$$\|u\|_{L^4(\mathbb{R}^3)} \le c\|u(0)\|_{L^2}$$

which implies the embedding $U^4_{KP} \subset L^4(\mathbb{R}^3)$ and

$$\|u\|_{L^4(\mathbb{R}^3)} \le c\|u\|_{U^4_{KP}} \le c\|u\|_{V^2_{KP}}. \tag{6.35}$$

There is the bilinear improvement

$$\|u_\lambda v_\mu\|_{L^2} \le c(\lambda/\mu)^{1/2}\|u_\lambda(0)\|_{L^2}\|v_\mu(0)\|_{L^2}. \tag{6.36}$$

which implies

$$\|u_\lambda v_\mu\|_{L^2} \le c(\mu/\lambda)^{1/2}\|u_\lambda\|_{U^2_{KP}}\|v_\mu\|_{U^2_{KP}}. \tag{6.37}$$

and together with the logarithmic interpolation

$$\|u_\lambda v_\mu\|_{L^2} \le c(\mu/\lambda)^{1/2}(\ln(2+\lambda/\mu))^2\|u_\lambda\|_{V^2_{KP}}\|v_\mu\|_{V^2_{KP}}. \tag{6.38}$$

Formally the L^2 norm is constant.

We use the norm

$$\|u\|_X = \left(\sum_{\lambda\in 2^{\mathbb{Z}}} \|u_\lambda\|^2_{V^2_{KP}}\right)^{1/2}$$

Theorem 6.8. *There exists $\varepsilon > 0$ such that for $u_0 \in \dot H^{-1/2,0}(\mathbb{R}^2)$ there exists a unique solution $u \in X$ with*

$$\|u\|_X \le c\|u_0\|_{H^{-1/2,0}}(\mathbb{R}^2).$$

If $u_0 \in L^2$ then there is a unique solution in $C(\mathbb{R};L^2)$ with

$$\|\chi_{[k,k+1]}(t)u\|_{U^2_{KP}} < C(\|u_0\|_{L^2})$$

Proof. By definition

$$\|S(t)u_0\|_X \le c\|u_0\|_{\dot H^{-1/2}}$$

We claim

$$\| \int_0^t S(t-s)\partial_x(uv)ds\|_X \le c\|u\|_X\|v\|_X. \tag{6.39}$$

With this information we set the fixed point argument and obtain a unique fixed point which is the solution. By duality (see) (6.39) follows from

$$\left|\int uvw dxdydt\right| \le c\|u\|_X\|v\|_X\|w\|_X. \tag{6.40}$$

We expand all factors and consider

$$\int u_{\lambda_1}v_{\lambda_2}w_{\lambda_3}dxdydt$$

The integral is symmetric with respect to the factors and we may assume that $\lambda_1 \le \lambda_2 \le \lambda_3$. If there are no $\lambda_1 \le |\xi_1| \le 2\lambda_1$, $\lambda_2 \le |\xi_2| \le 2\lambda_2$ and $\lambda_3 \le |\xi_3| \le 2\lambda_3$ which add up to zero then the integral vanishes. Thus

$$\lambda_3 \le 4\lambda_2$$

The integral vanishes unless there are such ξ_i, η_i and τ_i which add up to zero. Now

$$\xi_1^3+\xi_2^3+\xi_3^3-\frac{\eta_1^2}{\xi_1}-\frac{\eta_2^2}{\xi_2}-\frac{\eta_3^2}{\xi_3} = 3\xi_1\xi_2\xi_3\Big(1+\frac{|\eta_1\xi_2-\eta_2\xi_1|^2}{\xi_1\xi_2\xi_3}\Big).$$

We define Q_H by the Fourier multiplier $\chi_{|\tau-\xi^3+\eta^2/\xi|>|\xi_1||\xi_2||\xi_1+xi_2|/10}$ and $Q_L = 1-Q_H$. Then by the consideration of the supports

$$\int Q_Lu_{\lambda_1}Q_Lv_{\lambda_2}Q_Lw_{\lambda_3}dxdydt = 0.$$

It follows from the embedding (4.18) that

$$\|Q_Hu\|_{L^2} \le c(|\xi_1||\xi_2||\xi_1+\xi_2|)^{-1/2}\|u\|_{V^2_{KP}}$$

and

$$\|Q_Hu\|_{V^2_{KP}} \le c\|u\|_{V^2_{KP}}.$$

We estimate

$$\begin{aligned}
&\left|\int (u_{\lambda_1})v_{\lambda_2}Q_Hw_{\lambda_3}dxdydt\right| \\
&\quad\le \|u_{\lambda_1}v_{\lambda_2}\|_{L^2}\|Q_Hw_{\lambda_3}\|_{L^2} \\
&\quad\le c\left(\frac{\lambda_{min}}{\lambda_{max}}\right)^{1/2}(1+\ln(\lambda_2/\lambda_1))^2\lambda_{max}^{-1}\lambda_{min}^{-1/2}\|v_{\lambda_1}\|_{V^2_{KP}}\|v_{\lambda_2}\|_{V^2_{KP}}\|w_{\lambda_3}\|_{V^2_{KP}} \\
&\quad\le c\left(\frac{\lambda_{min}}{\lambda_{max}}\right)^{1/2}(1+\ln(\lambda_2/\lambda_1))\|v_{\lambda_1}\|_X\|v_{\lambda_2}\|_X\|w_{\lambda_3}\|_X
\end{aligned}$$

This is easy to sum with respect to all indices. Th case with $Q_Hu_{\lambda_1}$ is different since we don't gain a factor for the summation over the small frequencies. Here we need some orthogonality:

$$\left|\sum_{\lambda_1<\lambda_2}\int Q_H u_{\lambda_1} u_{\lambda_2} w_{\lambda_3} dxdydt\right| \le \left(\sum_{\lambda_1<\lambda_2} \|Q_H u_{\lambda_1}\|_{L^2}^2\right)^{1/2} \|v_{\lambda_2} w_{\lambda_3}\|_{L^2}$$
$$\le \left(\sum \lambda_1^{-1}\|u_{\lambda_1}\|_{V^2_{KP}}\right)^{\frac12} \lambda_{max}^{-1}\|v_{\lambda_2}\|_{V^2_{KP}}\|w_{\lambda_3}\|_{V^2_{KP}}$$

which can be summed.

Now consider data in $u_0 \in L^2$ with $\|u_0\|_{L^2} \le 1$. Let v be the solution to linear KP with initial data u_0. We search a solution in the form $u = v + w$. We need two estimates:

$$\|\chi_{[0,1]}\int_0^t S(t-s)\partial_x(uv)ds\|_{U^2} \le c\|u_{>1}\|_{U^2_{KP}}\|v_{>1}\|_{U^2_{KP}}$$

and

$$\|\chi_{[0,1]}\int_0^t S(t-s)\partial_x(u_{<1}v_{>1})ds\|_{U^2} \le c\|u\|_{U^2_{KP}}\|v\|_{U^2_{KP}}.$$

□

7. Appendix A: Young's inequality and interpolation

Young's inequality bounds convolutions in Lebesgue spaces gives bounds for the convolution of two functions. It is part of the statement that the integral exists for almost all arguments of the convolution. Let m^d denote the d dimensional Lebesgue measure.

Lemma 7.1. *Let $1 \le p, q, r \le \infty$ satisfy*

$$\frac{1}{p}+\frac{1}{q}+\frac{1}{r} = 2,$$

$$f \in L^p(\mathbb{R}^d), \quad g \in L^q(\mathbb{R}^n), \quad h \in L(\mathbb{R}^d).$$

Then

$$\int_{\mathbb{R}^d\times\mathbb{R}^d} f(x)g(x-y)h(y)dm^{2d}(x,y) \le \|f\|_{L^p}\|g\|_{L^q}\|h\|_{L^r}.$$

We assume that the Lemma holds and choose $f(x) = e^{-|x|^2} \in L^r(\mathbb{R}^d)$. It follows by Fubini's theorem that $g(x-y)h(y)$ is integrable with respect to y for almost all x. The estimate of the lemma shows that

$$L^p(\mathbb{R}^d) \ni f \to \int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d} h(y)g(x-y)dm^d(y)\right) f(x)dm^d(x) \in \mathbb{R}$$

defines a linear form of norm $\le \|g\|_{L^q}\|g\|_{L^r}$ on L^r. Thus

$$\|g*h\|_{L^{p'}} \le \|g\|_{L^q}\|h\|_{L^r}$$

for

$$\frac{1}{q}+\frac{1}{r} = 1+\frac{1}{p'}.$$

Proof of Lemma 7.1, as in [20]. Set

$$\frac{1}{\gamma_1} = 1 - \frac{1}{p}, \frac{1}{\gamma_2} = 1 - \frac{1}{q}, \frac{1}{\gamma_3} = 1 - \frac{1}{r}.$$

Then $1 \leq \gamma_1 \leq \infty$,

$$\frac{1}{\gamma_2} + \frac{1}{\gamma_3} = \frac{1}{p}, \frac{1}{\gamma_1} + \frac{1}{\gamma_3} = \frac{1}{q}, \frac{1}{\gamma_1} + \frac{1}{\gamma_2} = \frac{1}{r}$$

and

$$\frac{1}{\gamma_1} + \frac{1}{\gamma_2} + \frac{1}{\gamma_3} = 1.$$

Let

$$a(x,y) = |f(x)|^{p/\gamma_3} |g(x-y)|^{q/\gamma_3}, b(x,y) = |g(x-y)|^{q/\gamma_1} |h(y)|^{r/\gamma_1},$$

$$c(x,y) = |f(x)|^{p/\gamma_2} |h(y)|^{r/\gamma_2}.$$

Then

$$|f(x)g(x-y)h(y)| = a(x,y)b(x,y)c(x,y)$$

and, by applying Hölder's inequality twice

$$\int |f(x)g(x-y)h(y)| dm^{2d} \leq \|a\|_{L^{\gamma_3}} \|b\|_{L^{\gamma_1}} \|c\|_{L^{\gamma_2}} = \|f\|_{L^p} \|g\|_{L^q} \|h\|_{L^r}.$$

□

There is an improvement: the weak Young inequality. Let (X, μ) be a measure space. We will often suppress space and measure in the notation. The weak L^p spaces are defined by the quasi-norm

$$\|f\|_{L^p_w} = \sup_{t>0} t \left(\mu(\{x : |f(x)| > t\}) \right)^{1/p}.$$

If $1 < p < \infty$ then there is an equivalent norm on L^p_w,

$$\|f\|_{L^p_w} \sim \sup_{t>0} t \left(\int_{\{x:|f(x)|>t\}} |f(y)| d\mu(y) \right)^{1/p}.$$

It is not hard to see the equivalence, and that the term on the right hand side defines a norm.

Proposition 7.2. *Suppose that*

$$1 < p, q, r < \infty, \frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r},$$

$f \in L^p$ *and* $g \in L^q_w$. *Then* $f(x)g(x-y)$ *is integrable with respect to* x *for almost all* y *and*

$$\|f * g\|_{L^r} \leq c_{p,q} \|f\|_{L^p} \|g\|_{L^q_w}.$$

This is a consequence of the Markinkiewicz interpolation theorem. We state and prove the following version.

Let X and Y be normed linear spaces. We denote by $L(X, Y)$ the normed space of bounded linear operators from X to Y.

Lemma 7.3 (Markinciewicz interpolation)**.** *Let* (X,μ) *and* (Y,ν) *be measure spaces and* $1 \le p_1 < p_2 \le \infty$*,* $1 \le q_1, q_2 \le \infty$*,* $q_1 \ne q_2$*,* $0 < \lambda < 1$*,*

$$\frac{1}{p} = \frac{\lambda}{p_1} + \frac{1-\lambda}{p_2}, \qquad \frac{1}{q} = \frac{\lambda}{q_1} + \frac{1-\lambda}{q_2}.$$

Suppose that

$$T \in L(L^{p_1}(\mu), L^{q_1}_w(\nu)) \cap L(L^{p_2}(\mu), L^{q_2}_w(\nu)).$$

Then $T \in L(L^p_w(\mu), L^q_w(\nu))$*, and*

$$\|T\|_{L(L^p_w(\mu),L^q_w(\nu))} \le c \|T\|^{\lambda}_{L(L^{p_1}(\mu),L^{q_1}_w(\nu))} \|T\|^{1-\lambda}_{L(L^{p_2}(\mu),L^{q_2}_w(\nu))}$$

and, if $p \le q$*, then* $T \in L(L^p(\mu), L^q(\nu))$ *and*

$$\|T\|_{L(L^p(\mu),L^q(\nu))} \le c \|T\|^{\lambda}_{L(L^{p_1}(\mu),L^{q_1}_w(\nu))} \|T\|^{1-\lambda}_{L(L^{p_2}(\mu),L^{q_2}_w(\nu))}$$

with a constant c *depending only on the exponents.*

Proof of proposition 7.2 . Let $f \in L^p$ and $Tg : L^q \to L^r$ be the convolution with g. We interpolate the estimate with $p_1 = 1$ and $p_2 = p'$ and $q_1 = q$ and $q_2 = \infty$ to get the estimate in weak spaces

$$\|f * g\|_{L^r_w} \le \|g\|_{L^q_w} \|f\|_{L^p}.$$

Now we fix g and consider $T : f \to f * g$, and get

$$\|f * g\|_{L^r} \le c \|f\|_{L^p} \|g\|_{L^q_w}$$

by the second part of the Lemma. ☐

It is useful to generalize and sharpen the Markinciewiecz interpolation estimates before proving them.

Definition 7.4 (Lorentz spaces)**.** *Let* (A,μ) *be a measure space and* $1 \le p, q \le \infty$*. We define*

$$\|f\|_{L^{p,q}(\mu)} = \left(q \int_0^\infty \left(\mu(\{x : |f(x)| > t)^{1/p} t \right)^q \frac{dt}{t} \right)^{1/q}$$

with the obvious modification for $q = \infty$*. We denote by* $L^{pq}(\mu)$ *the set of all measurable functions* f *for which* $\|f\|_{L^{pq}(\mu)} < \infty$*.*

Properties:

(1) Since

$$\{x : |f(x) + g(x)| > t\} \subset \{x : |f(x)| > t/2\} \cup \{x : |g(x)| > t/2\}$$

it follows that

$$\mu(\{x : |f(x) + g(x)| > t\}) \le \mu(\{x : |f(x)| > t/2\}) + \mu(\{x : |g(x)| > t/2\})$$

and hence

$$\|f + g\|_{L^{pq}} \le c \left(\|f\|_{L^{pq}} + \|g\|_{L^{pq}} \right).$$

(2) For $q_1 \le q_2$

$$\|f\|_{L^{pq_2}} \le c \|f\|_{L^{pq_1}}.$$

We begin the proof with

$$\mu(\{|f| \ge t\}) t^q = q \int_0^t \mu(\{|f| \ge t\}) s^{q-1} ds \le q \int_0^t \mu(\{|f| \ge s\}) s^{q-1} ds \le \|f\|^q_{L^{pq}}.$$

Now, if $q_1 < q_2$,

$$q_2 \int_0^\infty [\mu(\{|f| \ge t\})^{1/p} t]^{q_2} \frac{dt}{t} \le \frac{q_2}{q_1} \|f\|_{L^{p,\infty}}^{q_2-q_1} \|f\|_{L^{p,q_1}}^{q_1} \le \frac{q_2}{q_1} \|f\|_{L^{p,q_1}}^{q_2}.$$

(3) If $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{p'} = \frac{1}{q} + \frac{1}{q'} = 1$ there exists $c > 0$ such that

$$\left| \int f g d\mu \right| \le c \|f\|_{L^{pq}} \|g\|_{L^{p'q'}}.$$

For the proof we define $f^* : (0,\infty) \to \mathbb{R}^+$ to be the unique function with

$$m^1(\{\tau : f^*(\tau) > t\}) = \mu(\{x : f(x) > t\})$$

for all $t > 0$. Then, using Fubini several times (with the Lebesgue measure $\mu = m^d$ for definiteness, but the argument holds for general measures)

$$\begin{aligned} \int |fg| dm^d =& m^{d+2}(\{(x,s,t) \in \mathbb{R}^d \times \mathbb{R} \times \mathbb{R} : 0 < s < |f(x)|, 0 < t < |g(x)|\}) \\ =& \int_{\mathbb{R}^+ \times \mathbb{R}^+} m^d(\{x : |f(x)| > s\} \cap \{x : |g(x)| > t\}) ds dt \\ \le& \int_{\mathbb{R}^+ \times \mathbb{R}^+} \min\{m^d(\{|f(x)| > s\}), m^d(\{|g(x)| > t\})\} ds dt \\ =& \int_{\mathbb{R}^+ \times \mathbb{R}^+} m^1(\{|f^*(x)| > s\} \cap \{|^* g(y)| > t\}) ds dt \\ =& \int_0^\infty f^*(\tau) g^*(\tau) d\tau \end{aligned}$$

which we use below,

$$\begin{aligned} \int f g d\mu \le& \int_0^\infty f^*(t) g^*(t) dt \\ =& \int_0^\infty (t^{1/p} f^*)(t^{1/p'} g^*(t)) dt/t \\ \le& \left(\int_0^\infty t^{(q/p)-1} (f^*)^q dt \right)^{1/q} \left(\int_0^\infty t^{(q'/p')-1} (g^*)^{q'} dt \right)^{1/q'} \end{aligned}.$$

The last inequality is an application of Hölder's inequality. The proof of the third part is completed by the equality

$$\frac{q}{p} \int_0^\infty t^{(q/p)-1} (f^*(t))^q dt = q \int_0^\infty (\mu(|f(x)| > s))^{q/p} s^{q-1} ds. \tag{7.1}$$

in one dimensional calculus. We observe that

$$s \to m^1(\{\tau : f^*(\tau) > s\})$$

is the inverse of f^*. Both functions are monotonically decreasing.

Let f and f^{-1} be inverse non-negative monotonically decreasing functions, and g and h non-negative monotonically increasing functions with antiderivatives G and H with

$$H(t) G \circ f(t) \to 0$$

as $t \to \infty$ and $t \to 0$. Then by an integration by parts and one substitution

$$\int_0^\infty hG \circ f dt == -\int_0^\infty Hg \circ f f' dt = \int_0^\infty H \circ f^{-1}(s) g(s) ds.$$

This specializes to (7.1). Moreover, checking the inequalities shows that

$$\|f\|_{L^{pq}} \le c \sup\{\int fg d\mu : \|g\|_{L^{p'q'}} \le 1\}.$$

(4) This pairing defines a duality isomorphism if $1 < p < \infty$ and $1 \le q < \infty$. In particular all spaces L^{pq} with $1 < p$ are Banach spaces.

$$L^{p'q'} \ni g \to (f \to \int fg d\mu) \in (L^{pq})^*$$

To prove it we choose B to be a measurable set of positive finite measure. There exists $\tilde{p} > p$ so that $L^{\tilde{p}}(B) \subset L^{pq}$. If l is a bounded linear functional on L^{pq} then it defines a bounded linear functional on $L^{\tilde{p}}$ which is represented by a function $g \in L^{\tilde{p}'}(\mu)$. The previous step gives a bound for $\|g\chi_B\|_{L^{p'q'}}$ in terms of l.

We order the measurable subsets of A by inclusion up to sets of measure zero. This defines a partial order on the subsets on which the duality statement holds. Every chain has an upper bound, the union of the chain. By the lemma of Zorn there is a maximal element. The procedure above allows to show that the maximal set is necessarily the full space.

In particular duality allows to define an equivalent norm on $L^{pq}(\mu)$ for $1 \le p \le \infty$ and $1 \le q \le \infty$. Completeness of dual spaces is obvious. Completeness of $L^{p1}(\mu)$ is left as an exercise.

Lemma 7.5. *Suppose that* $1 \le p_1, p_2, q_1, q_2 \le \infty$,

$$T \in L(L^{p_1 1}(\mu), L^{q_1 \infty}(\nu)) \cap L(L^{p_2 1}(\mu), L^{q_2 \infty}(\nu)),$$

$p_1 \ne p_2$, $q_1 \ne q_2$, $0 < \lambda < 1$ *and*

$$\frac{1}{p} = \frac{1-\lambda}{p_1} + \frac{\lambda}{p_2}, \frac{1}{q} = \frac{1-\lambda}{q_1} + \frac{\lambda}{q_2}$$

and $1 \le r \le \infty$.

Then the operator can be continuously extended to $T \in L(L^{pr}(\mu), L^{qr}(\nu))$. *Moreover*

$$\|T\|_{L(L^{pr}(\mu), L^{qr}(\nu))} \le c \|T\|^{\lambda}_{L(L^{p_1}(\mu), L^{q_1}_w(\nu))} \|T\|^{1-\lambda}_{L(L^{p_2}(\mu), L^{q_2}_w(\nu))}.$$

Proof. An easy calculation shows

$$\frac{1 - \frac{p}{p_2}}{1 - \frac{p}{p_1}} = \frac{1-\lambda}{\lambda} \tag{7.2}$$

This will be useful later on. Let $t > 0$ and

$$f_t(x) = \begin{cases} f(x) & \text{if } |f(x)| \le t \\ tf(x)/|f(x)| & \text{if } |f(x)| > t \end{cases}$$

and $f^t = f - f_t$. Then

$$f = f_t + f^t$$

and, if $p_1 < p < p_2$, which we assume in the sequel,

$$\|f^t\|_{L^{p_1}} \le (p - p_1)^{1/p_1} t^{1 - \frac{p}{p_1}} \|f\|^{\frac{p}{p_1}}_{L^p_w}$$

and

$$\|f_t\|_{L^{p_2}} \le (p_2-p)^{1/p_2} t^{1-\frac{p}{p_2}} \|f\|_{L^p_w}^{\frac{p}{p_2}}$$

with obvious modifications if $p_2 = \infty$.

Moreover, by the triangle inequality,

$$\{|Tf| > t\} \subset \{Tf^s > t/2\} \cup \{Tf_s > t/2\}.$$

Let

$$a_1 = \|T\|_{L(L^{p_1}, L^{q_1}_w)} \qquad a_2 = \|T\|_{L(L^{p_2}, L^{q_2}_w)}$$

and

$$s = t^{\dfrac{q_2-q_1}{q_2(1-\frac{p}{p_2}) - q_1(1-\frac{p}{p_1})}} a_1^{\frac{(1-\lambda)q/q_1-1}{1-p/p_1}} a_2^{\frac{\lambda q/q_2-1}{1-p/p_2}}.$$

Step 1. The bound in weak L^p space. We want to prove

$$\lambda \nu(\{|Tf(x)| > t\})^{1/q} \le c a_1^{1-\lambda} a_2^{\lambda}$$

for $\|f\|_{L^p_w} = 1$ with c depending only on the exponents. Then

$$\begin{aligned}
\lambda^q \mu(\{|Tf| > t\}) \le\, & c\left(t^{q-q_1}\|Tf^s\|_{L^{q_1}_w}^{q_1} + t^{q-q_2}\|Tf_s\|_{L^{q_2}_w}^{q_2}\right)\\
\le & c\left(t^{q-q_1}a_1^{q_1}\|f^s\|_{L^{p_1}}^{q_1} + t^{q-q_2}a_2^{q_2}\|f_s\|_{L^{p_2}}^{q_2}\right)\\
= & c\left(t^{q-q_1}s^{q_1-q_1p/p_1}\|f\|_{L^p_w}^{pq_1/p_1} + t^{q-q_2}s^{q_2-q_2p/p_2}\|f\|_{L^p_w}^{pq_2/p_2}\right)\\
= & c\left(t^{q-q_1-\frac{q_1(q_2-q_1)}{q_2\frac{1-\lambda}{\lambda}+q_1}} + t^{q-q_2-\frac{q_2(q_1-q_2)}{q_1\frac{\lambda}{1-\lambda}+q_2}}\right)a_1^{q(1-\lambda)}a_2^{q\lambda}\\
= & c\left(t^{q_1[q/q_1-1-(q/q_1-q/q_2)\lambda]} + t^{q_2[q/q_2-1-(q/q_2-q/q_1)(1-\lambda)]}\right)a_1^{q(1-\lambda)}a_2^{q\lambda}\\
= & c a_1^{q(1-\lambda)}a_2^{q\lambda}.
\end{aligned}$$

This completes the proof of the weak type estimate.

Step 2: The endpoints $L(L^{p1}, L^{q1})$ and $L(L^{p\infty}, L^{q\infty}$. We assume that $1 < p_1, p_2, q_1, q_2 < \infty$ which can be achieved by the first step.

By duality, with constant changing from line to line

$$\begin{aligned}
\|Tf\|_{L^{qr}} \le & c\sup\{\int (Tf)g d\nu : \|g\|_{L^{q'r'}}\\
= & c\sup\{\int fT^*g d\nu : \|g\|_{L^{q'r'}} \le 1\}\\
= & c\|f\|_{L^{pq}}\|T^*\|_{L(L^{q',r'}(\nu), L^{p',q'}(\mu))}
\end{aligned}$$

and hence, for $1 < p < \infty$,

$$\|T\|_{L(L^{pr}, L^{qr})} \le c\|T^*\|_{L(L^{q'r'}, L^{p'r'})}.$$

We apply this with $L^{p_1 1} \to L^{q_1\infty}$ to see that

$$\|T^*\|_{L(L^{q_i' 1}, L^{p_i'\infty})} \le c\|T\|_{L(L^{p_i 1}, L^{q_i\infty})}$$

for $i = 1, 2$. From Step 1

$$\|T^*\|_{L(L^{q'\infty}, L^{p'\infty})}$$

satisfies the desired bounds. Duality again gives the statement for $r = 1$.

Step 3: Interpolation in L^p.

Suppose that $T \in L(L^1(\mu), L^1(\nu)) \cap L(L^\infty(\mu), L^\infty(\mu))$ with norm $\le \frac{1}{2}$. Then

$$\|Tf\|_{L^p(\nu)} \le \left(\frac{p}{p-1}\right)^{1/p} \|f\|_{L^p(\mu)}$$

We begin the proof with the observation

$$\{|Tf| > t\} \subset \{Tf_t > t/2\} \cup \{Tf^t > t/2\}.$$

The first set is empty by assumption on the norm of T. Hence

$$\begin{aligned} p\int \nu(\{|Tf| > t\})t^{p-1}dt \le & p\int \nu(\{Tf^t > t/2)t^{p-1}dt \\ \le & p\int_0^\infty \|f^t\|_{L^1} t^{p-2}dt \\ = & p\int_0^\infty \int_t^\infty \mu(\{|f| \ge s\})ds t^{p-2}dt \\ = & p\int_0^\infty \int_0^s t^{p-2}dt\mu(\{|f| \ge s\})ds \\ = & \frac{p}{p-1}\|f\|_{L^p}^p \end{aligned}$$

Step 4: Conclusion

We have proven the bounds for $\|T\|_{L(L^{p,\infty},L^{q,\infty})}$ and $\|T\|_{L(L^{p,1},L^{q,1})}$

Let

$$f_t(x) = \begin{cases} f(x) & \text{if } \mu\{y : |f(y)| > |f(x)|)^{1/p}|f(x)|\} \le t \\ 0 & \text{otherwise} \end{cases}$$

and $f^t = f - f_t$. We assume that the bounds for T are $1/2$ as above. Since

$$\|f_t\|_{L^{p,\infty}} \le t$$

we have

$$\{|Tf(x)| \ge t\} \subset \{|Tf^t(x)| \ge t/2\}$$

Let

$$g^t(s) = \mu(\{|f^t| > s\})^{1/p}s \le \mu(\{|f| > s\})^{1/p}s$$

We proceed as in Step 3.

□

7.1. **Complex interpolation: The theorem of Riesz-Thorin.** The Riesz-Thorin interpolation theorem states the following.

Theorem 7.6. *Let $1 \le p_1, p_2, q_1, q_2 \le \infty$. Let T_λ, $0 \le \operatorname{Re}\lambda \le 1$ be an operator from $L^1 \cap L^\infty \to L^1 + L^\infty$. Suppose that*

$$\lambda \to \int T_\lambda f g$$

is continuous in $0 \le \operatorname{Re}\lambda \le 1$, holomorphic inside the strip, for all $f \in L^1 \cap L^\infty$ and $g \in L^1 \cap L^\infty$. Suppose that

$$\sup_{\operatorname{Re}\lambda=0} \|T_\lambda\|_{L(L^{p_0},L^{q_0})} = C_0$$

and

$$\sup_{\operatorname{Re}\lambda=1} \|T_\lambda\|_{L(L^{p_1},L^{q_1})} = C_1.$$

Then

$$\|T_\lambda\|_{L(L^p,L^q)} \le C_0^{1-\operatorname{Re}\lambda} C_1^{\operatorname{Re}\lambda}$$

if

$$\frac{1-\operatorname{Re}\lambda}{p_0} + \frac{\operatorname{Re}\lambda}{p_1} = \frac{1}{p} \qquad \frac{1-\operatorname{Re}\lambda}{q_0} + \frac{\operatorname{Re}\lambda}{q_1} = \frac{1}{q}$$

The proof relies on the three lines theorem in complex analysis:

Lemma 7.7 (Three lines theorem)**.** *Suppose that v is a bounded holomorphic function on the strip $C = \{z = x + iy : 0 < x < 1\}$ and that it is continuous on the closure. Then*

$$|v(x)| \le (\sup_y |v(iy)|)^{1-x} (\sup_y |v(1+iy)|)^x.$$

Proof. By the maximum principle of harmonic functions any harmonic function on a bounded open set, which is continuous on the closure, assumes the maximum of the modulus at the boundary. This is true for

$$u_\varepsilon(x,y) = e^{\varepsilon(x+iy)^2} u(x,y)$$

on $C \cap \overline{B_R(0)}$ for every R. This function tends to 0 as $y \to \infty$ hence

$$|u_\varepsilon(x+iy)| \le \max\{\sup_y |u(iy)|)^{1-x}, \sup_y |u(1+iy)|)^x\}$$

and $\varepsilon \to 0$ gives the result. □

Proof of Theorem 7.6. Let $f \in L^1(\mu) \cap L^\infty(\mu)$ and $g \in L^1(\nu) \cap L^\infty(\nu)$. Then, by assumption

$$v(\lambda) = \int T_\lambda f g d\nu$$

is a bounded analytic function. By the three lines theorem 7.7 we have

$$|v(\lambda)| \le \sup_t \max\{|v(it)|, |v(1+it)|\}$$

and

$$\left|\int T_{it} f g d\nu\right| \le \|T_{it} f\|_{L^{q_0}} \|g\|_{L^{q_0'}} \le C_0 \|f\|_{L^{p_0}} \|g\|_{L^{q_0'}}.$$

Similarly

$$\left|\int T_{1+it} f g d\nu\right| \le \|T_{1+it} f\|_{L^{q_1}} \|g\|_{L^{q_1'}} \le C_0 \|f\|_{L^{p_1}} \|g\|_{L^{q_1'}},$$

thus

$$\left|\int (T_\lambda f) g d\mu\right| \le \max\{C_0, C_1\} \left(\|f\|_{L^{p_0}} \|g\|_{L^{q_0'}} + \|f\|_{L^{p_1}} \|g\|_{L^{q_1'}} \right)$$

and we could derive that

$$\|T\|_{L(L^{p_0} \cap L^{p_1}, L^{q_0} + L^{q_1})} \le \max\{C_0, C_1\}$$

but we will avoid this step. Let $f \in L^p$ and $g \in L^{q'}$. We want to prove

$$\left|\int g T_\lambda f\right| \le \|f\|_{L^p} \|g\|_{L^{q'}} \sup_y \|T_{iy}\|^{1-\lambda}_{L(L^{p_1},L^{q_1})} \sup_y \|T_{1+iy}\|^{\lambda}_{L(L^{p_2},L^{q_2})}. \tag{7.3}$$

for $f \in L^p$ and $g \in L^{q'}$. The theorem follows then by an duality argument. Moreover it suffices to consider a dense set of functions, which are measurable, bounded, and for which there is $\varepsilon > 0$ such that either the functions vanish at a point, or else are at least of size ε. Moreover we may restrict to f and g with $\|f\|_{L^p} = \|g\|_{L^{q'}} = 1$.

Let

$$f_z(x) = \frac{f(x)}{|f(x)|} |f(x)|^{(1-z)\frac{p}{p_0} + z\frac{p}{p_1}},$$

$$g_z(x) = \frac{g(x)}{|g(x)|} |g(x)|^{(1-z)\frac{q'}{q'_0} + z\frac{q'}{q'_1}}$$

and

$$v(z) = \int g_z(y) T_z f_z(y) d\nu(y).$$

This is a bounded holomorphic map from the strip to $L^1 \cap L^\infty$ with values in $\mathbb{C}$. We claim that it is continuous on the closure of the strip at an arbitrary point λ. We write

$$v(z) - v(\lambda) = \int g_\lambda (T_z - T_\lambda) f_\lambda d\nu + \int (g_z - g_\lambda) T_z f_\lambda + g_z T_z (f_z - f_\lambda) d\nu.$$

The first term tends to zero as $z \to \lambda$ by assumption. Then

$$g_z - g_\lambda \to 0 \quad \text{ and } g_z - f_\lambda \to 0 \quad \text{ as } z \to \lambda$$

in $L^1 \cap L^\infty$. Continuity follows by the uniform bound above.

We turn to complex differentiability at an arbitrary point λ in the interior. Indeed

$$\frac{v(z) - v(\lambda)}{z - \lambda} = \frac{\int g_\lambda (T_z - T_\lambda) f_\lambda d\nu}{z - \lambda} + \int \frac{g_z - g_\lambda}{z - \lambda} T_z f_\lambda d\nu + \int g_z T_z \frac{f_z - f_\lambda}{z - \lambda} d\nu$$

The first term converges to a complex number by assumption. Moreover

$$\frac{g_z - g_\lambda}{z - \lambda}$$

converges to a function g'_λ in $L^1 \cap L^\infty$ as $z \to \lambda$. Let $\tilde{g}$ be the difference between the difference quotient and g'_λ. Then

$$\int \frac{g_z - g_\lambda}{z - \lambda} T_z f_\lambda d\nu = \int g' T_\lambda f_\lambda d\nu + \int \tilde{g} T_z f_\lambda d\nu + \int g'(T_z - T_\lambda) f_\lambda d\nu.$$

The second term tends to zero since $\tilde{g}$ tends to zero in $L^1 \cap L^\infty$ and the third one by the continuity assumption as $z \to \lambda$. Similarly we deal with the last term.

We turn to the behavior at the boundary.

$$|v(it)| = \int T_{it} f_{it} g_{it} d\nu \le \|T_{it}\|_{L(L^{p_0}, L^{p_1})} \|f_{it}\|_{L^{p_0}} \|g_{it}\|_{L^{q_0}}$$

and

$$\|f\|_{L^{p_0}} = \|f\|_{L^p}^{p_0/p} = 1 = \|g_{it}\|_{L^{q'_0}} = \|g\|_{L^{q'}}^{q'_0/q'}.$$

We apply the three lines theorem 7.7. Thus

$$|v(z)| \le \sup_y \|T_{iy}\|_{L(L^{p_1}, L^{q_1})}^{1-x} \sup_y \|T_{1+iy}\|_{L(L^{p_2}; L^{q_2})}^{x}.$$

We evaluate it at $z = \lambda$, which gives inequality (7.3). $\square$

8. Appendix B: Bessel functions

8.0.1. *Bessel functions.* The Bessel functions are confluent hypergeometric functions. They are solutions to confluent hypergeometric differential equations. Here is a very brief introduction. Consider a complex differential equation

$$x^{(n)} = \sum_{j=0}^{n-1} a_j(z) z^{(j)}$$

with initial data

$$x^{(j)}(z_0) = y_j$$

for $j = 1 \dots n-1$ and given complex numbers z_0 and y_j. If the coefficients are holomorphic in a neighborhood of z_0 then there is a unique solution which is holomorphic in z and the y_j.

Consider the scalar equation

$$\dot{x} = \frac{\lambda}{z - z_0} x$$

The space of solutions is at most 1 dimensional. Formally a solution is given by $x = (z - z_0)^\lambda$, which, unless z is an integer, is only defined in a set of the type $\mathbb{C} \backslash (-\infty, z_0]$ called slit domain. Similarly, if

$$\dot{x} = (\frac{\lambda}{z - z_0} + \phi(z))x$$

with a holomorphic function ϕ near z_0 there is a unique solution of the type

$$(z - z_0)^\lambda \left[1 + \sum_{k=1}^{\infty} a_k (z - z_0)^k \right]$$

again defined in the slit domain as above unless λ is an integer. The number λ is called characteristic number. It is not hard to see that there is a unique such solution, and the power series can be iteratively defined. The point z_0 is called a regular singular point. A point is called irregular singular point if the Laurent series of the coefficients contains terms below $(z - z_0)^{-1}$

We call ∞ regular point resp. regular singular resp. irregular singular point for

$$\dot{x} = a(z)x$$

if, when we express z in terms of z^{-1}, 0 is a regular resp. regular singular or irregular singular point of

$$\dot{x} = -z^{-2} a(z^{-1}) x$$

We use the same notation for systems of equations. The eigenvalues of A in

$$\dot{x} = \frac{1}{z - z_0} A(z)x + f(z)x$$

are called characteristic values. They play a very similar role as for scalar equations. Multiple characteristic values and/or resonances (a resonance denotes the situation when eigen values of A are linearly dependent over the integers) may lead to logarithmic terms.

We are interested in second order scalar equations

$$a(z)\ddot{x} + b(z)\dot{x} + c(z)x = 0$$

with meromorphic functions a, b and c. We may rewrite them as a 2×2 system, which we use to define the notion of a regular, regular singular, and irregular singular term. The point z_0 is regular if $b(z)/a(z)$ and $c(z)/a(z)$ have a holomorphic extension near z_0. It is a regular singular point if the Laurent expansion of $b(z)/a(z)$ begins with $c_0 z^{-1}$ and the one of $c(z)/a(z)$ begins with $c_1 z^{-2} + c_2 z^{-1}$. The characteristic numbers can be calculated in terms of the Laurent series. If they are independent over the integers then there are unique solutions of the type

$$z^\lambda \sum a_j z^j$$

where γ is one of the characteristic numbers.

Of particular importance is the case when there are only regular singular points. In that case there are exactly three of them, and applying a Moebius transform we may choose them to be 0, 1 and ∞. Moreover, multiplying by $z^\lambda(1-z)^\mu$ we can ensure that one of the characteristic values at 0 and 1 is 0. These are the hypergeometric differential equations

$$z(1-z)\frac{d^2}{dz^2}w + [c-(a+b+1)z]\frac{dw}{dz} - abw = 0$$

The characteristic numbers at $z=0$ are 0 and $1-c$, the ones at $z=1$ are 0 and $c-a-b$, and the ones at infinity are $-a$ and $-b$.

The regular solution near 0 with value 1 at zero is the hypergeometric function

$$ {}_2F_1(a,b;c;z).$$

The Bessel differential equation is

$$z^2\ddot{w} + w\dot{w} + (z^2-\nu^2)w = 0.$$

It has a regular singularity at $z=0$ with indices $\pm\nu$, and an irregular singularity at $z=\infty$. The Bessel function of the first kind is

$$J_\nu = (\frac{1}{2}z)^\nu \sum_{k=0}^\infty (-1)^k \frac{\left(\frac{1}{4}z^2\right)^k}{k!\Gamma(\nu+k+1)}$$

We have, unless ν is negative integer,

$$J_\nu(z) - (\frac{1}{2}z)^\nu/\Gamma(\nu+1) = O(|z|^{\operatorname{Re}\nu+1}) \text{ near } 0$$

$$J_\nu(z) - \sqrt{\frac{2}{\pi z}}\cos(z - \frac{1}{2}\nu\pi - \frac{1}{4}\pi) + e^{|\operatorname{Im} z|}o(1)$$

for $z \to \infty$ and $\nu \in \mathbb{R}$.

There are integral representation for $\nu > -\frac{1}{2}$,

$$\begin{aligned} J_\nu(z) =& \frac{2(\frac{1}{2}z)^\nu}{\pi^{1/2}\Gamma(\nu+\frac{1}{2})}\int_0^1 (1-t^2)^{\nu-\frac{1}{2}}\cos(zt)dt \\ =& \frac{(\frac{1}{2}z)^\nu}{\pi^{1/2}\Gamma(\nu+\frac{1}{2})}\int_0^\pi \cos(z\cos(\theta))\sin(\theta)^{2\nu}dt \end{aligned}$$

and if the absolute value of the argument of z is bounded by $\frac{1}{2}\pi$, the Schläfli-Sommerfeld formula

$$J_\nu(z) = \frac{1}{2\pi i}\int_{-\infty-\pi i}^{\infty+\pi i} e^{z\sinh t - \nu t}dt$$

$$
\begin{aligned}
J_\nu(z) =& \frac{2(\frac12 z)^\nu}{\pi^{1/2}\Gamma(\nu+\frac12)} \int_0^1 (1-t^2)^{\nu-\frac12}\cos(zt)dt \\
=& \frac{(\frac12 z)^\nu}{2\pi i}\int_{-\infty}^{0+} \exp(t-\frac{z^2}{4t}) t^{-\nu+1} dt
\end{aligned}
$$

The Bessel functions satisfy

$$
\left(\frac{d}{xdx}\right)^m (x^\nu J_\nu) = x^{\nu-m} J_{\nu-m}.
$$

See [24] for more information. We want to evaluate (with the Hausdorff measure of dimension s denoted by $\mathcal{H}^s$

$$
\begin{aligned}
H(\xi) = \int_{\mathbb{S}^{d-1}} e^{ix\xi} d\mathcal{H}^{d-1} =& \int_0^\pi \mathcal{H}^{d-2}(\mathbb{S}^{d-2}) \sin^{d-2}(\theta) e^{i|x|\cos(\theta)} d\theta \\
=& J_{\frac{d-2}{2}}(|x|)\pi^{\frac{d-1}{2}}(\frac12|x|)^{-\frac{d-2}{2}}
\end{aligned}
$$

which is seen by a substitution reducing the one dimensional integral to the formula of Schläfli-Sommerfeld. This function is real and radial. We choose a real function $\eta \in C^\infty(\mathbb{R})$, supported in $[-\frac12,\infty)$, with $\eta(x)+\eta(-x)=1$. Then $H(\xi)$ is the real part of

$$
\int_{-\pi}^{\pi} \mathcal{H}^{d-2}(\mathbb{S}^{d-2})\eta(\cos\theta)\sin^{d-2}(\theta) e^{i|x|\cos(\theta)} d\theta
$$

An application of stationary phase gives

Lemma 8.1. *For all $H(r)$ is the real part of a function $e^{-ir}\phi$ which satisfies*

$$
\Big|\big(\frac{d}{dr}\big)^k \phi\Big| \le c_k r^{-\frac{d-1}{2}-k}.
$$

Proof: Exercise.

9. Appendic C: The Fourier transform

Let f be an integrable complex valued function. We define its Fourier transform by

$$
\hat f(\xi) = \frac{1}{(2\pi)^{d/2}} \int e^{-ix\cdot\xi} f(x) dm^d(x). \tag{9.1}
$$

9.0.2. *The Fourier transform in L^1.* Properties are
1) The Fourier transform of an integrable function is a bounded continuous function which converges to 0 as $|\xi|\to\infty$. It satisfies

$$
\|\hat f\|_{sup} \le (2\pi)^{-d/2}\|f\|_{L^1}.
$$

The estimate is obvious, as is the continuity if f is compactly support. The limit as $x\to\infty$ follows by an integration by parts if the integrand is compactly supported and differentiable. Those functions are dense, and we obtain continuity and vanishing of the limit for compactly supported functions. The limit

$$
\lim_{R\to\infty}\int_{B_R(0)} e^{-ix\cdot\xi} f(x) dm^d(x)
$$

is uniform, and hence the Fourier transform is continuous and converges to 0 as $\xi\to\infty$.

2) For all η and y in $\mathbb{R}^d$

$$\hat{f}(\xi+\eta) = \widehat{e^{-i\eta\cdot x} f} \tag{9.2}$$

and

$$\widehat{f(.+y)} = e^{iy\xi}\hat{f}(\xi). \tag{9.3}$$

This follows by an simple calculation.
3) For $f, g \in L^1(\mathbb{R})$

$$\widehat{f * g}(\xi) = (2\pi)^{n/2}\hat{f}(\xi)\hat{g}(\xi).$$

which follows by application of Fubini's theorem:

$$\begin{aligned}
\frac{1}{(2\pi)^{d/2}} \int e^{-ix\xi} \int f(y)g(x-y)dm^d(y)dm^d(\xi)& \\
&= \int\int e^{-iy\xi} f(y) e^{-i(x-y)\xi} g(x-y)dm^d(y)dm^d(x)\\
&= \int\int e^{-iy\xi} f(y) e^{-iz\xi} g(z)dm^d(z)dm^d(y)\\
&=(2\pi)^{d/2}\hat{f}(\xi)\hat{g}(\xi)
\end{aligned}$$

4) For f and $g \in L^1$

$$\int f\hat{g}dm^d(x) = \int \hat{f} g dm^d \tag{9.4}$$

This is seen by applying Fubini to

$$\int\int e^{-iy\xi} f(y) e^{-i(x-y)\xi} g(y)dm^d(y)dm^d(x).$$

5)

$$\widehat{e^{-\frac{1}{2}|x|^2}} = e^{-\frac{1}{2}|\xi|^2}$$

We calculate as above

$$(2\pi)^{-d/2}\int e^{-ix\xi-\frac{1}{2}|x|^2}dm^d(x) = (2\pi)^{-d/2}\int e^{-i(x-i\eta)\xi-\frac{1}{2}(x-i\eta)^2}dm^d(x)$$

for $\eta \in \mathbb{R}^n$. We set $\eta = \xi$ and get

$$(2\pi)^{-d/2}e^{-\frac{|\xi|^2}{2}}\int e^{-\frac{1}{2}|x|^2}dx = e^{-\frac{|\xi|^2}{2}}.$$

9.0.3. *The Fourier transform of Schwartz functions.*

Definition 9.1. *We say $f \in C^\infty(\mathbb{R}^d)$ is a Schwartz function and write $f \in \mathcal{S}(\mathbb{R}^d)$ if for all multi-indices α and β*

$$\|x^\alpha \partial^\beta f\|_{sup} < \infty$$

We say $f_j \to f$ in $\mathcal{S}$ if for all multi-inidices

$$x^\alpha\partial^\beta f_j \to x^\alpha \partial^\beta f$$

uniformly.

We collect elementary properties.
1) $f \in \mathcal{S}$ if and only if $x^\alpha \partial^\beta f \in \mathcal{S}$ for all α and β.
2) $f \in \mathcal{S}$ implies f integrable.
3) $f \in \mathcal{S}$ and $g \in C^\infty$ with bounded derivatives implies $fg \in \mathcal{S}$.
4) $f \in \mathcal{S}$ and A an invertible $d \times d$ matrix implies $f \circ A \in \mathcal{S}$
5) $f \in \mathcal{S}$ and $x_0 \in \mathbb{R}^d$ implies $f(. + x_0) \in \mathcal{S}$.
6) We say that a distribution T has compact support, if there exists a ball $B_R(0)$ such that for all functions f in $C_0^\infty(\mathbb{R}^d)$ with support disjoint from $B_R(0)$ $Tf = 0$. We can easily extend such distributions to Schwartz functions (exercise).

We define the convolution with a Schwartz function by

$$T * f(x) = T(f(x - .))$$

This is well defined and $T * f$ is a Schwartz function whenever f is a Schwartz function. To see this we recall that by the definition of a distribution there exist $C > 0$ and $N > 0$ such that (since f has compact support)

$$|T(f)| \le c_N \|f\|_{C^N}.$$

Taking difference quotients shows that $x \to T * f(x)$ is differentiable and

$$\partial_i T * f = T * \partial_i f.$$

Recursively we see that $Tf \in C^\infty$. Morever

$$\|f(x - .)\|_{C^N(B_R(0))} \le c_M (1 + |x|)^{-M}$$

for Schwartz functions, and hence $T * f$ is a Schwartz function.

7) $f, g \in \mathcal{S}$ implies $f * g \in \mathcal{S}$ and

$$\widehat{f * g} = (2\pi)^{d/2} \hat{f} \hat{g} \tag{9.5}$$

If $f \in \mathcal{S}$ and S is a distribution with compact support then

$$S * f(x) := S(f(x - .)) \in \mathcal{S}.$$

8) All the operations above are continuous.

Theorem 9.2. *If $f \in \mathcal{S}$ then $\hat{f} \in \mathcal{S}$, and vice verse,*

$$\widehat{x_j f} = -i \partial_{\xi_j} \hat{f}$$

$$\widehat{-i\partial_{x_j} f} = \xi_j \hat{f}$$

and the Fourier inversion formula

$$f(x) = (2\pi)^{-d/2} \int e^{ix\xi} \hat{f}(\xi) dm^d(\xi)$$

and the Plancherel formula

$$\int \hat{f} \overline{\hat{g}} dm^d(\xi) = \int f \overline{g} dm^d(x)$$

hold. If A is a real invertible $d \times d$ matrix then

$$\widehat{f \circ A}(\xi) = (\det |A|)^{-1} \hat{f}(A^{-T} \xi).$$

Proof. According to property (1)

$$x^\alpha \partial^\beta f \in \mathcal{S}$$

and hence $x^\alpha \partial^\beta f$ is integrable. With the first calculation

$$\mathcal{F}(x^\alpha(-i\beta^\beta f)) = -i\partial^\alpha \xi^\beta \hat{f}$$

which is bounded by the second observation. Thus $\hat{f} \in \mathcal{S}$. We calculate

$$\mathcal{F}((2\pi)^{-d/2}\tau^{d/2}e^{-\frac{\tau}{2}x^2} * f) = e^{-\frac{1}{2\tau}\xi^2}\hat{f}(\xi)$$

and, with $\tau \to \infty$

$$f(0) = (2\pi)^{-d/2}\int \hat{f} d\xi.$$

Together with the formulas (9.3) we obtain the inversion formula

$$f(x) = (2\pi)^{-d/2}\int e^{ix\xi}\hat{f}(\xi)d\xi.$$

The Plancherel formula follows by (9.4). The last formula follows from

$$(2\pi)^{-d/2}\int e^{-ix\cdot\xi}f(Ax)dm^d(x) = (2\pi)^{-d/2}|\det A|^{-1}\int e^{-i(A^{-1}y)\cdot\xi}f(y)dm^d(y).$$

□

9.0.4. *Tempered distributions.*

Definition 9.3. *A tempered distribution T is a linear map*

$$T : \mathcal{S} \to \mathbb{C}$$

which is continuous, i.e. $f_j \to f \in \mathcal{S}$ implies

$$Tf_j \to Tf$$

We denote the set of tempered distributions as $\mathcal{S}^*$. We say T_j converges to T if $T_j f \to Tf$ for all $f \in \mathcal{S}$.

We list properties.

1) We call T bounded if there exists N such that

$$|Tf| \le C \sup_{|\alpha|+|\beta|\le N} \sup_x |x^\alpha \partial_x^\beta f|.$$

The linear $T : \mathcal{S} \to \mathbb{C}$ is bounded if and only if it is continuous.

2) Distributions with compact support are tempered distributions.

3) Let $T \in \mathcal{S}^*$ and $\phi \in C^\infty$ with bounded derivatives. We define

$$\phi T(f) = T(\phi f).$$

4) The derivative of a tempered distribution $\partial_j T$ is defined by

$$\partial_j T(f) = -T(\partial_j f)$$

5) Let $T \in \mathcal{S}^*$ and $\phi \in \mathcal{S}$. Then

$$T * \phi \in C^\infty(\mathbb{R}^d),$$

where we define $T * \phi$ as for distributions with compact support.

6) Let $T \in \mathcal{S}^*$ and S be a distribution with compact support. We define

$$S * T(f) = T(\tilde{S} * f)$$

where $\tilde{S}(f) = S(\tilde{f})$, $\tilde{f}(x) = f(-x)$. Then $S * T \in \mathcal{S}^*$.
7) Let $g \in L^p$ for one $1 \le p \le \infty$. It defines a unique distribution by

$$T_g(f) = \int gf dm^d.$$

The operations commute with this representation,

$$T_{\phi g} = \phi T_g$$

and we identify L^p with its image via the embedding.
8) We define the Fourier transform $\hat{T} \in \mathcal{S}^*$ by

$$\hat{T}(f) = T(\hat{f})$$

The inverse Fourier transform is defined similarly.

This is compatible with the interpretation for functions.
9)

$$\hat{\delta}_0 = (2\pi)^{d/2}$$

and

$$\hat{1} = (2\pi)^{d/2}\delta_0$$

The Euler relation

$$x \cdot \nabla f = mf$$

holds for every homogeneous function of degree m. We want to define homogeneous distributions.

Definition 9.4. *A tempered distribution is called homogeneous of degree $m \in \mathbb{C}$ if*

$$T(\phi) = \lambda^{-d-m} T(\phi(\lambda * .)).$$

Let $\operatorname{Re} m > -d$. Then $|x|^m$ is tempered distribution. Its Fourier transform is again a tempered distribution of homogeneity $-d-m$.

This can be seen from the Euler relation

$$x \cdot \nabla f = mf$$

for every homogeneous function of degree m.

Lemma 9.5. *Let $0 < \operatorname{Re} m < d$. The following identity holds*

$$\mathcal{F}\Big(\frac{1}{2^{m/2}\Gamma(m/2)}|x|^{m-d}\Big) = \frac{1}{2^{(d-m)/2}\Gamma(\frac{d-m}{2})}|x|^{-m}.$$

Proof. We claim that the Fourier transform of a homogeneous distribution of degree $m \in \mathbb{C}$ is a homogeneous distribution of degree $-d-m$. We denote by T_λ the distribution

$$T_\lambda(f) = \lambda^{-d} T f(\lambda.)$$

Then

$$\hat{T}_\lambda(f) = T_\lambda(\hat{f}) = T(\lambda^{-d}\hat{f}(\lambda.)) = T(\widehat{f(./\lambda)}) = \lambda^{-m-d}T(\hat{f}) = \lambda^{-m-d}\hat{T}(f).$$

Let f be a homogeneous function of degree m such that T_f is a homogeneous distribution. Let O be an orthogonal matrix with $f \circ O = f$. Then

$$\widehat{T}_f \circ O^T = \widehat{T}_f$$

where the term on the left hand side is defined by the action on Schwartz functions. In particular the Fourier transform of $|x|^{-m}$ is radial in the sense that it is invariant under the action of orthogonal matrices. This is equivalent to

$$Tf = T\left(\mathcal{H}^{d-1}(\mathbb{S}^{d-1})^{-1}\int_{\mathbb{S}^{d-1}} f(|x|\sigma)\,\mathcal{H}^{d-1}(\sigma)\right)$$

(a rigorous justification requires either an approximation, or a symmetrization argument). We denote the symmetrization operator by S.

Let T be a radial homogeneous distribution of degree m. We claim that We fix a non-negative function h with integral 1 with compact support and observe that

$$\begin{aligned} T(f) =& T\Big(\int_0^\infty \lambda^{-d-m-1}(Sf)(\lambda x)h(\ln(\lambda))d\lambda\Big) \\ =& T(\int_0^\infty \lambda^{-d-m-1}Sf(\lambda)|x|^{d+m}h(\ln(\lambda/|x|))d\lambda \\ =& T(|x|^{d+m}h(-\ln|x|)\int_0^\infty \lambda^{-d-m-1}Sf(\lambda)d\lambda \\ =& T\left(|x|^{d+m}h(-\ln|x|)\right)\int |y|^m f(y)dm^d(y) \end{aligned}$$

for all $f \in \mathcal{S}$ with 0 not in the support. This extends to Schwartz functions if $m > -d$.

By the consideration above

$$\widehat{|x|^{-m}} = c(n,m)|x|^{m-d}$$

and we have to determine $c(n,m)$. The Gaussian is its own Fourier transform. Let $T = |x|^m$ and denote by $\hat{T}$ its Fourier transform. Then, by the definition

$$T(e^{-\frac{|x|^2}{2}}) = \hat{T}(e^{-\frac{|\xi|^2}{2}})$$

We calculate

$$\begin{aligned} \int |x|^m e^{-\frac{|x|^2}{2}}dm^d(x) =& d\,m^d(B_1(0))\int_0^\infty e^{-r^2/2}r^{d-1+m}dr \\ =& d\,m^d(B_1(0))2^{-\frac{d+m}{2}-1}\int_0^\infty t^{\frac{d+m}{2}-1}e^{-t}dt \\ =& d\,m^d(B_1(0))2^{-\frac{d+m}{2}-1}\Gamma(\frac{d+m}{2}). \end{aligned}$$

Comparison with the calculation for $|x|^{-d-m}$ gives the formula. □

The formula extends to all $m \in \mathbb{C}\backslash(-\infty,-d] \cup [0,\infty)$. This requires however a proper definition of the homogeneous tempered distribution.

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