SUPERATOMIC BOOLEAN ALGEBRAS
CONSTRUCTED FROM MORASSES

PETER KOEPEKE AND JUAN CARLOS MARTÍNEZ

Abstract. By using the notion of a simplified \( (\kappa, 1) \)-morass, we construct \( \kappa \)-thin-tall, \( \kappa \)-thin-thick and, in a forcing extension, \( \kappa \)-very thin-thick superatomic Boolean algebras for every infinite regular cardinal \( \kappa \).

1. Introduction. A superatomic Boolean algebra (abbreviated sBa) is a Boolean algebra in which every subalgebra is atomic. It is known that a Boolean algebra \( B \) is superatomic iff its Stone space \( S(B) \) is scattered. A useful tool in the study of scattered spaces is the notion of the Cantor-Bendixson derivative. For every ordinal \( \alpha \), the \( \alpha \)-derivative of \( S(B) \) is defined by induction on \( \alpha \) as follows: \( S(B)^0 = S(B) \); if \( \alpha = \beta + 1 \), \( S(B)^\alpha \) is the set of accumulation points of \( S(B)^\beta \); and if \( \alpha \) is limit, \( S(B)^\alpha = \bigcap \{ S(B)^\beta : \beta < \alpha \} \). Then, \( S(B) \) is scattered iff \( S(B)^\alpha = 0 \) for some \( \alpha \). This Cantor-Bendixson process can be transferred to the Boolean algebra \( B \), obtaining in this way a sequence of ideals \( I_\alpha \), which are defined by induction on \( \alpha \) as follows: \( I_0 = \{ 0 \} \); if \( \alpha = \beta + 1 \), \( I_\alpha \) is the ideal generated by \( I_\beta \cup \{ b \in B : b/I_\beta \text{ is an atom in } B/I_\beta \} \); and if \( \alpha \) is limit, \( I_\alpha = \bigcup \{ I_\beta : \beta < \alpha \} \). Then \( B \) is a sBa iff \( B = I_\alpha \) for some \( \alpha \).

The height of a sBa \( B \), \( ht(B) \), is the least ordinal \( \alpha \) such that \( B/I_\alpha \) is finite (which means \( B = I_{\alpha+1} \)). For every ordinal \( \alpha < ht(B) \) let \( wd_\alpha(B) \) be the cardinality of the set of atoms in \( B/I_\alpha \). Then, for every infinite cardinal \( \kappa \):

(a) If \( \alpha \) is an ordinal with \( \alpha \neq 0 \), we say that \( B \) is a \( (\kappa, \alpha) \)-sBa, if \( ht(B) = \alpha \) and \( wd_\beta(B) \leq \kappa \) for \( \beta < \alpha \).

(b) \( B \) is \( \kappa \)-thin-tall, if \( B \) is a \( (\kappa, \kappa^+) \)-sBa.

(c) \( B \) is \( \kappa \)-thin-very tall, if \( B \) is a \( (\kappa, \kappa^{++}) \)-sBa.

(d) If \( \alpha \) is an ordinal with \( \alpha \neq 0 \) and \( \lambda \) is an infinite cardinal, we say that \( B \) is a \( (\kappa, \alpha, \lambda) \)-sBa, if \( ht(B) = \alpha + 1 \), \( wd_\beta(B) \leq \kappa \) for \( \beta < \alpha \) and \( wd_\alpha(B) = \lambda \).

(e) \( B \) is \( \kappa \)-thin-thick, if \( B \) is a \( (\kappa, \kappa, \kappa^+) \)-sBa.

(f) \( B \) is \( \kappa \)-very thin-thick, if \( B \) is a \( (\kappa, \kappa^+, \kappa^{++}) \)-sBa.

The question of the existence of an \( \omega \)-thin-tall sBa was posed by Telgársky, and answered in the affirmative by Rajagopalan and, independently, by Juhász and Weiss. In [4] it was even proved that there exists an \( (\omega, \alpha) \)-sBa for each \( \alpha < \omega_2 \), which in ZFC is the best we can hope for. By means of a forcing argument, Baumgartner and Shelah proved in [1] that the existence of an \( \omega \)-thin-very tall sBa is consistent with ZFC. A modification of this argument was introduced in [6] in

Received July 15, 1993; revised March 6, 1994.
The second-named author acknowledges financial support from DGICYT Grant PB91-0279.
order to show that if we consider a specific infinite regular cardinal $\kappa$, then the existence of a $\kappa$-thin-tall $sBa$ is consistent with ZFC.

By using the well-known fact that there is an almost disjoint family of $2^\omega$ subsets of $\omega$, it is very easy to construct an $\omega$-thin-thick $sBa$. Constructions of $\omega_1$-thin-thick $sBAs$ were carried out by Weese in [12] from the existence of a Canadian tree, and by Roitman in [7] using $MA + \neg CH$. A forcing construction of an $\omega$-very thin-thick $sBa$ was carried out by Roitman in [8].

The reader may find in [9] a survey of known results on superatomic Boolean algebras, as well as examples and basic facts.

In this paper we present some constructions of superatomic Boolean algebras, by using the notion of a simplified $(\kappa, 1)$-morass introduced by Velleman in [11]. This notion is a simplified version of Jensen's notion of a $(\kappa, 1)$-morass, which has been used to carry out a variety of constructions in set theory. Suppose that $\kappa$ is an infinite regular cardinal. Let $(\theta_\alpha : \alpha < \kappa)$ be a sequence of ordinals such that $0 < \theta_\alpha < \kappa$ for $\alpha < \kappa$ and $\theta_\kappa = \kappa^+$. Suppose that for every $\alpha, \beta$ with $\alpha < \beta \leq \kappa$, $\mathfrak{F}_{\alpha, \beta}$ is a nonempty set of strictly order-preserving functions from $\theta_\alpha$ into $\theta_\beta$. Then we say that $\langle \langle \theta_\alpha : \alpha < \kappa \rangle, \langle \mathfrak{F}_{\alpha, \beta} : \alpha < \beta \leq \kappa \rangle \rangle$ is a simplified $(\kappa, 1)$-morass, if the following conditions hold:

(M1) If $\alpha < \beta < \kappa$, $|\mathfrak{F}_{\alpha, \beta}| < \kappa$.
(M2) If $\alpha < \beta < \gamma < \kappa$, $\mathfrak{F}_{\alpha, \gamma} = \{g \circ f : f \in \mathfrak{F}_{\alpha, \beta}, g \in \mathfrak{F}_{\beta, \gamma}\}$.
(M3) If $\alpha < \kappa$, $\mathfrak{F}_{\alpha, \alpha+1}$ is a pair $\{f, g\}$ such that for some $\beta < \theta_\alpha$, $f \restriction \beta = g \restriction \beta$ and $f''(\beta) \subseteq g(\beta)$. We call such a pair of functions an amalgamation pair, and $\beta$ the split point of the pair.
(M4) If $\alpha \leq \kappa$ is a limit ordinal, $\beta_1, \beta_2 < \alpha$, $f_1 \in \mathfrak{F}_{\beta_1, \alpha}$ and $f_2 \in \mathfrak{F}_{\beta_2, \alpha}$, then there exists a $\gamma$ with $\beta_1, \beta_2 < \gamma < \alpha$, $f_1' \in \mathfrak{F}_{\beta_1, \gamma}$, $f_2' \in \mathfrak{F}_{\beta_2, \gamma}$, and $g \in \mathfrak{F}_{\gamma, \alpha}$, such that $f_1 = g \circ f_1'$ and $f_2 = g \circ f_2'$.
(M5) $\bigcup\{f''(\alpha) : \alpha < \kappa, f \in \mathfrak{F}_{\alpha, \kappa}\} = \kappa^+$.

We state the following three theorems for future reference.

**Theorem 1.1.** There exists a simplified $(\omega, 1)$-morass.

**Theorem 1.2** ($V = L$). For every infinite regular cardinal $\kappa$, there exists a simplified $(\kappa, 1)$-morass.

**Theorem 1.3.** Let $\kappa$ be an uncountable regular cardinal. If $\kappa^+$ is not inaccessible in $L$, then there exists a simplified $(\kappa, 1)$-morass.

The proof of 1.1 is given in [10]. The reader interested in the proofs of 1.2 and 1.3 should consult [2].

This paper is organized as follows. In Section 2 we show that for every infinite regular cardinal $\kappa$, the existence of a simplified $(\kappa, 1)$-morass implies the existence of a $\kappa$-thin-tall $sBa$. In Section 3 we show that for every uncountable regular cardinal $\kappa$, the existence of a simplified $(\kappa, 1)$-morass implies the existence of a $(\kappa, \alpha, \kappa^+)$-sBa for each $\alpha$ with $\kappa \leq \alpha < \kappa^+$ and $cf(\alpha) = \kappa$. In Section 4, by using the notion of a simplified $(\kappa, 1)$-morass, we generalize to larger cardinals the forcing construction presented by Roitman in [8], and prove in this way that if we consider a specific infinite regular cardinal $\kappa$, then the existence of a $\kappa$-very thin-thick $sBa$ is consistent with ZFC.
In this paper, the usual set-theoretic conventions are used. Undefined terms may be found in [3] or [5].

2. Construction of thin-tall sBAs. Our aim here is to prove the following result:

**Theorem 2.1.** For every infinite regular cardinal $\kappa$, the existence of a simplified $(\kappa, 1)$-morass implies the existence of a $\kappa$-thin-tall sBa.

Then, from 1.1, 1.2 and 2.1 we immediately obtain the following two results:

**Theorem 2.2.** There exists an $\omega$-thin-tall sBa.

**Theorem 2.3 ($V = L$).** For every infinite regular cardinal $\kappa$, there exists a $\kappa$-thin-tall sBa.

In order to prove 2.1 we need some preparations. If $C$ is a well-ordered set, we denote its order type by $\text{ot}(C)$. First we codify a method for construction of thin-tall superatomic Boolean algebras. Suppose that $\kappa$ is an infinite cardinal and $\eta$ is an ordinal with $\eta \neq 0$. Let $(X, \leq)$ be a partial order. If $s, t$ are elements of $X$ such that there is a $u \in X$ with $u \leq s$ and $u \leq t$, we say that $s$ and $t$ are compatible in $(X, \leq)$. Then suppose that there is a set $C$ of ordinals with $\text{ot}(C) = \eta$ and $X = \bigcup\{X^\alpha : \alpha \in C\}$ in such a way that for each $\alpha \in C$, $X^\alpha$ is a set of the form $\{\alpha\} \times Z$ where $Z$ is a nonempty set of ordinals of cardinality $\leq \kappa$. Then we say that $(X, \leq)$ is a $(\kappa, \eta)$-partial order, if the following two conditions hold:

1. If $s \in X^\alpha, t \in X^\beta$ and $s < t$, then $\alpha < \beta$.
2. Every pair $s, t$ of compatible elements in $(X, \leq)$ has an infimum, that is, there is a $v \in X$ with $v \leq s$ and $v \leq t$ and such that for every $u \in X$, $u \leq s$ and $u \leq t$ implies $u \leq v$.

Suppose that $(X, \leq)$ is a $(\kappa, \eta)$-partial order and $s, t$ are compatible elements in $(X, \leq)$ with $s \in X^\alpha, t \in X^\beta$ and $\alpha \leq \beta$. Let us denote by $i\{s, t\}$ the infimum of $s, t$ in $(X, \leq)$. Then it is easy to see that if $s \leq t$ we have $i\{s, t\} = s$, and if $s \leq t$ then $i\{s, t\} \in X^\tau$ for some $\tau < \alpha$.

If $(X, \leq)$ is a $(\kappa, \kappa^+)$-partial order with $X = \bigcup\{X^\alpha : \alpha < \kappa^+\}$ and such that whenever $\alpha < \beta < \kappa^+$ and $t \in X^\beta$ we have that $\{s \in X^\alpha : s < t\}$ is infinite, we say that $(X, \leq)$ is an admissible $(\kappa, \kappa^+)$-partial order.

**Lemma 2.4.** Let $\kappa$ be an infinite cardinal. If there is an admissible $(\kappa, \kappa^+)$-partial order, there exists a $\kappa$-thin-tall sBa.

**Proof.** Let $(X, \leq)$ be an admissible $(\kappa, \kappa^+)$-partial order with $X = \bigcup\{X^\alpha : \alpha < \kappa^+\}$. For each $t \in X$ we consider the cone of $t$, $C(t) = \{s \in X : s \leq t\}$. We define

$$B(X) = \{C(t) - (C(t_1) \cup \cdots \cup C(t_n)) : n < \omega, t, t_1, \ldots, t_n \in X, t_1, \ldots, t_n < t\}.$$ 

It is not difficult to verify that $B(X)$ is a clopen base for a topology $\sigma_\leq$ on $X$ in such a way that $(X, \sigma_\leq)$ is locally compact, Hausdorff and scattered. Let $\overline{X}$ be the one-point compactification of $X$. Note that for every ordinal $\alpha < \kappa^+$, $\overline{X}^\alpha - \overline{X}^{\alpha+1} = X^\alpha$. Thus $\text{Clop}(\overline{X})$, the algebra of clopen subsets of $\overline{X}$, is a $\kappa$-thin-tall sBa. -
It is also possible to give an algebraic proof of 2.4, by using an argument similar to that for [1, Lemma 7.3].

Let $\kappa$ be an infinite regular cardinal. Suppose that there exists a simplified $(\kappa, 1)$-morass. Then, our purpose is to construct an admissible $(\kappa, \kappa^{+})$-partial order. We put $T = \kappa^{+} \times \kappa$, and for each $\alpha < \kappa^{+}$, $T_{\alpha} = \{\alpha\} \times \kappa$. We define $\tilde{P} = \tilde{P}(T)$ as the set of all $p = (x_{p}, \leq_{p})$ such that $x_{p} \in [T]^{<\kappa}$ and $p$ is a $(\kappa, \eta)$-partial order for $\eta = ot(\{\alpha : x_{p} \cap T_{\alpha} \neq 0\})$.

If $p \in \tilde{P}$ and $s, t$ is a pair of compatible elements in $p$, we denote the infimum of $s, t$ in $p$ by $i_{p}\{s, t\}$.

Now we define the partial order relation $\leq$ on $\tilde{P}$ as follows. If $p, q \in \tilde{P}$ we put

\[
 p \leq q \text{ if and only if }
\]

(a) $x_{p} \supseteq x_{q}$;
(b) $\leq_{p} \mid x_{q} = \leq_{q}$; and
(c) if $s, t \in x_{q}$ and $s, t$ are compatible in $p$, then $s, t$ are compatible in $q$ and $i_{p}\{s, t\} = i_{q}\{s, t\}$.

For every $p \in \tilde{P}$ we set $\gamma_{p} = \{\alpha : x_{p} \cap T_{\alpha} \neq 0\}$.

Now we define $P$ as the set of all $p = (x_{p}, \leq_{p}) \in \tilde{P}$ such that for every $\alpha, \beta \in \gamma_{p}$ with $\alpha < \beta$ and $t \in x_{p} \cap T_{\beta}$, we have $\{s \in T_{\alpha} : s <_{p} t\} \neq 0$. And, for every $\xi < \kappa$, we define $P_{\xi}$ as the set of all $p = (x_{p}, \leq_{p}) \in P$ such that for any $\alpha, \beta \in \gamma_{p}$ with $\alpha < \beta$ and $t \in x_{p} \cap T_{\beta}$ there is a $\zeta$ such that $\xi < \zeta < \kappa$ and $(\alpha, \zeta) <_{p} t$. The following lemma will be needed later.

**Lemma 2.5.** For every $p \in \tilde{P}$ and every ordinal $\xi_{0} < \kappa$, there is an $r \in P_{\xi_{0}}$ such that $r \leq p$ and $\gamma_{r} = \gamma_{p}$.

**Proof.** Consider an ordinal $\xi_{0} < \kappa$. Then, for every $p = (x_{p}, \leq_{p}) \in \tilde{P}$ and every $v \in x_{p}$ we define by transfinite induction on the $\alpha$ such that $v \in T_{\alpha}$ an element $p_{v} \in \tilde{P}$ such that if we put $p_{v} = q = (x_{q}, \leq_{q})$, the following hold:

(i) $q \leq p$ and $\gamma_{q} = \gamma_{p}$.
(ii) If $v \in T_{\beta}$, then for any $\alpha \in \gamma_{q}$ with $\alpha < \beta$, we have $\{s \in T_{\alpha} : s <_{q} v\} \neq 0$.
(iii) If $u \in x_{q} - x_{p}$, then $u <_{q} v$ and $u$ is of the form $(\alpha, \zeta)$ for some $\alpha \in \gamma_{q}$ and $\xi_{0} < \zeta < \kappa$.

Let $\gamma_{p} = \{\alpha_{n} : \eta < \delta\}$ where $\alpha_{n} < \alpha_{n'}$ for $\eta < \eta' < \delta$. We consider $\xi < \delta$ such that $v \in T_{\alpha_{n}}$. If $\xi = 0$ we put $p_{v} = p$. If $\xi > 0$ we choose for every $\mu < \xi$ an element $s_{\mu} = (\alpha_{\mu}, \zeta)$ with $\zeta < \xi < \kappa$ and $s_{\mu} \notin x_{p}$, and then we define $p' = (x', \leq')$ by $x' = x_{p} \cup \{s_{\mu} : \mu < \xi\}$ and $\leq' = \leq_{p} \cup \{(s_{\mu}, t) : \mu < \xi, v \leq_{p} t\}$. Note that if $s, t$ are compatible in $p'$, then $i_{p'}\{s, t\} = i_{p}\{s, t\}$.

Now, for $p \leq q$, we put $p'_{s_{\mu}} = (x_{\mu}, \leq_{p})$. Since $|x_{p}| < \kappa$ and $\kappa$ is regular, if $\mu_{1} \neq \mu_{2}$ we may assume that whenever $u_{1} \in x_{\mu_{1}} - x'$ and $u_{2} \in x_{\mu_{2}} - x'$, we have $u_{1} \neq u_{2}$. Then we define $p_{v} = q = (x_{q}, \leq_{q})$ where $x_{q} = \bigcup\{x_{\mu} : \mu < \xi\}$ and $\leq_{q} = \bigcup\{\leq_{\mu} : \mu < \xi\}$. Note that if $s, t$ are compatible in $q$, we have that $s, t$ are compatible in $p'_{s_{\mu}}$ for some $\mu < \xi$ and then $i_{q}\{s, t\} = i_{p'_{s_{\mu}}}\{s, t\}$.

Then, considering the elements $p_{v}$ for $v \in x_{p}$, we can easily construct an $r \in P_{\xi_{0}}$ with $r \leq p$ and $\gamma_{r} = \gamma_{p}$.
Let us consider a simplified \((\kappa, 1)\)-morass \(\langle \langle \theta_\xi : \xi \leq \kappa \rangle, \mathfrak F_{\mu, \xi} : \mu < \xi \leq \kappa \rangle\). Let 
\(\mu < \xi \leq \kappa, p = (x_p, \leq_p) \in P\) such that \(\gamma_p \subseteq \theta_\mu\) and \(f \in \mathfrak F_{\mu, \xi}\). Then we define 
\(p^f = q = (x_q, \leq_q) \in P\) as follows. We put 
\(\delta = f''(\gamma_p)\) and for each \(\alpha \in \delta, \)
\(x^{(\alpha)} = \{(\alpha, \beta): (f^{-1}(\alpha), \beta) \in x_p\}\). We set 
\(x_q = \bigcup \{x^{(\alpha)} : \alpha \in \delta\}\). Now we define \(\pi_{pq} : x_p \rightarrow x_q\) by \(\pi_{pq}((\alpha, \beta)) = (f(\alpha), \beta)\) for any \((\alpha, \beta) \in x_p\), and we put 
\(\pi_{pq} = \pi_{pq}^{-1}\). If \(s, t \in x_q\), we set \(s \leq q t\) iff \(\pi_{pq}(s) \leq p \pi_{pq}(t)\). Clearly, if \(s, t\) are compatible elements in \(q\), we have that \(\pi_{pq}(s), \pi_{pq}(t)\) are compatible in \(p\) and 
\(i_q\{s, t\} = \pi_{pq}(i_p\{\pi_{pq}(s), \pi_{pq}(t)\})\).

Note that if \(\xi < \mu \leq \xi \leq \kappa, f \in \mathfrak F_{\mu, \xi}, g \in \mathfrak F_{\mu, \xi}, p, q \in P\) with \(\gamma_p, \gamma_q \subseteq \theta_\xi\), then 
\(p^g \circ f = (p^f)^g\), and \(p \leq q\) implies \(p^f \leq q^f\).

Our goal is to define by transfinite induction on \(\xi < \kappa\) a \(p_\xi = (x_\xi, \leq_\xi) \in P_\xi\) such that \(\gamma_{p_\xi} = \theta_\xi\) and for every \(\mu < \xi\) and \(f \in \mathfrak F_{\mu, \xi}\), \(p_\mu \leq p_\xi^f\). Each \(p_\xi\) will be an approximation to the admissible \((\kappa, \kappa^+)\)-partial order we want to construct. In order to define \(p_0\), consider an \(r \in \mathfrak P\) such that \(\gamma_r = \theta_0\) and then apply 2.5. Next suppose that \(\xi = \mu + 1\) and we have constructed \(p_\mu\). To construct \(p_\xi\) we proceed as follows. Let \(\{g, h\}\) be the amalgamation pair of \(\mathfrak F_{\mu+1, \xi}\) and \(r\) the split point of \(\{g, h\}\), i.e., \(g \upharpoonright h = h \upharpoonright r\) and \(g'' \theta_\mu \subseteq h(r)\). Put \(p_\mu^g = p = (x_p, \leq_p)\) and 
\(p_\mu^h = q = (x_q, \leq_q)\). Then we define:

\[
\begin{align*}
x &= \bigcup \{x_p \cap T_\alpha : \alpha < g(r)\} = \bigcup \{x_q \cap T_\alpha : \alpha < h(r)\}, \\
y_1 &= \bigcup \{x_p \cap T_\alpha : \alpha \geq g(r)\}, \\
y_2 &= \bigcup \{x_q \cap T_\alpha : \alpha \geq h(r)\}.
\end{align*}
\]

Now we define an \(r = (x_r, \leq_r) \in \mathfrak P\) such that \(r \leq p\) and \(r \leq q\). We put \(x_r = x \cup y_1 \cup y_2\). If \(s, t \in x\), we set 
\(s \leq_r t\) iff \(s \leq_p t\) or \(s \leq_q t\) or \((s \in y_1, t \in y_2\) and \(\pi_{pp_p}(s) \leq \pi_{pp_q}(t)\)).

Note that \(\leq_r\) is a partial order on \(x_r\). Suppose that \(s, t\) are compatible elements in \(r\). We show that the pair \(s, t\) has an infimum in \(r\). If \(s, t \in x_p\), it is clear that 
\(i_r\{s, t\} = i_p\{s, t\}\). If \(s, t \in x_q\), by the definition of \(\leq_r\), it is not difficult to see that 
\(s, t\) are compatible in \(q\), and then we can prove that \(i_r\{s, t\} = i_q\{s, t\}\). To check this 
point, suppose that \(s, t \in y_2, u \in y_1\) and \(u \leq_r s, t\). Let \(u' = \pi_{pp_p}(u)\), \(s' = \pi_{pp_p}(s)\) and 
\(t' = \pi_{pp_q}(t)\). Let \(v' = i_p\{s', t'\}\). From \(u \leq_r s, t\) we infer that \(u' \leq \mu s', t'\), and thus 
\(u' \leq \mu v'\). Let \(v = \pi_{pp_q}(v')\). Then \(v = i_q\{s, t\}\), and \(u' \leq \mu v'\) implies \(u \leq_r v\). Now, if \(s \in y_1\) and \(t \in y_2\), we have \(i_r\{s, t\} = \pi_{pp_p}(i_p\{\pi_{pp_p}(s), \pi_{pp_q}(t)\})\). To prove 
this fact, let us set \(s' = \pi_{pp_p}(s)\) and \(t' = \pi_{pp_q}(t)\). Since \(s\) and \(t\) are compatible 
in \(r\), it is easy to see that \(s'\) and \(t'\) are compatible in \(p_p\). Let \(v' = i_p\{s', t'\}\) and 
\(v = \pi_{pp_p}(v')\). We have \(v' \leq \mu s', t'\). If \(v \in x\) we infer that \(v \leq_p s\) and \(v \leq_q t\). And 
if \(v \in y_1\), then \(v' \leq \mu t'\) implies that \(v \leq_r t\). Thus, we obtain \(v \leq_r s, t\). Now assume 
that \(u \leq_r s, t\). Consider \(u' = \pi_{pp_p}(u)\). Then we infer that \(u' \leq \mu s', t'\), and thus 
\(u' \leq \mu v'\), and hence \(u \leq_p v\).

Therefore, \(r \in \mathfrak P\) and \(r \leq p, q\). Now consider an \(r' \in \mathfrak P\) such that \(r' \leq r\) and 
\(\gamma_{r'} = \theta_\xi\). Then, by 2.5, we can construct an \(r'' \in \mathfrak P\) such that \(r'' \leq r'\) and \(\gamma_{r''} = \theta_\xi\). We put \(p_\xi = r''\). Clearly, \(p_\xi \leq p_\mu^g, p_\mu^h\). Now consider \(p < \mu\) and \(f \in \mathfrak F_{\mu, \xi}\). Without 
loss of generality we may assume that \(f = g \circ f^*\) for some \(f^* \in \mathfrak F_{\mu, \xi}\). By the 
induction hypothesis \(p_\mu \leq p_\mu^f\), and thus \(p_\xi \leq p_\mu^g \leq (p_\mu^f)^g = p_\mu^g = p_\mu^g \\
= p_\mu^f = p_\mu^f\).
Next assume that $\xi$ is limit and that we have chosen $p_\mu$ for $\mu < \xi$. If $f, g \in \mathcal{F}_{\mu, \xi}$, we write $p^f_\mu = (x^f_\mu, \leq^f_\mu)$. We define $q = (x_q, \leq_q)$ by $x_q = \bigcup \{x^f_\mu : \mu < \xi, f \in \mathcal{F}_{\mu, \xi}\}$ and $\leq_q = \bigcup \{\leq^f_\mu : \mu < \xi, f \in \mathcal{F}_{\mu, \xi}\}$. Note that if $\xi \leq \mu < \xi$, $f \in \mathcal{F}_{\mu, \xi}$, $g \in \mathcal{F}_{\mu, \xi}$ and $s, t$ are compatible in $p^f_\mu$ and also in $p^g_\mu$, then by applying (M4) to $\{f, g\}$ and by using the induction hypothesis, we obtain $i_{p^f_\mu} \{s, t\} = i_{p^g_\mu} \{s, t\}$. Also, if $s, t$ are compatible in $q$, then by using again (M4), we obtain that $s, t$ are compatible in $p^h_\mu$ for some $\mu < \xi$ and $h \in \mathcal{F}_{\mu, \xi}$, and thus $i_q \{s, t\} = i_{p^h_\mu} \{s, t\}$. Now, it is easy to see that $q \in \bar{P}$. Then, proceeding as above, we can find an $r \in P_\xi$ such that $r \leq q$ and $\gamma_r = \theta_\xi$. We put $p_\xi = r$.

Now let us define $X = \bigcup \{x^f_\xi : \xi < \kappa, f \in \mathcal{F}_{\xi, \kappa}\}$ and $\leq = \bigcup \{\leq^f_\xi : \xi < \kappa, f \in \mathcal{F}_{\xi, \kappa}\}$. Note that if $\alpha < \kappa^+$, by (M5), there are $\xi < \kappa$, $\alpha' < \theta_\xi$ and $f \in \mathcal{F}_{\xi, \kappa}$ such that $f(\alpha') = \alpha$. Then since $\gamma_{p_\xi} = \theta_\xi$, we infer that $(\alpha', \xi) \in x^f_\xi$ for some $\xi < \kappa$, and hence $(\alpha, \xi) \in x^f_\xi$, whence $X \cap T_\alpha \neq \emptyset$. Then one can verify that $(X, \leq)$ is a $(\kappa, \kappa^+)$-partial order. In order to prove that $(X, \leq)$ is admissible, consider $\alpha < \beta < \kappa^+$ and $t = (\beta, \tau) \in X$. We show that $\{s \in X \cap T_\alpha : s < t\}$ has cardinality $\kappa$. Let $\xi < \kappa$. Consider $\mu_1 < \kappa$ and $f \in \mathcal{F}_{\mu_1, \kappa}$ such that $t \in x^f_\mu$. Then there is a $\beta' < \theta_{\mu_1}$ such that $f(\beta') = \beta$ and $(\beta', \tau) \in x^f_{\mu_1}$. On the other hand, by using (M5), there are $\mu_2 < \kappa$, $\alpha' < \theta_{\mu_2}$ and $g \in \mathcal{F}_{\mu_2, \kappa}$ such that $g(\alpha') = \alpha$. Now, applying (M2) and (M4), we can find $\mu \geq \max\{\mu_1, \mu_2, \xi\}$, $f' \in \mathcal{F}_{\mu, \kappa}$, $g' \in \mathcal{F}_{\mu, \kappa}$, and $h \in \mathcal{F}_{\mu, \kappa}$ such that $f = h \circ f'$ and $g = h \circ g'$. Let $\beta^* = f'(\beta')$ and $\alpha^* = g'(\alpha')$. Since $\mu \geq \xi$, $\mu \in P_{\mu}$ and $\gamma_{\mu} = \theta_{\mu}$, it follows that $(\alpha^*, \xi) <_{\mu} (\beta^*, \tau)$ for some $\xi$ with $\xi < \kappa$. But this implies $(\alpha, \xi) < (\beta, \tau)$.

Then, using 2.4, we can construct a $\kappa$-thin-tall sBa. This completes the proof of 2.1.

Therefore, applying 1.3, we obtain that if there is no inaccessible cardinal in $L$, then there exists a $\kappa$-thin-tall sBa for every infinite regular cardinal $\kappa$.

In [4], Juhász and Weiss construct in ZFC an $(\omega, \alpha)$-sBa for every ordinal $\alpha$ such that $\omega_1 \leq \alpha < \omega_2$. We do not know whether, under $V = L$, there is a $(\kappa, \alpha)$-sBa for uncountable regular cardinal and $\alpha$ such that $\kappa^+ < \alpha < \kappa^{++}$. The construction carried out in [4] is topological, and it uses the well-known fact that every regular Lindelöf space is paracompact.

3. Construction of thin-thick sBAs. In this section we prove the following result:

**Theorem 3.1.** For every uncountable regular cardinal $\kappa$, the existence of a simplified $(\kappa, 1)$-morass implies the existence of a $(\kappa, \eta, \kappa^+)$-sBa for each ordinal $\eta$ with $\kappa \leq \eta < \kappa^+$ and $cf(\eta) = \kappa$.

Since $\omega$-thin-thick sBAs have little interest, we omit the case $\kappa = \omega$ in 3.1.

It was proved in [1] by Baumgartner that the consistency of ZFC + “there exists an inaccessible cardinal” implies the consistency of ZFC + “there is no $\omega_1$-thin-thick sBa.” Then, by using 1.3 and 3.1, we immediately deduce that these theories are equiconsistent. This fact can also be proved by other means (see [9, Section 5]).

As a corollary of 1.2 and 3.1 we obtain:
Theorem 3.2 ($V = L$). For every uncountable regular cardinal $\kappa$, there exists a $(\kappa, \eta, \kappa^+)$-sBa for each ordinal $\eta$ with $\kappa \leq \eta < \kappa^+$ and $\text{cf}(\eta) = \kappa$.

In order to prove 3.1, we need some previous notions. Suppose that $\kappa$ is an infinite cardinal and $\eta$ is an ordinal such that $\kappa \leq \eta < \kappa^+$. Let $(X, \leq)$ be a partial order with $X = \bigcup\{X(\alpha) : \alpha \leq \eta\}$ where $X(\alpha)$ is an infinite subset of $\{\alpha\} \times \kappa$ for each $\alpha < \eta$, and $X(\eta) = \{\eta\} \times \kappa^+$. Then we say that $(X, \leq)$ is an admissible $(\kappa, \eta, \kappa^+)$-partial order, if the following conditions hold.

1. If $s \in X(\alpha), t \in X(\beta)$ and $s < t$, then $\alpha < \beta$.
2. For every pair $s, t$ of compatible elements in $(X, \leq)$ there is a $v \in X$ with $v \leq s$ and $v \leq t$ and such that for every $u \in X$, $u \leq s$ and $u \leq t$ implies $u \leq v$.
3. If $\alpha < \beta \leq \eta$ and $t \in X(\beta)$, then $\{s \in X(\alpha) : s < t\}$ is infinite.

The proof of the following result is similar to the one given in 2.4.

Lemma 3.3. Let $\kappa$ be an infinite cardinal and $\eta$ an ordinal with $\kappa \leq \eta < \kappa^+$. If there is an admissible $(\kappa, \eta, \kappa^+)$-partial order, there exists a $(\kappa, \eta, \kappa^+)$-sBa.

Let $\kappa$ be an uncountable regular cardinal and $\eta$ an ordinal with $\kappa \leq \eta < \kappa^+$ and $\text{cf}(\eta) = \kappa$. Suppose that there exists a simplified $(\kappa, 1)$-morass $\langle \{\theta_\xi : \xi \leq \kappa\}, \langle \mathcal{F}_{\mu, \xi} : \mu < \xi \leq \kappa \rangle\rangle$. Our goal is to construct an admissible $(\kappa, \eta, \kappa^+)$-partial order.

We put $T_\alpha = \{\alpha\} \times \kappa$ for each $\alpha < \eta$, $T_\eta = \{\eta\} \times \kappa^+$ and $T = \bigcup\{\alpha < \eta\}$. We define $\tilde{P} = \tilde{P}(T)$ and the binary transitive relation $\leq$ on $\tilde{P}$ as in §2. If $p = (x_\mu, \leq_p) \in \tilde{P}$, we put $\gamma_p = \{\alpha : x_\mu \cap T_\alpha \neq 0\}$ and $\delta_\mu = \{\xi : (\eta, \xi) \in x_\mu\}$.

Now we define $P$ as the set of all $p = (x_\mu, \leq_p) \in \tilde{P}$ such that whenever $\alpha_\beta \in \gamma_p, \alpha_\beta < \beta$ and $t \in x_\mu \cap T_\beta$, we have $\{s \in T_\alpha : s \leq_p t\}$ is infinite. If $p = (x_\mu, \leq_p) \in P$, we set $x_\mu^0 = \{s \in x_\mu : s \in T_\alpha$ for some $\alpha < \eta\}$ and $x_\mu^1 = x_\mu \cap T_\eta$.

Consider $\mu < \xi \leq \eta$, $p = (x_\mu, \leq_p) \in P$ such that $\delta_\mu \subseteq \theta_\mu$ and $f \in \mathcal{F}_{\mu, \xi}$. Then we define $p^f = q = (x_\mu, \leq_q) \in P$ as follows. We put $x_q = x_\mu^0 \cup x_\mu^1$, where $x_\mu^0 = x_\mu^0$ and $x_\mu^1 = \{(\eta, f(\xi)) : (\eta, \xi) \in x_\mu\}$.

We define $\pi_{pq} : x_\mu \rightarrow x_q$ by $\pi_{pq}(s) = s$ if $s \in x_\mu^0$, and $\pi_{pq}(x_\mu^1) = (\eta, f(\xi))$ if $(\eta, \xi) \in x_\mu^1$, and we set $\pi_{pq} = \pi_{pq}^{-1}$. If $s, t \in x_q$ we put $s \leq q$ iff $\pi_{pq}(s) \leq_p \pi_{pq}(t)$. Obviously, if $s$ and $t$ are compatible in $q$ and $s \neq t$, we have $i_q(s, t) = i_p(\pi_{pq}(s), \pi_{pq}(t))$.

We fix a 1-1 function $b$ from $\kappa$ onto $\eta$. Our purpose is to construct by induction on $\xi < \kappa$ a $p_\xi = (x_\xi, \leq_\xi) \in P$ such that $\delta_{p_\xi} = \theta_\xi$, $x_\xi \cap T_{\beta(\xi)} \neq 0$ and in such a way that for every $\mu < \xi$ and every $f \in \mathcal{F}_{\mu, \xi}$, we have $p_\mu \subseteq p_\xi$. The definition of $p_0$ is clear. Suppose that $\xi = \mu + 1$ and we have constructed $p_\mu$. To construct $p_{\xi}$ let us consider the amalgamation pair $(g, h)$ of $\mathcal{F}_{\mu, \xi}$, and let us write $p_\mu^g = p = (x_\mu, \leq_p)$ and $p_\mu^h = q = (x_q, \leq_q)$. Since $|x_\mu| < \kappa$ and $\text{cf}(\eta) = \kappa$, we may choose a $\beta < \eta$ such that for every $s \in x_\mu^0$, if $s \in T_\alpha$ then $\alpha < \beta$. Let $\tau$ be the split point of $(g, h)$. Let $\xi$ be the order type of the set $\{\rho \in \theta_\mu : \rho \geq \tau\}$. Then we define:

$x = x_\mu^0$,
$y = \{(\beta, \rho) : \rho < \xi\}$,
$y_1 = x_\mu^1$.

The proof of the following result is similar to the one given in 2.4.
We consider \( y, y_1, y_2 \) as well-ordered sets. So, \( ot(y) = ot(y_1) = ot(y_2) \). Let \( g_1 : y \rightarrow y_1 \) and \( g_2 : y \rightarrow y_2 \) be the corresponding order-preserving bijections. Then we define an \( r = (x_r, \leq_r) \in \tilde{P} \) such that \( r \leq p, q \). We put \( x_r = x_p \cup x_q \cup y \).

If \( s, t \in x_r \), then \( s < r, t \) iff one of the following conditions holds:

(a) \( s <_p t \),
(b) \( s <_q t \),
(c) \( s \in x, t \in y \) and \( s <_p g_1(t) \),
(d) \( s \in y \) and \( t = g_1(s) \),
(e) \( s \in y \) and \( t = g_2(s) \).

Now suppose that \( s, t \) are compatible elements in \( r \). If \( s, t \in x_p \), we have \( i_r \{ s, t \} = i_p \{ s, t \} \). If \( s \in x \) and \( t \in y \), then \( i_r \{ s, t \} = i_p \{ s, g_1(t) \} \). If \( s, t \in y \) and \( s \neq t \), then \( i_r \{ s, t \} = i_q \{ g_1(s), g_1(t) \} \).

It follows that \( r \in \tilde{P} \), and by means of an argument similar to the one given in §2, one can find an \( r' = (x_{r'}, \leq_{r'}) \in \tilde{P} \) such that \( r' \leq r, \delta_{r'} = \delta_\xi \) and \( x_{r'} \cap T_{b(\xi)} \neq 0 \). We put \( p_\xi = r' \). Note that if \( \xi < \zeta \) and \( f \in \mathcal{G}_{\xi, \zeta} \), we have \( p_\xi \leq p_{\xi}^f \).

If \( \xi \) is limit, the construction of \( p_\xi \) is similar to the one given in §2.

Now we define \( X = \bigcup \{ x_\xi : \xi < \kappa, f \in \mathcal{G}_{\xi, \kappa} \} \) and \( < = \bigcup \{ \xi < \xi : \xi < \kappa, f \in \mathcal{G}_{\xi, \kappa} \} \). Then \( (X, <) \) is an admissible \( (\kappa, \eta, \kappa^+) \)-partial order. Therefore, by using 3.3, we can find a \( (\kappa, \eta, \kappa^+)-sBa \). This completes the proof of 3.1.

4. Construction of very thin-thick sBas. In this section our aim is to show the following result:

**Theorem 4.1.** Let \( \kappa \) be an infinite cardinal with \( \kappa = \kappa^{<\kappa} \) and such that there exists a simplified \( (\kappa^+, 1) \)-morass. Then there is a cardinal-preserving notion of forcing that forces the existence of a \( \kappa \)-very thin-thick sBa.

Since simplified morasses exist in the constructible universe, if we consider a specific infinite regular cardinal \( \kappa \), we obtain from 4.1 that the existence of a \( \kappa \)-very thin-thick sBa is consistent with the axioms of set theory.

To prove 4.1, we use the generalization to larger cardinals of the "new \( \Delta \)-property," which was employed by Roitman in [8] to construct by forcing an \( \omega \)-very thin-thick sBa. Let \( \kappa \) be an infinite cardinal. We assign to every function \( F : [\kappa^{++}]^2 \rightarrow \kappa^+ \) the function \( F^* : [\kappa^{++}]^{<\kappa} \rightarrow \kappa^+ \) defined as follows. If \( \{ a, b \} \in [\kappa^{++}]^{<\kappa} \), we put \( F^\star \{ a, b \} = \inf \{ F \{ \alpha, \beta \} : \alpha \in a, \beta \in b \} \). Then we say that \( F : [\kappa^{++}]^2 \rightarrow \kappa^+ \) is a new \( \Delta_\kappa \)-function if for every \( A \subseteq [\kappa^{++}]^{<\kappa} \) with \( |A| = \kappa^+ \) and \( A \) consisting of pairwise disjoint sets, we have that the set \( F^* \upharpoonright |A|^2 \) is unbounded in \( \kappa^+ \). We write \( NDP_\kappa \) to denote the existence of a new \( \Delta_\kappa \)-function.

**Lemma 4.2.** Let \( \kappa \) be an infinite cardinal such that \( \kappa = \kappa^{<\kappa} \). Assume that there exists a simplified \( (\kappa^+, 1) \)-morass. Then, \( NDP_\kappa \) holds.
PROOF. Suppose \( \langle \theta_\alpha : \alpha \leq \kappa^+ \rangle, \langle \delta_{\alpha \beta} : \alpha < \beta \leq \kappa^+ \rangle \) is a simplified \((\kappa^+, 1)\)-morass. We shall use without explicit mention the well-known fact proved in [11, Lemma 3.2] that if \( \alpha < \beta \leq \kappa^+, \tau_1, \tau_2 < \theta_\alpha, f_1, f_2 \in \delta_{\alpha \beta} \) and \( f_1(\tau_1) = f_2(\tau_2) \), then \( \tau_1 = \tau_2 \).

Note that, by using (M5) and (M4), we can infer that for every \( \mu, \xi < \kappa^+ \) there are \( \mu < \kappa^+ \) and \( f \in \delta_{\alpha \kappa^+} \) such that \( \mu, \xi \in \text{ran}(f) \). Then we define \( F : [\kappa^+]^2 \rightarrow \kappa^+ \) as follows. If \( \{ \mu, \xi \} \in [\kappa^+]^2 \), we put \( F(\mu, \xi) = \) the least ordinal \( \alpha < \kappa^+ \) such that there is an \( f \in \delta_{\alpha \kappa^+} \) with \( \mu, \xi \in \text{ran}(f) \). Our goal is to show that \( F \) is a new \( \Delta_\kappa \)-function. Let us consider \( A \subseteq [\kappa^+]^{< \kappa} \) with \( |A| = \kappa^+ \) and \( A \) consisting of pairwise disjoint sets. Let \( \delta < \kappa^+ \). If there is \( \{ a, b \} \in [A]^2 \) with \( F^* \{ a, b \} \geq \delta \), we are done. Suppose instead that for every \( \{ a, b \} \in [A]^2 \), \( F^* \{ a, b \} < \delta \). We first control the possible preimages of the sets \( a \in A \) on the \( \delta \)-th level. For every \( \mu < \kappa^+ \) and \( \gamma < \kappa^+ \) we define \( \mu_\gamma \) as follows: if there are \( \mu < \theta_\gamma \) and \( f \in \delta_{\gamma \kappa^+} \) such that \( f(\mu) = \mu \) we put \( \mu_\gamma = \mu \), and we set \( \mu_\gamma = 0 \) otherwise. Now, for every \( a \in A \) we define \( a_\delta = \{ \mu_\delta : \mu \in a \} \). Since \( a_\delta \in [\theta_\delta]^{< \kappa} \), \( \theta_\delta \leq \kappa \) and \( \kappa^{< \kappa} = \kappa \), there are at most \( \kappa \) many possible \( a_\delta \)'s for \( a \in A \). Then since \( |A| = \kappa^+ \), by thinning out if necessary, we can assume that there is an \( X \in [\theta_\delta]^{< \kappa} \) such that \( a_\delta = X \) for every \( a \in A \). Clearly, we can also assume that for every \( a \in A \), \( \text{ot}(a) \equiv \tau \) for some fixed \( \tau < \kappa \). Let \( \{ a_\xi : \xi < \kappa^+ \} \) be an enumeration of \( A \) without repetitions. Then we define \( G : [\kappa^+]^2 \rightarrow X^2 \times \tau^2 \) as follows. If \( \{ \xi, \eta \} \in [\kappa^+]^2 \) and \( \xi < \eta \), we choose elements \( a_\xi \in a_\zeta \) and \( a_\eta \in a_\zeta \) such that \( F(a_\xi, a_\eta) < \delta \) and a function \( f \in \delta_{\beta \kappa^+} \) such that \( a_\xi, a_\eta \in \text{ran}(f) \), and then we put \( G(\xi, \eta) = (f^{-1}(a_\xi), f^{-1}(a_\eta), \text{ot}(a_\xi \cap a_\eta), \text{ot}(a_\eta \cap a_\eta)) \).

Recall that if \( \kappa_1, \kappa_2, \kappa_3 \) are cardinals, then the notation \( \kappa_1 \rightarrow (\kappa_2)^n \) means that for any \( f : [\kappa_1]^2 \rightarrow \kappa_2 \) there is a set \( A \subseteq \kappa_1 \) and such that \( A \) is homogeneous for \( f \), i.e., \( A \subseteq \kappa_1 \) and \( f \) is constant on \( [A]^2 \). Then we can show that there is an infinite homogeneous set \( H \) for \( G \). If \( \kappa = \omega \), it is enough to apply Ramsey’s property \( \omega \rightarrow (\omega)^2(n < \omega) \). And if \( \kappa > \omega \), we apply the Erdős-Rado property \( (2^\omega)^+ \rightarrow (\lambda^+)^2 \) for an infinite cardinal \( \lambda \) such that \( X \times X \leq \lambda < \kappa \).

Now consider \( \xi_1, \xi_2, \xi_3 \in H \) with \( \xi_1 < \xi_2 < \xi_3 \). Put \( G(\xi_1, \xi_2) = (\alpha, \beta, \xi, \eta) \). Since \( a_{\xi_1}, a_{\xi_2} \in \text{ran}(f) \) and \( a_{\xi_1} \neq a_{\xi_2} \), we have \( \alpha \neq \beta \). Suppose \( \alpha < \beta \). Let \( \alpha = \) the \( \xi \)-th element of \( a_{\xi_1} \), \( \alpha' = \) the \( \xi \)-th element of \( a_{\xi_2} \) and \( \beta = \) the \( \eta \)-th element of \( a_{\xi_1} \). Since \( G(\xi_1, \xi_2) = G(\xi_2, \xi_3) = (\alpha, \beta, \xi, \eta) \), there are functions \( f_1, f_2 \in \delta_{\beta \kappa^+} \) such that \( f_1(\alpha) = \alpha, f_1(\beta) = \beta, f_2(\alpha) = \alpha' \) and \( f_2(\beta) = \beta' \). Thus, we have \( \alpha_\delta = \alpha_\delta = \alpha \) and \( \beta_\delta = \beta \). Applying (M4) to \( \{ f_1, f_2 \} \), it is easy to see that there is a \( \gamma < \kappa^+ \) such that \( \alpha_\gamma \neq \alpha_\gamma' \). Let \( \gamma \) be the least \( \gamma \) with this property. Using (M2) and (M4), we deduce that \( \gamma \neq \xi + 1 \). It follows that \( \alpha_{\xi} = \alpha_{\xi}' \).

Now put \( \delta_{\beta \xi} = \{ g, h \} \). Then, \( f_1 = f_{1}^{*} \circ g \circ f_{1}^{*} \) and \( f_2 = f_{2}^{*} \circ h \circ f_{2}^{*} \) for some \( f_{1}^{*}, f_{2}^{*} \in \delta_{\beta \xi}, f_{1}^{*}, f_{2}^{*} \in \delta_{\xi \kappa^+} \). Hence, \( \alpha_{\xi} = g(\alpha_{\xi}) \) and \( \alpha_{\xi}' = h(\alpha_{\xi}) \). Let \( \sigma_{\xi} \) be the split point of \( \{ g, h \} \). Since \( g(\alpha_{\xi}) \neq h(\alpha_{\xi}) \), we infer that \( \sigma_{\xi} < \gamma \). But since \( f_1(\beta) = f_2(\beta) = \beta \), it follows that \( g(\beta_{\xi}) = h(\beta_{\xi}) = \beta_{\xi} \), which contradicts the fact that \( \sigma_{\xi} < \beta_{\xi} \).

If \( \beta < \alpha \) we set \( \alpha = \) the \( \xi \)-th element of \( a_{\xi_3} \), \( \beta = \) the \( \eta \)-th element of \( a_{\xi_3} \), and \( \beta' = \) the \( \eta \)-th element of \( a_{\xi_3} \), and then we proceed in a symmetric fashion.
PROOF of 4.1. We use a refinement of the argument given in [6]. Suppose that \( \kappa \) is an infinite cardinal with \( \kappa = \kappa^{<\kappa} \) and such that there exists a simplified \((\kappa^+, 1)\)-morass. By 4.2, there is a new \( \Delta_\kappa \)-function \( F : [\kappa^{++}]^2 \rightarrow \kappa^+ \). We put \( T_\alpha = \{ \alpha \} \times \kappa \) for every \( \alpha < \kappa^+ \), \( T_{\kappa^+} = \{ \kappa^+ \} \times \kappa^{++} \), and \( T = \bigcup \{ T_\alpha : \alpha \leq \kappa^+ \} \).

We define \( \tilde{P} = \tilde{P}(T) \) and the binary transitive relation \( \leq \) on \( \tilde{P} \) as in §2. Now we define \( P_\kappa \) as the set of all \( p = (x_p, \leq_p) \in \tilde{P} \) such that for every pair \( s, t \) of compatible elements in \( p \) with \( s, t \in T_{\kappa^+} \) and \( s \neq t \), if \( s = (\kappa^+, \mu) \) and \( t = (\kappa^+, \xi) \) then \( i_p(s, t) \in T_\tau \) for some \( \tau \in F(\mu, \xi) \). We put \( \leq_\kappa = \leq | P_\kappa \). Suppose that \( P_\kappa \) preserves cardinals. Then, if \( G \) is a \( P_\kappa \)-generic filter, we put \( \leq_\kappa \leq \bigcup \{ \leq_p : p \in G \} \).

It is easy to see that \( \leq \) is a partial order on \( T \), and by means of a density argument one can show that if \( \alpha < \beta \leq \kappa^+ \) and \( t \in T_\beta \), then \( \{ s \in T_\alpha : s < t \} \) is infinite (see [1, Lemma 7.2]). Now, by using an argument similar to the one given in 2.4, we can construct a \( \kappa \)-very thin-thick \( sBa \).

Note that \( P_\kappa \) is \( \kappa \)-closed. So, in order to prove that \( P_\kappa \) preserves cardinals, it is enough to show that \( P_\kappa \) satisfies the \( \kappa^+ \)-chain condition. Suppose on the contrary that there is an antichain \( A \) of cardinality \( \kappa^+ \). For every \( p \in A \), we put \( y_p = \{ a \in \kappa : x_p \cap T_\alpha \neq 0 \} \). By the \( \Delta \)-system lemma, we may assume that the \( y_p \) form a \( \Delta \)-system with kernel \( y* \). Without loss of generality we may assume that \( \kappa^+ \in y* \). Then, by thinning out \( A \) again if necessary, we may suppose that \( y_0 \) is an initial segment of \( y_\alpha \) for any \( \alpha \in A \), and also that there is an ordinal \( \gamma(1) < \kappa \) such that the order type of \( y_\alpha - y^* \) is \( \gamma(1) \) for any \( \alpha \in A \). We define \( \gamma(0) = \sup \{ \alpha + 1 : \alpha \in y_\alpha \} \) and \( \gamma = (\gamma(0) + \gamma(1) + 1) - \gamma(0) \). Then, we may assume that \( \gamma_\alpha \cap \gamma = 0 \) for each \( \alpha \in A \). Now, for every \( \alpha \in A \), we define \( z_\alpha = x_p \cap T_{\kappa^+} \). By the \( \Delta \)-system lemma, we may assume that the \( z_\alpha \) form a \( \Delta \)-system with kernel \( z^* \).

We consider in \( T_{\kappa^+} \) the well-order induced by \( \kappa^{++} \). Then we may suppose that there is an ordinal \( \zeta < \kappa \) such that the order type of \( z_\alpha - z^* \) is \( \zeta \) for any \( \alpha \in A \). Now, for every \( \alpha, \beta \in A \), we consider the unique order-preserving bijection \( n_{pq} : y_\alpha - y^* \rightarrow y_\beta - y^* \). Then we may also assume that \( n_{pq} \) lifts to an isomorphism of \( x_\alpha \) with \( x_\beta \) satisfying the following:

(a) For every \( \alpha \in y_\alpha - \{ \kappa^+ \}, \pi_{pq}(\alpha, \beta) = (\pi_{pq}(\alpha), \beta) \).
(b) \( n_{pq} \) is the identity on \( z^* \).
(c) For every ordinal \( \xi < \zeta \), if \( s \) is the \( \xi \)-element in \( z_\alpha - z^* \) and \( t \) is the \( \xi \)-element in \( z_\beta - z^* \), then \( \pi_{pq}(s) = t \).

Moreover, we may suppose that if \( \alpha, \beta \in A \) and \( s, t \in x_\alpha \), then:

(a) \( s \leq \rho \) if \( \pi_{pq}(s) \leq \pi_{pq}(t) \), and
(b) If \( s, t \) are compatible in \( p \), then \( \pi_{pq}(s), \pi_{pq}(t) \) are compatible in \( q \) and \( i_q(\pi_{pq}(s), \pi_{pq}(t)) = \pi_{pq}(i_p(s, t)) \).

Now since \( F : [\kappa^{++}]^2 \rightarrow \kappa^+ \) is a new \( \Delta_\kappa \)-function, there are \( p, q \in A \) such that if we put \( a = \{ \xi \in \kappa^{++} : (\kappa^+, \xi) \in z_\alpha - z^* \} \) and \( b = \{ \xi \in \kappa^{++} : (\kappa^+, \xi) \in z_\beta - z^* \} \), then \( F^*(a, b) > \gamma(0) + \gamma(1) + 1 \). Our aim is to prove that \( p \) and \( q \) are compatible. So, we construct an \( r \in P_\kappa \) with \( r \leq \kappa^+ p, q \). Let \( g_1 : \gamma \rightarrow (\gamma_\rho - \gamma) \) and \( g_2 : \gamma \rightarrow (\gamma_q - \gamma) \) be the corresponding order-preserving bijections. Set \( \eta = \gamma(0) + \gamma(1) \). For \( \alpha \in \gamma - \{ \eta \} \), we put \( x^{(\alpha)} = \{ (\alpha, \beta) \in T_\alpha : (g_1(\alpha), \beta) \in x_\rho \} = \{ (\alpha, \beta) \in T_\alpha : (g_2(\alpha), \beta) \in x_q \} \). And we put \( x^{(\eta)} = \{ (\eta, \beta) \in T_\eta : \beta < \zeta \} \). Then
we set \( x_r = x_p \cup x_q \cup \bigcup \{ x^{(\alpha)} : \alpha \in \gamma \} \). Now we define:

\[
\begin{align*}
    x &= \bigcup \{ x_p \cap T_\alpha : \alpha \in \gamma \} = \bigcup \{ x_q \cap T_\alpha : \alpha \in \gamma \}, \\
y &= \bigcup \{ x^{(\alpha)} : \alpha \in \gamma \}, \\
y_1 &= \bigcup \{ x_p \cap T_\alpha : \alpha \in \gamma_p - \gamma^* \} \cup (z_p - z^*), \\
y_2 &= \bigcup \{ x_q \cap T_\alpha : \alpha \in \gamma_q - \gamma^* \} \cup (z_q - z^*).
\end{align*}
\]

Now we define:

\[
\begin{align*}
y &= u_{\{x^{(\alpha)} : \alpha \in \gamma \}}, \\
y_1 &= u_{\{x_p \cap T_\alpha : \alpha \in \gamma_p - \gamma^* \} \cup (z_p - z^*)}, \\
y_2 &= u_{\{x_q \cap T_\alpha : \alpha \in \gamma_q - \gamma^* \} \cup (z_q - z^*)}.
\end{align*}
\]

Let us consider the extensions of \( g_1 \) and \( g_2 \) to the corresponding isomorphisms of \( y \) with \( y_1 \) and \( y \) with \( y_2 \), respectively. Thus, for each \( s \in y \), we have \( \pi_{pq}(g_1(s)) = g_2(s) \). Then we define \( \leq_r \) as follows:

\[
s \leq_r t \text{ iff } s \leq_p t \text{ or } s \leq_q t \text{ or one of the following conditions holds:}
\]

(a) \( s \in x, t \in y \) and \( s \leq_p g_1(t) \),
(b) \( s, t \in y \) and \( g_1(s) \leq_p g_1(t) \),
(c) \( s \in y, t \in y_1 \cup z^* \) and \( g_1(s) \leq_p t \),
(d) \( s \in y, t \in y_2 \) and \( g_2(s) \leq_q t \).

We have that \( \leq_r \) is a partial order on \( x_r \). Now suppose that \( s, t \) are compatible elements in \( r \) with \( s \neq t \). Note that if \( s, t \in z^* \) with \( s = (\kappa^+, \mu) \) and \( t = (\kappa^+, \xi) \), then since \( i_{p'}\{s, t\} \in x_{p'} \cap \bigcup \{ T_\tau : \tau \in F\{\mu, \xi\} \} \) for every \( p' = (x_{p'}, \leq_{p'}) \in A \), we infer that \( v \leq_r s, t \) implies \( v \in x \). Then it is easy to see that if \( s, t \in x_p \) we have \( i_r\{s, t\} = i_p\{s, t\} \), and if \( s, t \in x_q \) we have \( i_r\{s, t\} = i_q\{s, t\} \). Now if \( s \in x \) and \( t \in y \), then \( i_r\{s, t\} = i_{\pi}\{s, \pi_{pq}(t)\} \). If \( s, t \in y \), we put \( v = i_p\{g_1(s), g_1(t)\} \), and then we have \( i_r\{s, t\} = v \) if \( s \in y \), and \( i_r\{s, t\} = g_1^{-1}(v) \) otherwise. Analogously we can determine \( i_r\{s, t\} \), if \( s \in y \) and \( t \in y_1 \cup z^* \) or if \( s \in y \) and \( t \in y_2 \). And if \( s \in y_1 \) and \( t \in y_2 \), we consider \( v = i_p\{s, \pi_{pq}(t)\} \), and then we can easily check that \( i_r\{s, t\} = v \) if \( v \in x \), and \( i_r\{s, t\} = g_1^{-1}(v) \) otherwise.

Then we obtain that \( r = (x_r, \leq_r) \in P_\kappa \) and \( r \leq_\kappa p, q \). This completes the proof that \( P_\kappa \) has the \( \kappa^+ \)-chain condition.

REFERENCES


MATHEMATISCHES INSTITUT
BERINGSTRASSE 4
D-5300 BONN 1, GERMANY

FACULTAD DE MATEMÁTICAS
UNIVERSIDAD DE BARCELONA, GRAN VÍA 585
08007 BARCELONA, SPAIN
You have printed the following article:

**Superatomic Boolean Algebras Constructed from Morasses**
Peter Koepke; Juan Carlos Martínez
Stable URL: [http://links.jstor.org/sici?sici=0022-4812%28199509%2960%3A3%3C940%3ASBACFM%3E2.0.CO%3B2-K](http://links.jstor.org/sici?sici=0022-4812%28199509%2960%3A3%3C940%3ASBACFM%3E2.0.CO%3B2-K)

This article references the following linked citations. If you are trying to access articles from an off-campus location, you may be required to first logon via your library web site to access JSTOR. Please visit your library's website or contact a librarian to learn about options for remote access to JSTOR.

References

7 *Height and Width of Superatomic Boolean Algebras*
Judy Roitman
Stable URL: [http://links.jstor.org/sici?sici=0002-9939%28198505%2994%3A1%3C9%3AHAWOSB%3E2.0.CO%3B2-K](http://links.jstor.org/sici?sici=0002-9939%28198505%2994%3A1%3C9%3AHAWOSB%3E2.0.CO%3B2-K)

10 *#-Morasses, and a Weak form of Martin's Axiom Provable in ZFC*
Dan Velleman
Stable URL: [http://links.jstor.org/sici?sici=0002-9947%28198410%29285%3A2%3C617%3AAAWFOM%3E2.0.CO%3B2-2](http://links.jstor.org/sici?sici=0002-9947%28198410%29285%3A2%3C617%3AAAWFOM%3E2.0.CO%3B2-2)

11 *Simplified Morasses*
Dan Velleman
Stable URL: [http://links.jstor.org/sici?sici=0022-4812%28198403%2949%3A1%3C257%3ASM%3E2.0.CO%3B2-L](http://links.jstor.org/sici?sici=0022-4812%28198403%2949%3A1%3C257%3ASM%3E2.0.CO%3B2-L)

**NOTE:** The reference numbering from the original has been maintained in this citation list.