Moduli Spaces

Two Riemann surfaces of the same topological type can, of course, be conformally inequivalent; but how many conformal structures are there, and how can one deform them ?

Take for example an annulus $A(r,R) = \{z \in \mathbb{C} \,|\, 0 < r < |z| < R\}$. Two annuli are conformally equivalent if and only if their ratio R/r of outer and inner radius coincide. Thus the space of all such conformal structures, called the *moduli space*, is the interval $]1,\infty[$ in this case. For a torus $F_1 = \mathbb{S}^1 \times \mathbb{S}^1$ the moduli space is the upper half-plane $\mathbb{H}$ divided by the action of the group $\mathrm{SL}_2(\mathbb{Z})$ of Möbius transformations; this moduli space is therefore the open unit disc.

In general, the moduli space $\mathfrak{M}_g$ for a surface F_g of genus g is a complicated object. Already Riemann noticed that its dimension is $6g-6$ for $g \geq 2$; but a satisfactory definition of the space and its metric was possible only many years later via Teichmüller theory. The moduli space is a quotient of the contractible *Teichmüller space* $\mathfrak{T}_g$ by the action of the *mapping class group* $\Gamma_g = \pi_0(\mathrm{Diff}(F_g))$, the isotopy classes of orientation preserving diffeomorphisms of F_g.

Teichmüller and moduli spaces of Riemann surfaces are studied by several fields of mathematics: complex analysis, real analysis, algebraic geometry, differential geometry, dynamical systems, geometric group theory, topology — and theoretical physics.

We are interested in the homotopy type and the homology of $\mathfrak{M}^m_{g,n}$, the moduli space of the surface $F^m_{g,n}$ of genus g with $n \leq 1$ boundary curves and $m \leq 0$ punctures. In these cases the mapping class group $\Gamma^m_{g,n}$ is torsion-free, the action on $\mathfrak{T}^m_{g,n}$ is free, and the moduli spac $\mathfrak{M}^m_{g,n}$ are manifolds homotopy equivalent to the classifying space $B\Gamma^m_{g,n}$.

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