The Affine Grassmannian with a View Towards Geometric Satake

Timm Peerenboom

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Advisor: Prof. Dr. Catharina Stroppel

Second Advisor: Dr. Jacob Matherne

Mathematisches Institut

Mathematisch-Naturwissenschaftliche Fakultät der

Rheinischen Friedrich-Wilhelms-Universität Bonn

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Introduction

The connections between representation theory and geometry run far and deep.

In geometry we often consider the projective space $\mathbb{P}^1(\mathbb{C}) = \mathbb{C} \amalg \{\infty\}$. One way to construct $\mathbb{P}^1(\mathbb{C})$ is as the quotient $\mathrm{GL}_2(\mathbb{C})/P$ where

$$P = \left\{ \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \,\middle|\, a, c \in \mathbb{C}^*, b \in \mathbb{C} \right\}.$$

More generally, we can take the group

$$P := \left\{ \left(\begin{array}{c|c} A_1 & 0 \\ \hline * & A_2 \end{array} \right) \,\middle|\, A_1 \in \mathrm{GL}_k(\mathbb{C}), A_2 \in \mathrm{GL}_{n-k}(\mathbb{C}) \right\}$$

in $\mathrm{GL}_n(\mathbb{C})$ to obtain the classical Grassmannian of k-dimensional subspaces in $\mathbb{C}^n$ as the quotient $\mathrm{GL}_n(\mathbb{C})/P$, which is a projective variety. Even more generally, for a reductive group G over the complex numbers together with a parabolic subgroup P, one can observe that the quotient G/P has the structure of a projective variety — called the partial flag variety. These geometric objects are hugely important in representation theory. For example, one can recover the irreducible rational representations of G as cohomology groups of line bundles on a partial flag variety by the Borel–Weil–Bott Theorem, see [Dem76].

The partial flag variety G/P has a standard decomposition into locally closed subsets called the Schubert cells, which give standard bases of its cohomology ring. In the example of $\mathbb{P}^n$ this is given by

$$\mathbb{P}^n(\mathbb{C}) = \mathbb{C}^n \amalg \mathbb{C}^{n-1} \amalg \cdots \amalg \{pt\}.$$

The Schubert cells in partial flag varieties are always of the form $\mathbb{C}^n$, but their closures in G/P — the Schubert varieties — have rich geometric structure.

In this thesis, we consider an infinite-dimensional analogue of a partial flag variety, called the Affine Grassmannian Gr_G. We consider the ring of Laurent series $\mathbb{C}((t))$ and the ring of power series $\mathbb{C}[[t]]$. Then $\mathrm{Gr}_G(\mathbb{C})$ is given by $G(\mathbb{C}((t)))/G(\mathbb{C}[[t]])$. This object has the structure of an ind-projective ind-variety, a notion that is very similar to that of a projective variety, but which also captures the infinite-dimensionality of Gr_G. It turns out the Affine Grassmannian has a decomposition into finite-dimensional subvarieties, which we also call Schubert cells. Their closures, the Affine Grassmannian's Schubert varieties, are projective varieties. Unlike the classical case of partial flag varieties, the

Schubert cells of the Affine Grassmannian have a much richer structure. It turns out that these Schubert cells are bundles over partial flag varieties, see Theorem 2.3.15. In particular, classical Grassmannians appear as Schubert cells in the Affine Grassmannian of GL_n for some n.

The Affine Grassmannian is linked to representation theory in two ways.

The Beĭlinson-Bernstein Localization Theorem [BB81], which links the geometry of partial flag varieties with the representation theory of (finite-dimensional) Lie algebras, has an analogue for the Affine Grassmannian. This result, due to Frenkel and Gaitsgory [FG09], links the Affine Grassmannian to representations of affine Kac–Moody Lie algebras.

We will however focus on the other central result: the Geometric Satake Equivalence. This theorem states an equivalence of Tannakian categories between the representation category of $G^\vee$, the Langlands dual group of G, and perverse sheaves on Gr_G.

We define the Langlands dual of G via the classification of complex reductive groups in terms of their root datum, which is a purely combinatorial object. This notion comes with an inherent duality and so $G^\vee$ is defined as the reductive group corresponding to the dual root datum of G.

By the Tannakian Reconstruction Theorem we obtain $G^\vee$ from $\mathrm{Rep}(G^\vee)$ and thus the Geometric Satake Equivalence recovers the group $G^\vee$ without making reference to the classification of reductive groups via their root data.

A precursor of the Geometric Satake Equivalence was the Satake isomorphism, see [Sat63] and [Mac68], which relates the center of the extended affine Hecke algebra with the Grothendieck group of $\mathrm{Rep}(G^\vee)$. Geometric Satake can be thought of as a categorification of this result. It also explains the name Geometric Satake Equivalence as the word "geometric" means "sheaf-theoretic."

Another step on the way to the Geometric Satake Equivalence was done by Lusztig in [Lus83], where he observed a relationship between products of intersection complexes on Schubert varieties in the Affine Grassmannian and representations of $G^\vee$.

Geometric Satake was first proven by Ginzburg for complex representations in [Gin95] and later generalized by Mirković and Vilonen in [MV00].

Both proofs make use of the Tannakian formalism. One can construct a certain category of perverse sheaves $\mathsf{P}_\Lambda(\mathrm{Gr}_G)$ on Gr_G and then endow this category with a monoidal structure $\star$ as well as a fiber functor to the category of finite-dimensional vector spaces to deduce that

$$(\mathsf{P}_\Lambda(\mathrm{Gr}_G), \star) \cong (\mathrm{Rep}(\widetilde{G}), \otimes)$$

as Tannakian categories for some algebraic group $\widetilde{G}$. Then one can show that $\widetilde{G} \cong G^\vee$ by showing that $\widetilde{G}$ is reductive and then calculating its root datum.

In this thesis, we will only give a brief sketch of the proof of Geometric Satake and focus mainly on understanding the geometry of the Affine Grassmannian in the type A cases $\mathrm{GL}_n, \mathrm{SL}_n$ and PGL_n.

The thesis is structured as follows: In Chapter 1 we first give a reminder on complex reductive algebraic groups and their classification via root data. We also recall the definition and basic properties of the projective linear group PGL_n. Afterwards we introduce ind-schemes, which are an essential tool to understand the natural structure of the Affine Grassmannian. They can be thought of as formal colimits of schemes and capture the infinite-dimensionality of Gr_G.

In Chapter 2 we begin by introducing lattices, which are in our context $R[[t]]$-submodules of $R((t))^n$ subject to certain conditions, where R is a $\mathbb{C}$-algebra. Using the functor of points perspective, we then define the R-valued points of the Affine Grassmannian of GL_n as the set of lattices in $R((t))^n$. Our first proposition will be to recover the $\mathbb{C}$-points of $\mathrm{Gr}_{\mathrm{GL}_n}$ as

$$\mathrm{Gr}_{\mathrm{GL}_n}(\mathbb{C}) = \mathrm{GL}_n(\mathbb{C}((t)))/\mathrm{GL}_n(\mathbb{C}[[t]]).$$

Sticking with GL_n, we show that $\mathrm{Gr}_{\mathrm{GL}_n}$ is an ind-projective ind-variety. Thereafter we are able to define Gr_G for general reductive groups G and show the analogous statement for Gr_G using the GL_n-case. Subsequently, we construct the Schubert cell decomposition in the cases $\mathrm{GL}_n, \mathrm{SL}_n, \mathrm{PGL}_n$ explicitly. We state properties of the Schubert cells and varieties, providing proofs in the aforementioned cases.

In Chapter 3 we give a short introduction to Verdier duality, a generalization of Poincaré duality. Afterwards we define perverse sheaves on a space X and summarize well-known facts about them and their category $\mathsf{P}(X)$. This category is an abelian subcategory of $D^b(X)$, the derived category of sheaves on X. Perverse sheaves were first introduced in [BBD82] and serve as a powerful tool in geometric representation theory. Notably, they appear in the proof of the Kazhdan-Lusztig conjectures, see [BB81].

We finish the thesis by formulating the Geometric Satake Equivalence and briefly sketching its proof.

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Chapter 1

Preliminaries

In this chapter we summarize preliminaries about the foundations of the thesis — the classical theory of complex reductive algebraic groups and ind-schemes which is the natural structure on the Affine Grassmannian.

Throughout the thesis all rings and algebras are assumed to be unital and commutative.

1.1 Reductive Algebraic Groups

We start by giving a brief reminder about complex reductive algebraic groups. We follow mostly [Mil17] and [Hum75]. For the original treatment in arbitrary characteristic, see [GR03].

Let GL_n be the affine algebraic group(-scheme) over $\mathbb{C}$. A linear algebraic group is a closed subgroup(-scheme) of GL_n, We write $G(R)$ for the R-points of the linear algebraic group G where R is a $\mathbb{C}$-algebra.

Definition 1.1.1. We call a connected linear algebraic group G **reductive**, if the category of rational representations $\mathrm{Rep}(G)$ of G is semi-simple.

The main examples of reductive groups are GL_n and SL_n. Another important example of a reductive group is PGL_n, which turns out to be the so-called Langlands dual group of SL_n. In order to define PGL_n consider the canonical inclusion of algebraic groups $\mathbb{G}_m = \mathrm{GL}_1 \hookrightarrow \mathrm{GL}_n$ given by

$$\mathbb{G}_m(R) = R^* \to \mathrm{GL}_n(R), \quad r \mapsto \begin{pmatrix} r & & \\ & \ddots & \\ & & r \end{pmatrix},$$

which identifies $\mathbb{G}_m$ with the center $Z(\mathrm{GL}_n)$ of GL_n.

Definition 1.1.2. The group PGL_n is defined as the quotient

$$\mathrm{PGL}_n := \mathrm{GL}_n/\mathbb{G}_m$$

of algebraic groups.

The next two propositions summarize well-known algebraic properties of PGL_n, after which we analyze the representation theory of this group.

Proposition 1.1.3. *The group* PGL_n *is a connected linear algebraic group.*

Note that the R-points $\mathrm{PGL}_n(R) = (\mathrm{GL}_n/\mathbb{G}_m)(R)$ are not necessarily equal to $\mathrm{GL}_n(R)/\mathbb{G}_m(R)$. However, this equality does hold for all rings that we will ever consider in this thesis (namely $\mathbb{C}, \mathbb{C}((t))$, and $\mathbb{C}[[t]]$) by the following proposition.

Proposition 1.1.4. *If* $\mathrm{Pic}(R) = \{1\}$*, then*

$$\mathrm{PGL}_n(R) = \mathrm{GL}_n(R)/\mathbb{G}_m(R).$$

In particular, we have $\mathrm{PGL}_n(R) = \mathrm{GL}_n(R)/\mathbb{G}_m(R)$ *if* R *is a field or even a local ring.*

Proof. By [Mil17, Chapter 3.k] the exact sequence

$$0 \to \mathbb{G}_m \to \mathrm{GL}_n \to \mathrm{PGL}_n \to 0$$

of algebraic groups yields an exact sequence

$$0 \to \mathbb{G}_m(R) \to \mathrm{GL}_n(R) \to \mathrm{PGL}_n(R) \to H^1(R, (\mathbb{G}_m)_R),$$

which comes from the long exact cohomology sequence. Now recall that

$$H^1(R, (\mathbb{G}_m)_R) = \mathrm{Pic}(R),$$

c.f. [Sta21, Tag 09NU]. We therefore have

$$0 \to \mathbb{G}_m(R) \to \mathrm{GL}_n(R) \to \mathrm{PGL}_n(R) \to 0,$$

if $\mathrm{Pic}(R)$ is trivial. □

Corollary 1.1.5. *The composition*

$$\varphi\colon \mathrm{SL}_n(\mathbb{C}) \hookrightarrow \mathrm{GL}_n(\mathbb{C}) \twoheadrightarrow \mathrm{PGL}_n(\mathbb{C})$$

is surjective with kernel the group of n*-th roots of unity.*

Proof. It is clear that the kernel is given by

$$\left\{ \begin{pmatrix} \xi & & \\ & \ddots & \\ & & \xi \end{pmatrix} \;\middle|\; \xi \text{ is an } n\text{-th root of unity} \right\}.$$

Now every element of $\mathrm{PGL}_n(\mathbb{C}) = \mathrm{GL}_n(\mathbb{C})/\mathbb{G}_m(\mathbb{C})$ can be written as $[A]$ with $A \in \mathrm{GL}_n(\mathbb{C})$. But because $\mathbb{C}$ is algebraically closed, $\det A \neq 0$ has an n-th root $\sqrt[n]{\det A}$. Now the matrix

$$A' := \left(\sqrt[n]{\det A}\right)^{-1} \cdot A = \begin{pmatrix} \left(\sqrt[n]{\det A}\right)^{-1} & & \\ & \ddots & \\ & & \left(\sqrt[n]{\det A}\right)^{-1} \end{pmatrix} \cdot A \in \mathrm{SL}_n(\mathbb{C})$$

has image $[A'] = [A]$ in $\mathrm{PGL}_n(\mathbb{C})$ under φ. □

Using this corollary we obtain a fully faithful functor

$$\mathrm{Rep}(\mathrm{PGL}_n) \to \mathrm{Rep}(\mathrm{SL}_n),$$

where a representation M of PGL_n becomes a representation of SL_n via

$$A \cdot m := [A] \cdot m \quad \text{for } m \in M \text{ and } A \in \mathrm{SL}_n(\mathbb{C})$$

where $[A]$ denotes the image of A in $\mathrm{PGL}(\mathbb{C})$.

Proposition 1.1.6. *The functor* $\mathrm{Rep}(\mathrm{PGL}_n) \to \mathrm{Rep}(\mathrm{SL}_n)$ *identifies* $\mathrm{Rep}(\mathrm{PGL}_n)$ *with*

$$\{M \in \mathrm{Rep}(\mathrm{SL}_n) \mid \xi \textit{ acts trivially on } M \textit{ for all } n\textit{-th roots of unity } \xi\}.$$

Proof. The image of the fully-faithful functor $\mathrm{Rep}(\mathrm{PGL}_n) \to \mathrm{Rep}(\mathrm{SL}_n)$ is precisely the right-hand side as $\mathrm{PGL}_n(\mathbb{C}) = \mathrm{SL}_n(\mathbb{C})/\{n\text{-th roots of unity}\}$. □

From this proposition we conclude:

Corollary 1.1.7. *The group* PGL_n *is reductive.*

Proof. The category $\mathrm{Rep}(\mathrm{PGL}_n)$ is a full subcategory of $\mathrm{Rep}(\mathrm{SL}_n)$ and closed under direct summands in $\mathrm{Rep}(\mathrm{SL}_n)$. But $\mathrm{Rep}(\mathrm{SL}_n)$ is semi-simple and therefore also $\mathrm{Rep}(\mathrm{PGL}_n)$. □

Remark 1.1.8. The notion of a connected group with semi-simple representation category is not the correct notion for reductive groups in positive characteristic. For example both GL_n and SL_n do not satisfy Definition 1.1.1 in positive characteristic for general n. For the general notion of reductive groups, one needs to consider groups such that their unipotent radical is trivial, see [Mil17, Chapter 6.h]. Then by [Mil17, Theorem 22.42], the two definitions of reductive coincide for groups over a field of characteristic zero.

In the following, we will give the classification of reductive groups over $\mathbb{C}$ in terms of their so-called root datum which is a generalization of root systems. The classification is true in fact for all algebraically closed fields using the more general definition of reductive from Remark 1.1.8.

First we define the root datum of a reductive group as well as abstract root data, which will turn out to be equivalent notions by Theorem 1.1.17.

Definition 1.1.9 (Root Datum of a Reductive Algebraic Group)**.** Let G be a reductive group. Choose a maximal torus $T \subseteq G$, i.e. a group of the form $\mathbb{G}_m^n$ for some n. We define

$$X^*(G,T) := \mathrm{Hom}(T, \mathbb{G}_m),$$
$$X_*(G,T) := \mathrm{Hom}(\mathbb{G}_m, T).$$

We call $X^* = X^*(G,T)$ the character lattice and $X_* = X_*(G,T)$ the cocharacter lattice of G. Note that the cocharacter lattice $X_* = (X^*)^\vee := \mathrm{Hom}_{\mathbb{Z}}(X^*, \mathbb{Z})$ is the dual abelian group of the character lattice X^* using the perfect pairing

$$X^* \times X_* = \mathrm{Hom}(T, \mathbb{G}_m) \times \mathrm{Hom}(\mathbb{G}_m, T) \to \mathrm{Hom}(\mathbb{G}_m, \mathbb{G}_m) = \mathbb{Z},$$
$$(x, y) \mapsto \langle x, y \rangle := x \circ y.$$

Importantly, there are subsets $\Phi \subseteq X^*$ and $\Phi^\vee \subseteq X_*$ — of roots and coroots — defined as the weights of the adjoint action of G on the Lie algebra $\mathfrak{g}$. Then the root datum is defined as the quadruple

$$(X^*, \Phi, X_*, \Phi^\vee).$$

We follow this definition by gathering the root data for $\mathrm{GL}_n, \mathrm{SL}_n$, and PGL_n, our most important examples.

Example 1.1.10 (Root Datum of GL_n)**.** A maximal torus in GL_n is given by the diagonal matrices

$$\left\{ \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix} \in \mathrm{GL}_n(\mathbb{C}) \right\} \cong \mathbb{G}_m(\mathbb{C}) = (\mathbb{C}^*)^n.$$

Therefore we have $X^*(\mathrm{GL}_n, T) \cong \mathbb{Z}^n$. The roots are $e_i - e_j$ for $i \neq j$, where e_i is the i-basis vector in $X^* \cong \mathbb{Z}^n$. For the cocharacters we have by the perfect pairing $X_* = (X^*)^\vee \cong \mathbb{Z}^n$ with dual basis $(\varepsilon_1, \dots, \varepsilon_n)$. The coroots are precisely $\varepsilon_i - \varepsilon_j$ for $i \neq j$. Hence, the root datum is given by

$$(\mathbb{Z}^n, \{e_i - e_j \mid i \neq j\}, \mathbb{Z}^n, \{\varepsilon_i - \varepsilon_j \mid i \neq j\}).$$

Note that neither the roots form a $\mathbb{Z}$-spanning set of the character lattice, nor the coroots of the cocharacter lattice.

Example 1.1.11 (Root Datum of SL_n)**.** A maximal torus in SL_n is given by the diagonal matrices

$$\left\{ \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix} \in \mathrm{SL}_n(\mathbb{C}) \right\} \cong \mathbb{G}_m^{n-1}(\mathbb{C}).$$

By embedding this torus into the diagonal matrices T' of $\mathrm{GL}_n(\mathbb{C})$ we obtain a surjection $\mathbb{Z}^n \cong X^*(\mathrm{GL}_n, T) \twoheadrightarrow X^*(\mathrm{SL}_n, T)$. This yields an identification

$$X^*(\mathrm{SL}_n, T) = \mathbb{Z}^n / \langle e_1 + \dots + e_n \rangle.$$

The roots of SL_n are precisely the images of the roots $e_i - e_j$ of GL_n, which we also denote by $e_i - e_j$. The cocharacter lattice will be a subset of $X_*(\mathrm{GL}_n, T')$. Explicitly, we have

$$X_*(\mathrm{SL}_n, T) = \left\{ y = (y_1, \dots, y_n) \in \mathbb{Z}^n = X_*(\mathrm{GL}_n, T') \;\middle|\; \sum_{i=1}^n y_i = 0 \right\}.$$

The coroots $\varepsilon_i - \varepsilon_j$ of GL_n are also the coroots of SL_n. Hence, the root datum of SL_n is given by

$$\left(\mathbb{Z}^n / \langle e_1 + \dots + e_n \rangle, \{e_i - e_j\}, \left\{ y = (y_1, \dots, y_n) \in \mathbb{Z}^n \;\middle|\; \sum_{i=1}^n y_i = 0 \right\}, \{\varepsilon_i - \varepsilon_j\} \right).$$

The case $n = 2$ specializes to

$$(\mathbb{Z}, \{\pm 2\}, \mathbb{Z}, \{\pm 1\}).$$

where $2 = e_1 - e_2$ and $1 = \varepsilon_1 - \varepsilon_2$.

Example 1.1.12 (Root Datum of PGL_n)**.** A maximal torus T in $\mathrm{PGL}_n = \mathrm{GL}_n/\mathbb{G}_m$ is given by the image of the diagonal matrices T' of $\mathrm{GL}_n(\mathbb{C})$. Concretely, we have

$$T = \left\{ \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix} \in \mathrm{GL}_n(\mathbb{C}) \right\} \Big/ \left\{ \begin{pmatrix} a & & \\ & \ddots & \\ & & a \end{pmatrix} \right\} \cong \mathbb{G}_m^{n-1}(\mathbb{C}).$$

The homomorphism $T' \twoheadrightarrow T$ induces maps

$$X^*(\mathrm{PGL}_n, T) \hookrightarrow X^*(\mathrm{GL}_n, T') \text{ and } X_*(\mathrm{GL}_n, T') \twoheadrightarrow X_*(\mathrm{PGL}_n, T).$$

Explicitly, we have

$$\begin{aligned} X^*(\mathrm{PGL}, T) &= \left\{ x = (x_1, \dots, x_n) \in \mathbb{Z}^n \;\middle|\; \sum_{i=1}^n x_i = 0 \right\}, \\ X_*(\mathrm{PGL}_n, T) &= \mathbb{Z}^n / \langle \varepsilon_1 + \cdots + \varepsilon_n \rangle. \end{aligned}$$

The roots of GL_n live in $X^*(\mathrm{PGL}_n, T)$ and are the roots of PGL_n, and the coroots of PGL_n are precisely the images of the coroots of GL_n. Hence, the root datum of PGL_n is given by

$$\left(\left\{ x = (x_1, \dots, x_n) \in \mathbb{Z}^n \;\middle|\; \sum_{i=1}^n x_i = 0 \right\}, \{e_i - e_j\}, \mathbb{Z}^n / \langle \varepsilon_1 + \cdots + \varepsilon_n \rangle, \{\varepsilon_i - \varepsilon_j\} \right).$$

The case $n = 2$ specializes to

$$(\mathbb{Z}, \{\pm 1\}, \mathbb{Z}, \{\pm 2\}).$$

It will turn out the similarity between the root data for SL_n and PGL_n is an example for a general duality of reductive algebraic groups see Definition 1.1.18.

Recall that semi-simple complex Lie algebras are classified by the their root systems which can be defined axiomatically. The following definition is a refinement of the concept of an abstract root system.

Definition 1.1.13 (Abstract Root Datum)**.** An abstract root datum is an ordered quadruple $(X, \Phi, X^\vee, \Phi^\vee)$ where

- $X, X^\vee$ are finite free $\mathbb{Z}$-modules in duality by a perfect pairing

$$\langle -, - \rangle \colon X \times X^\vee \to \mathbb{Z},$$

- $\Phi, \Phi^\vee$ are finite subsets of X and $X^\vee$ in bijection via $\alpha \mapsto \alpha^\vee$, such that

(R1) $\langle \alpha, \alpha^\vee \rangle = 2$,
(R2) $s_\alpha(\Phi) \subseteq \Phi, s_{\alpha^\vee}(\Phi^\vee) \subseteq \Phi^\vee$, where

$$\begin{aligned} s_\alpha(x) &= x - \langle x, \alpha^\vee \rangle \alpha, \quad \text{for } x \in X, \alpha \in \Phi, \\ s_{\alpha^\vee}(y) &= y - \langle \alpha, y \rangle \alpha^\vee, \quad \text{for } y \in X^\vee, \alpha \in \Phi. \end{aligned}$$

We call Φ the roots and $\Phi^\vee$ the coroots.

Proposition 1.1.14. *The root datum of a reductive group is an abstract root datum.*

Proof. This can be found in [Mil17, Corollary 21.21]. □

Remark 1.1.15. Note that these definitions are refinements of the definitions of root system and abstract root system. Indeed, we can recover the root system of G from its root datum $(X^*, \Phi, X_*, \Phi^\vee)$ as $(X^* \otimes_\mathbb{Z} \mathbb{R}, \Phi)$.

Example 1.1.16. The root systems of PGL_n and SL_n are equal. Indeed, by Remark 1.1.15 Example 1.1.12 the root system of PGL_n is given by

$$(X^*(\mathrm{PGL}_n) \otimes_\mathbb{Z} \mathbb{R}, \Phi(\mathrm{PGL}_n)) = \left(\{x \in \mathbb{R}^n \mid \sum x_i = 0\}, \{e_i - e_j\}\right)$$

and by Example 1.1.11 the root system of SL_n is given by

$$(X^*(\mathrm{SL}_n) \otimes_\mathbb{Z} \mathbb{R}, \Phi(\mathrm{SL}_n)) = (\mathbb{R}^n / \langle e_1 + \cdots + e_n \rangle_\mathbb{R}, \{e_i - e_j\}).$$

The obvious homomorphism

$$\begin{aligned} X^*(\mathrm{PGL}_n) \otimes_\mathbb{Z} \mathbb{R} &\longrightarrow X^*(\mathrm{SL}_n) \otimes_\mathbb{Z} \mathbb{R} \\ x &\longmapsto x + \langle e_1 + \cdots + e_n \rangle_\mathbb{R} \end{aligned}$$

is an injective map between vector spaces of the same dimension and therefore an isomorphism. However, the corresponding map on the level of lattices $X^*(\mathrm{PGL}_n) \to X^*(\mathrm{SL}_n)$ is not surjective for $n > 1$.

One can show by hand that $\mathfrak{pgl}_n$, the complex Lie algebra of PGL_n, is isomorphic to $\mathfrak{sl}_n$. This also follows from the classification theorem for complex semi-simple Lie algebras in terms of their root systems.

The next theorem is the classification theorem for reductive algebraic groups in terms of abstract root data.

Theorem 1.1.17 (Chevalley, Demazure–Grothendieck)**.** *Reductive algebraic groups over $\mathbb{C}$ are classified by their root data. To be precise, two reductive algebraic groups are isomorphic if and only if they have the same root datum. Conversely, for every given abstract root datum there exists a reductive group which has this root datum.*

This was first proven by Chevalley for complex algebraic groups over $\mathbb{C}$. In [DG11] Demazure extended this to the case of algebraically closed fields of arbitrary characteristic.

The next definition captures the duality we observed between the root data of SL_n and PGL_n. One can go from one root datum to the other by switching the roles of characters and cocharacters, roots and coroots. In general, for an abstract root datum $\mathcal{R} = (X, \Phi, X^\vee, \Phi^\vee)$ we observe that $\mathcal{R}^\vee = (X^\vee, \Phi^\vee, X, \Phi)$ is also an abstract root datum. We call $\mathcal{R}^\vee$ the Langlands dual root datum of $\mathcal{R}$.

Definition 1.1.18 (Langlands Dual)**.** Given a reductive group G with root datum $\mathcal{R}$, consider the Langlands dual root datum $\mathcal{R}^\vee$. By Theorem 1.1.17 there is a (up to ismorphism) unique reductive group $G^\vee$ with associated root datum $\mathcal{R}^\vee$. This group $G^\vee$ is called the Langlands dual group of G.

The following is an immediate observation from this definition.

Corollary 1.1.19. *If G is a reductive group with dual $G^\vee$, we have*

$$(G^\vee)^\vee \cong G.$$

Proof. Let G have root datum $(X, \Phi, X^\vee, \Phi^\vee)$. Then the root datum of $G^\vee$ will be $(X^\vee, \Phi^\vee, X, \Phi)$. Therefore the root datum of $(G^\vee)^\vee$ is equal to $(X, \Phi, X^\vee, \Phi^\vee)$, the root datum of G. By Theorem 1.1.17 we must have $G \cong (G^\vee)^\vee$. $\square$

Example 1.1.20. By Example 1.1.10 the root datum of GL_n is

$$(\mathbb{Z}^n, \{e_i - e_j \mid i \neq j\}, \mathbb{Z}^n, \{\varepsilon_i - \varepsilon_j \mid i \neq j\}).$$

Observe that this is invariant under taking the Langlands dual. We therefore have $\mathrm{GL}_n^\vee = \mathrm{GL}_n$.

Example 1.1.21. By Example 1.1.11 the root datum of SL_n is

$$\left(\mathbb{Z}^n/\langle e_1 + \cdots + e_n\rangle, \{e_i - e_j\}, \left\{ y = (y_1, \ldots, y_n) \in \mathbb{Z}^n \;\middle|\; \sum_{i=1}^n y_i = 0 \right\}, \{\varepsilon_i - \varepsilon_j\}\right),$$

while by Example 1.1.12 the root datum of PGL_n is

$$\left(\left\{ x = (x_1, \ldots, x_n) \in \mathbb{Z}^n \;\middle|\; \sum_{i=1}^n x_i = 0 \right\}, \{e_i - e_j\}, \mathbb{Z}^n/\langle \varepsilon_1 + \cdots + \varepsilon_n\rangle, \{\varepsilon_i - \varepsilon_j\}\right).$$

These root data are dual to each other and we obtain

$$(\mathrm{SL}_n)^\vee = \mathrm{PGL}_n \quad \text{and } (\mathrm{PGL}_n)^\vee = \mathrm{SL}_n.$$

The classification theorem 1.1.17 tells us, that a reductive algebraic group is completely determined by its root datum. We expect to be able to read off information about the group from its root datum. The next theorem gives the celebrated classification of finite dimensional simple representation of G in terms of the root datum. We fix a Borel subgroup and a torus $T \subseteq B \subseteq G$. The choice of Borel yields a choice of positive roots $\Phi_+ \subseteq \Phi$.

Theorem 1.1.22. *Let G be a reductive algebraic group with root datum $(X^*, \Phi, X_*, \Phi^\vee)$. Then there is one-to-one correspondence between the (isomorphism classes of) finite dimensional simple representations of G and the set*

$$(X^*)_+ \coloneqq \{x \in X^* \mid \langle x, \alpha^\vee \rangle \geq 0 \text{ for all } \alpha \in \Phi_+\}$$

of dominant integral weights.

For a reference see [Mil17, Theorem 22.2].

Example 1.1.23. For GL_n we can choose $\Phi_+ = \{e_i - e_j \mid i < j\}$. The simple representations are then labeled by the set

$$\{a \in \mathbb{Z}^n \mid \langle a, \varepsilon_i - \varepsilon_j \rangle \geq 0 \ \forall i > j\} = \{(a_1, \ldots, a_n) \in \mathbb{Z}^n \mid a_1 \geq a_2 \geq a_3 \geq \cdots \geq a_n\}.$$

Example 1.1.24. For SL_n we can choose $\Phi_+ = \{e_i - e_j \mid i < j\}$. The simple representations are labeled by the set

$$\begin{aligned}(X^*)_+ &= \{(a_1, \ldots, a_n) \in \mathbb{Z}^n \mid a_1 \geq a_2 \geq \cdots \geq a_n\}/\{(a, a, \ldots, a)\} \\ &= \{(a_1, \ldots, a_{n-1}, 0) \in \mathbb{Z}^n \mid a_1 \geq a_2 \geq \cdots \geq 0\}.\end{aligned}$$

Example 1.1.25. For PGL_n we can choose $\Phi_+ = \{e_i - e_j \mid i < j\}$. The simple representations of PGL_n are labeled by the set

$$(X^*)_+ = \left\{(a_1, \ldots, a_n) \in \mathbb{Z}^n \;\middle|\; a_1 \geq a_2 \geq \cdots \geq a_n \text{ and } \sum_{i=1}^n a_i = 0\right\}.$$

Next we note that the dimensions of the simple representations can also be read off from the root datum.

Theorem 1.1.26 (Weyl's Dimension Formula)**.** *Let $T \subseteq B \subseteq G$ be as above. Let $L_G(x)$ be the simple representation with highest weight $x \in (X^*)_+$. Then*

$$\dim L_G(x) = \prod_{\alpha \in \Phi_+} \frac{\langle x + \rho, \alpha^\vee \rangle}{\langle \rho, \alpha^\vee \rangle}.$$

Here, ρ is defined as the half-sum of all positive roots.

Example 1.1.27. The simple representations of PGL_2 are given by the set

$$\{(a_1, a_2) \in \mathbb{Z}^2 \mid a_1 \geq a_2, a_1 + a_2 = 0\} = \{(k, -k) \mid k \in \mathbb{N}_0\}.$$

We obtain $\rho = \frac{1}{2} \cdot (1, -1)$ and therefore

$$\dim L_{\mathrm{PGL}_2}((k, -k)) = \frac{\langle (k, -k) + (\frac{1}{2}, -\frac{1}{2}), (1, -1) \rangle}{\langle (\frac{1}{2}, \frac{1}{2}), (1, -1) \rangle} = 2k + 1.$$

Alternatively, recall that by Proposition 1.1.6

$$\mathrm{Rep}(\mathrm{PGL}_2) = \{M \in \mathrm{Rep}(\mathrm{SL}_2) \mid I \text{ and } -I \text{ operate trivially}\},$$

where I denotes the unit matrix. Recall that the finite dimensional simple SL_2 represenations are given by $\mathbb{C}[x_0, x_1]_d$. Note that $-I$ acts as $(-1)^d$ on this representation. We can therefore identify $L_{\mathrm{PGL}_2}((k, -k))$ with $L_{\mathrm{SL}_2}(2k)$, the unique simple SL_2-representation of dimension $2k + 1$.

The next proposition allows us to study the geometry of G from its root datum.

Proposition 1.1.28 (Fundamental Group of G)**.** *The fundamental group of G can be computed as*

$$\pi_1(G) = X_*(G, T)/\langle \Phi^\vee \rangle.$$

For a proof see [Mil17, Chapter 18].

Example 1.1.29. We can compute

- $\pi_1(\mathrm{GL}_n) = X_*(\mathrm{GL}_n, T)/\langle \Phi^\vee \rangle = \mathbb{Z}^n/\langle \varepsilon_i - \varepsilon_j \mid i \neq j \rangle \cong \mathbb{Z}$,
- $\pi_1(\mathrm{SL}_n) = X_*(\mathrm{SL}_n, T)/\langle \Phi^\vee \rangle = \{a \in \mathbb{Z}^n \mid a_1 + \cdots + a_n = 0\}/\langle \varepsilon_i - \varepsilon_j \rangle = \{0\}$,
- $\pi_1(\mathrm{PGL}_n) = X_*(\mathrm{PGL}_n, T)/\langle \Phi^\vee \rangle = \mathbb{Z}^n/\langle \varepsilon_1 + \cdots + \varepsilon_n, \varepsilon_i - \varepsilon_j \rangle \cong \mathbb{Z}/n$.

1.2 ind-Schemes

We begin with the definition of an ind-scheme, because our main object of study, the Affine Grassmannian, will turn out to be such an object. We loosely follow [Ric19]. Recall that any scheme X over a ring k (in our case k will always be the field of complex numbers, but one can also take $k = \mathbb{Z}$ to recover the total situation) yields a corresponding functor of points

$$X \colon k\text{-Alg} \to \mathrm{Sets}, \quad R \mapsto X(R) \coloneqq \mathrm{Hom}_{\mathrm{Sch}/k}(\mathrm{Spec}(R), X),$$

where k-Alg is the category of commutative k-algebras with 1.

Taking the functor of points is a functor

$$\mathsf{Y} \colon \mathrm{Sch} \to \mathrm{Set}^{k\text{-Alg}}.$$

Lemma 1.2.1. *The functor* Y *is fully-faithful.*

Proof. This is just the Yoneda lemma plus the glueing property for maps of schemes. □

Example 1.2.2.

- If we take the affine scheme $\mathbb{A}^I = \mathrm{Spec}(k[x_i \mid i \in I])$ for some set I we obtain $R \mapsto R^I$ as its functor of points. Notice that for infinite I this does not coincide with

$$\mathbb{A}^{(I)}(R) \coloneqq R^{(I)} \coloneqq \{(r_i)_{i \in I} \in R^I \mid r_i = 0 \text{ for all but finitely many } i\}.$$

- The scheme $\mathbb{P}^I := \operatorname{Proj}(k[x_i \mid i \in I])$ represents the functor
$$R \mapsto \left\{ R^{(I)} \overset{\alpha}{\twoheadrightarrow} L \mid L \text{ is an invertible } R\text{-module} \right\} / \sim$$
where
$$\alpha \sim \alpha' :\Leftrightarrow \text{there exists some } \beta \colon L \xrightarrow{\sim} L' \text{ such that } \alpha' = \beta \circ \alpha.$$
If R is local, this degenerates to $\{(r_i) \in R^I \mid r_i \in R^* \text{ for at least one } i\}/R^*$, see [Sta21, Tag 01NA]. Notice that if I is infinite this is again quite different from the functor
$$\mathbb{P}^{(I)}(R) := \left\{ R^{(I)} \overset{\alpha}{\twoheadrightarrow} L \;\middle|\; \begin{array}{c} L \text{ is an invertible } R\text{-module and} \\ \alpha(e_i) = 0 \text{ for all but finitely many } i \end{array} \right\} / \sim,$$
which for R local becomes $\left\{ (r_i) \in R^{(I)} \mid r_i \in R^* \text{ for at least one } i \right\} / R^*$.

Both functors $\mathbb{A}^{(\mathbb{N})}$ and $\mathbb{P}^{(\mathbb{N})}$ are not representable by schemes (see Proposition 1.2.15), but rather have the structure of ind-schemes.

We now introduce ind-schemes.

Definition 1.2.3. We call a functor $X \colon k\text{-Alg} \to \text{Set}$ an **ind-scheme**, if X is an $\mathbb{N}$-filtered colimit of representable functors along closed immersions in the category $\text{Set}^{k\text{-Alg}}$, i.e. there is a diagram of schemes
$$X_1 \hookrightarrow X_2 \hookrightarrow X_3 \hookrightarrow \cdots$$
where all transition maps are closed immersions, such that $X(R) = \varinjlim X_i(R)$ in Set. Morphisms of ind-schemes are natural transformations of functors.

What we call an ind-scheme is in the literature sometimes referred to as a strict ind-scheme. These authors use the term ind-scheme also for colimits over general filtered categories.

Example 1.2.4. The following are standard examples of ind-schemes.

- The functor $\mathbb{A}^{(\mathbb{N})}$ can be written as the colimit of
$$\mathbb{A}^1 \hookrightarrow \mathbb{A}^2 \hookrightarrow \mathbb{A}^3 \hookrightarrow \ldots$$
where the transition maps are the coordinate embeddings.
- The functor $\mathbb{P}^{(\mathbb{N})}$ can be written as the colimit of
$$\mathbb{P}^1 \hookrightarrow \mathbb{P}^2 \hookrightarrow \mathbb{P}^3 \hookrightarrow \cdots$$
- Any scheme X has a trivial ind-scheme structure as the colimit of the diagram
$$X \xhookrightarrow{\mathrm{id}_X} X \xhookrightarrow{\mathrm{id}_X} X \xhookrightarrow{\mathrm{id}_X} \cdots$$

- There are schemes which have a non-trivial ind-scheme structure, such as $\coprod_{\mathbb{N}}\{pt\}$. It can be written as the colimit of

$$\{pt\} \hookrightarrow \{pt\} \amalg \{pt\} \hookrightarrow \{pt\} \amalg \{pt\} \amalg \{pt\} \hookrightarrow \cdots$$

Remark 1.2.5. Giving a scheme the structure of an ind-scheme defines a functor

$$\mathrm{Sch} \to \text{ind-Sch}.$$

This functor is in fact fully-faithful, because for two schemes S, T we have

$$\mathrm{Hom}_{\text{ind-Sch}}(S,T) = \mathrm{Hom}_{\mathrm{Set}^{k\text{-Alg}}}(S,T) = \mathrm{Hom}_{\mathrm{Sch}}(S,T).$$

Definition 1.2.6. We call an ind-scheme ind-projective, ind-reduced, ind-finite type, ind-quasi-compact, etc., if we can write X as the colimit of schemes which are projective, reduced, finite type, quasi-compact, etc.

Example 1.2.7. The ind-schemes from Example 1.2.4 have the following ind-properties:

- The ind-scheme $\mathbb{A}^{(\mathbb{N})}$ is ind-affine, ind-reduced, ind-finite type, ind-quasi-compact.
- The ind-scheme $\mathbb{P}^{(\mathbb{N})}$ is ind-projective, ind-reduced, ind-finite type, ind-quasi-compact.
- If X is a scheme with some property, then X has the ind-version of this property as an ind-scheme, but the converse does not hold, by the next example.
- The ind-scheme $\varinjlim \coprod_{1,\dots,n}\{pt\} = \coprod_{\mathbb{N}}\{pt\}$ is ind-quasi-compact, ind-finite type and ind-affine even though it is a scheme and neither quasi-compact, nor finite type, nor affine.

In the following, we examine morphisms of ind-schemes.

If $X = \varinjlim X_i$ is an ind-scheme and S is a scheme, the morphism of ind-schemes $X_i \to X$ induces a map $\mathrm{Hom}_{\mathrm{Sch}}(S, X_i) \to \mathrm{Hom}_{\text{ind-Sch}}(S,X)$ for $i \in \mathbb{N}$. From these we obtain a natural map $\varinjlim \mathrm{Hom}_{\text{ind-Sch}}(S,X_i) \to \mathrm{Hom}_{\text{ind-Sch}}(S,X)$. It turns out that this map is not far from being a bijection in general.

Proposition 1.2.8. *The map $\varinjlim \mathrm{Hom}_{\text{ind-Sch}}(S,X_i) \to \mathrm{Hom}_{\text{ind-Sch}}(S,X)$ is injective. If S is additionally assumed to be quasi-compact, the map is bijective.*

Proof. First let S be arbitrary. For every ring R the map $X_i(R) \to X_{i+1}(R)$ is injective, because the morphisms of schemes $X_i \to X_{i+1}$ are closed immersions. We obtain that also $X_i(R) \to \varinjlim X_i(R)$ is injective. It follows that the morphism of ind-schemes $X_i \to X$ is a monomorphism of functors and therefore a monomorphism of ind-schemes. Thus $\mathrm{Hom}(S,X_i) \to \mathrm{Hom}(S,X)$ is injective and we conclude that $\varinjlim \mathrm{Hom}(S,X_i) \to \mathrm{Hom}(S,X)$ is injective, too.

Now let S be quasi-compact. We need to show that for every morphism $\alpha\colon S \to X$ there is an $i_0 \in \mathbb{N}$ such that α factors through $X_{i_0} \to X$. We have that S is a finite union of affine schemes $S = U_1 \cup \cdots \cup U_r$ with embeddings $\iota_j\colon U_j \hookrightarrow S$. Note that if all morphisms $\alpha \circ \iota_j \colon U_j \to X$ factor through some $X_{i_j} \to X$ we also have that α factors through $X_{\max(i_1,\dots i_r)} \to X$. We can therefore assume $S = \operatorname{Spec} A$ to be affine. Let $\alpha\colon \operatorname{Spec} A \to X$ be a natural transformation of functors. We want to show that there is some i_0 such that α factors as $\alpha\colon \operatorname{Spec} A \to X_{i_0} \hookrightarrow X$. Consider $\mathrm{id}_A \in \mathrm{Hom}_{\mathrm{Ring}}(A,A) = \mathrm{Hom}(\operatorname{Spec} A, \operatorname{Spec} A)$. This element gets mapped by α_A to some element in $X(A) = \bigcup_i X_i(A)$, and thus $\alpha_A(\mathrm{id}_A) \in X_{i_0}(A)$ for some i_0. We show that α factors through $X_{i_0} \to X$. Let $f \in S(R) = \mathrm{Hom}(\operatorname{Spec} R, S) = \mathrm{Hom}(A,R)$. The following diagram commutes by naturality of α

$$\begin{tikzcd}
\mathrm{Hom}(A,A) \arrow[r, "\alpha_A"] \arrow[d, "f_*"] & \varinjlim X_i(A) = X(A) \arrow[d, "X(f)"] \\
\mathrm{Hom}(A,R) \arrow[r, "\alpha_R"] & \varinjlim X_i(R) = X(R)
\end{tikzcd}$$

and so $\alpha_R(f) = \alpha_R(f_*(\mathrm{id}_A)) = X(f)(\alpha_A(\mathrm{id}_A)) \in X_{i_0}(R)$. It follows that α factors through X_{i_0}. □

Corollary 1.2.9. *If $S = \operatorname{Spec} A$ is affine we have*

$$X(A) = \varinjlim X_i(A) = \mathrm{Hom}(\operatorname{Spec} A, X).$$

In lieu of this corollary, we are able to write

$$X(S) := \mathrm{Hom}_{\text{ind-Sch}}(S, X)$$

without confusion.

Example 1.2.10. Let $X_i = \coprod_{1,\dots,n}\{pt\}$ and $S = X = \varinjlim X_i = \coprod_{\mathbb{N}}\{pt\}$. Then $\varinjlim X_i(S) \neq X(S)$, because the identity on $X = S$ does not factor through any X_i. We see that the quasi-compactness assumption in Proposition 1.2.8 is indeed necessary.

Corollary 1.2.11. *If X, Y are ind-schemes and $Y = \varinjlim Y_j$ is ind-quasi-compact with Y_j quasi-compact, we have*

$$\begin{aligned}
\mathrm{Hom}(Y,X) &= \mathrm{Hom}(\varinjlim_j Y_j, \varinjlim_i X_i) \\
&= \varprojlim_j \mathrm{Hom}(Y_j, \varinjlim_i X_i) \\
&= \varprojlim_j \varinjlim_i \mathrm{Hom}(Y_j, X_i). && \text{(Proposition 1.2.8)}
\end{aligned}$$

Next, we define the underlying topological space of an ind-scheme, which generalizes the notion of underlying topological space of a scheme.

Definition 1.2.12. We define the topological space $|X|$ of an ind-scheme $X = \varinjlim X_i$ as the colimit

$$|X| := \varinjlim \left(|X_1| \hookrightarrow |X_2| \hookrightarrow \cdots\right)$$

in the category of topological spaces, where $|X_i|$ denotes the underlying topological space of the scheme X_i. This means that $Z \subseteq |X|$ is closed, respectively open, if $Z \cap |X_i|$ is closed, respectively open for all i.

We will see that this definition is independent of the presentation of X as a specific colimit of schemes in Lemma 1.2.14.

Example 1.2.13. Let k be a field. The underlying topological space of the ind-scheme given by the colimit of the diagram

$$\operatorname{Spec} k \hookrightarrow \operatorname{Spec} k[t]/(t^2) \hookrightarrow \operatorname{Spec} k[t]/(t^3) \hookrightarrow \cdots$$

has underlying topological space $\{pt\}$, because $k[t]/(t^n)$ only has a single prime ideal.

This ind-scheme is not (representable by) a scheme: If this ind-scheme was a scheme, it follows that it is affine, because its underlying topological space is a singleton. But a colimit of the above diagram exists in the category of affine schemes, namely $\operatorname{Spec} k[[t]]$ which has two points. This kind of example is also referred to as a formal scheme, see [Har77, Chapter II.9].

Lemma 1.2.14. *We have*

$$|X| = \varinjlim_{K \text{field}} X(K).$$

We see that $|X|$ is independent of a presentation as a diagram of schemes.

Proof. The statement is true for schemes by [Sta21, Tag 01J9] and we can exchange colimits. □

We are finally able to show that our first and most natural examples of ind-schemes — namely $\mathbb{A}^{(\mathbb{N})}$ and $\mathbb{P}^{(\mathbb{N})}$ — are in fact not representable by schemes

Proposition 1.2.15. *The ind-schemes $\mathbb{A}^{(\mathbb{N})}$ and $\mathbb{P}^{(N)}$ cannot be represented by schemes.*

Proof. Assume $\mathbb{A}^{(\mathbb{N})}$ was a scheme. Then there is an open affine subscheme $S \subseteq \mathbb{A}^{(\mathbb{N})}$. Since S is quasi-compact, the inclusion $S \hookrightarrow \mathbb{A}^{(\mathbb{N})}$ factors by Proposition 1.2.8 through some $\mathbb{A}^n$. The affine scheme S therefore has dimension n. It follows that S cannot be open in $\mathbb{A}^{n+1}$, which contradicts $S \subseteq X$ open. Exactly the same argument also works for $\mathbb{P}^{(N)}$. □

We would like to extend some constructions defined for schemes to ind-quasi-compact ind-schemes. To guarantee independence from a chosen presentation of our ind-scheme, we need the following technical lemma.

Lemma 1.2.16. *Any functor $F\colon \mathrm{Sch} \to \mathcal{C}$, where $\mathcal{C}$ is category which has $\mathbb{N}$-filtered colimits, induces a functor*

$$F\colon \{\textit{ind-quasi-compact ind-schemes}\} \to \mathcal{C}$$

via $F(\varinjlim X_i) = \varinjlim F(X_i)$ for X_i quasi-compact.

Proof. Given two presentations $X = \varinjlim_i X_i = \varinjlim_j X'_j$ by quasi-compact schemes, we obtain two objects $F(X) := \varinjlim_i F(X_i)$ and $F(X)' := \varinjlim_j F(X'_j)$ in $\mathcal{C}$. By Lemma 1.2.8, for any i there exists an j_i and a map $X_i \to X'_{j_i}$ such that $X_i \to X$ factors as $X_i \to X'_{j_i} \to X$ and similar for j. We then obtain maps $F(X_i) \to F(X'_{j_i}) \to F(X)'$ which induce $F(X) \to F(X)'$. Similarly, we obtain a map $F(X)' \to F(X)$. The composition $F(X) \to F(X)' \to F(X)$ is the identity since $F(X_i) \to F(X)' \to F(X)$ factors as

$$\Big(F(X_i) \to F(X'_{j_i}) \to F(X_{i_{j_i}}) \to F(X)\Big) = F(X_i) \to F(X). \qquad \square$$

Definition 1.2.17. For any quasi-compact ind-scheme $X = \varinjlim X_i$ we can define its global sections $\Gamma(X, \mathcal{O}_X)$ as

$$\Gamma(X, \mathcal{O}_X) = \varprojlim \Big(\Gamma(X_1, \mathcal{O}_{X_1}) \leftarrow \Gamma(X_2, \mathcal{O}_{X_2}) \leftarrow \Gamma(X_3, \mathcal{O}_{X_3}) \leftarrow \cdots\Big)$$

If X is an ind-scheme over $\mathbb{C}$, we denote this also by $\mathbb{C}[X] = \varprojlim \mathbb{C}[X_i]$.

Remark 1.2.18. Definition 1.2.17 can also be applied to open sub-ind-schemes of X in order to define a sheaf of rings on $|X|$. See [Ric19] for the definition of open sub-ind-scheme, and the equivalence between open sub-ind-scheme and open subsets of $|X|$.

Definition 1.2.19. For any ind-quasi-compact ind-scheme $X = \varinjlim X_i$ we define its reduction X_{red} as $\varinjlim (X_i)_{\mathrm{red}}$.

Definition 1.2.20. For any ind-finite type (so in particular ind-quasi-compact) ind-scheme $X = \varinjlim X_i$ over $\mathbb{C}$ we define its analytification X^{an} as $\varinjlim X_i^{\mathrm{an}}$. Recall that the analytification of a finite type scheme is defined as the set of $\mathbb{C}$-points together with the analytic topology.

Note that $X^{\mathrm{an}} = (X_{\mathrm{red}})^{\mathrm{an}}$, because this also holds for schemes.

Recall that reduced finite type schemes over $\mathbb{C}$ (varieties) are completely determined by their $\mathbb{C}$-valued points since $\mathbb{C}$ is algebraically closed. The same will also hold for ind-reduced ind-finite type ind-schemes (ind-varieties).

Proposition 1.2.21. *If X is of ind-finite type and ind-reduced, its $\mathbb{C}$-points are identified with the closed points of $|X|$. Additionally, the set of closed points are very dense in $|X|$. Recall that this means that the closed points of $|X|$ that lie in a closed subset $Z \subseteq |X|$ are dense in Z. We also have*

$$\mathrm{Hom}(X, Y) = \varprojlim_i \varinjlim_j \mathrm{Hom}(X_i, Y_j) = \varprojlim_i \varinjlim_j \mathrm{Hom}(X_i(\mathbb{C}), Y_j(\mathbb{C}))$$

where by $\mathrm{Hom}(X_i(\mathbb{C}), Y_j(\mathbb{C}))$ *we mean the classically defined regular maps between algebraic sets.*

Proof. A point $x \in |X| = \varinjlim |X_i|$ is closed if and only if $x \in |X_i|$ is closed. We obtain

$$\begin{aligned}\{\text{closed points of } |X|\} &= \varinjlim \{\text{closed points in } X_i\} \\ &= \varinjlim X_i(\mathbb{C}) \\ &= X(\mathbb{C}).\end{aligned}$$

Next, let $Z \subseteq |X|$ be closed. Then $Z \cap \{\text{closed points in } |X_i|\}$ is dense in $|X_i|$. Therefore, $Z \cap \{\text{closed points of } |X|\}$ is dense in $|X|$ because $|X|$ has the weak colimit topology. $\square$

Chapter 2

The Affine Grassmannian

For a reductive algebraic group G over $\mathbb{C}$ we are interested in the infinite-dimensional complex manifold $\mathrm{Gr}_G(\mathbb{C}) := G(\mathbb{C}((t)))/G(\mathbb{C}[[t]])$ called the Affine Grassmannian. This space actually has the structure of an ind-projective ind-scheme and so we will begin with the definition of its functor of points

$$\mathrm{Gr}_G \colon \mathbb{C}\text{-Alg} \to \text{Sets}.$$

Since we are interested in linear algebraic groups $G \hookrightarrow \mathrm{GL}_n$ and the construction will be functorial, we start with the special case of $G = \mathrm{GL}_n$.

2.1 The Affine Grassmannian of GL_n

For a ring R (commutative with 1) denote by $R[[t]]$ the ring of formal power series with coefficients in R and by $R((t))$ the ring of formal Laurent series.

Definition 2.1.1. A lattice $L \subseteq R((t))^n$ is a finite locally free $R[[t]]$-submodule of $R((t))^n$ such that $L \otimes_{R[[t]]} R((t)) \to R((t))^n \otimes_{R[[t]]} R((t)) = R((t))^n$ is an isomorphism.

Remark 2.1.2. Recall that a module M over a commutative ring R is locally free if there are elements $f_1, \dots, f_r \in R$ such that $M_{f_i} := M[f_i^{-1}]$ is a free R_{f_i}-module (where $R_{f_i} := R[f_i^{-1}]$) and the ideal generated by the ring elements f_i is equal to R. Then the following conditions are equivalent, see [Sta21, Tag 00NX].

- M is finite locally free, i.e. locally free and finitely generated.
- M is finitely generated and projective as an R-module.
- M is finitely presented and flat.

Finite locally free modules are also called vector bundles.

Example 2.1.3. The $R[[t]]$-module $R[[t]]^n \subseteq R((t))^n$ is a lattice: It is locally free as a free $R[[t]]$-module and the map

$$R[[t]]^n \otimes_{R[[t]]} R((t)) \to R((t))^n$$

is obviously an isomorphism.

We call this the standard lattice and also denote it by Γ_R.

Example 2.1.4. The $R[[t]]$-module $\langle t^1 e_1, t^2 e_2\rangle_{R[[t]]} \subseteq R((t))^2$ is a lattice, which is free with basis $t^1 e_1, t^2 e_2$. More generally, any $(k_1, \ldots, k_n) \in \mathbb{Z}^n$ defines a free lattice in $R((t))^n$ with basis $(t^{k_1} e_1, \ldots, t^{k_n} e_n)$. Putting $k_1 = \cdots = k_n = 0$ we recover the standard lattice.

Example 2.1.5. If $L \subseteq R((t))^n$ is a lattice, then $t^N L \subseteq R((t))^n$ will also be a lattice for any $N \in \mathbb{Z}$. Indeed, $L \to t^N L, m \mapsto t^N m$ is an isomorphism with inverse $t^N L \to L, m' \mapsto t^{-N} m'$ and so $t^N L$ is finite locally free. Also note that if $N \geq 0$ the above map $t^N L \otimes_{R[[t]]} R((t)) \to R(((t))^n$ factors as

$$t^N L \otimes_{R[[t]]} R((t)) \to L \otimes_{R[[t]]} R((t)) \to R((t))^n,$$

where the first map is an isomorphism induced from the embedding $t^N L \hookrightarrow L, m \mapsto m$. If $N < 0$, we have the factorization

$$L \otimes_{R[[t]]} R((t)) \to t^N L \otimes_{R[[t]]} R((t)) \to R((t))^n$$

where the first map is an isomorphism induced from the embedding $L \hookrightarrow t^N L, m \mapsto m$.

Example 2.1.6. Let $R = R_1 \times R_2$ be a commutative ring with disconnected spectrum and corresponding orthogonal idempotents p_1, p_2. Then

$$L = R_1 t^{-1} \oplus R[[t]] \subseteq R((t))^1$$

is the $R[[t]]$-submodule of $R((t))$ generated by the elements $p_1 t^{-1}, p_2 t^0$. It is locally free as an $R[[t]]$-module, because we can localize at the idempotents $p_1, p_2 \in R \subseteq R[[t]]$ to obtain

$$\begin{aligned} L \otimes_{R[[t]]} R[[t]]_{p_1} = R_1 t^{-1} \oplus R[[t]]_{p_1} &= t^{-1}(R[[t]])_{p_1} \cong (R[[t]])_{p_1}, \\ L \otimes_{R[[t]]} R[[t]]_{p_2} &= \{0\} t^{-1} \oplus R[[t]]_{p_2} \cong (R[[t]])_{p_2}. \end{aligned}$$

It also satisfies

$$\begin{aligned} L \otimes_{R[[t]]} R((t)) &= \langle p_1 t^{-1}, p_2 t^0\rangle_{R[[t]]} \otimes_{R[[t]]} R((t)) \\ &= \langle p_1 t^{-1}, p_2 t^0\rangle_{R((t))} \\ &= \langle p_1, p_2\rangle_{R((t))} \\ &= R((t))^1. \end{aligned}$$

Hence, L is a lattice in $R((t))^1$.

The following proposition gives a useful criterion to check whether a locally free $R[[t]]$-submodule of $R((t))^n$ is a lattice.

Proposition 2.1.7. *For a finitely generated $R[[t]]$-submodule L of $R((t))^n$, we have that the condition $L \otimes_{R[[t]]} R((t)) = R((t))^n$ is equivalent to*

$$t^N R[[t]]^n = t^N \Lambda_R \subseteq L \subseteq t^{-N} \Lambda_R \text{ for some } N \in \mathbb{N}_0.$$

It follows that any lattice in $R((t))^n$ is of constant rank n.

Proof. If L is finitely generated, we have $L = \langle l_1, \dots, l_r \rangle_{R[[t]]} \subseteq R((t))^n$. Now every $l_i \in R((t))^n$ can be written as $t^{-N_i} l_i'$ where l_i' lies in $R[[t]]^n$. Picking

$$N := \max(N_1, N_2, \dots, N_r)$$

we always have $L \subseteq t^{-N} R[[t]]$ without any additional condition.

Now the condition $t^N R[[t]]^n \subseteq L$ implies that

$$R((t))^n = t^N R[[t]]^n \otimes_{R[[t]]} R((t)) \subseteq L \otimes_{R[[t]]} R((t)) \subseteq R((t))^n$$

and so $L \otimes_{R[[t]]} R((t)) = R((t))^n$.

Conversely, let $L \otimes_{R[[t]]} R((t)) = R((t))^n$. But from $L = \langle l_1, \dots, l_r \rangle_{R[[t]]}$ we obtain that

$$L \otimes_{R[[t]]} R((t)) = \langle l_1, \dots, l_r \rangle_{R((t))}.$$

By the assumption, there are equations

$$\sum_i r_{i,j} l_i = e_j$$

with $r_{i,j} \in R((t))$ where e_j is the j-th standard basis vector in $R((t))^n$. Now write $r_{i,j} = t^{-N_{i,j}} r_{i,j}'$ with $r_{i,j} \in R[[t]]$ and set

$$N := \max(N_{i,j} \mid i = 1, \dots, r, j = 1, \dots, n).$$

We get $t^N R[[t]]^n \subseteq L$. $\square$

The following example is a more general version of Example 2.1.6.

Example 2.1.8. Let M be a finite locally free R-module. By Remark 2.1.2 M is a projective module and therefore a direct summand of some $R^n = M \oplus K$. We write $M \hookrightarrow R^n$ for the inclusion. Now consider the $R[[t]]$-module

$$L := Mt^{-1} \oplus R[[t]]^n \subseteq R((t))^n.$$

This is the $R[[t]]$-submodule of $R((t))^n$ generated by $R[[t]]^n$ and

$$Mt^{-1} \subseteq R^n t^{-1} \subseteq t^{-1} R[[t]]^n.$$

This module is a lattice: We immediately see that

$$t^1 R[[t]]^n \subseteq R[[t]]^n \subseteq L \subseteq t^{-1} R[[t]]^n.$$

To check that L is a lattice, it remains to show that L is a finite locally free $R[[t]]$-module. For this, let $f_1, \dots, f_r \in R$ be chosen as in Remark 2.1.2, such that $M_{f_i} \cong R_{f_i}^{n_i}$. Note that the inclusion $M \hookrightarrow R^n$ splits and so $R_f^{n_i} \cong M_{f_i} \hookrightarrow R_f^n$ is still the inclusion of a direct summand with complement K_{f_i} in $R_{f_i}^n = M_{f_i} \oplus K_{f_i}$. By localizing further we can assume that K_{f_i} is also free, i.e. $K_{f_i} \cong R^{n-n_i}$. Then

$$\begin{aligned} L \otimes_{R[[t]]} R[[t]]_{f_i} &\cong M_{f_i} t^{-1} \oplus R[[t]]_{f_i}^n \\ &\cong R_{f_i}^{n_i} t^{-1} \oplus R[[t]]_{f_i}^{n_i + (n-n_i)} \\ &\cong R[[t]]_{f_i}^{n_i} t^{-1} \oplus R[[t]]_{f_i}^{n-n_i} \end{aligned}$$

as $R[[t]]_{f_i}$-submodules of $R((t))_{f_i}^n$.

The following lemma states, that the local freeness of a lattice is never a too complicated condition. We write R_f for the ring $R[f^{-1}]$ and $R_f[[t]]$ for the power series ring of R_f. Note that $R_f[[t]] \neq R[[t]]_f$ since a general element in $R_f[[t]]$ looks like $\sum_{i\geq 0} \frac{a_i}{f^{n_i}} t^i$ with $n_i \in \mathbb{N}_0$ for all i, whereas a general element in $R[[t]]_f$ has the form $\frac{\sum_{i\geq 0} a_i t^i}{f^n}$ for a single $n \in \mathbb{N}_0$.

Lemma 2.1.9. *Assume L is a lattice. Then there are elements $f_1, \dots, f_r \in R$ such that:*

- *The elements $f_1, \dots, f_r$ generate R as an ideal, i.e. there are elements $a_i \in R$ such that $a_1 f_1 + \dots + a_r f_r = 1$.*
- *The $R_{f_i}[[t]]$-module $L \otimes_{R[[t]]} R_{f_i}[[t]]$ is free for all i.*

Proof. We follow [Gör10, Lemma 2.11]. By definition of locally free, there are elements $\widetilde{f}_1, \dots, \widetilde{f}_r \in R[[t]]$ generating the unit ideal such that

$$L[\widetilde{f}_i^{-1}] = L \otimes_{R[[t]]} R[[t]]_{\widetilde{f}_i}$$

is free as an $R[[t]]_{\widetilde{f}_i}$-module. Plugging in $t = 0$ into an equation $\sum_{i=1}^r a_i \widetilde{f}_i = 1$ in $R[[t]]$ we see that the elements $f_i := \widetilde{f}_i(0)$ generate the unit ideal in R. Now observe that $\widetilde{f}_i$ is invertible in $R_{f_i}[[t]]$ such that $R[[t]]_{\widetilde{f}_i} \subseteq R_{f_i}[[t]]$. We obtain

$$\begin{aligned} L \otimes_{R[[t]]} R_{f_i}[[t]] &= L \otimes_{R[[t]]} R[[t]]_{\widetilde{f}_i} \otimes_{R[[t]]_{\widetilde{f}_i}} R_{f_i}[[t]] \\ &\cong R[[t]]^n_{\widetilde{f}_i} \otimes_{R[[t]]_{\widetilde{f}_i}} R_{f_i}[[t]] = R_{f_i}[[t]]^n. \end{aligned}$$ □

The following is a nice lemma giving us a condition for a lattice to be free.

Lemma 2.1.10. *A lattice $L \subseteq R((t))^n$ is free as an $R[[t]]$-module if and only if $L \otimes_{R[[t]]} R$ is free as an R-module.*

Proof. We only need to show that L is free if $L \otimes_{R[[t]]} R$ is free.

We have a surjective map $\varphi\colon L \twoheadrightarrow L \otimes_{R[[t]]} R$, because R is a quotient of $R[[t]]$. Let e_j be the standard basis vectors in $L \otimes_{R[[t]]} R$ and choose elements $\widetilde{e}_j \in L$ that map to e_j under φ.

Claim: The elements $\widetilde{e}_j$ are $R[[t]]$-linearly independent.

Let $\sum_j r_j \widetilde{e}_j = 0$ with $r_j \in R[[t]]$ with some $r_j \neq 0$. We may assume that some $r_j \in R[[t]] \setminus tR[[t]]$, because otherwise $\sum_j t^{-1} r_j \widetilde{e}_j = 0$ is another $R[[t]]$-linear dependence. We apply φ to this equation and obtain $\sum_j r_j(0) e_j = 0$ with $r_j(0) \neq 0$ for some j, which contradicts the assumption that the e_j are an R-basis.

Claim: The elements $\widetilde{e}_j$ are an $R[[t]]$-spanning set of L.

Let $l \in L$. We can find $r_j^{(0)} \in R$ such that $\varphi(l) = \sum_j r_j^{(0)} e_j$. The element $l - \sum_j r_j^{(0)} \widetilde{e}_j$ lies in the kernel of φ and must therefore be of the form

$$l - \sum_j r_j^{(0)} \widetilde{e}_j = t l^{(1)}$$

with $l^{(1)} \in L$. We next construct $r_j^{(1)} \in R$ such that $l^{(1)} - \sum_j r_j^{(1)} \widetilde{e}_j$ is a multiple of t. We repeat this procedure to obtain sequences $(r_j^{(i)})_i$ in R such that

$$l - \sum_j \sum_{i=0}^{N} r_j^{(i)} t^i \in t^{N+1} L.$$

The elements $\widetilde{r}_j := \sum_{i=0}^{\infty} r_j^{(i)} t^i$ therefore satisfy

$$l = \sum_j \widetilde{r}_j \widetilde{e}_j.$$

We have now shown that $(\widetilde{e}_j)_j$ forms an $R[[t]]$-basis of L. $\square$

We now come to the definition of the Affine Grassmannian — our central object of study.

Definition 2.1.11. The Affine Grassmannian of GL_n is the functor

$$\begin{aligned} \mathrm{Gr}_{\mathrm{GL}_n} \colon \mathbb{C}\text{-Alg} &\longrightarrow \text{Sets} \\ R &\longmapsto \{\text{lattices in } R((t))^n\} \end{aligned}$$

sending a ring R to the set of lattices in $R((t))^n$.

We next compute the $\mathbb{C}$-points of this functor. Later we will prove that $\mathrm{Gr}_{\mathrm{GL}_n}$ is represented by an ind-projective ind-scheme and therefore we will be able to identify its reduction with its $\mathbb{C}$-points.

Proposition 2.1.12. *The $\mathbb{C}$-points of $\mathrm{Gr}_{\mathrm{GL}_n}$ are precisely*

$$\mathrm{Gr}_{\mathrm{GL}_n}(\mathbb{C}) = \mathrm{GL}_n(\mathbb{C}((t)))/\mathrm{GL}_n(\mathbb{C}[[t]]).$$

Proof. As $\mathbb{C}[[t]]$ is a principal ideal domain, any finitely generated projective module is automatically free. Any lattice in $\mathbb{C}((t))^n$ is therefore given by n $\mathbb{C}[[t]]$-linearly independent vectors in $\mathbb{C}((t))^n$, which means that they are a basis of the vector space $\mathbb{C}((t))^n$. Hence any lattice is given as the column span of an element of $\mathrm{GL}_n(\mathbb{C}((t)))$. Two such matrices yield the same lattice if and only if they differ by an element of $\mathrm{GL}_n(\mathbb{C}[[t]])$. $\square$

Example 2.1.13. Consider the $\mathbb{C}[[t]]$-module generated by the columns of the matrix $A = \begin{pmatrix} t^2 & 0 \\ 0 & t^{-3} \end{pmatrix}$. This is a lattice. It is the same as the $\mathbb{C}[[t]]$-module generated by the columns of $\begin{pmatrix} t^2 & t^3 \\ 0 & t^{-3} \end{pmatrix} = A \cdot \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$.

Example 2.1.14. The $\mathbb{C}$-points of $\mathrm{Gr}_{\mathrm{GL}_1}$ are given by

$$\begin{aligned}\mathrm{Gr}_{\mathrm{GL}_1}(\mathbb{C}) &= \mathrm{GL}_1(\mathbb{C}((t)))/\mathrm{GL}_1(\mathbb{C}[[t]]) \\ &= \mathbb{C}((t))^*/\mathbb{C}[[t]]^* \\ &= (\mathbb{C}((t)) \setminus \{0\}) \Big/ \left\{ a = \sum_{i\geq 0} a_i t^i \in \mathbb{C}[[t]] \;\middle|\; a_0 \neq 0 \right\} \\ &= \{[t^n] \mid n \in \mathbb{Z}\} \\ &= \mathbb{Z}.\end{aligned}$$

In the next example we compute the R points of $\mathrm{Gr}_{\mathrm{GL}_1}$ for the ring $R = \mathbb{C}[x]/(x^2)$.

Example 2.1.15. Consider $\mathbb{C}[\varepsilon] = \mathbb{C}[x]/(x^2)$. This is a local ring and so all locally free modules are free. We write $\mathbb{C}[\varepsilon][[t]]$ for its ring of power series. We obtain

$$\begin{aligned}\mathrm{Gr}_{\mathrm{GL}_1}(\mathbb{C}[\varepsilon]) &= \{\text{free rank 1 } \mathbb{C}[\varepsilon][[t]]\text{-submodules of } \mathbb{C}[\varepsilon]((t))\} \\ &= \left(\mathbb{C}[\varepsilon]((t))\right)^*/(\mathbb{C}[\varepsilon][[t]])^*\end{aligned}$$

Note that the invertible elements of $\mathbb{C}[\varepsilon]((t))$ are precisely the preimage of the invertible elements of $\mathbb{C}((t))$ under

$$\mathbb{C}[\varepsilon]((t)) \to \mathbb{C}((t))$$

mapping ε to 0. Therefore

$$\left(\mathbb{C}[\varepsilon]((t))\right)^* = \Big(\mathbb{C}[\varepsilon]((t)) \setminus \varepsilon\mathbb{C}[\varepsilon]((t))\Big).$$

We observe that we can multiply an element

$$x = a_{n-r}\varepsilon t^{n-r} + \cdots + a_{n-2}\varepsilon t^{n-2} + a_{n-1}\varepsilon t^{n-1} + a_n t^n + \cdots \in \mathbb{C}[\varepsilon]((t))^*$$

with $a_n \notin \varepsilon\mathbb{C}[\varepsilon]$ by a unique element $y \in \mathbb{C}[[t]]^*$ to obtain an element of the form

$$xy = a'_{n-r}\varepsilon + \cdots + a'_{n-2}\varepsilon t^{n-2} + a'_{n-1}\varepsilon t^{n-1} + t^n.$$

Therefore we have

$$\begin{aligned}\mathrm{Gr}_{\mathrm{GL}_1}(\mathbb{C}[\varepsilon]) &= \left(\mathbb{C}[\varepsilon]((t))\right)^*/(\mathbb{C}[\varepsilon][[t]])^* \\ &= \left\{ [a_{n-r}\varepsilon t^{n-r} + \cdots + a_{n-1}\varepsilon t^{n-1} + t^n] \;\middle|\; \begin{matrix} n \in \mathbb{Z}, r \in \mathbb{N}_0, \\ a_i \in \mathbb{C} \end{matrix} \right\} \\ &= \mathbb{C}^{(\infty)} \times \mathbb{Z},\end{aligned}$$

where $\mathbb{C}^{(\infty)} = \{(\ldots, a_{-2}, a_{-1}, a_0) \in \mathbb{C}^\infty \mid a_i \neq 0 \text{ finitely often}\}$.

Theorem 2.1.16. *The functor $\mathrm{Gr}_{\mathrm{GL}_n}$ is representable by an ind-projective ind-scheme, which means that there are projective schemes $\mathrm{Gr}^{(N)}_{\mathrm{GL}_n}$ together with closed immersions $\mathrm{Gr}^{(N)}_{\mathrm{GL}_n} \hookrightarrow \mathrm{Gr}^{(N+1)}_{\mathrm{GL}_n}$ such that for all $R \in \mathbb{C}$-Alg*

$$\mathrm{Gr}_{\mathrm{GL}_n}(R) = \varinjlim_{N} \mathrm{Gr}^{(N)}_{\mathrm{GL}_n}(R).$$

We follow the proof explained in [Zhu16]. We begin the proof with the definition of the $\mathrm{Gr}^{(N)}_{\mathrm{GL}_n}$, which we abbreviate by $\mathrm{Gr}^{(N)}$.

Definition 2.1.17. We define $\mathrm{Gr}^{(N)} \colon \mathbb{C}\text{-Alg} \to \text{Sets}$ via

$$\mathrm{Gr}^{(N)}(R) \coloneqq \{L \subseteq R((t))^n \mid L \text{ is a lattice and } t^N R[[t]]^n \subseteq L \subseteq t^{-N} R[[t]]^n\},$$

together with embeddings of functors

$$\mathrm{Gr}^{(N)}(R) \hookrightarrow \mathrm{Gr}^{(N+1)}(R), \quad L \mapsto L.$$

Example 2.1.18. In the case $R = \mathbb{C}$, we have that any finite $\mathbb{C}[[t]]$-submodule of $\mathbb{C}((t))^n$ is automatically free, since it is torsion free. And so

$$\begin{aligned} \mathrm{Gr}^{(N)}(\mathbb{C}) &= \{\mathbb{C}[[t]]\text{-modules } L \text{ such that } t^N \mathbb{C}[[t]]^n \subseteq L \subseteq t^{-N}\mathbb{C}[[t]]^n\} \\ &= \left\{ L/t^N\mathbb{C}[[t]]^n \subseteq t^{-N}\mathbb{C}[[t]]^n/t^N\mathbb{C}[[t]]^n \cong \mathbb{C}^{2Nn} \right\} \\ &\subseteq \mathrm{Grass}(2Nn)(\mathbb{C}) = \{\text{linear subspaces of } \mathbb{C}^{2Nn}\}. \end{aligned}$$

We can identify $\mathrm{Gr}^{(N)}(\mathbb{C})$ with those subspaces of $M_{\mathbb{C}} \coloneqq t^{-N}\mathbb{C}[[t]]^n/t^n\mathbb{C}[[t]]^n \cong \mathbb{C}^{2Nn}$ which are stable under the action of the nilpotent endomorphism $M_{\mathbb{C}} \to M_{\mathbb{C}}$ given by multiplication with t.

In the case GL_1 we have

$$\begin{aligned} \mathrm{Gr}^{(N)}_{\mathrm{GL}_1}(\mathbb{C}) &= \{t^k\mathbb{C}[[t]] \mid -N \leq k \leq N\} \\ &= \{\text{vector subspaces of } \mathbb{C}^{2N} \text{ with basis of the form } e_k, e_{k+1}, \dots, e_{N-1}\} \end{aligned}$$

where the $(e_{-N}, \dots, e_{N-1})$ denotes a fixed basis of $\mathbb{C}^{2N}$, which correspond to $t^{-N}, \dots, t^{N-1}$ under $t^{-N}\mathbb{C}[[t]]/t^N\mathbb{C}[[t]] \cong \mathbb{C}^{2N}$. We see that

$$\#\mathrm{Gr}^{(N)}_{\mathrm{GL}_1}(\mathbb{C}) = 2N + 1;$$

in particular $\mathrm{Gr}_{\mathrm{GL}_1}$ is the colimit of zero dimensional schemes. In Example 2.1.15, we saw that $\mathrm{Gr}_{\mathrm{GL}_1}(\mathbb{C}[\varepsilon]) \neq \mathrm{Gr}_{\mathrm{GL}_1}(\mathbb{C})$. Therefore $\mathrm{Gr}_{\mathrm{GL}_1}$ cannot be ind-reduced.

We write $\mathrm{Grass}_d(n)$ for the classical Grassmannian of linear d-dimensional subspaces of $\mathbb{C}^n$. We write $\mathrm{Grass}(n)$ for the disjoint union of all $\mathrm{Grass}_d(n)$ with $d \leq n$.

Proof of Theorem 2.1.16. As defined these $\mathrm{Gr}^{(N)}$ satisfy

$$\mathrm{Gr}_{\mathrm{GL}_n}(R) = \varinjlim_N \mathrm{Gr}^{(N)}(R)$$

by Proposition 2.1.7. If these subfunctors are represented by projective schemes, the transition maps $\mathrm{Gr}^{(N)} \hookrightarrow \mathrm{Gr}^{(N+1)}$ will be proper. These maps are injective on R-points and so define proper monomorphisms of schemes. However, proper monomorphisms are closed immersions by [Sta21, Tag 04XV]. So we now need to show that $\mathrm{Gr}^{(N)}$ is in fact (represented by) a projective scheme. For this, let $M_R := (t^{-N}R[[t]]^n)/(t^N R[[t]]^n) = M_{\mathbb{C}} \otimes_{\mathbb{C}} R \cong R^{2Nn}$. Consider the map

$$\begin{aligned}\mathrm{Gr}^{(N)}(R) &\longrightarrow \{U \subseteq M_R \cong R^{2Nn} \mid M_R/U \text{ is a locally free } R\text{-module}\},\\ L &\longmapsto L/(t^N R[[t]]^n).\end{aligned}$$

To show this map is well-defined, we need to prove that $M_R/(L/t^N R[[t]]^n) \cong t^{-N}R[[t]]^n/L$ is a finite locally free R-module. By Lemma 2.1.9, we find $f_1, \dots, f_r$ generating the unit ideal in R such that $L \otimes_{R[[t]]} R_{f_i}[[t]]$ is free. So, since we need to work locally on R, we can just assume that L is free. Observe now that

$$R((t))^n/L = L[t^{-1}]/L \cong R((t))^n/R[[t]]^n \cong \bigoplus_{i<0} t^i R^n$$

is a free R-module. Therefore we obtain the short exact sequence

$$0 \to t^{-N}R[[t]]^n/L \to R((t))^n/L \to R((t))/t^{-N}R[[t]]^n \to 0.$$

Note that the term on the right is R-free and therefore projective. Thus, this sequence splits and the term on the left is projective as a direct summand of the free middle term. So this map is well-defined.

Notice that the codomain is precisely the functor represented by the classical Grassmannian $\mathrm{Grass}(2Nn)$, see [GW10, Chapter 8.4], which is a projective scheme. So we need to show that this map identifies $\mathrm{Gr}^{(N)}$ with a closed subscheme of the classical Grassmannian of subspaces in $\mathbb{C}^{2Nn}$.

Notice how multiplication by t induces a nilpotent endomorphism on M_R and that every element in the image of the map $\mathrm{Gr}^{(N)} \to \mathrm{Grass}(2Nn)$ will be fixed by this endomorphism. It is the content of the next lemma that $\mathrm{Gr}^{(N)}(R)$ gets identified with

$$\mathrm{Grass}(2Nn)^t(R) := \{U \subseteq M_R \mid M_R/U \text{ locally free}, tU \subseteq U\}.$$

This is a closed subscheme of $\mathrm{Grass}(2Nn)$ and we only need to show the lemma to conclude the proof. $\square$

Lemma 2.1.19. *The map*

$$\mathrm{Gr}^{(N)}(R) \to \mathrm{Grass}(2Nn)^t(R), \quad L \mapsto L/t^N R[[t]]^n$$

is a natural bijection for all R and therefore gives an isomorphism of functors between $\mathrm{Gr}^{(N)}$ and a closed subscheme of the classical Grassmannian. In particular, we obtain that the functor $\mathrm{Gr}^{(N)}$ is represented by a projective scheme.

Proof. The map is clearly injective as $t^N R[[t]]^n \subseteq L$ by assumption.

For surjectivity we reduce first to the Noetherian case. Let $U \in \mathrm{Grass}(2Nn)^t(R)$. Then U will be a direct summand of M_R and so will be generated by finitely many elements of the form $\sum_{i=1}^n a_{i,j}e_i$ finite. Now, U will already be defined over $\mathbb{C}[a_{i,j} \mid i,j]$. So we can assume that R is Noetherian.

Noetherianness implies that $R[t] \to R[[t]]$ is a flat ring map (see [Sta21, Tag 00MB]). Also observe that $t^{-N}R[t]^n/t^N R[t]^n \cong t^{-N}R[[t]]^n/t^N R[[t]]^n$.
Now consider the composition

$$\varphi\colon t^{-N}R[t]^n \to t^{-N}R[t]^n/t^N R[t]^n \cong t^{-N}R[[t]]^n/t^N R[[t]]^n \to t^{-N}R[[t]]^n/U$$

and set $L_f \coloneqq \ker(\varphi)$. Then we get by the flatness of $R[t] \to R[[t]]$ that

$$\left(L_f \otimes_{R[t]} R[[t]]\right)/t^N R[[t]]^n = U.$$

So if we show that L_f is a finite locally free $R[t]$-module, we have found the required preimage of U and we are done. Since we are in the Noetherian case, L_f is finitely presented, so we only need to check that L_f is flat. Using [Sta21, Tag 00MH] we can reduce this to the case where R is a field, and thus $R[t]$ a principal ideal domain. However in this case, since $L_f \subseteq t^{-N}R[[t]]^n$, it is torsion free and thus free. □

We have proved on the way the following:

Corollary 2.1.20. *For all rings R we have*

$$\mathrm{Gr}_{\mathrm{GL}_n}(R) = \left\{ \begin{matrix} \textit{finite locally free } R[t]\textit{-submodules } L_f \subseteq R[t,t^{-1}]^n \\ \textit{such that } L_f[t^{-1}] = R[t,t^{-1}]^n \end{matrix} \right\}.$$

We make this corollary explicit in the case $R = \mathbb{C}$.

Example 2.1.21. Any lattice $L \subseteq \mathbb{C}((t))^n$ is given as the column span of a matrix $A \in \mathrm{GL}_n(\mathbb{C}((t)))$, see Proposition 2.1.18. Now Corollary 2.1.20 translates to the following: For any $A \in \mathrm{GL}_n(\mathbb{C}((t)))$ there is a matrix $B \in \mathrm{GL}_n(\mathbb{C}[[t]])$ such that $AB \in \mathrm{GL}_n(\mathbb{C}[t,t^{-1}])$, because we can identify the right hand side in the corollary with $\mathrm{GL}_n(\mathbb{C}[t,t^{1-}])/\mathrm{GL}_n(\mathbb{C}[t])$ as in Proposition 2.1.12. We can even choose B in such a way that AB is obtained from A by deleting all terms of a high enough degree: Let $A \in \mathrm{GL}_n(\mathbb{C}((t)))$ such that the smallest degree term appearing in any entry of A^{-1} is of degree N, i.e. such that $t^N A^{-1} \in \mathbb{C}[[t]]^{n\times n}$. Write $A = A_0 + t^{N+1}\widetilde{A}$ with $A_0 \in \mathbb{C}[t,t^{-1}]^{n\times n}$ and $\widetilde{A} \in \mathbb{C}[[t]]^{n\times n}$. Let $B = I_n - t^{N+1}A^{-1}\widetilde{A}$ where I_n is denotes the unit matrix. By construction, B lies in $\mathbb{C}[[t]]^{n\times n}$ and even in $\mathrm{GL}_n(\mathbb{C}[[t]])$ as it is of the form $I_n + t\widetilde{B}$. This matrix satisfies

$$A \cdot B = A \cdot (I_n - t^{N+1}A^{-1}\widetilde{A}) = A - t^{N+1}\widetilde{A} = A_0.$$

Example 2.1.22. Consider a matrix of the form

$$A = \begin{pmatrix} t^k a & b \\ t^k c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{C}((t)))$$

with $k \geq 0$ and $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \sum_{i\geq 0} a_i t^i & \sum_{i\geq 0} b_i t^i \\ \sum_{i\geq 0} c_i t^i & \sum_{i\geq 0} d_i t^i \end{pmatrix} \in \mathrm{GL}_2(\mathbb{C}[[t]])$. The residue class $[A] \in \mathrm{Gr}_{\mathrm{GL}_2}(\mathbb{C}) = \mathrm{GL}_2(\mathbb{C}((t)))/\mathrm{GL}_2(\mathbb{C}[[t]])$ lies in $\mathrm{Gr}^{(k)}_{\mathrm{GL}_2}(\mathbb{C})$ and gets identified with the subspace of $\mathbb{C}^{2\cdot 2k}$

$$\begin{aligned}
\left\langle \begin{pmatrix} t^k a \\ t^k c \end{pmatrix}, \begin{pmatrix} b \\ d \end{pmatrix} \right\rangle_{\mathbb{C}[[t]]} \Big/ t^k\mathbb{C}[[t]]^2 &= \left\langle t^i \begin{pmatrix} t^k a \\ t^k c \end{pmatrix}, t^j \begin{pmatrix} b \\ d \end{pmatrix} \;\middle|\; i,j \in \mathbb{N}_0 \right\rangle_{\mathbb{C}} \Big/ t^k\mathbb{C}[[t]]^2 \\
&= \left\langle t^j \begin{pmatrix} b \\ d \end{pmatrix} \;\middle|\; j = 0,\dots k-1 \right\rangle_{\mathbb{C}} \Big/ t^k\mathbb{C}[[t]]^2 \\
&= \left\langle \sum_{i=0}^{k-1} b_i e_{i+j,1} + \sum_{i=0}^{k-1} d_i e_{i+j,2} \;\middle|\; j = 0,\dots k-1 \right\rangle_{\mathbb{C}} \\
&\subseteq \mathbb{C}^{2\cdot 2k} = t^{-k}\mathbb{C}[[t]]^2 / t^k\mathbb{C}[[t]]^2
\end{aligned}$$

with basis elements $e_{i,j}$ corresponding to $t^i e_j \in \mathbb{C}((t))^2$.

2.2 The Affine Grassmannian of General Groups

Definition 2.2.1. Let G be a group scheme over R. We call a scheme $\mathcal{E} \to \operatorname{Spec} R$ together with an operation of G on $\mathcal{E}$ a G-torsor, if G operates on $\mathcal{E}$ such that $\mathcal{E}$ is fppf-locally trivial. Recall that fppf-local tirivality means that there is a faithfully flat ring map of finite presentation $R \to R'$ such that after base change to R'

$$\mathcal{E} \times_{\operatorname{Spec} R} \operatorname{Spec} R' \cong G \times_{\operatorname{Spec} R} \operatorname{Spec} R'$$

and all involved maps are G-equivariant.

Remark 2.2.2. Assume that $\mathcal{E}$ is Zariski-locally trivial, meaning that there is an open covering $\operatorname{Spec} R = \bigcup_i U_i$ such that $\mathcal{E} \times_{\operatorname{Spec} R} U_i \cong G \times_{\operatorname{Spec} R} U_i$. It is automatically fppf-locally trivial, since we can assume that all $U_i = \operatorname{Spec} R_i$ are affine and that there are finitely many of them. Then $R \to R' = \bigtimes_i R_i$ is a fppf-local trivialization.

In the language of Grothendieck topologies, this is just to say that the fppf topology is finer than the Zariski topology. Note that the concept does not refer literally to a topology on some set, but rather to a certain family of covering morphisms in a category, c.f. [Sta21, Tag 020K].

In order to generalize the definition of the Affine Grassmannian to other groups than GL_n we rewrite $\mathrm{GL}_{\mathrm{GL}_n}(R) = \{\text{lattices in } R((t))^n\}$ in terms of GL_n-torsors. Recall that a lattice is a vector bundle on $R[[t]]$.

Lemma 2.2.3. *The category of* GL_n*-torsors on* R *is identified with vector bundles of rank* n *on* R.

Proof. Let V be a rank n vector bundle which is trivial on $\operatorname{Spec} R_i \subseteq \operatorname{Spec} R$. V consists of all elements $\{(x_i)_i \mid x_i|_{R_{ij}} \overset{\varphi_{ij}}{\mapsto} x_j|_{R_{ij}}\}$ where the φ_{ij} satisfy the cocycle condition. On the other hand, by [GR03, Exposé XI, Propositon 5.1] any GL_n-torsor is Zariski-locally trivial and so the elements of a GL_n-torsor are given by matrices $\psi_i \in \mathrm{GL}_n(R_i)$ such that $\psi_i|_{R_{ij}} \overset{\varphi_{ij}}{\mapsto} \psi_j|_{R_{ij}}$. ∎

Theorem 2.2.4. *The functor* $\mathbb{C}$-Alg $\to$ Sets *sending a ring* R *to the set*

$$\left\{ (\mathcal{E}, \beta) \;\middle|\; \begin{array}{c} \mathcal{E} \textit{ is a } \mathrm{GL}_n\textit{-torsor on } \operatorname{Spec} R[[t]] \textit{ such that} \\ \beta\colon \mathcal{E} \times_{R[[t]]} \operatorname{Spec} R((t)) \xrightarrow{\sim} \mathrm{GL}_n(R[[t]]) \times_{R[[t]]} \operatorname{Spec} R((t)) \end{array} \right\}$$

is isomorphic to $\mathrm{Gr}_{\mathrm{GL}_n}$.

Proof. This follows immediately from Lemma 2.2.3 as $\mathcal{E}$ gets identified with a finite locally free $R[[t]]$-module L and β defines the embedding $L \subseteq R((t))^n$. $\square$

We now finally come to the definition of the Affine Grassmannian for general groups. Note that all affine algebraic groups over $\mathbb{C}$ are smooth.

Definition 2.2.5. For G an affine algebraic group over $\mathbb{C}$ we set

$$\mathrm{Gr}_G(R) \coloneqq \left\{ (\mathcal{E}, \beta) \;\middle|\; \begin{array}{c} \mathcal{E} \text{ is a } G_{R[[t]]}\text{-torsor on } \operatorname{Spec} R[[t]] \text{ such that} \\ \beta\colon \mathcal{E} \times_{R[[t]]} \operatorname{Spec} R((t)) \xrightarrow{\sim} G(R[[t]]) \times_{R[[t]]} \operatorname{Spec} R((t)) \end{array} \right\}.$$

Proposition 2.2.6. *If every* $G_{R[[t]]}$*-torsor on* $\operatorname{Spec} R[[t]]$ *is trivial, then*

$$\mathrm{Gr}_G(R) = G(R((t)))/G(R[[t]]).$$

Proof. We have

$$\mathrm{Gr}_G(R) = \{G_{R((t))}\text{-equivariant automorphisms } \beta\colon G(R((t))) \to G(R((t)))\}/\sim$$

where $\beta \sim \beta'$ if they differ by an automorphism which lives on the level of $G(R[[t]])$, since $(\mathcal{E}, \beta) = (\mathcal{E}', \beta')$ if there is an isomorphism $\alpha\colon \mathcal{E} \to \mathcal{E}'$ such that

$$\begin{tikzcd} \mathcal{E} \otimes_{R[[t]]} R((t)) \arrow[rr, "\alpha"] \arrow[dr, "\beta"'] & & \mathcal{E}' \otimes_{R[[t]]} R((t)) \arrow[dl, "\beta'"] \\ & \mathcal{E}_0. & \end{tikzcd}$$

By the equivariance, any automorphism of the trivial torsor is just given by a group element. And so

$$\mathrm{Gr}_G(R) = G(R((t)))/G([[t]])$$

as claimed. $\square$

The following subsumes Proposition 2.1.12.

Proposition 2.2.7. *If* G *is an affine algebraic group over* $\mathbb{C}$ *then*

$$\mathrm{Gr}_G(\mathbb{C}) = G(\mathbb{C}((t)))/\mathrm{Gr}(\mathbb{C}[[t]]).$$

Proof. Since G is smooth, there is an étale cover $\operatorname{Spec} R' \to \operatorname{Spec} R$ such that $\mathcal{E}$ becomes trivial after base change to $R'[[t]]$ by [Zhu16, Lemma 1.3.7]. But as $\mathbb{C}$ is separably closed there are no non-trivial étale extensions $\mathbb{C} \to R'$ and therefore any G-torsor on $\mathbb{C}[[t]]$ is trivial and the assertion follows from Proposition 2.2.6. $\square$

Note that the definition of the Affine Grassmannian is functorial in G. The next lemmas allow us to deduce from this functorality that Gr_G is an ind-projective ind-scheme if G is reductive.

Theorem 2.2.8. *If G is reductive, then Gr_G is an ind-projective ind-scheme.*

Proof. If G is reductive, G can be realized as a closed subgroup of GL_n such that GL_n/G is affine by [Alp14, Corollary 9.7.7]. By the following lemma, $\mathrm{Gr}_G \hookrightarrow \mathrm{Gr}_{\mathrm{GL}_n}$ is a closed immersion, but $\mathrm{Gr}_{\mathrm{GL}_n}$ is ind-projective and then so is Gr_G. $\square$

Lemma 2.2.9. *If $G \hookrightarrow \mathrm{GL}_n$ is a closed immersion of affine group schemes such that GL_n/G is quasi-affine, then the induced map $\mathrm{Gr}_G \to \mathrm{Gr}_{\mathrm{GL}_n}$ is an immersion. If GL_n/G is affine, then $\mathrm{Gr}_G \to \mathrm{Gr}_{\mathrm{GL}_n}$ is a closed immersion.*

The proof can be found in [Zhu16, Proposition 1.2.5].

Example 2.2.10. The group $\mathbb{G}_a$ is not reductive and so we do not expect $\mathrm{Gr}_{\mathbb{G}_a}$ to be ind-projective. We can show that every $\mathbb{G}_a$ on an affine scheme is trivial, as by [GR03, Exposé XI, Propositon 5.1] every $\mathbb{G}_a$-torsor is Zariski-locally trivial. However, every Zariski-locally trivial $\mathbb{G}_a$-torsor on an affine scheme is already trivial, since Zariski-locally trivial G-torsors on a scheme X for an abelian group G are classified by the first sheaf cohomology group $H^1(X, G)$ of the abelian sheaf G by [Sta21, Tag 02FQ]. But $H^1(\operatorname{Spec} A, \mathbb{G}_a) = 0$ as $\mathbb{G}_a$ is quasi-coherent. Therefore,

$$\begin{aligned}
\mathrm{Gr}_{\mathbb{G}_a}(R) &= \mathbb{G}_a(R((t)))/\mathbb{G}_a(R[[t]]) = (R((t)), +)/(R[[t]], +) \\
&= \{(a_{-1}, a_{-2}, \dots) \in R^{\mathbb{N}} \mid \text{only finitely many } a_i \text{ are non-zero}\} \\
&= \mathbb{A}_{\mathbb{C}}^{(\infty)}(R).
\end{aligned}$$

We therefore obtain $\mathrm{Gr}_{\mathbb{G}_a} = \mathbb{A}^{(\mathbb{N})}$, which is not ind-projective.

Note that later on we will only work with the analytification of Gr_G. But the analytification factors through the reduction $(\mathrm{Gr}_G)_{\mathrm{red}}$. We will therefore prefer to work with $(\mathrm{Gr}_G)_{\mathrm{red}}$ over Gr_G.

As we now know that Gr_G is an ind-finite type ind-scheme over the algebraically closed field $\mathbb{C}$, we may identify its reduction with its $\mathbb{C}$-valued points $G(\mathbb{C}((t)))/G(\mathbb{C}[[t]])$.

Proposition 2.2.11. *For the product $G = G_1 \times G_2$ we have*

$$(\mathrm{Gr}_G)_{\mathrm{red}} = (\mathrm{Gr}_{G_1})_{\mathrm{red}} \times (\mathrm{Gr}_{G_2})_{\mathrm{red}}.$$

Proof. We check this on $\mathbb{C}$-valued points, which can be computed using Proposition 2.2.7. We have

$$\begin{aligned}
\mathrm{Gr}_G(\mathbb{C}) &= G(\mathbb{C}((t)))/G(\mathbb{C}[[t]]) \\
&= \Big(G_1(\mathbb{C}((t))) \times G_2(\mathbb{C}((t)))\Big)\Big/\Big(G_1(\mathbb{C}[[t]]) \times G_2(\mathbb{C}[[t]])\Big) \\
&= (G_1(\mathbb{C}((t)))/G_1(\mathbb{C}[[t]])) \times (G_2(\mathbb{C}((t)))/G_2(\mathbb{C}[[t]])) \\
&= \mathrm{Gr}_{G_1}(\mathbb{C}) \times \mathrm{Gr}_{G_2}(\mathbb{C}).
\end{aligned}$$

The result for $(\mathrm{Gr}_G)_{\mathrm{red}}$ follows from Proposition 1.2.21. $\square$

Example 2.2.12. If $G = T = \mathbb{G}_m^n$ is a torus, $\mathrm{Gr}_G(\mathbb{C})$ equals $X_*(G) = \mathbb{Z}^n$. Note that all tori are split, as we are working over $\mathbb{C}$.

We now consider what the functorality of the Affine Grassmannian does in type A. We only look at $\mathbb{C}$-points.

Recall that $\mathrm{Gr}_{\mathrm{GL}_1}(\mathbb{C})$ is a disjoint union of $\mathbb{Z}$-many points. The determinant map $\mathrm{GL}_n \to \mathrm{GL}_1$ induces a surjective map $\mathrm{Gr}_{\mathrm{GL}_n}(\mathbb{C}) \to \mathbb{Z}$, which maps a coset $[A] \in \mathrm{Gr}_{\mathrm{GL}_n}$ to the unique integer $\nu_t(\det A)$ satisfying

$$t^{-\nu_t(\det A)} \det A \in \mathbb{C}[[t]]^*.$$

Proposition 2.2.13. *The fibers of the map* $\mathrm{Gr}_{\mathrm{GL}_n}(\mathbb{C}) \to \mathrm{Gr}_{\mathrm{GL}_1}(\mathbb{C}) = \mathbb{Z}$ *are the connected components of* $\mathrm{Gr}_{\mathrm{GL}_n}(\mathbb{C})$.

Proof. Obviously, the fibers are open and closed. So we only need to show that they are connected. Let $[A] = A\mathrm{GL}_n(\mathbb{C}[[t]]) \in \mathrm{Gr}_{\mathrm{GL}_n}(\mathbb{C})$ be given. We show that there is a path in $\mathrm{Gr}(\mathbb{C})$ from $[A]$ to

$$\left[\begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & t^{\nu_t(\det A)} \end{pmatrix}\right].$$

We first show that any element can be moved to a diagonal element of the form $\mathrm{Diag}(t^{k_1}, \dots t^{k_n})$. Indeed, for any $A' \in \mathbb{C}[[t]]^{n\times n}$ there is a matrix $B \in \mathrm{GL}_n(\mathbb{C}[[t]])$ such that $A'B$ is upper triangular. So we apply this to $A' = t^N A \in \mathbb{C}[[t]]^{n\times n}$ for the given $A \in \mathrm{GL}_n(\mathbb{C}((t)))$. Now write $A = T + C$ where T is an invertible diagonal matrix and C is strictly upper diagonal. Then $T + rC$ for $r \in [0, 1]$ is a path from A to T in $\mathrm{GL}_n(\mathbb{C}((t)))$ and thus from $[A]$ to $[T]$ in $\mathrm{Gr}_{\mathrm{GL}_n}(\mathbb{C})$. But we can assume that T is of the form $T = \begin{pmatrix} t^{k_1} & & \\ & \ddots & \\ & & t^{k_n} \end{pmatrix}$ by multiplying with a diagonal matrix in $\mathrm{GL}_n(\mathbb{C}[[t]])$. We now show that there is a path in Gr between the 2×2 matrices $\begin{pmatrix} t^k & 0 \\ 0 & t^l \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 0 & t^{k+l} \end{pmatrix}$. By concatenating paths like this we obtain a path from T to a matrix of the required form.

There is a path $r \mapsto \left[\begin{pmatrix} t^k + r^2 & r \\ rt^l & t^l \end{pmatrix}\right]$ from $\left[\begin{pmatrix} t^k & 0 \\ 0 & t^l \end{pmatrix}\right]$ to $\left[\begin{pmatrix} t^k + 1 & 1 \\ t^l & t^l \end{pmatrix}\right]$ in Gr. Notice that $(t^k+1)^{-1}$ always lies in $\mathbb{C}[[t]]$. We get

$$\begin{aligned}\left[\begin{pmatrix} t^k+1 & 1 \\ t^l & t^l \end{pmatrix}\right] &= \left[\begin{pmatrix} t^k+1 & 1 \\ t^l & t^l \end{pmatrix} \cdot \begin{pmatrix} 1 & -(t^k+1)^{-1} \\ 0 & 1 \end{pmatrix}\right] \\ &= \left[\begin{pmatrix} t^k+1 & 0 \\ t^l & (t^k+1)^{-1}t^{k+l} \end{pmatrix}\right] \\ &= \left[\begin{pmatrix} t^k+1 & 0 \\ t^l & (t^k+1)^{-1}t^{k+l} \end{pmatrix} \cdot \begin{pmatrix} (t^k+1)^{-1} & 0 \\ 0 & t^k+1 \end{pmatrix}\right] \\ &= \left[\begin{pmatrix} 1 & 0 \\ * & t^{k+l} \end{pmatrix}\right].\end{aligned}$$

From this there is a path to $\left[\begin{pmatrix} 1 & 0 \\ 0 & t^{k+l} \end{pmatrix}\right]$. □

Observe that multiplication from the left by t is an automorphism of $\mathrm{Gr}_{\mathrm{GL}_n}(\mathbb{C})$ which induces an isomorphism between the i-th connected component and the $(i+n)$-th one. Therefore $\mathrm{Gr}_{\mathrm{GL}_n}$ is the infinite disjoint union of its first n components.

We now examine the 0-th component of $\mathrm{Gr}_{\mathrm{GL}_n}$, after which we will consider the union of the first n-components.

Theorem 2.2.14. *The inclusion $\mathrm{SL}_n \hookrightarrow \mathrm{GL}_n$ induces an isomorphism between $\mathrm{Gr}_{\mathrm{SL}_n}$ and the (reduction of the) 0-th component of $\mathrm{Gr}_{\mathrm{GL}_n}$. In particular, $\mathrm{Gr}_{\mathrm{SL}_n}$ is connected.*

Proof. We only need to look at the inclusion on the level of $\mathbb{C}$-points

$$\mathrm{SL}_n(\mathbb{C}((t)))/\mathrm{SL}_n(\mathbb{C}[[t]]) \hookrightarrow \mathrm{GL}_n(\mathbb{C}((t)))/\mathrm{GL}_n(\mathbb{C}[[t]]).$$

It is clear that $\mathrm{Gr}_{\mathrm{SL}_n}(\mathbb{C})$ maps injectively into the 0-th component of $\mathrm{Gr}_{\mathrm{GL}_n}$, since for $A \in \mathrm{SL}_n(\mathbb{C}((t)))$ we have $\nu_t(\det A) = \nu_t(1) = 0$. But it is also clear that this map is surjective as any $A \in \mathrm{GL}_n(\mathbb{C}((t)))$ with $\det A \in \mathbb{C}[[t]]^*$ can be multiplied with a fitting matrix in $\mathrm{GL}_n(\mathbb{C}[[t]])$ such that the product lies in $\mathrm{SL}_n(\mathbb{C}((t)))$. □

As before consider the multiplication by t map on $\mathrm{Gr}_{\mathrm{GL}_n}(\mathbb{C})$. We can let $\mathbb{Z}$ act on $\mathrm{Gr}_{\mathrm{GL}_n}(\mathbb{C})$ via $n.[A] = [t^n A]$. Consider the quotient space $\mathrm{Gr}_{\mathrm{GL}_n}(\mathbb{C})/\mathbb{Z}$.

Theorem 2.2.15. *We have $\mathrm{Gr}_{\mathrm{PGL}_n}(\mathbb{C}) \cong \mathrm{Gr}_{\mathrm{GL}_n}(\mathbb{C})/\mathbb{Z}$ and the natural morphism $\mathrm{GL}_n \twoheadrightarrow \mathrm{PGL}_n$ induces the quotient map $\mathrm{Gr}_{\mathrm{GL}_n}(\mathbb{C}) \to \mathrm{Gr}_{\mathrm{GL}_n}(\mathbb{C})/\mathbb{Z}$. This implies that $\mathrm{Gr}_{\mathrm{PGL}_n}$ is the disjoint union of the first n-connected components of $\mathrm{Gr}_{\mathrm{GL}_n}$.*

Proof. Let π be the quotient map $\mathrm{GL}_n \to \mathrm{PGL}_n$. We have the element

$$\pi(A\mathrm{GL}_n(\mathbb{C}[[t]])) = \pi(A)\mathrm{PGL}_n(\mathbb{C}[[t]])$$

is equal to $\pi(A'\mathrm{GL}_n(\mathbb{C}[[t]]))$ if and only if there is an element $B \in \mathrm{PGL}_n(\mathbb{C}[[t]])$ such that $\pi(A)B = \pi(A')$. This happens precisely if there is a matrix $B \in \mathrm{GL}_n(\mathbb{C}[[t]])$ and an element $a \in \mathbb{C}((t))^*$ such that $A = A' \cdot a \cdot B$. We can write $a = t^k a'$ with $a' \in \mathbb{C}[[t]]^*$ and therefore assume that $a = t^k$. So the map $\mathrm{Gr}_{\mathrm{GL}_n}(\mathbb{C}) \to \mathrm{Gr}_{\mathrm{PGL}_n}(\mathbb{C})$ identifies connected components which differ by the index n.

The second statement follows as we have identified $\mathrm{Gr}_{\mathrm{PGL}_n}(\mathbb{C})$ with the connected components 0 to $n-1$. □

Corollary 2.2.16. *We have closed-open immersions*

$$\mathrm{Gr}_{\mathrm{SL}_n}(\mathbb{C}) \hookrightarrow \mathrm{Gr}_{\mathrm{PGL}_n}(\mathbb{C}) \hookrightarrow \mathrm{Gr}_{\mathrm{GL}_n}(\mathbb{C}),$$

where the map $\mathrm{Gr}_{\mathrm{SL}_n} \hookrightarrow \mathrm{Gr}_{\mathrm{PGL}_n}$ *is induced by the composition*

$$\mathrm{SL}_n \to \mathrm{GL}_n \to \mathrm{PGL}_n.$$

Proof. The map $\mathrm{Gr}_{\mathrm{SL}_n}(\mathbb{C}) \to \mathrm{Gr}_{\mathrm{GL}_n}(\mathbb{C})$ is the inclusion of the zeroth connected component of $\mathrm{Gr}_{\mathrm{GL}_n}(\mathbb{C})$ by Theorem 2.2.14 and the map $\mathrm{Gr}_{\mathrm{PGL}_n}(\mathbb{C}) \to \mathrm{Gr}_{\mathrm{GL}_n}(\mathbb{C})$ is the inclusion of the first n connected component by Theorem 2.2.15. It follows that we have a closed-open immersion $f\colon \mathrm{Gr}_{\mathrm{SL}_n}(\mathbb{C}) \hookrightarrow \mathrm{Gr}_{\mathrm{PGL}_n}(\mathbb{C})$. Now we show that this map f is induced by the morphism $\mathrm{SL}_n \to \mathrm{GL}_n \to \mathrm{PGL}_n$. The composition

$$\mathrm{Gr}_{\mathrm{SL}_n}(\mathbb{C}) \xrightarrow{f} \mathrm{Gr}_{\mathrm{PGL}_n}(\mathbb{C}) \hookrightarrow \mathrm{Gr}_{\mathrm{GL}_n}(\mathbb{C})$$

is the inclusion of the zeroth connected component and therefore induced by the map $\mathrm{SL}_n \to \mathrm{GL}_n$. This coincided with the composition

$$\mathrm{Gr}_{\mathrm{SL}_n}(\mathbb{C}) \to \mathrm{Gr}_{\mathrm{GL}_n}(\mathbb{C}) \to \mathrm{Gr}_{\mathrm{PGL}_n}(\mathbb{C}) \hookrightarrow \mathrm{Gr}_{\mathrm{GL}_n}(\mathbb{C}),$$

where $\mathrm{Gr}_{\mathrm{SL}_n}(\mathbb{C}) \to \mathrm{Gr}_{\mathrm{GL}_n}(\mathbb{C})$ and $\mathrm{Gr}_{\mathrm{GL}_n}(\mathbb{C}) \to \mathrm{Gr}_{\mathrm{PGL}_n}(\mathbb{C})$ are the natural maps. As the map $\mathrm{Gr}_{\mathrm{PGL}_n}(\mathbb{C}) \hookrightarrow \mathrm{Gr}_{\mathrm{GL}_n}(\mathbb{C})$ is an immersion, it is in particular a monomorphism. Therefore the map f coincides with the map induced by functorality. □

Remark 2.2.17. For a complex reductive group G there is a general way to see how many connected components Gr_G has. It follows from [Zhu16, Theorem 1.3.11] that $\pi_0(\mathrm{Gr}_G) = \pi_1(G) = \pi_1(G(\mathbb{C}))$. But the fundamental group can be read off from root datum $(X^*, \Phi, X_*, \Phi^\vee)$ of G as $X_*/\langle \Phi^\vee(G)\rangle_{\mathbb{Z}}$, by Proposition 1.1.28. Using this labeling of the connected components, the element $[t^\lambda]$ lies in the component with the label $\lambda + \langle \Phi^\vee(G)\rangle_{\mathbb{Z}}$. Indeed

$$\begin{aligned}
\pi_1(\mathrm{GL}_n) =& X_*(\mathrm{GL}_n)/\langle \Phi^\vee(\mathrm{GL}_n)\rangle = \mathbb{Z}^n/\langle \varepsilon_i - \varepsilon_j\rangle \\
&\cong \mathbb{Z} = \pi_0(\mathrm{Gr}_{\mathrm{GL}_n}), \\
\pi_1(\mathrm{SL}_n) =& X_*(\mathrm{SL}_n)/\langle \Phi^\vee(\mathrm{SL}_n)\rangle = \left\{ a \in \mathbb{Z}^n \;\middle|\; \textstyle\sum_i a_i = 0 \right\} / \langle \varepsilon_i - \varepsilon_j\rangle \\
&\cong \{0\} = \pi_0(\mathrm{Gr}_{\mathrm{SL}_n}), \\
\pi_1(\mathrm{PGL}_n) =& X^*(\mathrm{SL}_n)/\langle \Phi(\mathrm{SL}_n)\rangle = (\mathbb{Z}^n/\langle e_1 + \dots + e_n\rangle)/\langle e_i - e_j\rangle \\
&\cong \mathbb{Z}/n = \pi_0(\mathrm{Gr}_{\mathrm{PGL}_n}).
\end{aligned}$$

2.3 The Schubert Cells of Gr

In this section we want to find a cell decomposition of Gr_G. We rather work with its reduction which is identified with $\mathrm{Gr}_G(\mathbb{C}) = G(\mathbb{C}((t)))/G(\mathbb{C}[[t]])$. This cell decomposition is analogous to the classical Schubert cell decomposition of partial flag varieties. The Schubert cells in Gr, however, will be more complicated than the affine cells in partial flag varieties.

Warning. In this section, we will state the results for general reductive groups G, but the proofs will be given only in the cases $\mathrm{GL}_n, \mathrm{SL}_n$, and PGL_n.

We consider a reductive group G with fixed Borel and torus $G \supseteq B \supseteq T$ and corresponding root datum

$$(X^*(G,T), \Phi(G,T), X_*(G,T), \Phi^\vee(G,T)) = (X^*, \Phi, X_*, \Phi^\vee).$$

Recall that $X^*(G,T)$ denotes the characters $\mathrm{Hom}(T, \mathbb{G}_m)$ of G and $X_*(G,T)$ denotes the cocharacters $\mathrm{Hom}(\mathbb{G}_m, T)$. We write

$$(X_*)_+ \coloneqq (X_*(G,T))_+ \coloneqq \{\lambda \in X_* \mid \langle \alpha, \lambda \rangle \geq 0 \text{ for all positive roots } \alpha \in \Phi^+\}.$$

Notice that there is a natural left action of the group $G(\mathbb{C}[[t]])$ on $\mathrm{Gr}_G(\mathbb{C})$ by left multiplication. We obtain a decomposition of $\mathrm{Gr}_G(\mathbb{C})$ into its $G(\mathbb{C}[[t]])$-orbits.

Theorem 2.3.1 (Cartan Decompostion). *There is a bijection*

$$\begin{aligned} X_*(G,T)_+ &\xleftrightarrow{1:1} \{G(\mathbb{C}[[t]])\text{-}\textit{orbits of } \mathrm{Gr}_G(\mathbb{C})\}, \\ \lambda &\longmapsto G(\mathbb{C}[[t]]).[t^\lambda]. \end{aligned}$$

We think of this statement as a decomposition of the Affine Grassmannian into its orbits

$$\mathrm{Gr}_G(\mathbb{C}) = \coprod_{\lambda \in X_*(G,T)_+} G(\mathbb{C}[[t]])[t^\lambda].$$

Alternatively, the name decomposition can also refer to a matrix decomposition. Thinking of G as a group of matrices, we obtain the following.

Theorem 2.3.2 (Cartan Decomposition — Matrix Version). *For any group element $A \in G(\mathbb{C}((t)))$ there is a unique $\lambda \in X_*(G,T)_+$ and some elements $X, Y \in G(\mathbb{C}[[t]])$ such that $A = Xt^\lambda Y$.*

The equivalence of theses two formulations follows immediately from the equality $\mathrm{Gr}_G(\mathbb{C}) = G(\mathbb{C}((t)))/G(\mathbb{C}[[t]])$, Proposition 2.2.7.

Definition 2.3.3. The $G(\mathbb{C}[[t]])$-orbits of $\mathrm{Gr}_G(\mathbb{C})$ are called the Schubert cells. We denote the Schubert cell corresponding to the element $\lambda \in (X_*(G,T))_+$ by Gr_G^λ.

Remark 2.3.4. The finite-dimensional irreducible representations of G are labeled by

$$X^*_+ = \{\lambda \in X^* \mid \langle \lambda, \alpha^\vee \rangle \geq 0 \text{ for all positive roots } \alpha \in \Phi^+\}$$

and so $(X_*)_+$ is in bijection with the irreducible representations of $G^\vee$, the Langlands dual group of G, see Theorem 1.1.22 and Definition 1.1.18. We therefore have identified irreducible representations of $G^\vee$ with $G(\mathbb{C}[[t]])$ orbits on $\mathrm{Gr}_G(\mathbb{C})$. We can think of Theorem 2.3.1 as a "set-theoretic" Satake equivalence, see also Example 4.1.4.

The general proof of Theorem 2.3.1 can be found in [Tit79]. We give elementary proofs in the cases $\mathrm{GL}_n, \mathrm{SL}_n$, and PGL_n. We need the following theorem form linear algebra, known as the elementary divisor theorem, Gaussian elimination, or Smith normal form.

Theorem 2.3.5 (Smith Normal Form)**.** *Let R be a principal ideal domain and $A \in R^{n\times n}$ a square matrix. Then there are matrices $X, Y \in \mathrm{GL}_n(R)$ such that*

$$XAY = \begin{pmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_n \end{pmatrix},$$

where d_{i+1} divides d_i. In addition, the d_i are uniquely determined up to unit in R.

Proof of Theorem 2.3.1, Case GL_n. We are interested in the double cosets

$$\mathrm{GL}_n(\mathbb{C}[[t]])\backslash\mathrm{Gr}_{\mathrm{GL}_n}(\mathbb{C}) = \mathrm{GL}_n(\mathbb{C}[[t]])\backslash\mathrm{GL}_n(\mathbb{C}((t)))/\mathrm{GL}_n(\mathbb{C}[[t]]).$$

Let $A \in \mathrm{GL}_n(\mathbb{C}((t)))$. Then there is an $N \in \mathbb{N}_0$ such that $\widetilde{A} := t^N A$ has coefficients in $\mathbb{C}[[t]]$. We apply the Smith normal form to $\widetilde{A} \in \mathbb{C}[[t]]^{n\times n}$. We obtain matrices $\widetilde{X}, \widetilde{Y} \in \mathrm{GL}_n(\mathbb{C}[[t]])$ such that $\widetilde{X}\widetilde{A}\widetilde{Y}$ is diagonal with entries d_i and d_{i+1} divides d_i. As the determinant of $\widetilde{A}$ is not 0, all d_i are non-zero. We get that $d_i = t^{\widetilde{\lambda}_i} a_i$ with $a_i \in \mathbb{C}[[t]]^*$ and $\widetilde{\lambda}_i \geq \widetilde{\lambda}_{i+1} \geq 0$. We can therefore find matrices $X, Y \in \mathrm{GL}_n(\mathbb{C}[[t]])$ such that

$$X\widetilde{A}Y = \begin{pmatrix} t^{\widetilde{\lambda}_1} & & & \\ & t^{\widetilde{\lambda}_2} & & \\ & & \ddots & \\ & & & t^{\widetilde{\lambda}_n} \end{pmatrix}.$$

Define $\lambda_i := \widetilde{\lambda}_i - N$. We obtain

$$XAY = Xt^{-N}\widetilde{A}Y = t^{-N}X\widetilde{A}Y = \begin{pmatrix} t^{\lambda_1} & & & \\ & t^{\lambda_2} & & \\ & & \ddots & \\ & & & t^{\lambda_n} \end{pmatrix}.$$

with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \in \mathbb{Z}$.

It is clear from the uniqueness properties of the Smith normal form that given two such sequences $\lambda_1 \geq \cdots \lambda_n$ and $\lambda'_1 \geq \cdots \geq \lambda'_n$ in $\mathbb{Z}$ there cannot be matrices $X, Y \in \mathrm{GL}(\mathbb{C}[[t]])$ such that

$$X \begin{pmatrix} t^{\lambda_1} & & & \\ & t^{\lambda_2} & & \\ & & \ddots & \\ & & & t^{\lambda_n} \end{pmatrix} Y = \begin{pmatrix} t^{\lambda'_1} & & & \\ & t^{\lambda'_2} & & \\ & & \ddots & \\ & & & t^{\lambda'_n} \end{pmatrix}$$

unless $\lambda_i = \lambda'_i$ for all i. We therefore have shown that

$$\left\{ \begin{pmatrix} t^{\lambda_1} & & & \\ & t^{\lambda_2} & & \\ & & \ddots & \\ & & & t^{\lambda_n} \end{pmatrix} \,\middle|\, \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \in \mathbb{Z} \right\}$$

is a system of representatives of the double coset $\mathrm{GL}_n(\mathbb{C}[[t]]) \backslash \mathrm{Gr}_{\mathrm{GL}_n}(\mathbb{C})$. However, the root datum of GL_n with the standard Borel and Cartan is

$$(\mathbb{Z}^n, \{e_i - e_j \mid i \neq j\}, \mathbb{Z}^n, \{\varepsilon_i - \varepsilon_j \mid i \neq j\})$$

and positive roots $e_i - e_j$ for $i < j$, see Example 1.1.10. And therefore

$$\begin{aligned} (X_*)_+ &= \{\lambda \in \mathbb{Z}^n \mid \langle e_i - e_j, \lambda \rangle \geq 0 \text{ for all } i > j\} \\ &= \{(\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n \mid \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n\}, \end{aligned}$$

which is identified with $\mathrm{GL}_n(\mathbb{C}[[t]]) \backslash \mathrm{Gr}_{\mathrm{GL}_n}(\mathbb{C})$ via $\lambda \mapsto [t^\lambda]$. □

Proof of Theorem 2.3.1, Case SL_n. The argument in case SL_n is very similar to the GL_n case. Just notice that the X and Y in the Smith normal form can be chosen to lie in $\mathrm{SL}_n(\mathbb{C}[[t]])$. We can see this by carefully going through the steps of Gaussian elimination or by noticing that in $\mathbb{C}[[t]]$ any element in $\mathbb{C}[[t]]^*$ has an n-th root by Hensel's lemma [Eis95, Theorem 7.3] and then multiplying X and Y with $\left(\sqrt[n]{\det X}\right)^{-1}$ and $\sqrt[n]{\det X}$, respectively. In any rate, we see that for any $A \in \mathrm{SL}_n(\mathbb{C}((t)))$ there are matrices $X, Y \in \mathrm{SL}_n(\mathbb{C}[[t]])$ such that

$$XAY = \begin{pmatrix} t^{\lambda_1} & & & \\ & t^{\lambda_2} & & \\ & & \ddots & \\ & & & t^{\lambda_n} \end{pmatrix}$$

with $\lambda_i \geq \lambda_{i+1}$ and the additional condition that $\sum_i \lambda_i = 0$. As before it is clear that such matrices form a system of representatives of the double coset $\mathrm{SL}_n(\mathbb{C}[[t]]) \backslash \mathrm{Gr}_{\mathrm{SL}_n}(\mathbb{C})$.

The root datum of SL_n with the standard Borel and Cartan is

$$\left(\mathbb{Z}^n / \langle e_1 + \cdots + e_n \rangle, \{e_i - e_j \mid i \neq j\}, \left\{\lambda \in \mathbb{Z}^n \,\middle|\, \sum\nolimits_i \lambda_i = 0\right\}, \{\varepsilon_i - \varepsilon_j \mid i \neq j\}\right)$$

with positive roots $e_i - e_j$ for $i < j$, see Example 1.1.11. Therefore

$$\begin{aligned}(X_*)_+ &= \left\{\lambda \in \mathbb{Z}^n \;\middle|\; \langle e_i - e_j, \lambda\rangle \geq \text{ for } i > j \text{ and } \sum\nolimits_i \lambda_i = 0\right\} \\ &= \{(\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n \mid \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n, \lambda_1 + \dots + \lambda_n = 0\}.\end{aligned}$$

We again have that $\lambda \mapsto [t^\lambda]$ identifies $(X_*)_+$ with $\mathrm{SL}_n(\mathbb{C}[[t]])\backslash \mathrm{Gr}_{\mathrm{SL}_n}(\mathbb{C})$. ☐

Proof of Theorem 2.3.1, Case PGL_n. We can deduce this from the already proven GL_n case. Notice that

$$\begin{aligned}\mathrm{PGL}_n(\mathbb{C}[[t]])\backslash \mathrm{Gr}_{\mathrm{PGL}_n}(\mathbb{C}) &= \mathrm{PGL}_n(\mathbb{C}[[t]])\backslash \mathrm{PGL}_n(\mathbb{C}((t)))/\mathrm{PGL}_n(\mathbb{C}[[t]]) \\ &= \Big(\mathrm{GL}_n(\mathbb{C}[[t]])\backslash \mathrm{GL}_n(\mathbb{C}((t)))/\mathrm{GL}_n(\mathbb{C}[[t]])\Big)\Big/\mathbb{C}((t))^* \\ &\overset{1:1}{\leftrightarrow} \{\lambda \in \mathbb{Z}^n \mid \lambda_1 \geq \dots \geq \lambda_n\}/\mathbb{Z}(1, \dots, 1) \\ &= \{\lambda \in \mathbb{Z}^n/\mathbb{Z}(1, \dots, 1) \mid \lambda_1 \geq \dots \geq \lambda_n\}.\end{aligned}$$

The root datum of PGL_n with standard Borel and Cartan is

$$\left(\left\{a \in \mathbb{Z}^n \;\middle|\; \sum\nolimits_i a_i = 0\right\}, \{e_i - e_j \mid i \neq j\}, \mathbb{Z}^n/\langle \varepsilon_1 + \dots + \varepsilon_n\rangle, \{\varepsilon_i - \varepsilon_j \mid i \neq j\}\right)$$

with positive roots $\varepsilon_i - \varepsilon_j$ for $i < j$, see Example 1.1.12. Therefore

$$\begin{aligned}(X_*)_+ &= \{\lambda \in \mathbb{Z}^n/\langle e_1 + \dots + e_n\rangle \mid \langle e_i - e_j, \lambda\rangle \geq 0\} \\ &= \{\lambda \in \mathbb{Z}^n/\mathbb{Z}(1, \dots, 1) \mid \lambda_1 \geq \dots \geq \lambda_n\}.\end{aligned}$$ ☐

The following proposition shows that many statements about the Schubert cells in $\mathrm{Gr}_{\mathrm{PGL}_n}(\mathbb{C})$ and $\mathrm{Gr}_{\mathrm{SL}_n}(\mathbb{C})$ can be deduced from the corresponding statements for $\mathrm{Gr}_{\mathrm{GL}_n}(\mathbb{C})$.

Proposition 2.3.6. *The embeddings* $\mathrm{Gr}_{\mathrm{SL}_n}(\mathbb{C}) \hookrightarrow \mathrm{Gr}_{\mathrm{PGL}_n}(\mathbb{C}) \hookrightarrow \mathrm{Gr}_{\mathrm{GL}_n}(\mathbb{C})$ *from Corollary 2.2.16 induce bijections and therefore isomorphisms on Schubert cells.*

Proof. It is clear that the map $\mathrm{Gr}_{\mathrm{SL}_n}(\mathbb{C}) \hookrightarrow \mathrm{Gr}_{\mathrm{PGL}_n}$ injects any $\mathrm{SL}_n(\mathbb{C}[[t]])$-orbit in $\mathrm{Gr}_{\mathrm{SL}_n}(\mathbb{C})$ into a single $\mathrm{PGL}_n(\mathbb{C}[[t]])$-orbit in $\mathrm{Gr}_{\mathrm{PGL}_n}(\mathbb{C})$, because this map is induced by $\mathrm{SL}_n \to \mathrm{GL}_n \to \mathrm{PGL}_n$. Now note that the natural map $\mathrm{SL}_n(\mathbb{C}[[t]]) \to \mathrm{PGL}_n(\mathbb{C}[[t]])$ is surjective, because any element $[A] \in \mathrm{PGL}_n(\mathbb{C}[[t]]) = \mathrm{GL}_n(\mathbb{C}[[t]])/Z(\mathrm{GL}_n(\mathbb{C}[[t]]))$ can be written as $[A] = \left[A \cdot (\sqrt[n]{\det A})^{-1}\right]$, since any invertible element in $\mathbb{C}[[t]]$ has an n-th root. It follows that the map $\mathrm{Gr}_{\mathrm{SL}_n}(\mathbb{C}) \to \mathrm{Gr}_{\mathrm{PGL}_n}(\mathbb{C})$ induces bijections on orbits.

Now note that the natural map $\mathrm{Gr}_{\mathrm{GL}_n}(\mathbb{C}) \twoheadrightarrow \mathrm{Gr}_{\mathrm{PGL}_n}(\mathbb{C})$ induces bijections on Schubert cells, using Theorem 2.2.15. It follows that the same is true for $\mathrm{Gr}_{\mathrm{PGL}_n}(\mathbb{C}) \hookrightarrow \mathrm{Gr}_{\mathrm{GL}_n}(\mathbb{C})$, which on orbits is just the inverse. ☐

Remark 2.3.7. On the level of $(X_*)_+$ the first map is given by

$$(X_*(\mathrm{SL}_n))_+ \ni \lambda = (\lambda_1, \dots \lambda_n) \mapsto [(\lambda_1, \dots, \lambda_n)] \in (X_*(\mathrm{PGL}_n))_+.$$

The map $\mathrm{Gr}_{\mathrm{GL}_n} \twoheadrightarrow \mathrm{Gr}_{\mathrm{PGL}_n}$ similarly corresponds to mapping $\lambda \in (X_*(\mathrm{GL}_n))_+$ to its residue class $[\lambda] \in (X_*(\mathrm{PGL}_n))_+$. Its split $\mathrm{Gr}_{\mathrm{PGL}_n} \hookrightarrow \mathrm{Gr}_{\mathrm{GL}_n}$ sends the class $[\lambda] \in (X_*(\mathrm{PGL}_n))_+$ to the unique $\lambda' \in [\lambda]$ such that $\sum_i \lambda'_i \in \{0, 1, \dots, n-1\}$.

Recall from Remark 2.2.17 that the connected components of Gr_G are labeled by the fundamental group $\pi_1(G) = X_*(G,T)/\langle \Phi^\vee \rangle_{\mathbb{Z}}$.

Proposition 2.3.8. *Let $\lambda \in (X_*)_+$. The element $[t^\lambda]$ is contained in the connected component with label $\lambda + \langle \Phi^\vee(G,T)\rangle_{\mathbb{Z}} \in X_*(G,T)/\langle \Phi^\vee(G,T)\rangle_{\mathbb{Z}} = \pi_1(G)$. It follows that its $G(\mathbb{C}[[t]])$-orbit Gr_G^λ must be completely contained in this component. We see that Gr_G^λ and Gr_G^μ lie in the same connected component of $\mathrm{Gr}_G(\mathbb{C})$ if and only if we can write $\lambda - \mu = \sum_i n_i\alpha_i$ for $\alpha_i \in \Phi^\vee(G)$ and $n_i \in \mathbb{Z}$.*

Proof. We verify that $[t^\lambda]$ lies in the connected component with label $\lambda + \langle \Phi^\vee(G,T)\rangle_{\mathbb{Z}}$ in the case GL_n. The cases SL_n and PGL_n work the same.

Indeed $\det(t^\lambda) = t^{\sum_i \lambda_i}$. But $\sum_i \lambda_i$ is precisely the image of $\lambda \in X_* = \mathbb{Z}^n$ in $X_*/\langle \Phi^\vee(G,T)\rangle_{\mathbb{Z}} = \mathbb{Z}^n/\langle e_i - e_j\rangle_{\mathbb{Z}} \cong \mathbb{Z}$. □

Proposition 2.3.9. *The sets $\mathrm{Gr}^{(N)}_{\mathrm{GL}_n}(\mathbb{C})$ as in Theorem 2.1.16 are $\mathrm{GL}_n(\mathbb{C}[[t]])$-invariant and so the Schubert cell Gr_G^λ will be completely contained in $\mathrm{Gr}_G^{(N)}$ for $N \gg 0$. Explicitly, we can choose N such that $N \geq \lambda_1$ and $\lambda_n \geq -N$.*

Proof. The invariance follows from Lemma 2.1.19. With the choice of N as above, we have $[t^\lambda] \in \mathrm{Gr}^{(N)}(\mathbb{C})$ and therefore $\mathrm{Gr}_G^\lambda \subseteq \mathrm{Gr}^{(N)}(\mathbb{C})$. □

For any $\lambda \in (X_*)_+$, let $P_\lambda \subseteq G(\mathbb{C})$ be the parabolic subgroup corresponding to λ. It can be defined as the group generated by T as well as those $U_\alpha \subseteq G$ such that $\langle \alpha, \lambda\rangle \leq 0$. Explicitly, for $\lambda = (\lambda_1, \ldots, \lambda_n) \in (X_*(\mathrm{GL}_n, T))_+$ consider the partition $n = \sum_{i=1}^l k_i$ such that $\lambda_1 = \lambda_2 = \cdots \lambda_{k_1} > \lambda_{k_1+1} = \lambda_{k_1+2} = \cdots > \lambda_{k_1+\cdots+k_{l-1}+1} = \cdots = \lambda_n$. Then

$$P_\lambda = \left\{ \begin{pmatrix} \boxed{A_1} & & & 0 \\ & \boxed{A_2} & & \\ & & \ddots & \\ * & & & \boxed{A_l} \end{pmatrix} \in \mathrm{GL}_n \;\middle|\; A_i \in \mathbb{C}^{k_i \times k_i} \right\}$$
$$= \{A = (a_{ij})_{ij} \in \mathrm{GL}_n(\mathbb{C}) \mid a_{ij} = 0 \text{ if } \lambda_i > \lambda_j\}.$$

Also consider the evaluation at zero map $\mathrm{ev}_0 \colon \mathbb{C}[[t]] \to \mathbb{C}$ and its induced group homomorphism $\mathrm{ev}_0 \colon G(\mathbb{C}[[t]]) \to G(\mathbb{C})$.

Lemma 2.3.10. *The inverse image of $P_\lambda \subseteq G(\mathbb{C})$ in $G(\mathbb{C}[[t]])$ contains*

$$P_\lambda^{\mathrm{aff}} := G(\mathbb{C}[[t]]) \cap t^\lambda G(\mathbb{C}[[t]]) t^{-\lambda}$$

and $\mathrm{ev}_0^{-1}(P_\lambda)/P_\lambda^{\mathrm{aff}} \cong \mathbb{C}^l$. If $G = \mathrm{GL}_n$ we find

$$l = \sum_{\lambda_i > \lambda_j} (\lambda_i - \lambda_j - 1).$$

Note that while P_λ only depends on the underlying partition of λ, we can have $P_\lambda^{\mathrm{aff}} \neq P_{\lambda'}^{\mathrm{aff}}$ even if λ and λ' have the same partition.

Proof. We focus only on the case $G = \mathrm{GL}_n$. The arguments for $G = \mathrm{SL}_n$ and $G = \mathrm{PGL}_n$ are exactly the same.

It is clear that the inverse image of P_λ in $\mathrm{GL}_n(\mathbb{C}[[t]])$ contains the elements of the form $A + tB$, where $A \in P_\lambda$ and $B \in \mathbb{C}[[t]]^{n\times n}$, i.e. elements (a_{ij}) of $\mathrm{GL}_n(\mathbb{C}[[t]])$ such that the constant term of a_{ij} is zero if $\lambda_i < \lambda_j$. Now notice that

$$\begin{aligned}\mathrm{GL}_n(\mathbb{C}[[t]]) \cap t^\lambda \mathrm{GL}_n(\mathbb{C}[[t]])t^{-\lambda} &= \left\{ t^\lambda A t^{-\lambda} \in \mathbb{C}[[t]]^{n\times n} \;\middle|\; A = (a_{ij})_{i,j} \in \mathrm{GL}_n(\mathbb{C}[[t]]) \right\} \\ &= \left\{ (t^{\lambda_i - \lambda_j} a_{ij})_{i,j} \in \mathbb{C}[[t]]^{n\times n} \;\middle|\; (a_{ij})_{i,j} \in \mathrm{GL}_n(\mathbb{C}[[t]]) \right\}.\end{aligned}$$

We see that the constant term of a_{ij} is only allowed to be non-zero if $\lambda_i \geq \lambda_j$.

That the quotient has the form $\mathbb{C}^l$ for $l = \sum_{\lambda_i > \lambda_j} (\lambda_i - \lambda_j - 1)$ is now clear from the above description of P_λ^{aff}. □

Remark 2.3.11. If $G = \mathrm{GL}_n$, notice that $P_\lambda^{\mathrm{aff}} = \mathrm{ev}_0^{-1}(P_\lambda)$ if and only if

$$\lambda = (l+1, l+1, \ldots, l+1, l, l \ldots, l),$$

which are precisely those λ that are minuscule, i.e. $\lambda - \alpha^\vee \notin (X_*(G))_+$ for all positive coroots $\alpha^\vee$. The same holds for arbitrary G.

Now we compute what P_λ^{aff} can look like for GL_2 and GL_3.

Example 2.3.12. Let $\lambda \in (X_*(\mathrm{GL}_2))_+$. So $\lambda = (k, l) \in \mathbb{Z}^2$ with $k \geq l$. We have that a matrix $A \in \mathrm{GL}_2(\mathbb{C}[[t]])$ lies in P_λ^{aff} if and only if there is a matrix $A' \in \mathrm{GL}_2(\mathbb{C}[[t]])$ such that $t^\lambda A' t^{-\lambda} = A$. We have

$$t^\lambda \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} t^{-\lambda} = \begin{pmatrix} a' & t^{k-l}b' \\ t^{-(k-l)}c' & d' \end{pmatrix}.$$

It follows that any $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{C}[[t]])$ lies in P_λ^{aff} if and only if b is divisible by t^{k-l} in $\mathbb{C}[[t]]$; and so

$$P_\lambda^{\mathrm{aff}} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\middle|\; t^{k-l} \text{ divides } b \right\} = \left\{ \begin{pmatrix} a & t^{k-l}b \\ c & d \end{pmatrix} \right\}.$$

Note that there is no condition on c because for every $c \in \mathbb{C}[[t]]$ there is a $c' \in \mathbb{C}[[t]]$ such that $c = t^{-(k-l)}c'$.

- If $k = l$, we have $P_\lambda = \mathrm{GL}_2(\mathbb{C})$ and $P_\lambda^{\mathrm{aff}} = \mathrm{GL}_2(\mathbb{C}[[t]])$. Therefore $\mathrm{ev}_0^{-1}(P_\lambda) = P_\lambda^{\mathrm{aff}}$.
- If $k = l+1$, we have $P_\lambda = \left\{ \begin{pmatrix} a & 0 \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{C}) \right\}$ and

$$P_\lambda^{\mathrm{aff}} = \left\{ \begin{pmatrix} a & tb \\ c & d \end{pmatrix} \right\} = \mathrm{ev}_0^{-1}(P_\lambda).$$

- If $k > l+1$, we again have $P_\lambda = \left\{\begin{pmatrix} a & 0 \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{C})\right\}$ which also satisfies $\mathrm{ev}_0^{-1}(P_\lambda) = \left\{\begin{pmatrix} a & tb \\ c & d \end{pmatrix}\right\}$. However, now we have

$$P_\lambda^{\mathrm{aff}} = \left\{\begin{pmatrix} a & t^{k-l}b \\ c & d \end{pmatrix}\right\} \subsetneq \mathrm{ev}_0^{-1}(P_\lambda) = \left\{\begin{pmatrix} a & tb \\ c & d \end{pmatrix}\right\}$$

with quotient

$$\mathrm{ev}_0^{-1}(P_\lambda)/P_\lambda^{\mathrm{aff}} = \mathbb{C}^{(k-l)-1}.$$

Example 2.3.13. Let $\lambda \in (X_*(\mathrm{GL}_3))_+$. So $\lambda = (k_1, k_2, k_3) \in \mathbb{Z}^3$ with $k_1 \geq k_2 \geq k_3$. We have

$$t^\lambda \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} t^{-\lambda} = \begin{pmatrix} b_{11} & t^{k_1-k_2}b_{12} & t^{k_1-k_3}b_{13} \\ t^{k_2-k_1}b_{21} & b_{22} & t^{k_2-k_3}b_{23} \\ t^{k_3-k_1}b_{31} & t^{k_3-k_2}b_{32} & b_{33} \end{pmatrix}.$$

and thus

$$P_\lambda^{\mathrm{aff}} = \left\{ \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \;\middle|\; t^{k_1-k_2}|a_{12},\ t^{k_1-k_3}|a_{13}, \text{ and } t^{k_2-k_3}|a_{23} \right\}.$$

As in the previous example note that there is no condition on any entry on or below the diagonal. We distinguish four cases for λ.

- If $k_1 = k_2 = k_3$, we have $P_\lambda = \mathrm{GL}_3(\mathbb{C})$, $P_\lambda^{\mathrm{aff}} = \mathrm{GL}_3(\mathbb{C}[[t]])$, and

$$\mathrm{ev}_0^{-1}(P_\lambda) = \mathrm{GL}_3(\mathbb{C}[[t]]) = P_\lambda^{\mathrm{aff}}.$$

- If $k_1 > k_2 = k_3$, we have $P_\lambda = \left\{\begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \in \mathrm{GL}_3(\mathbb{C})\right\}$ and

$$\mathrm{ev}_0^{-1}(P_\lambda) = \left\{\begin{pmatrix} a_{11} & ta_{12} & ta_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\right\}.$$

We also have

$$P_\lambda^{\mathrm{aff}} = \left\{\begin{pmatrix} a_{11} & t^{k_1-k_2}a_{12} & t^{k_1-k_3}a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\right\}$$

and therefore $\mathrm{ev}_0^{-1}(P_\lambda)/P_\lambda^{\mathrm{aff}} \cong \mathbb{C}^{k_1-k_2-1+k_1-k_3-1} = \mathbb{C}^{2k_1-2k_2-2}$.

- If $k_1 = k_2 > k_3$, we have $P_\lambda = \left\{ \begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \in \mathrm{GL}_3(\mathbb{C}) \right\}$ and

$$\mathrm{ev}_0^{-1}(P_\lambda) = \left\{ \begin{pmatrix} a_{11} & a_{12} & ta_{13} \\ a_{21} & a_{22} & ta_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \right\}.$$

  We also have

$$P_\lambda^{\mathrm{aff}} = \left\{ \begin{pmatrix} a_{11} & a_{12} & t^{k_1-k_3}a_{13} \\ a_{21} & a_{22} & t^{k_2-k_3}a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \right\}$$

  and therefore $\mathrm{ev}_0^{-1}(P_\lambda)/P_\lambda^{\mathrm{aff}} \cong \mathbb{C}^{(k_1-k_3-1)+(k_2-k_3-1)} = \mathbb{C}^{2k_1-2k_3-2}$.

- If $k_1 > k_2 > k_3$, we have $P_\lambda = \left\{ \begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \in \mathrm{GL}_3(\mathbb{C}) \right\}$ and

$$\mathrm{ev}_0^{-1}(P_\lambda) = \left\{ \begin{pmatrix} a_{11} & ta_{12} & ta_{13} \\ a_{21} & a_{22} & ta_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \right\}.$$

  We also have

$$P_\lambda^{\mathrm{aff}} = \left\{ \begin{pmatrix} a_{11} & t^{k_1-k_2}a_{12} & t^{k_1-k_3}a_{13} \\ a_{21} & a_{22} & t^{k_2-k_3}a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \right\}$$

  and therefore $\mathrm{ev}_0^{-1}(P_\lambda)/P_\lambda^{\mathrm{aff}} \cong \mathbb{C}^{(k_1-k_2-1)+(k_1-k_3-1)+(k_2-k_3-1)}$.

Definition 2.3.14. An affine bundle on X is map $p\colon Y \to X$ of varieties such that for some covering $X = \bigcup_i U_i$ we have $p^{-1}(U_i) \cong \mathbb{A}^k \times U_i$.

Typical examples of affine bundles are vector bundles. These concepts differ in that there are no linearity assumption on transition maps.

Theorem 2.3.15. *The Schubert cell Gr_G^λ is an affine bundle over the partial flag variety G/P_λ with fibers $\mathrm{ev}_0^{-1}(P_\lambda)/P_\lambda^{\mathrm{aff}}$. It follows that the Schubert cell is a smooth, quasi-projective variety of dimension $\langle 2\rho, \lambda\rangle$, where 2ρ is the sum of all positive roots in the root datum of (G, B, T).*

Proof. We consider the natural map $G(\mathbb{C}[[t]]) \twoheadrightarrow \mathrm{Gr}_G^\lambda, g \mapsto [g.t^\lambda]$, as Gr_G^λ is defined as the $G(\mathbb{C}[[t]])$-orbit of $[t^\lambda] \in \mathrm{Gr}_G(\mathbb{C})$. We get an identification

$$\mathrm{Gr}_G^\lambda = G(\mathbb{C}[[t]])/\{A \in G(\mathbb{C}[[t]]) \mid [At^\lambda] = [t^\lambda]\}.$$

We can calculate

$$\begin{aligned} G(\mathbb{C}[[t]])_{[t^\lambda]} &= \{A \in G(\mathbb{C}[[t]]) \mid [At^\lambda] = [t^\lambda]\} \\ &= \{A \in G(\mathbb{C}[[t]]) \mid \text{there is a } B \in G(\mathbb{C}[[t]]) \text{ such that } At^\lambda = t^\lambda B\} \\ &= \{A \in G(\mathbb{C}[[t]]) \mid \text{there is a } B \in G(\mathbb{C}[[t]]) \text{ such that } A = t^\lambda B t^{-\lambda}\} \\ &= G(\mathbb{C}[[t]]) \cap t^\lambda G(\mathbb{C}[[t]]) t^{-\lambda} \\ &= P_\lambda^{\mathrm{aff}}. \end{aligned}$$

As by Lemma 2.3.10, we have $P_\lambda^{\mathrm{aff}} \subseteq \mathrm{ev}_0^{-1}(P_\lambda)$, we obtain a map

$$\mathrm{Gr}_G^\lambda = G(\mathbb{C}[[t]])/(G(\mathbb{C}[[t]]) \cap t^\lambda G(\mathbb{C}[[t]])t^{-\lambda}) \longrightarrow G(\mathbb{C}[[t]])/\mathrm{ev}_0^{-1}(P_\lambda) = G(\mathbb{C})/P_\lambda,$$

which endows Gr_G^λ with the structure of an affine bundle over $G(\mathbb{C})/P_\lambda$. We need to check that this map is locally trivial. Indeed, the map $\overline{\pi}\colon G(\mathbb{C}[[t]]) \overset{\mathrm{ev}_0}{\to} G(\mathbb{C}) \overset{\pi}{\twoheadrightarrow} G(\mathbb{C})/P_\lambda$ is a fiber bundle with fibers $\mathrm{ev}_0^{-1}(P_\lambda)$. Consider the diagram

$$\begin{tikzcd} G(\mathbb{C}[[t]]) \arrow[rr, "\overline{\pi}"] \arrow[dr, "\kappa"'] & & G(\mathbb{C})/P_\lambda \\ & G(\mathbb{C}[[t]])/P_\lambda^{\mathrm{aff}}. \arrow[ur, "\varphi"'] & \end{tikzcd}$$

Now let $U \subseteq G(\mathbb{C})/P_\lambda$ such that $\overline{\pi}^{-1}(U) \cong U \times \mathrm{ev}_0^{-1}(P_\lambda)$. Then

$$\varphi^{-1}(U) = \kappa(\overline{\pi}^{-1}(U)) = \kappa(U \times \mathrm{ev}_0^{-1}(P_\lambda)) \cong U \times \mathrm{ev}_0^{-1}/P_\lambda^{\mathrm{aff}}$$

and we have shown local triviality of $\mathrm{Gr}_G^\lambda \twoheadrightarrow G(\mathbb{C})/P_\lambda$.

It follows that Gr_G^λ is smooth. It is quasi-projective, as any Gr_G^λ lies completely in some finite-dimensional projective variety $\mathrm{Gr}_G^{(N)}$ by Proposition 2.3.9.

Now we compute the dimension in the case $G = \mathrm{GL}_n$: We have shown that $\mathrm{Gr}_{\mathrm{GL}_n}^\lambda$ is a bundle on $G(\mathbb{C})/P_\lambda$ with fibers isomorphic to $\mathbb{C}^l$ with $l = \sum_{\lambda_i > \lambda_j}(\lambda_i - \lambda_j - 1)$ by Lemma 2.3.10. However, we know that $G(\mathbb{C})/P_\lambda$ is a smooth projective variety of dimension $\#\{(i,j) \in \{1,\dots,n\}^2 \mid \lambda_i > \lambda_j\}$. Therefore

$$\begin{aligned} \dim \mathrm{Gr}_{\mathrm{GL}_n}^\lambda &= l + \#\{(i,j) \in \{1,\dots,n\}^2 \mid \lambda_i > \lambda_j\} \\ &= \sum_{\lambda_i > \lambda_j} (\lambda_i - \lambda_j - 1) + \#\{(i,j) \in \{1,\dots,n\}^2 \mid \lambda_i > \lambda_j\} \\ &= \sum_{\lambda_i > \lambda_j} (\lambda_i - \lambda_j) = \sum_{i>j} (\lambda_i - \lambda_j) = \sum_{i>j} \langle e_i - e_j, \lambda\rangle \\ &= \langle 2\rho, \lambda \rangle. \end{aligned}$$

□

Remark 2.3.16. In [Zhu16, Proposition 2.1.5] a different argument for the dimension count is given, which works for arbitrary G. There the tangent space at $[t^\lambda]$ in Gr_G^λ is computed as

$$\begin{aligned}\mathfrak{g}(\mathbb{C}[[t]])/(\mathfrak{g}(\mathbb{C}[[t]]) \cap t^{\lambda}\mathfrak{g}(\mathbb{C}[[t]])t^{-\lambda}) &= \bigoplus_{\text{roots } \alpha} \mathfrak{g}_\alpha(\mathbb{C}[[t]])/(\mathfrak{g}_\alpha(\mathbb{C}[[t]]) \cap t^{\lambda}\mathfrak{g}(\mathbb{C}[[t]])t^{-\lambda}) \\ &= \bigoplus_{\text{positive roots } \alpha} \mathfrak{g}_\alpha(\mathbb{C}[[t]])/t^{\langle\alpha,\lambda\rangle}\mathfrak{g}_\alpha(\mathbb{C}[[t]]).\end{aligned}$$

which is a $\sum_{\alpha\in\Phi^+}\langle\alpha,\lambda\rangle = \langle 2\rho,\lambda\rangle$ dimensional vector space, since $\mathfrak{g}_\alpha$ is one dimensional.

We now take some corollaries from Theorem 2.3.15.

Corollary 2.3.17. *The Schubert cells are irreducible.*

Proof. By Theorem 2.3.15 we have $\pi_0(\mathrm{Gr}_G^\lambda) = \pi_0(G(\mathbb{C})/P_\lambda)$. However, the partial partial flag variety $G(\mathbb{C})/P_\lambda$ is connected. Now, Gr_G^λ is smooth connected, and therefore irreducible. □

Corollary 2.3.18. *The Schubert cells Gr_G^λ are simply-connected.*

Proof. By Theorem 2.3.15 we have $\pi_1(\mathrm{Gr}_G^\lambda) = \pi_1(G(\mathbb{C})/P_\lambda)$. However, $G(\mathbb{C})/P_\lambda$ is simply-connected, because the classical Schubert cells of $G(\mathbb{C})/P_\lambda$ give $G(\mathbb{C})/P_\lambda$ the structure of a CW-complex, where all cells appear in even dimension, since these classical Schubert cells have the form $\mathbb{C}^k$. □

Note that in the classical case of Schubert cells in partial flag varieties both of these corollaries are trivial, as the classical Schubert cells are always of the form $\mathbb{C}^k$ for some k.

Corollary 2.3.19. *The Schubert cell Gr_G^λ is projective if and only if λ is minuscule, i.e. $\lambda - \alpha^\vee \notin (X_*(G,T))_+$ for all positive coroots $\alpha^\vee$.*

Proof. The cell Gr_G^λ is an affine bundle of rank $\dim \mathrm{ev}_0^{-1}(P_\lambda)/P_\lambda^{\mathrm{aff}}$ over the projective variety $G(\mathbb{C})/P_\lambda$ and is therefore projective if and only if $P_\lambda^{\mathrm{aff}} = \mathrm{ev}_0^{-1}(P_\lambda)$. This is equivalent to λ minuscule by Remark 2.3.11. □

Remark 2.3.20. In the case of partial flag varieties, the only projective Schubert cell is the singleton. This is precisely the Schubert cell labeled by the minimal element 1.

The following corollary gives us a condition for the Schubert cell to not just be projective, but a point.

Corollary 2.3.21. *The following are equivalent:*

(1) The Schubert cell Gr_G^λ is a singleton.

(2) We have $P_\lambda^{\mathrm{aff}} = G(\mathbb{C}[[t]])$.

(3) We have $P_\lambda = G(\mathbb{C})$.

(4) We have $\langle\alpha,\lambda\rangle = 0$ for all roots $\alpha \in \Phi$.

Proof. We show (1) $\Leftrightarrow$ (2) $\Rightarrow$ (3) $\Leftrightarrow$ (4) $\Rightarrow$ (2).

We begin with the equivalence of (1) and (2). We have that the Schubert cell Gr_G^λ is a singleton if and only if $G(\mathbb{C}[[t]])$ acts trivially on the element

$$[t^\lambda] \in \mathrm{Gr}_G(\mathbb{C}) = G(\mathbb{C}((t)))/G(\mathbb{C}[[t]]).$$

This is the case if and only if for every $A \in G(\mathbb{C}[[t]])$ there is a matrix $B \in G(\mathbb{C}[[t]])$ such that $At^\lambda B = t^\lambda$, i.e. such that $A = t^\lambda B^{-1} t^{-\lambda}$. But this is now equivalent to $P_\lambda^{\mathrm{aff}} = G(\mathbb{C}[[t]]) \cap t^\lambda G(\mathbb{C}[[t]]) t^{-\lambda}$ being equal to $G(\mathbb{C}[[t]])$.

The equivalence of (3) and (4) is classical and follows from [Mil17, Corollary 21.92].

For (2) implies (3) recall that $P_\lambda^{\mathrm{aff}} \subseteq \mathrm{ev}_0^{-1}(P_\lambda)$ by Lemma 2.3.10. We deduce

$$P_\lambda = \mathrm{ev}_0(\mathrm{ev}_0^{-1}(P_\lambda)) \supseteq \mathrm{ev}_0(P_\lambda^{\mathrm{aff}}) = \mathrm{ev}_0(G(\mathbb{C}[[t]])) = G(\mathbb{C}).$$

For (4) implies (2), note that we have $\langle \alpha, \lambda \rangle = 0$ for all $\alpha \in \Phi$. This implies that λ is minuscule, because $\langle \alpha, \lambda - \alpha^\vee \rangle < 0$ for all simple coroots $\alpha^\vee$. Hence, Gr_G^λ is projective by Corollary 2.3.19. But by Theorem 2.3.15 we know that Gr_G^λ is an affine bundle over $G(\mathbb{C})/P_\lambda$ with fibers $\mathrm{ev}_0^{-1}(P_\lambda)/P_\lambda^{\mathrm{aff}}$. However, an affine bundle can never be projective unless it has trivial fibers. We therefore have

$$P_\lambda^{\mathrm{aff}} = \mathrm{ev}_0^{-1}(P_\lambda) = \mathrm{ev}_0^{-1}(G(\mathbb{C})) = G(\mathbb{C}[[t]]). \qquad \square$$

Example 2.3.22. In the case $G = \mathrm{GL}_1$ we have that $P_\lambda = \mathrm{GL}_1(\mathbb{C})$ is the only possible parabolic subgroup of $\mathrm{GL}_1(\mathbb{C})$. Therefore every Schubert cell is a singleton, which again shows that $\mathrm{Gr}_{\mathrm{GL}_1}(\mathbb{C}) = \mathbb{Z}$, as we have already seen in Example 2.1.14.

Proposition 2.3.23. *The Schubert cell corresponding to*

$$\lambda = (\underbrace{1, \dots, 1}_{r \text{ many}}, 0, \dots 0) \in X_*(\mathrm{GL}_n, T)_+$$

is the classical Grassmannian $\mathrm{Grass}_r(n)(\mathbb{C})$ *of r-dimensional subspaces in $\mathbb{C}^n$.*

Proof. We have $P_\lambda = \left\{ \left(\begin{array}{c|c} A & 0 \\ \hline C & D \end{array} \right) \in \mathrm{GL}_n(\mathbb{C}) \text{ where } A \in \mathrm{GL}_r(\mathbb{C}) \right\}$ and so

$$G(\mathbb{C})/P_\lambda = \mathrm{Grass}_r(n)(\mathbb{C}).$$

The map $\mathrm{Gr}_{\mathrm{GL}_n}^\lambda \twoheadrightarrow \mathrm{Grass}_r(n)(\mathbb{C})$ has fibers $\mathrm{ev}_0^{-1}(P_\lambda)/P_\lambda^{\mathrm{aff}}$, but

$$\mathrm{ev}_0^{-1}(P_\lambda) = \left\{ \left(\begin{array}{c|c} A & tB \\ \hline C & D \end{array} \right) \in \mathrm{GL}_n(\mathbb{C}[[t]]) \text{ where } A \in \mathrm{GL}_r(\mathbb{C}[[t]]) \right\} = P_\lambda^{\mathrm{aff}}. \qquad \square$$

Next we will give a description of the Schubert cells in the Affine Grassmannian of GL_2. There are two possibilities for P_λ:

- If $\lambda = (k, k)$ we have $P_\lambda = \mathrm{GL}_2(\mathbb{C})$ and so $G(\mathbb{C})/P_\lambda = \{pt\}$. In fact all such $\mathrm{Gr}_{\mathrm{GL}_2}^\lambda$ are isomorphic to each other and are equal to $\{pt\}$.

- If $k > l$, we have that $P_\lambda = B^- = \left\{ \begin{pmatrix} a & 0 \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{C}) \right\}$ and $\mathrm{GL}_2(\mathbb{C})/B^- = \mathbb{P}^1(\mathbb{C})$.

To give a description of $\mathrm{Gr}^\lambda_{\mathrm{GL}_2}$ in the second case, we need the following definition:

Definition 2.3.24. Let X be a scheme of finite type over $\mathbb{C}$. The k-th jet-scheme $J_k(X)$ of X represents the functor

$$J_k(X) \colon \mathbb{C}\text{-Alg} \to \mathrm{Sets}, A \mapsto \mathrm{Hom}\left(\mathrm{Spec}\, A[t]/(t^{k+1}), X\right).$$

Remark 2.3.25. If $X = \mathrm{Spec}\,(\mathbb{C}[x_1, \dots, x_r]/(f_1, \dots, f_k))$ is affine, one can check that $J_k(X)$ is representable by an affine scheme of finite type over $\mathbb{C}$. If X is arbitrary the representability of $J_k(X)$ follows from the affine case by [GW10, Theorem 8.9].

Example 2.3.26. Notice that $J_0(X) = X$ and $J_1(X) = \mathcal{T}_{X/\mathbb{C}}$ is the tangent bundle on X.

Example 2.3.27. We want to compute $J_k(\mathbb{P}^1_\mathbb{C})$. To do so, we cover take the standard cover $\mathbb{P}^1_\mathbb{C} = \mathbb{A}^1_\mathbb{C} \cup \mathbb{A}^1_\mathbb{C}$ and obtain

$$J_k(\mathbb{P}^1_\mathbb{C}) = J_k(\mathbb{A}^1_\mathbb{C}) \cup J_k(\mathbb{A}^1_\mathbb{C}).$$

We therefore begin by computing $J_k(\mathbb{A}^1_\mathbb{C})$ and $J_k(\mathbb{A}^1_\mathbb{C} \setminus \{0\})$.

$$\begin{aligned} J_k(\mathbb{A}^1_\mathbb{C})(R) &= \mathrm{Hom}(\mathrm{Spec}\, R[t]/(t^{k+1}), \mathbb{A}^1_\mathbb{C}) = \mathrm{Hom}(\mathbb{C}[x], R[t]/(t^{k+1})) \\ &= R[t]/(t^{k+1}) = R^{k+1} \\ &= \mathbb{A}^{k+1}_\mathbb{C}(R) \end{aligned}$$

and therefore $J_k(\mathbb{A}^1_\mathbb{C}) = \mathbb{A}^{k+1}_\mathbb{C}$. We also have

$$\begin{aligned} J_k(\mathbb{A}^1_\mathbb{C} \setminus \{0\})(R) &= J_k(\mathrm{Spec}\, \mathbb{C}[x,y]/(xy-1))(R) \\ &= \mathrm{Hom}(\mathrm{Spec}\, R[t]/(t^{k+1}), \mathrm{Spec}\, \mathbb{C}[x,y]/(xy-1)) \\ &= \mathrm{Hom}(\mathbb{C}[x,y]/(xy-1), R[t]/(t^{k+1}) \\ &= \left\{ (\alpha, \beta) \in \left(R[t]/(t^{k+1})\right)^2 \;\middle|\; \alpha \cdot \beta = 1 \right\} \\ &= \left\{ \alpha = \alpha_0 + \alpha_1 t + \dots + \alpha_k t^k \;\middle|\; \alpha_0 \in R^* \right\} \\ &= \left((\mathbb{A}^1 \setminus \{0\}) \times \mathbb{A}^k \right)(R) \end{aligned}$$

and therefore $J_k(\mathbb{A}^1_\mathbb{C} \setminus \{0\}) \cong (\mathbb{A}^1_\mathbb{C} \setminus \{0\}) \times \mathbb{A}^k_\mathbb{C}$ with the embedding into $J_k(\mathbb{A}^1_\mathbb{C})$ the obvious map. It follows that $J_k(\mathbb{P}^1_\mathbb{C})$ consists of two affine open subsets $\mathbb{A}^{k+1}_\mathbb{C}$, glued along

$$\begin{tikzcd}[column sep=small] & \left\{ (\alpha, \beta) \in \left(R[t]/(t^{k+1})\right)^2 \;\middle|\; \alpha \cdot \beta = 1 \right\} \arrow[dl, hook'] \arrow[dr, hook] & \\ \left\{ \alpha \in R[t]/(t^{k+1}) \right\} & & \left\{ \beta \in R[t]/(t^{k+1}) \right\}. \end{tikzcd}$$

We have introduced the jet schemes because they describe the Schubert cells in $\mathrm{Gr}_{\mathrm{GL}_2}$.

Proposition 2.3.28. *If $\lambda = (k,l)$ with $k > l$ then $\mathrm{Gr}^{\lambda}_{\mathrm{GL}_2} \cong J_{k-l-1}(\mathbb{P}^1_{\mathbb{C}})$.*

Proof. As $\mathrm{Gr}^{\lambda}_{\mathrm{GL}_2} \cong \mathrm{Gr}^{\lambda-(1,1)}_{\mathrm{GL}_2}$, we can assume that $k > l = 0$. The elements of the Schubert cell $\mathrm{Gr}^{\lambda}_{\mathrm{GL}_2}$ are of the form

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \left[\begin{pmatrix} t^k & 0 \\ 0 & t^l \end{pmatrix}\right] = \left[\begin{pmatrix} t^k a & t^l b \\ t^k c & t^l d \end{pmatrix}\right] = \left[\begin{pmatrix} t^k a & b \\ t^k c & d \end{pmatrix}\right],$$

where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{C}[[t]])$ and we write $[B]$ for the residue class of a matrix $B \in \mathrm{GL}_2(\mathbb{C}((t)))$ in $\mathrm{Gr}_{\mathrm{GL}_2}(\mathbb{C}) = \mathrm{GL}_2(\mathbb{C}((t)))/\mathrm{GL}_2(\mathbb{C}[[t]])$. Let b_0 and d_0 in $\mathbb{C}$ be the constant term of the elements b and d in $\mathbb{C}[[t]]$. We must have $b_0 \neq 0$ or $d_0 \neq 0$, as the matrix A would otherwise not be invertible. Write $U_0 := \{b_0 \neq 0\}$ and $U_1 := \{d_0 \neq 0\}$. These are Zariski-open subsets of $\mathrm{Gr}^{\lambda}_{\mathrm{GL}_2}$ by Example 2.1.22. We therefore have a cover $\mathrm{Gr}^{\lambda}_{\mathrm{GL}_2} = U_0 \cup U_1$.

Let $A \cdot \left[\begin{pmatrix} t^k & 0 \\ 0 & 1 \end{pmatrix}\right] = \left[\begin{pmatrix} t^k a & b \\ t^k c & d \end{pmatrix}\right] \in U_0$. Then

$$\begin{aligned}
\left[\begin{pmatrix} t^k a & b \\ t^k c & d \end{pmatrix}\right] &= \left[\begin{pmatrix} t^k a & b \\ t^k c & d \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ -t^k b^{-1} a & 1 \end{pmatrix}\right] \\
&= \left[\begin{pmatrix} 0 & b \\ -t^k b^{-1}\det(A) & d \end{pmatrix}\right] = \left[\begin{pmatrix} 0 & b \\ -t^k b^{-1}\det(A) & d \end{pmatrix} \cdot \begin{pmatrix} b\det(A)^{-1} & 0 \\ 0 & b^{-1} \end{pmatrix}\right] \\
&= \left[\begin{pmatrix} 0 & 1 \\ -t^k & b^{-1}d \end{pmatrix}\right] = \left[\begin{pmatrix} 0 & 1 \\ -t^k & b^{-1}d \end{pmatrix} \cdot \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}\right] \\
&= \left[\begin{pmatrix} 0 & 1 \\ -t^k & \beta_0 + \beta_1 t + \dots + \beta_{k-1}t^{k-1} \end{pmatrix}\right].
\end{aligned}$$

Notice that matrices of this form form a system of representatives, because

$$\left[\begin{pmatrix} 0 & 1 \\ -t^k & \beta_0 + \beta_1 t + \dots + \beta_{k-1}t^{k-1} \end{pmatrix}\right] = \left[\begin{pmatrix} 0 & 1 \\ -t^k & \beta'_0 + \beta'_1 t + \dots + \beta'_{k-1}t^{k-1} \end{pmatrix}\right]$$

if and only if the product

$$\begin{aligned}
\begin{pmatrix} 0 & 1 \\ -t^k & \beta_0 + \dots \end{pmatrix}^{-1} \cdot \begin{pmatrix} 0 & 1 \\ -t^k & \beta'_0 + \dots \end{pmatrix} &= \begin{pmatrix} \beta_0 t^{-k} + \dots \beta_{k-1}t^{-1} & -t^{-k} \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ -t^k & \beta'_0 + \dots \end{pmatrix} \\
&= \begin{pmatrix} 1 & (\beta_0 - \beta'_0)t^{-k} + \dots + (\beta_{k-1} - \beta'_{k-1})t^{-1} \\ 0 & 1 \end{pmatrix}
\end{aligned}$$

lies in $\mathrm{GL}_2(\mathbb{C}[[t]])$. This only happens if $\beta = \beta'$. We therefore have

$$U_0 = \left\{ \left[\begin{pmatrix} 0 & 1 \\ -t^k & \beta_0 + \dots \beta_{k-1}t^{k-1} \end{pmatrix}\right] \;\middle|\; \beta_i \in \mathbb{C} \right\} \cong \mathbb{C}^k.$$

Now consider $A \cdot \left[\begin{pmatrix} t^k & 0 \\ 0 & 1 \end{pmatrix}\right] = \left[\begin{pmatrix} t^k a & b \\ t^k c & d \end{pmatrix}\right] \in U_1$, that is $d_0 \neq 0$. Then

$$\begin{aligned}
\left[\begin{pmatrix} t^k a & b \\ t^k c & d \end{pmatrix}\right] &= \left[\begin{pmatrix} t^k a & b \\ t^k c & d \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ -t^k d^{-1} c & 1 \end{pmatrix}\right] \\
&= \left[\begin{pmatrix} t^k d^{-1} \det(A) & b \\ 0 & d \end{pmatrix}\right] = \left[\begin{pmatrix} t^k d^{-1} \det(A) & b \\ 0 & d \end{pmatrix} \cdot \begin{pmatrix} d \det(A)^{-1} & 0 \\ 0 & d^{-1} \end{pmatrix}\right] \\
&= \left[\begin{pmatrix} t^k & bd^{-1} \\ 0 & 1 \end{pmatrix}\right] = \left[\begin{pmatrix} t^k & bd^{-1} \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}\right] \\
&= \left[\begin{pmatrix} t^k & \alpha_0 + \alpha_1 t + \cdots + \alpha_{k-1} t^{k-1} \\ 0 & 1 \end{pmatrix}\right].
\end{aligned}$$

Similarly as for U_0 we have

$$\left[\begin{pmatrix} t^k & \alpha_0 + \alpha_1 t + \cdots + \alpha_{k-1} t^{k-1} \\ 0 & 1 \end{pmatrix}\right] = \left[\begin{pmatrix} t^k & \alpha'_0 + \alpha'_1 t + \cdots + \alpha'_{k-1} t^{k-1} \\ 0 & 1 \end{pmatrix}\right]$$

if and only if the matrix

$$\begin{aligned}
\begin{pmatrix} t^k & \alpha_0 + \dots \\ 0 & 1 \end{pmatrix}^{-1} \cdot \begin{pmatrix} t^k & \alpha'_0 + \dots \\ 0 & 1 \end{pmatrix} &= \begin{pmatrix} t^{-k} & -\alpha_0 t^{-k} - \dots \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} t^k & \alpha'_0 + \dots \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & (\alpha'_0 - \alpha_0) t^{-k} + \dots \\ 0 & 1 \end{pmatrix}
\end{aligned}$$

lies in $\mathrm{GL}_2(\mathbb{C}[[t]])$. This only happens if $\alpha = \alpha'$. We therefore have

$$U_1 = \left\{ \left[\begin{pmatrix} t^k & \alpha_0 + \dots \alpha_{k-1} t^{k-1} \\ 0 & 1 \end{pmatrix}\right] \;\middle|\; \alpha_i \in \mathbb{C} \right\} \cong \mathbb{C}^k.$$

We now calculate $U_0 \cap U_1$. Notice from the calculations above that $\beta = \beta_0 + \cdots + \beta_{k-1} t^{k-1}$ can be seen as the image of $b^{-1} d$ in the ring $\mathbb{C}[[t]]/(t^k) = \mathbb{C}[t]/(t^k)$ and α as bd^{-1} in $\mathbb{C}[t]/(t^k)$. Therefore $U_0 = \mathbb{C}^k$ and $U_1 = \mathbb{C}^k$ are glued along the open subset $\mathbb{C}^* \times \mathbb{C}^{k-1} = U_0 \cap U_1$. The transition map is given by

$$U_0 \cap U_1 = \{\alpha_0 \neq 0\} \ni \alpha \longmapsto \alpha^{-1} \in \{\beta_0 \neq 0\},$$

seen as an elements in $\mathbb{C}[t]/(t^k)$.

We thus recognize $\mathrm{Gr}^{\lambda}_{\mathrm{GL}_2}$ as $J_{k-1}(\mathbb{P}^1_{\mathbb{C}})$ from Example 2.3.27. □

Let $\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(n) \to \mathbb{P}^1_{\mathbb{C}}$ be (the total space of) the line bundle $\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(1)^{\otimes n}$.

Corollary 2.3.29. *If $\lambda = (l+2, l)$, we have $\mathrm{Gr}^{\lambda}_{\mathrm{GL}_2} \cong \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(2)$.*

Proof. We have

$$\begin{aligned}\mathrm{Gr}_{\mathrm{GL}_2}^{(l+2,l)} &\cong J_1(\mathbb{P}^1_{\mathbb{C}}) = \mathcal{T}_{\mathbb{P}^1_{\mathbb{C}}/\mathbb{C}} = \left(\Omega^1_{\mathbb{P}^1_{\mathbb{C}}/\mathbb{C}}\right)^{\vee} = \left(\bigwedge^1 \Omega^1_{\mathbb{P}^1_{\mathbb{C}}/\mathbb{C}}\right)^{\vee} = \omega^{\vee}_{\mathbb{P}^1_{\mathbb{C}}/\mathbb{C}} \\ &\cong \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(-1-1)^{\vee} = \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(2),\end{aligned}$$

where $\Omega^1_{X/\mathbb{C}}$ is the sheaf of Kähler differentials and $\omega_{X/\mathbb{C}} = \bigwedge^{\dim X} \Omega^1_{X/\mathbb{C}}$ is the canonical sheaf of a smooth X, see [Har77, Chapter II.8]. □

Example 2.3.30. We glue $\mathbb{C}^k \setminus \{x_0 = 0\} = (\mathbb{C}[t]/(t^{k+1}))^*$ We can calculate the inverse β of an invertible $\alpha \in \mathbb{C}[t]/(t^{k+1})$ inductively using the formulas $\alpha_0\beta_0 = 1$ and

$$\sum_{j=0}^{l} \alpha_{l-j}\beta_j = 0$$

which is equivalent to $\beta_l = -\alpha_0^{-1}\sum_{j=0}^{l-1}\alpha_{l-j}\beta_j$. We therefore have that the gluing map in $\mathrm{Gr}_{\mathrm{GL}_2}^{(l+2,l)} = J_1(\mathbb{P}^1_{\mathbb{C}})$ is given by

$$\mathbb{C}^* \times \mathbb{C} \ni \alpha_0 + \alpha_1 t \mapsto \alpha_0^{-1} + \left(-\frac{\alpha_1}{\alpha_0^2}\right) t \in \mathbb{C}^* \times \mathbb{C}.$$

However, for $\mathrm{Gr}_{\mathrm{GL}_2}^{(l+3,l)} = J_2(\mathbb{P}^1_{\mathbb{C}})$ we have the following transition map

$$\mathbb{C}^* \times \mathbb{C}^2 \ni \alpha_0 + \alpha_1 t + \alpha_2 t^2 \mapsto \alpha_0^{-1} + \left(-\frac{\alpha_1}{\alpha_0^2}\right) t + \left(-\frac{\alpha_2}{\alpha_0^2} + \frac{\alpha_1}{\alpha_3}\right) t^2 \in \mathbb{C}^* \times \mathbb{C}.$$

As we expect from Theorem 2.3.15, this is indeed an affine bundle on $\mathbb{P}^1_{\mathbb{C}}$, but in fact not a vector bundle. The same thing happens for all higher $J_k(\mathbb{P}^1_{\mathbb{C}})$.

2.4 The Schubert Varieties

We now examine the closures of the Schubert cells in the Affine Grassmannian. In the case of partial flag varieties, the closures of classical Schubert cells were called Schubert varieties. These were irreducible projective varieties with a nice decomposition into Schubert cells.

The closures of the Schubert cells in the Affine Grassmannian have analogous structure. It turns out that they are also interesting projective varieties with a decomposition into Schubert cells. We will therefore refer to these as Schubert varieties as well:

Definition 2.4.1. The closure $\overline{\mathrm{Gr}_G^{\lambda}}$ of the Schubert cell Gr_G^{λ} is called the Schubert variety of $\lambda \in (X_*(G))_+$.

The following proposition captures the properties of the Schubert varieties that follow immediately from our earlier discussion of Schubert cells in the Affine Grassmannian.

Proposition 2.4.2. *The Schubert variety* $\overline{\mathrm{Gr}_G^\lambda}$ *of* λ *is an irreducible projective variety of dimension* $\langle 2\rho, \lambda\rangle$.

Proof. The Schubert cell Gr^λ lies completely in some $\mathrm{Gr}^{(N)}(\mathbb{C})$ by Proposition 2.3.6. Therefore the Schubert variety $\overline{\mathrm{Gr}^\lambda}$ will be projective as a closed subvariety of the projective $\mathrm{Gr}^{(N)}(\mathbb{C})$. The Schubert cell is irreducible by Corollary 2.3.17 and therefore the Schubert variety must be irreducible, too. Now, by Theorem 2.3.15 we have

$$\dim \overline{\mathrm{Gr}_G^\lambda} = \dim \mathrm{Gr}_G^\lambda = \langle 2\rho, \lambda\rangle. \qquad \square$$

Proposition 2.4.3. *We have* $\mathrm{Gr}_G^\lambda = \overline{\mathrm{Gr}_G^\lambda}$ *if and only if* λ *is minuscule.*

Proof. Immediate from Proposition 2.4.2 and Corollary 2.3.19. $\square$

Remark 2.4.4. In the case of partial flag varieties, we have that a classical Schubert cell equals its own closure if and only if it is projective, which happens only if the cell is a single point, i.e. the cell is labeled with 1.

Before we come to the aforementioned decomposition of the Schubert variety, we make an observation about how the Schubert varieties of in the case $\mathrm{SL}_n, \mathrm{PGL}_n$, and GL_n relate to each other.

Proposition 2.4.5. *The embeddings*

$$\mathrm{Gr}_{\mathrm{SL}_n}(\mathbb{C}) \hookrightarrow \mathrm{Gr}_{\mathrm{PGL}_n}(\mathbb{C}) \hookrightarrow \mathrm{Gr}_{\mathrm{GL}_n}(\mathbb{C})$$

from Corollary 2.2.16 induce isomorphisms on Schubert varieties.

Proof. By Proposition 2.3.6 these embeddings induce isomorphisms on Schubert cells. But these maps are closed-open immersions and so induce isomorphisms on the closures of Schubert cells as well. $\square$

Recall that a classical Schubert cell in a partial flag variety is a finite union of Schubert cells and that the cells appearing in a Schubert variety are given by some combinatorially defined partial order. Something similar will be true for the Schubert varieties in the Affine Grassmannian. First we need to define the partial order.

Definition 2.4.6. We write $\lambda \geq \mu$ if $\lambda - \mu$ can be written as an $\mathbb{N}_0$-linear combination of simple coroots $\alpha^\vee \in \Phi(G)^\vee$.

With this partial order, the Schubert varieties decompose in the following way.

Theorem 2.4.7. *The Schubert variety is the union*

$$\overline{\mathrm{Gr}_G^\lambda} = \coprod_{\mu \leq \lambda} \mathrm{Gr}_G^\mu.$$

We only prove the case $G = \mathrm{GL}_n$, but in light of Proposition 2.4.5 the cases $G = \mathrm{SL}_n$ and $G = \mathrm{PGL}_n$ follow immediately. For the case of arbitrary G, confer [Zhu16, Proposition 2.1.5], which takes a different approach with an alternative definition of the Schubert cells.

Before we prove Theorem 2.4.7, we need the following lemma, explaining how the Schubert cells of $\mathrm{Gr}_{\mathrm{GL}_n}(\mathbb{C})$ fit into the ind-scheme structure of $\mathrm{Gr}_{\mathrm{GL}_n}$.

Lemma 2.4.8. *Let $N \in \mathbb{N}_0$ be such that $\mathrm{Gr}^{\lambda}_{\mathrm{GL}_n} \subseteq \mathrm{Gr}^{(N)}_{\mathrm{GL}_n}(\mathbb{C})$ as in Proposition 2.3.9. Recall the map*

$$\begin{aligned}\mathrm{Gr}^{(N)}_{\mathrm{GL}_n}(\mathbb{C}) &\xrightarrow{\sim} \{t\text{-invariant subspaces of } t^{-N}\mathbb{C}[[t]]^n/t^N\mathbb{C}[[t]]^n \cong \mathbb{C}^{2Nn}\},\\ [A] &\mapsto \left\langle A_{-,1}, tA_{-,1}, \dots, t^{N-1}A_{-,1}, A_{-,2}, \dots, t^{N-1}A_{-,2}, \dots, t^{N-1}A_{-,n}\right\rangle_{\mathbb{C}}.\end{aligned}$$

from Lemma 2.1.19. The image of $\mathrm{Gr}^{\lambda}_{\mathrm{GL}_n}$ under this map is

$$\left\{ V \subseteq t^{-N}\mathbb{C}[[t]]^n/t^N\mathbb{C}[[t]]^n \;\middle|\; \begin{array}{l} V \text{ is } t\text{-invariant and} \\ \dim(t^iV) = \sum_{j:N-i>\lambda_j}(N-\lambda_j-i) \text{ for } i \geq 0 \end{array} \right\}.$$

Proof. The image of $[At^{\lambda}]$ with $A \in \mathrm{GL}_n(\mathbb{C}[[t]])$ is

$$V := \left\langle t^{\lambda_1}A_{-,1}, t^{\lambda_1+1}A_{-,1}, \dots, t^{N-1}A_{-,1}, t^{\lambda_2}A_{-,2}, \dots, t^{N-1}A_{-,2}, \dots, t^{N-1}A_{-,n}\right\rangle_{\mathbb{C}}.$$

The generating set

$$t^{\lambda_1+i}A_{-,1}, \dots t^{N-1}A_{-,1}, \dots t^{N-1}A_{-,n}$$

of t^iV, where we ignore any $t^jA_{-,i}$ if $j \geq N$, is in fact a basis. Indeed, given a $\mathbb{C}$-linear dependence of these vectors, we obtain a $\mathbb{C}[[t]]$-linear dependence between the columns $A_{-,i}$ of $A \in \mathrm{GL}_n(\mathbb{C}[[t]])$. It follows that t^iV has the required dimension.

Conversely, let V be a t-invariant subspace of $t^{-N}\mathbb{C}[[t]]^n/t^N\mathbb{C}[[t]]^n$ such that the above dimension conditions are satisfied. Taking the generalized eigenspace decomposition of V with respect to the linear endomorphism t gives us a basis of the required form. ☐

We can now come back to Theorem 2.4.7.

Proof of Theorem 2.4.7. Observe that the set

$$X := \left\{ V \subseteq t^{-N}\mathbb{C}[[t]]^n/t^N\mathbb{C}[[t]]^n \;\middle|\; \begin{array}{l} V \text{ is } t\text{-invariant and} \\ \dim(V) = \sum_{j=1}^{n}(N-\lambda_j) \text{ and} \\ \dim(t^iV) \leq \sum_{j:N-i>\lambda_j}(N-\lambda_j-i) \text{ for } i \geq 0 \end{array} \right\}$$

is a closed subset of $\mathrm{Grass}(2Nn)^t_d(\mathbb{C})$ for $d = \sum_j(N-\lambda_j)$. We can therefore identify X with a closed subset of $\mathrm{Gr}^{(N)}_{\mathrm{GL}_n}(\mathbb{C})$. But X contains $\mathrm{Gr}^{\lambda}_{\mathrm{GL}_n}$ by Lemma 2.4.8 and so X also contains the Schubert variety $\overline{\mathrm{Gr}^{\lambda}_{\mathrm{GL}_n}}$.

Claim: We have $X \subseteq \coprod_{\mu\leq\lambda}\mathrm{Gr}^{\mu}_{\mathrm{GL}_n}$.

Let $V \in X$. The vector space V will lie in some $\mathrm{Gr}^{\mu}_{\mathrm{GL}_n}$ for $\mu = (\mu_1, \ldots, \mu_n) \in (X_*(G))_+$ as $\mathrm{Gr}_{\mathrm{GL}_n}(\mathbb{C}) = \bigcup_{\mu} \mathrm{Gr}^{\mu}_{\mathrm{GL}_n}$. Therefore V is the vector space generated by the columns of At^{μ} for some $A \in \mathrm{GL}_n(\mathbb{C}[[t]])$. If $\mu \neq \lambda$, let i be minimal such that $\dim(t^i V) < \sum (N - \lambda_j - i)$. This corresponds to a $\mu_j < \lambda_j$ and $\mu_k = \lambda_k$ for $k < j$. But $\sum_j \mu_k = nN - \dim V = \sum_j \lambda_j$. Therefore there must be some $j' > j$ such that $\lambda_{j'} < \mu_{j'}$. Adding $\varepsilon_j - \varepsilon_{j'}$ to μ, we obtain a $\mu^{(1)}$ such that $\mu^{(1)} > \mu$ and the columns of $At^{\mu^{(1)}}$ still lie in X. Repeating this argument, we obtain a sequence $\mu < \mu^{(1)} < \mu^{(2)} < \ldots$ which approximates λ and we must therefore have $\lambda = \mu^{(m)} > \mu$ for some $m \gg 0$. The claim follows.
Claim: We have $\mathrm{Gr}^{\mu}_{\mathrm{GL}_n} \subseteq \overline{\mathrm{Gr}^{\lambda}_{\mathrm{GL}_n}}$ for $\mu \leq \lambda$.
To simplify the notation, we only work in the case $n = 2$, but the case of general n is exactly the same.

Let $\lambda = (k, l)$ with $k \geq l + 2$. Consider $\left[\begin{pmatrix} t^k & t^l + at^{l+1} \\ 0 & t^l \end{pmatrix}\right] \in \mathrm{Gr}^{(k,l)}_{\mathrm{GL}_2}$ with $a \in \mathbb{C}$. Elements of this type form a subvariety of $\mathrm{Gr}^{\lambda}_{\mathrm{GL}_2}$ isomorphic to $\mathbb{A}^1_{\mathbb{C}}$, see the proof of Proposition 2.3.28. We have for $a \neq 0$

$$\begin{aligned}
\left[\begin{pmatrix} t^k & t^l + at^{l+1} \\ 0 & t^l \end{pmatrix}\right] &= \left[\begin{pmatrix} t^k & t^l + at^{l+1} \\ 0 & t^l \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ -a^{-1}t^{k-l-1} & 1 \end{pmatrix}\right] \\
&= \left[\begin{pmatrix} -a^{-1}t^{k-1} & t^l + at^{l+1} \\ -a^{-1}t^{k-1} & t^l \end{pmatrix}\right] \\
&= \left[\begin{pmatrix} -a^{-1}t^{k-1} & t^l + at^{l+1} \\ -a^{-1}t^{k-1} & t^l \end{pmatrix} \cdot \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}\right] \\
&= \left[\begin{pmatrix} -t^{k-1} & a^{-1}t^l + t^{l+1} \\ -t^{k-1} & a^{-1}t^l \end{pmatrix}\right] \\
&\xrightarrow{a \to \infty} \left[\begin{pmatrix} -t^{k-1} & t^{l+1} \\ -t^{k-1} & 0 \end{pmatrix}\right].
\end{aligned}$$

Note that the limit must exist in $\overline{\mathrm{Gr}^{\lambda}_{\mathrm{GL}_2}}$ as this is a projective variety. Therefore the element $\left[\begin{pmatrix} t^{k-1} & t^{l+1} \\ -t^{k-1} & 0 \end{pmatrix}\right] \in \mathrm{Gr}^{(k-1,l+1)}_{\mathrm{GL}_2}$ lies in $\overline{\mathrm{Gr}^{(k,l)}_{\mathrm{GL}_2}}$ and so also $\mathrm{Gr}^{(k-1,l+1)}_{\mathrm{GL}_2} \subseteq \overline{\mathrm{Gr}^{(k,l)}_{\mathrm{GL}_2}}$.

By the same argument for GL_n it follows that $\mathrm{Gr}^{\mu}_{\mathrm{GL}_n} \subseteq \overline{\mathrm{Gr}^{\lambda}_{\mathrm{GL}_n}}$ if $\lambda - \mu$ is a simple coroot. By iteration we deduce $\mathrm{Gr}^{\mu}_{\mathrm{GL}_n} \subseteq \overline{\mathrm{Gr}^{\lambda}_{\mathrm{GL}_n}}$ if $\mu \leq \lambda$.

Summing up, we have shown that

$$\coprod_{\mu \leq \lambda} \mathrm{Gr}^{\mu}_{\mathrm{GL}_n} \subseteq \overline{\mathrm{Gr}^{\lambda}_{\mathrm{GL}_n}} \subseteq X \subseteq \coprod_{\mu \leq \lambda} \mathrm{Gr}^{\mu}_{\mathrm{GL}_n}.$$

The theorem follows. $\square$

The following is an easy corollary of Theorem 2.4.7.

Corollary 2.4.9. *If the Schubert cell Gr_G^μ lies in the Schubert variety $\overline{\mathrm{Gr}_G^\lambda}$, then*

$$\dim \mathrm{Gr}_G^\mu \equiv \dim \overline{\mathrm{Gr}_G^\lambda} \mod 2.$$

Proof. By Theorem 2.3.15 we have $\dim \mathrm{Gr}_G^\lambda = \dim \overline{\mathrm{Gr}_G^\lambda} = \langle 2\rho, \lambda\rangle$ and therefore

$$\dim \overline{\mathrm{Gr}_G^\lambda} - \dim \mathrm{Gr}_G^\mu = \langle 2\rho, \lambda - \mu\rangle.$$

Now by Theorem 2.4.7 we have the $\mathrm{Gr}_G^\mu \subseteq \overline{\mathrm{Gr}_G^\lambda}$ if and only if $\lambda - \mu$ is a sum of simple coroots. But $\langle \rho, \alpha^\vee\rangle \in \mathbb{Z}$ for all coroots $\alpha^\vee \in \Phi^\vee$. Hence, $\dim \overline{\mathrm{Gr}_G^\lambda} - \dim \mathrm{Gr}_G^\mu \in 2\mathbb{Z}$. $\square$

The remainder of this chapter is dedicated to calculations in GL_2.

Example 2.4.10. The Schubert variety $\overline{\mathrm{Gr}_{\mathrm{GL}_2}^{(k,l)}}$ has the decomposition

$$\overline{\mathrm{Gr}_{\mathrm{GL}_2}^{(k,l)}} = \begin{cases} \mathrm{Gr}_{\mathrm{GL}_2}^{(k,l)} \amalg \mathrm{Gr}_{\mathrm{GL}_2}^{(k-1,l+1)} \amalg \cdots \amalg \mathrm{Gr}_{\mathrm{GL}_2}^{(\frac{k-l}{2},\frac{k-l}{2})}, & \text{if } k-l \text{ is even;} \\ \mathrm{Gr}_{\mathrm{GL}_2}^{(k,l)} \amalg \mathrm{Gr}_{\mathrm{GL}_2}^{(k-1,l+1)} \amalg \cdots \amalg \mathrm{Gr}_{\mathrm{GL}_2}^{(\frac{k-l-1}{2}+1,\frac{k-l-1}{2})}, & \text{if } k-l \text{ is odd.} \end{cases}$$

In particular,

$$\overline{\mathrm{Gr}_{\mathrm{GL}_2}^{(1,-1)}} = \mathrm{Gr}_{\mathrm{GL}_2}^{(1,-1)} \amalg \mathrm{Gr}_{\mathrm{GL}_2}^{(0,0)} = \mathrm{Gr}_{\mathrm{GL}_2}^{(1,-1)} \amalg \{pt\}.$$

We have seen in Proposition 2.3.28 that $\mathrm{Gr}_{\mathrm{GL}_2}^{(k,l)}$ is isomorphic to the $(k-l-1)$-th jet bundle on $\mathbb{P}^1_{\mathbb{C}}$ and in particular that $\mathrm{Gr}_{\mathrm{GL}_2}^{(l+2,l)}$ is the tangent bundle of $\mathbb{P}^1_{\mathbb{C}}$.

Proposition 2.4.11. *The Schubert variety of $(k+2,k)$ in $\mathrm{Gr}_{\mathrm{GL}_2}$ has the form*

$$\overline{\mathrm{Gr}_{\mathrm{GL}_2}^{(k+2,k)}} \cong \mathbb{P}(1,1,2),$$

where $\mathbb{P}(1,1,2)$ is the weighted projective space of weight $(1,1,2)$.

For a general definition of weighted projective space see for instance [GW10, Exercise 13.1]. We do not recall it here, as we give a description of the Schubert variety in terms of affine open subsets and transition maps and recognize this afterwards as $\mathbb{P}(1,1,2)$.

Proof. We have $\overline{\mathrm{Gr}_{\mathrm{GL}_2}^{(k+2,k)}} \cong \overline{\mathrm{Gr}_{\mathrm{GL}_2}^{(1,-1)}} = \mathrm{Gr}_{\mathrm{GL}_2}^{(1,-1)} \amalg \mathrm{Gr}_{\mathrm{GL}_2}^{(0,0)} = \mathrm{Gr}_{\mathrm{GL}_2}^{(1,-1)} \amalg \{pt\}$. By the proof of Proposition 2.3.28 $\mathrm{Gr}_{\mathrm{GL}_2}^{(1,-1)} = \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(2) = U_0 \cup U_1$ is glued from the affine spaces

$$U_0 = \left\{ \left[\begin{pmatrix} 0 & t^{-1} \\ -t & at^{-1} + b \end{pmatrix} \right] \right\} \cong \mathbb{C}^2,$$
$$U_1 = \left\{ \left[\begin{pmatrix} t & ct^{-1} + d \\ 0 & t^{-1} \end{pmatrix} \right] \right\} \cong \mathbb{C}^2$$

along the transition map

$$U_0 \supseteq \{a \neq 0\} \xrightarrow{(a,b)\mapsto(a^{-1},-\frac{b}{a^2})} \{c \neq 0\} \subseteq U_1.$$

The spaces U_0, U_1 are open in $\mathrm{Gr}_{\mathrm{GL}_2}^{(1,-1)} \amalg \{pt\}$ and so we only need to produce a third open subset which contains the point of $\mathrm{Gr}_{\mathrm{GL}_2}^{(0,0)}$.

We see that $\{b = 0\} \cup \{d = 0\}$ is a closed subset of $U_0 \cup U_1$. It is isomorphic to $\mathbb{P}^1_{\mathbb{C}}(\mathbb{C})$ as we glue two copies of $\mathbb{C}^1$ along $\mathbb{C} \setminus \{0\} \ni x \mapsto x^{-1} \in \mathbb{C} \setminus \{0\}$. As this subset is a projective variety, it is closed even in $\overline{\mathrm{Gr}_{\mathrm{GL}_2}^{(1,-1)}}$. Let its complement in the Schubert variety be U_2. This open subset of $\overline{\mathrm{Gr}_{\mathrm{GL}_2}^{(1,-1)}}$ can be written as

$$U_2 = \{pt\} \cup (U_0 \cap \{b \neq 0\}) \cup (U_1 \cap \{d \neq 0\}).$$

An element in $U_0 \cap \{b \neq 0\}$ is of the form

$$\begin{aligned}
\left[\begin{pmatrix} 0 & t^{-1} \\ -t & at^{-1}+b \end{pmatrix}\right] &= \left[\begin{pmatrix} 0 & t^{-1} \\ -t & at^{-1}+b \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ b^{-1}t & 1 \end{pmatrix}\right] \\
&= \left[\begin{pmatrix} b^{-1} & t^{-1} \\ ab^{-1} & at^{-1}+b \end{pmatrix}\right] = \left[\begin{pmatrix} b^{-1} & t^{-1} \\ ab^{-1} & at^{-1}+b \end{pmatrix} \cdot \begin{pmatrix} b & 0 \\ 0 & b^{-1} \end{pmatrix}\right] \\
&= \left[\begin{pmatrix} 1 & b^{-1}t^{-1} \\ a & ab^{-1}t^{-1}+1 \end{pmatrix}\right] = \left[\begin{pmatrix} 1 & b^{-1}t^{-1} \\ a & ab^{-1}t^{-1}+1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ -a & 1 \end{pmatrix}\right] \\
&= \left[\begin{pmatrix} -ab^{-1}t^{-1}+1 & b^{-1}t^{-1} \\ -a^2b^{-1}t^{-1} & ab^{-1}t^{-1}+1 \end{pmatrix}\right] \\
&= \left[\begin{pmatrix} -ab^{-1}t^{-1}+1 & b^{-1}t^{-1} \\ -a^2b^{-1}t^{-1} & ab^{-1}t^{-1}+1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right] \\
&= \left[\begin{pmatrix} -b^{-1}t^{-1} & -ab^{-1}t^{-1}+1 \\ -ab^{-1}t^{-1}-1 & -a^2b^{-1}t^{-1} \end{pmatrix}\right].
\end{aligned}$$

An element in $U_1 \cap \{d \neq 0\}$ is of the form

$$\begin{aligned}
\left[\begin{pmatrix} t & ct^{-1}+d \\ 0 & t^{-1} \end{pmatrix}\right] &= \left[\begin{pmatrix} t & ct^{-1}+d \\ 0 & t^{-1} \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ -d^{-1}t & 1 \end{pmatrix}\right] \\
&= \left[\begin{pmatrix} -cd^{-1} & ct^{-1}+d \\ -d^{-1} & t^{-1} \end{pmatrix}\right] = \left[\begin{pmatrix} -cd^{-1} & ct^{-1}+d \\ -d^{-1} & t^{-1} \end{pmatrix} \cdot \begin{pmatrix} d & 0 \\ 0 & d^{-1} \end{pmatrix}\right] \\
&= \left[\begin{pmatrix} -c & cd^{-1}t^{-1}+1 \\ -1 & d^{-1}t^{-1} \end{pmatrix}\right] = \left[\begin{pmatrix} -c & cd^{-1}t^{-1}+1 \\ -1 & d^{-1}t^{-1} \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}\right] \\
&= \left[\begin{pmatrix} c^2d^{-1}t^{-1} & cd^{-1}t^{-1}+1 \\ cd^{-1}t^{-1}-1 & d^{-1}t^{-1} \end{pmatrix}\right].
\end{aligned}$$

We can also write $\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right] = \left[\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right]$. Therefore all the elements of U_2 are of the form

$$\left[\begin{pmatrix} xt^{-1} & yt^{-1}+1 \\ yt^{-1}-1 & zt^{-1} \end{pmatrix}\right],$$

with $xz = y^2$.

Now observe that if $\left[\begin{pmatrix} xt^{-1} & yt^{-1}+1 \\ yt^{-1}-1 & zt^{-1} \end{pmatrix}\right] = \left[\begin{pmatrix} x't^{-1} & y't^{-1}+1 \\ y't^{-1}-1 & z't^{-1} \end{pmatrix}\right]$, then

$$\begin{aligned} &\begin{pmatrix} z't^{-1} & -y't^{-1}-1 \\ -y't^{-1}+1 & x't^{-1} \end{pmatrix} \cdot \begin{pmatrix} xt^{-1} & yt^{-1}+1 \\ yt^{-1}-1 & zt^{-1} \end{pmatrix} \\ &= \begin{pmatrix} (xz'-yy')t^{-2}+(y'-y)t^{-1}+1 & (yz'-y'z)t^{-2}+(z'-z)t^{-1} \\ (x'y-xy')t^{-2}+(x-x')t^{-1} & (x'z-yy')t^{-2}+(y-y')t^{-1}+1 \end{pmatrix} \in \mathbb{C}[[t]]^{2\times 2}, \end{aligned}$$

i.e. $(x, y, z) = (x', y', z')$. We have therefore found

$$U_2 \cong \{(x, y, z) \in \mathbb{C}^3 \mid xz = y^2\}.$$

All together we have $\mathrm{Gr}_1 \amalg \mathrm{Gr}_1 = U_0 \cup U_1 \cup U_2$ where

$$U_0 \cong \mathbb{C}^2, \quad U_1 \cong \mathbb{C}^2, \quad U_2 \cong \operatorname{Spec} \mathbb{C}[x, y, z]/(xz - y^2) \cong \operatorname{Spec} \mathbb{C}[x^2, xy, y^2],$$

with intersections

$$U_0 \cap U_1 \cong \mathbb{C}^* \times \mathbb{C}, \quad U_0 \cap U_2 \cong \mathbb{C} \times \mathbb{C}^*, \quad U_1 \cap U_2 \cong \mathbb{C} \times \mathbb{C}^*$$

and transition functions

$$\begin{aligned} U_0 \supseteq U_0 \cap U_1 \cong \mathbb{C}^* \times \mathbb{C} &\xrightarrow{\sim} \mathbb{C}^* \times \mathbb{C} \cong U_0 \cap U_1 \subseteq U_1 \\ (a, b) &\longmapsto (a^{-1}, -a^{-2}b), \end{aligned}$$

$$\begin{aligned} U_0 \supseteq U_0 \cap U_2 \cong \mathbb{C} \times \mathbb{C}^* &\xrightarrow{\sim} \mathbb{C} \times \mathbb{C}^* \cong U_0 \cap U_2 \subseteq U_2 \\ (a, b) &\longmapsto (-b^{-1}, -ab^{-1}), \end{aligned}$$

$$\begin{aligned} U_1 \supseteq U_1 \cap U_2 \cong \mathbb{C} \times \mathbb{C}^* &\xrightarrow{\sim} \mathbb{C}^* \times \mathbb{C} \cong U_1 \cap U_2 \subseteq U_2 \\ (c, d) &\longmapsto (cd^{-1}, d^{-1}). \end{aligned}$$

We can recognize this as the weighted projective space $\mathbb{P}(1, 1, 2)$. $\square$

Chapter 3

Perverse Sheaves

We give a very quick introduction to Verdier duality and perverse sheaves without many proofs. Our main reference for perverse sheaves is [BBD82]. For Verdier duality see the original paper [Ver95]. Also confer [GM03, Chapter III.8] for generalities on sheaves.

Given a sheaf $\mathcal{F}$ on the space X, we denote by $H^i(X, \mathcal{F})$ the i-th sheaf cohomology group of $\mathcal{F}$ for $i \geq 0$. Recall that this is defined as $R^i(\Gamma(X, -))(\mathcal{F})$ where $\Gamma(X, \mathcal{F})$ denotes the global sections of $\mathcal{F}$. For a complex $\mathcal{F}^\bullet \in D^b(X)$ we write $\mathcal{H}^i(\mathcal{F}^\bullet)$ for the i-th cohomology sheaf, with $i \in \mathbb{Z}$.

3.1 Local Systems

For now, let X be (the complex points of) a finite-dimensional complex variety. All sheaves will be sheaves with respect to the complex-analytic topology, whereas closed, open, and locally closed subsets will almost always refer to subsets in the Zariski topology. Note that X has nice topological properties: X is Hausdorff, locally path connected, has only finitely many connected components, etc.

We will generalize this afterwards to ind-varieties.

Definition 3.1.1. The constant sheaf A_X of an abelian group A on X is the sheafification of the constant presheaf on X, which is given by $X \supseteq U \mapsto A$. That means that

$$A_X(U) = \{A\text{-valued locally constant functions on } U\}.$$

Proposition 3.1.2. *The constant sheaf A_X is given by $X \supseteq U \mapsto A^{\pi_0(U)}$ and the stalk at $x \in X$ is given by $(A_X)_x = A$.*

It follows from this proposition that any map between abelian groups $A \to B$ defines a map between the constant sheaves $A_X \to B_X$, via $A_X(U) = A^{\pi_0(U)} \to B^{\pi_0(U)} = B_X(U)$. It turns out that a map between abelian groups is the same as a map between the corresponding constant sheaves.

Proposition 3.1.3. *The map*

$$\begin{aligned} \mathrm{Hom}(A,B) &\longrightarrow \mathrm{Hom}_{\mathrm{Sh(X)}}(A_X,B_X), \\ f &\longmapsto \Big(f^{\pi_0(U)}\colon A_X(U)=A^{\pi_0(U)}\to B^{\pi_0(U)}=B_X(U)\Big) \end{aligned}$$

is a bijection.

The following needs some weak topological conditions on X, which are satisfied in the context of varieties.

Proposition 3.1.4. *If we apply the sheaf cohomology functors to the constant sheaf A_X on X we obtain the classically defined singular cohomology groups*

$$H^i(X,A_X)=H^i_{\mathrm{sing}}(X;A).$$

We denote this from now on by $H^i(X;A)$.

Proof. See for example [Bre97, Theorem 3.1.1]. □

From now on, let k be a field.

Definition 3.1.5. A sheaf $\mathcal{L}$ is a local system if there is an open cover $X=\bigcup_i U_i$ such that $\mathcal{L}|_{U_i}$ is a constant sheaf and all stalks are finite-dimensional k-vector spaces. We denote the the full subcategory of $\mathrm{Sh}(X)$ of local systems by $\mathrm{Loc}_f(X;k)$.

Definition 3.1.6. Let $\mathcal{A}$ be an abelian category. We call a full subcategory $\mathcal{B}$ a weak Serre subcategory, if it is closed under kernels, cokernels and extensions.

Proposition 3.1.7. *The full subcategory $\mathcal{B}\subseteq\mathcal{A}$ is weak Serre if and only if for every exact sequence*

$$X_1\to X_2\to X\to X_3\to X_4$$

in $\mathcal{A}$ such that X_i lies in $\mathcal{B}$ for $i=1,\dots,4$ also X lies in $\mathcal{B}$.

Proof. It is clear that a category satisfying the second condition is closed under kernels, cokernels, and extensions.

Conversely, let $\mathcal{B}$ be a weak Serre subcategory of $\mathcal{A}$ and let

$$X_1\overset{\varphi}{\to}X_2\overset{\alpha}{\to}X\overset{\beta}{\to}X_3\overset{\psi}{\to}X_4$$

be an exact sequence in $\mathcal{A}$ with $X_i\in\mathcal{B}$. We can rewrite X as the extension

$$0\to\ker(\beta)\to X\to\mathrm{im}(\beta)\to 0,$$

with $\ker(\beta)=\mathrm{im}(\alpha)=X_2/\ker(\alpha)=X_2/\mathrm{im}(\varphi)\in\mathcal{B}$ and $\mathrm{im}(\beta)=\ker(\psi)\in\mathcal{B}$. So X lies in $\mathcal{B}$ as an extension of objects in $\mathcal{B}$. □

Proposition 3.1.8. *The category* $\mathrm{Loc}_f(X;k)$ *is a weak Serre subcategory of* $\mathrm{Sh}(X,k)$.

We give a proof of this to indicate how one reduces statements about local systems to statements about constant sheaves and finite-dimensional vector spaces.

Proof. Kernels and cokernels of a map $A_X \to B_X$ between constant sheaves can be computed as $\ker(A \to B)_X$ and $\operatorname{coker}(A \to B)_X$, respectively. It follows that kernels and cokernels between locally constant sheaves are also locally constant. The finiteness condition on stalks follows from the fact that k is Noetherian as a field. We now show the closure under extensions. Let $0 \to \mathcal{L}' \hookrightarrow \mathcal{L} \to \mathcal{L}'' \to 0$ be an exact sequence in $\mathrm{Sh}(X;k)$ such that $\mathcal{L}'$ and $\mathcal{L}''$ are local systems. We can find a common trivialization of $\mathcal{L}'$ and $\mathcal{L}''$ and consider the exact sequence

$$0 \to \mathcal{L}'|_U \to \mathcal{L}|_U \to \mathcal{L}''|_U \to 0$$

of sheaves on $U \subseteq X$. We may therefore assume that $\mathcal{L}'$ and $\mathcal{L}''$ are constant sheaves on X.

For a connected open subset $U \subseteq X$ consider the long exact cohomology sequence

$$0 \to \mathcal{L}'(U) \to \mathcal{L}(U) \to \mathcal{L}''(U) \to H^1(U, \mathcal{L}'|_U) \to \cdots$$

We have that $\mathcal{L}'(U)$ and $\mathcal{L}''(U)$ are finite-dimensional vector spaces, because they are local systems and U is connected. It follows that $\mathcal{L}(U)$ must also be finite-dimensional for every connected $U \subseteq X$.

Next, we localize and obtain for every $x \in X$ the exact sequence

$$0 \to \mathcal{L}'_x \to \mathcal{L}_x \to \mathcal{L}''_x \to 0.$$

But

$$\mathcal{L}_x = \varinjlim_{x \in U} \mathcal{L}(U) = \varinjlim_{\substack{x \in U \\ U \text{ connected}}} \mathcal{L}(U),$$

and so, by properties of direct limits of finite-dimensional vector spaces, there must be some connected open U_x containing x such that $\mathcal{L}(U_x) \to \mathcal{L}_x$ is an isomorphism. As $\mathcal{L}'$ and $\mathcal{L}''$ are constant, we have $\mathcal{L}'(U_x) = \mathcal{L}'_x$ and $\mathcal{L}''(U_x) = \mathcal{L}''_x$. We therefore have

$$0 \to \mathcal{L}'(U_x) \to \mathcal{L}(U_x) \to \mathcal{L}''(U_x) \to 0$$

is exact.

The constant sheaf $(\mathcal{L}(U_x))_{U_x}$ on U_x fits into an exact sequence

$$0 \to \mathcal{L}'|_{U_x} \to (\mathcal{L}(U_x))_{U_x} \to \mathcal{L}''|_{U_x} \to 0.$$

Now we can construct a map from the constant sheaf $(\mathcal{L}(U_x))_{U_x}$ to $\mathcal{L}|_{U_x}$. For a connected $U \subseteq U_x$ we define $(\mathcal{L}(U_x))(U) = \mathcal{L}(U_x) \to \mathcal{L}(U)$ as the restriction map of the sheaf $\mathcal{L}$ from U_x to U. This is fits into the diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{L}'|_{U_x} \arrow[r] \arrow[d, equal] & (\mathcal{L}(U_x))_{U_x} \arrow[r] \arrow[d] & \mathcal{L}''|_{U_x} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathcal{L}'|_{U_x} \arrow[r] & \mathcal{L}|_{U_x} \arrow[r] & \mathcal{L}''|_{U_x} \arrow[r] & 0.
\end{tikzcd}$$

By the five-lemma, the middle arrow is an isomorphism and $\mathcal{L}$ is constant on U_x with finite fibers. As $X = \bigcup_{x \in X} U_x$, $\mathcal{L}$ is locally constant and we have shown that the extension of local systems is a local system. □

Theorem 3.1.9. *If X is connected and $x_0 \in X$, there is an equivalence of categories*

$$\mathrm{Loc}_f(X; k) \simeq \mathrm{Rep}_k(\pi_1(X, x_0))$$

between the category of local systems on X and the finite-dimensional k-linear representations of the fundamental group of X.

Proof. For a full proof see [Sza09, Theorem 2.5.14].

We will only sketch the construction of the functor $\mathrm{Loc}_f(X; k) \to \mathrm{Rep}_k(\pi_1(X, x_0))$. For a local system $\mathcal{L}$ we take the vector space $\mathcal{L}_{x_0}$ and endowing it with an action of $\pi_1(X, x_0)$ in the following way: Take a loop α in X from x_0 to itself and a trivializing open cover of X for $\mathcal{L}$. This covers the image of α in X, which is compact and we may therefore assume the cover to be finite. Then we obtain isomorphisms

$$\mathcal{L}_{x_0} = \mathcal{L}(U_0) \to \mathcal{L}(U_1) \to \cdots \to \mathcal{L}(U_0) = \mathcal{L}_{x_0},$$

the composition of which is independent of the chosen cover and the chosen representative of the homotopy class $[\alpha] \in \pi_1(X, x_0)$. This defines the action of the fundamental group $\pi_1(X, x_0)$ on $\mathcal{L}_{x_0}$. □

One can leave out the assumption that X is connected, by replacing the fundamental group of X with its fundamental groupoid.

It follows that $\mathrm{Loc}_f(X; k)$ is an abelian category.

Corollary 3.1.10. *If X is connected and simply-connected,* $\mathrm{Loc}_f(X; k)$ *is equivalent to* Vect_k*, the category of finite-dimensional k-vector spaces, via the functor $L \mapsto L_{x_0}$, and so k_X^n are the only local systems.*

A nice independent observation is the following:

Corollary 3.1.11. *Simply-connected manifolds are orientable.*

Proof. The orientation sheaf is a local system. It must however already be constant, as all local systems are constant on a simply-connected space. Now a space is orientable if its orientation sheaf is constant. □

3.2 Verdier Duality and Constructible Complexes

Let $f : X \to Y$ be a morphism. On the level of sheaves we obtain the direct image $f_* : \mathrm{Sh}(X; k) \to \mathrm{Sh}(Y; k)$ by $f_*\mathcal{F}(V) = \mathcal{F}(f^{-1}(V))$ for $V \subseteq Y$ and the inverse image functor $f^* : \mathrm{Sh}(Y; k) \to \mathrm{Sh}(X; k)$ by sheafifying the mapping

$$U \mapsto \varinjlim_{V \supseteq f(U)} \mathcal{G}(V).$$

These functors form an adjoint pair (f^*, f_*). What is more, we have that f^* is both left and right exact, while f_* is in general only left exact. Note that our inverse image is called f^{-1} by [Har77], while he uses f^* for a morphism between categories of $\mathcal{O}_X$-modules.

We also have a functor $f_! \colon \mathrm{Sh}(X;k) \to \mathrm{Sh}(Y;k)$ given by

$$f_!\mathcal{F}(V) = \left\{\sigma \in f_*\mathcal{F}(V) = \mathcal{F}(f^{-1}(V)) \;\middle|\; \operatorname{supp}\sigma \to f^{-1}(V) \to V \text{ is a proper morphism}\right\}$$

where proper means that the inverse image of any compact set is compact.

We give a description of $f_!$ in the cases f is proper and $f = j$ is an open immersion.

Proposition 3.2.1. *If f is proper, $f_* = f_!$.*

Proof. This is immediate from the definition as for all sheaves $\mathcal{F}$, open subsets $V \subseteq X$, and sections $\sigma \in \mathcal{F}(V)$ the composition $\operatorname{supp}\sigma \to f^{-1}(V) \to V$ is proper. □

If $j \colon U \hookrightarrow X$ is an open embedding and $\mathcal{F} \in \mathrm{Sh}(U)$, then $j_!\mathcal{F}$ coincides with the extension by zero functor defined in [Har77, II, Ex.1.19b] as the sheafification of the presheaf $j_!^{pre}\mathcal{F}$ defined by

$$j_!^{pre}\mathcal{F}(V) := \begin{cases} \mathcal{F}(V), & V \subseteq U \\ 0, & \text{otherwise.} \end{cases}$$

One can see this by calculating the stalks and then using the fact that for every sheaf $\mathcal{F}$ on U there is a unique sheaf $\mathcal{G}$ on X with the property that $\mathcal{G}|_U = \mathcal{F}$ and $\mathcal{G}_x = 0$ for $x \notin U$, see [Har77]. The following is now immediate.

Proposition 3.2.2. *If $j \colon U \to X$ is an open embedding, then the functor j^* is right adjoint to $j_!$.*

Proof. We have

$$\mathrm{Hom}_{\mathrm{Sh}(X)}(j_!\mathcal{F}, \mathcal{G}) = \mathrm{Hom}_{\mathrm{PreSh}(X)}(j_!^{pre}\mathcal{F}, \mathcal{G}) = \mathrm{Hom}_{\mathrm{Sh}(U)}(\mathcal{F}, \underbrace{\mathcal{G}|_U}_{=j^*\mathcal{G}}).$$ □

Recall that $\mathrm{Sh}(\{pt\}) = \mathrm{Vect}_k$.

Proposition 3.2.3. *Every complex in $D^b(\{pt\};k)$ is the direct sum of its cohomology sheaves.*

Proof. The category $\mathrm{Sh}(\{pt\}) = \mathrm{Vect}_k$ is semi-simple and so every object is projective. But every bounded complex of projective objects is quasi-isomorphic to its cohomology groups. □

We consider now the constant map $p \colon X \to \{pt\}$. We can recover from p the global sections functor as

$$p_*(\mathcal{F}) = \Gamma(X, \mathcal{F}) = \mathcal{F}(X)$$

and also define the functor of global sections with compact support

$$p_!\mathcal{F} = \{\sigma \in \mathcal{F}(X) \mid \operatorname{supp}\sigma \hookrightarrow X \text{ is proper}\} =: \Gamma_c(X, \mathcal{F}).$$

We write $H^i_c(X; k)$ for cohomology with compact support, which is defined as the right derived functor of the left exact $\Gamma_c(X, -)$.

Unlike f_*, the functor $f_!$ does not have an adjointness properties on the level of sheaves in general. However there is a functor $f^!\colon D^b(Y; k) \to D^b(X; k)$ which is right adjoint to the derived version of $f_!$, which will be the statement of the next proposition.

From now on the functors $f_*, f^*, f_!$ refer to the derived versions of the above defined functors, unless specified otherwise.

Proposition 3.2.4 ([Ver95])**.** *If $f\colon X \to Y$ is a morphism, the functor*

$$f_!\colon D^b(X; k) \to D^b(Y; k)$$

has a right adjoint $f^!$.

Proof. This was first proven in [Ver95]. For an English reference, see [GM03, Theorem III.8.16]. □

Definition 3.2.5. The dualizing sheaf or dualizing complex ω_X of X is defined as $p^!(k_{\{pt\}})$ where $p\colon X \to \{pt\}$ is the constant map. We define the Verdier duality functor as

$$\mathbb{D}\colon D^b(X; k)^{op} \to D^b(X; k), \quad \mathcal{F}^\bullet \mapsto \mathcal{H}om(\mathcal{F}^\bullet, \omega_X).$$

Here $\mathcal{H}om$ denotes the internal Hom-functor in $D^b(X; k)$.

The following statement is a version of Poincaré duality. We will explain the connection to the classical Poincaré duality statement after the full statement of Verdier duality 3.2.23.

Proposition 3.2.6. *If X is smooth connected of dimension n, the dualizing complex ω_X is just the shifted stalk complex $k_X[2n]$.*

Proof. See [GM03, Corollary III.8.27]. □

Note that $2n$ is the real dimension of the complex variety X.

Example 3.2.7. If $X = \{pt\}$, we have that $p\colon \{pt\} \to \{pt\}$ is the equality and so $p_! = p_* = p^* = p^!$ is the identity functor on $D^b(\{pt\}; k)$. We see that

$$\omega_{\{pt\}} = p^!(k_{\{pt\}}) = k_{\{pt\}}$$

and therefore $\mathbb{D} = \mathcal{H}om(-, k_{\{pt\}})$. By Proposition 3.2.3 it suffices to determine what $\mathbb{D}$ does on (shifted) stalk complexes to completely describe $\mathbb{D}$. We compute for a sheaf $\mathcal{F} \in \operatorname{Sh}(\{pt\}; k)$ corresponding to the vector space $V = \mathcal{F}(\{pt\})$

$$\begin{aligned}\mathbb{D}(\mathcal{F}[i]) &= \mathcal{H}om(\mathcal{F}[i], k_{\{pt\}}) = \mathcal{H}om(\mathcal{F}, k_{\{pt\}})[-i] \\ &= \operatorname{Hom}(\mathcal{F}(\{pt\}), k)[-i] \\ &= V^\vee[-i],\end{aligned}$$

which is just the vector space dual.

Next we define constructible sheaves with respect to some stratification of the complex variety X. First we recall the definition of stratifications.

Definition 3.2.8. A stratification of X is a finite disjoint union decomposition

$$X = \coprod_{\lambda \in \Lambda} X_\lambda$$

into locally closed subspaces such that each X_λ is a smooth connected variety and such that the closure $\overline{X_\lambda}$ of any stratum is a (finite) union of strata. We write $i_\lambda \colon X_\lambda \hookrightarrow X$ for the inclusion of the stratum X_λ.

Note that smoothness means that each X_λ has the structure of a manifold. Additionally, it guarantees finite global dimension of $\mathrm{Sh}(X_\lambda)$ which is important in order for all the derived functors to send bounded complexes to bounded complexes.

Remark 3.2.9. Setting $\mu \leq \lambda$ if $X_\mu \subseteq X_\lambda$ defines a partial order on Λ. Strata which are minimal with respect to $\leq$ are closed and strata which are maximal with respect to $\leq$ are open.

Example 3.2.10. If X is a smooth connected variety, then we have the trivial stratification $X = X_\lambda$.

Example 3.2.11. If X is a singular connected variety such that its singular points $\mathrm{Sing}(X)$ form a smooth connected variety, then

$$X = \mathrm{Sing}(X) \amalg (X \setminus \mathrm{Sing}(X))$$

is a stratification.

Example 3.2.12. The decomposition

$$\mathbb{P}^n(\mathbb{C}) = \mathbb{C}^n \amalg \mathbb{C}^{n-1} \amalg \cdots \amalg \{pt\}$$

is a stratification.

Example 3.2.13. The classically defined Schubert cells are a stratification of the partial flag variety.

In fact, a stratification exists for all varieties X, see for example [Kal05].

Definition 3.2.14. A sheaf $\mathcal{F}$ is called constructible with respect to the stratification $X = \coprod_{\lambda \in \Lambda} X_\lambda$ if the sheaf $\mathcal{F}|_{X_\lambda} := i_\lambda^* \mathcal{F}$ is a local system for all $\lambda \in \Lambda$. We call an object $\mathcal{F}^\bullet \in D^b(X;k)$ constructible with respect to Λ if all cohomology sheaves are constructible with respect to Λ and we write $D^b_\Lambda(X;k)$ for the full subcategory of $D^b(X;k)$ of constructible complexes. We write $D^b_{cb}(X;k)$ for the full subcategory of $D^b(X;k)$ of objects which are constructible with respect to some stratification, which contains all $D^b_\Lambda(X;k) \subseteq D^b_{cb}(X;k)$.

Proposition 3.2.15. *The category of constructible sheaves forms a weak Serre subcategory of* $\mathrm{Sh}(X;k)$*.*

Proof. Let $\mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F} \to \mathcal{F}_3 \to \mathcal{F}_4$ be an exact sequence such that $\mathcal{F}_i$ is constructible for $i = 1, \ldots, 4$. We can apply the exact functor i_λ^* to obtain the exact sequence

$$i_\lambda^* \mathcal{F}_1 \to i_\lambda^* \mathcal{F}_2 \to i_\lambda^* \mathcal{F} \to i_\lambda^* \mathcal{F}_3 \to i_\lambda^* \mathcal{F}_4.$$

Now $i_\lambda^* \mathcal{F}_i$ is a local system, but then so must be $i_\lambda^* \mathcal{F}$, since $\mathrm{Loc}_f(X;k)$ is a weak Serre subcategory of $\mathrm{Sh}(X;k)$ by Proposition 3.1.8. □

Proposition 3.2.16. *Consider the derived categories* $D^*(\mathcal{A})$ *for* $* \in \{\emptyset, +, -, b\}$*. If* $\mathcal{B} \subseteq \mathcal{A}$ *is a weak Serre subcategory, then the category*

$$D^*_{\mathcal{B}}(\mathcal{A}) := \left\{ X^\bullet \in D^*(\mathcal{A}) \;\middle|\; H^i(X^\bullet) \in \mathcal{B} \right\}$$

is closed under extensions and therefore a triangulated subcategory of $D^*(\mathcal{A})$*.*

Proof. If $X'^\bullet \to X^\bullet \to X''^\bullet \to X'^\bullet[1]$ is a distinguished triangle in $D^*(\mathcal{A})$ such that $X'^\bullet$ and $X''^\bullet$ lie in $D^*_{\mathcal{B}}(\mathcal{A})$, then we obtain the exact sequence

$$H^{i-1}(X''^\bullet) \to H^i(X'^\bullet) \to H^i(X^\bullet) \to H^i(X''^\bullet) \to H^{i+1}(X'^\bullet).$$

As $\mathcal{B}$ is a weak Serre subcategory, $H^i(X^\bullet)$ lies in $\mathcal{B}$ by Proposition 3.1.7 and so $X^\bullet \in D^*_{\mathcal{B}}(\mathcal{A})$. □

Corollary 3.2.17. *The extension of constructible complexes of sheaves is constructible and so* $D^b_\Lambda(X;k)$ *is a triangulated subcategory of* $D^b(X;k)$*. The same follows also for* $D^b_{cb}(X;k)$*, because this can be thought of as the union of all* $D^b_\Lambda(X;k) \subseteq D^b(X;k)$*.*

Proof. By Proposition 3.1.8 the category of constructible sheaves is a weak Serre subcategory of $\mathrm{Sh}(X;k)$. Now the assertion follows from Proposition 3.2.16. The statement for $D^b_{cb}(X;k)$ follows from the fact that any two stratifications have a common refinement. □

Example 3.2.18. If we choose the trivial stratification $X = X$, we obtain

$$D^b_{\{X\}}(X;k) = D^b_{const}(X;k),$$

where $D^b_{const}(X;k) := D^b_{\mathrm{Loc}_f(X;k)}(\mathrm{Sh}(X;k))$ denotes the category of complexes whose cohomology sheaves are local systems.

We now come to the question of when the functors $f_*, f_!, f^*, f^!$ preserve constructible complexes. A main source for constructibility results in our context is [BS84]. For a more general discussion about conditions that guarantee constructibility see [Sch03, Chapter 4].

Proposition 3.2.19. *If we have a fixed stratification $X = \coprod_{\lambda \in \Lambda} X_\lambda$ and a subset $Z = \coprod_{\lambda \in \Lambda'} X_\lambda$ which is a union of strata for some $\Lambda' \subseteq \Lambda$ with inclusion $i\colon Z \hookrightarrow X$, then the functors i^* and $i_!$ preserve constructibility*

$$\begin{tikzcd} D^b_{\Lambda'}(Z;k) \arrow[r, bend left, "i_!"] & D^b_{\Lambda}(X;k). \arrow[l, bend left, "i^*"] \end{tikzcd}$$

It follows that $i_!$ and i^* are also functors between $D^b_{cb}(Z;k)$ and $D^b_{cb}(X;k)$. Additionally, the derived internal Hom*-functor $\mathcal{H}om(-,-)$ also preserve constructibility.*

Proof. The non-derived versions of i^* and $i_!$ are exact and map constructible sheaves to constructible sheaves. For a proof of constructibility of $\mathcal{H}om(-,-)$, see [BS84, Theorem 8.6]. ☐

Unfortunately, to guarantee that i_* and $i^!$ also preserve constructibility, we need the following additional technical assumption on our stratification.

Definition 3.2.20 ([Whi65])**.** Let $X = \coprod_{\lambda \in \Lambda} X_\lambda$ be stratified. Let $x \in X_\mu \subseteq X$. We say x satisfies the Whitney condition, if for all λ such that $x \in X_\mu \subseteq \overline{X_\lambda}$ and all sequences $(x_n) \subseteq X_\mu$ and $(y_n) \subseteq X_\lambda$ converging to x the secant lines between x_n and y_n converge to some $v \in T_x X_\mu$. We say X satisfies the Whitney condition, if all $x \in X$ satisfy the Whitney condition.

Observe that the set of points in X satisfying the Whitney condition are fixed by automorphisms of X that restrict to automorphisms on all X_λ.

Any stratification can be refined to a Whitney stratification, see [Whi65].

Proposition 3.2.21. *Let $X = \coprod_{\lambda \in \Lambda} X_\lambda$ be stratified and $Z \subseteq X$ be a union of strata with inclusion $i\colon Z \hookrightarrow X$. If X satisfies the Whitney condition and, then $i_*, i^*, i_!$, and $i^!$ preserve constructibility.*

Proof. See for example [Sch03, Proposition 4.0.2]. ☐

Proposition 3.2.22. *The duality functor $\mathbb{D}$ preserves $D^b_\Lambda(X;k)$ if Λ is a Whitney stratification and so it also preserves $D^b_{cb}(X;k)$.*

Proof. See for example [BS84, Proposition 8.3]. ☐

We are now finally ready to formulate the Verdier duality theorem which justifies the name "duality functor" for $\mathbb{D}$.

Theorem 3.2.23 (Verdier Duality)**.** *The contravariant functor $\mathbb{D}\colon D^b_{cb}(X;k) \to D^b_{cb}(X;k)$ satisfies the following natural isomorphisms:*

(1) $\mathbb{D}^2 = \mathrm{id}$.

(2) *For any $f \colon X \to Y$ we have $f_* \mathbb{D}_X = \mathbb{D}_Y f_!$ and $f^* \mathbb{D}_Y = \mathbb{D}_X f^!$. (The same statements with $!$'s and $*$'s swapped follow immediately.)*

(3) *If X is smooth connected of dimension n and $\mathcal{L}$ is a local system on X, then we have $\mathbb{D}\mathcal{L} = \mathcal{H}om(\mathcal{L}, k_X)[2n]$.*

Proof. This is mostly a consequence of Proposition 3.2.4. For the proof see [BS84, Theorem 8.10]. □

The last statement is just a reformulation of Proposition 3.2.6.

Corollary 3.2.24 (Poincaré Duality)**.** *If X is a smooth connected complex variety of (complex) dimension n we obtain an isomorphism*

$$H^i(X;k)^* \cong H_c^{2n-i}(X;k).$$

If additionally X is projective, this simplifies further to $H^i(X;k)^* \cong H^{2n-i}(X;k)$.

Proof. Consider $p_* k_X$ in $D^b(\{pt\};k) = D^b(\mathrm{Vect}_k)$. By Proposition 3.2.3 this complex is just the direct sum of its cohomology groups. We have seen in Example 3.2.7 $\mathbb{D}_{\{pt\}}(-) = (-)^*$ is the vector space dual. We get that the i-th cohomology group of $\mathbb{D}(p_* k_X)$ is just the dual of the $(-i)$-th cohomology group of $p_* k_X$ by Example 3.2.7. Equivalently, $H^i(X;k)^* \cong H^{-i}(\mathbb{D} p_* k_X)$ and therefore

$$\begin{aligned} H^i(X;k)^* &\cong H^{-i}(\mathbb{D}_{\{pt\}}(p_* k_X)) \\ &\cong H^{-i}(p_!(\mathbb{D}_X k_X)) && \text{(Theorem 3.2.23(2))} \\ &\cong H^{-i}(p_! k_X[2n]) && \text{(Theorem 3.2.23(3))} \\ &= H^{2n-i}(p_! k_X) \\ &\cong H_c^{2n-i}(X;k). && \text{(Definition } H_c^i\text{)} \end{aligned}$$

If X is projective, p is proper and $p_* = p_!$ by Proposition 3.2.1. In this case the global sections and the global sections with compact support coincide, and we obtain $H_c^i(X;k) = H^i(X;k)$. □

Remark 3.2.25. The theory of Verdier and Poincaré duality is formally very similar to that of classical Grothendieck and Serre duality; there we also have a dualizing sheaf which induces a duality. The difference however is that we are working with constructible and locally constant sheaves in the complex-analytic topology, whereas classical Grothendieck and Serre duality are statements about coherent sheaves in the Zariski topology on X.

3.3 Constructible Sheaves for ind-Varieties

We would like to claim that the Cartan Decomposition 2.3.1

$$\mathrm{Gr}_G(\mathbb{C}) = \coprod_{\lambda \in (X_*)_+} \mathrm{Gr}_G^\lambda$$

is a stratification. However thus far, we have only defined finite stratifications for varieties, while $\mathrm{Gr}_G(\mathbb{C})$ is an ind-variety.

If $i\colon Z \hookrightarrow X$ is a closed embedding, Proposition 3.2.19 guarantees that $i_* = i_!$ maps constructible sheaves to constructible sheaves. However, the image of this functor turns out to have a explicit description in terms of sheaves that are supported on Z. The support of a complex of sheaves is defined as the union of the supports of its cohomology sheaves.

Lemma 3.3.1. *The non-derived functor i_* is an equivalence of categories between constructible sheaves on Z and constructible sheaves on X which are supported on Z. Therefore the derived version of the functor i_* is an equivalence between $D^b_{cb}(Z;k)$ and $\{\mathcal{F}^\bullet \in D^b_{cb}(X;k) \mid \operatorname{supp}\mathcal{F} \subseteq Z\}$. Given a fixed stratification of $X = \coprod_{\lambda\in\Lambda} X_\lambda$ and a closed subset $Z \subseteq X$ which is a union of strata $Z = \coprod_{\lambda\in\Lambda'} X_\lambda$ for $\Lambda' \subseteq \Lambda$, we even have an equivalence between $D^b_{\Lambda'}(Z;k)$ and $\{\mathcal{F}^\bullet \in D^b_\Lambda(X;k) \mid \operatorname{supp}\mathcal{F}^\bullet \subseteq Z\}$.*

Proof. One can easily check that $i^*i_* = \mathrm{id}$ as functors on $\mathrm{Sh}(Z)$. For $\mathcal{G} \in \mathrm{Sh}(X)$, one considers the adjunction unit $\mathcal{G} \to i_*i^*\mathcal{G}$ which induces an isomorphism on stalks in Z. However, if $\mathcal{G}$ is supported on Z, this map is an isomorphism of sheaves. The statements for constructible sheaves and constructible complexes of sheaves follow. □

Remark 3.3.2. The Riemann–Hilbert correspondence, see [HTT08, Chapter 7], gives an equivalence of categories between $D^b_{cb}(X;k)$ and a certain subcategory of the bounded derived category of $\mathcal{D}$-modules on X. Under this equivalence, the above lemma gets translated to Kashiwara's theorem, see [HTT08, Chapter 1.6]. It is surprising that the above lemma is almost trivial, while Kashiwara's theorem has a very difficult proof.

Let now $X = \bigcup_N X^{(N)}$ be an ind-variety where $X^{(N)} \hookrightarrow X^{(n+1)}$ is a closed embedding of (finite-dimensional) complex varieties. We consider sheaves with respect to the analytic topology on X.

Definition 3.3.3. A stratification of X is a family of stratifications $X^{(N)} = \coprod_{\lambda\in\Lambda^{(N)}} X^{(N)}_\lambda$ such that the embedding $X^{(N)} \hookrightarrow X^{(N+1)}$ induces an isomorphism as varieties between $X^{(N)}_\lambda$ and some $X^{(N+1)}_{\lambda'}$. We can interpret $\Lambda^{(N)}$ as a subset of $\Lambda^{(N+1)}$ and may therefore write $\Lambda = \bigcup_N \Lambda^{(N)}$ and $X = \coprod X_\lambda$ where $X_\lambda = X^{(N)}_\lambda$ for N such that $\lambda \in \Lambda^{(N)}$.

Write $i^{(N)}$ for the inclusion $X^{(N)} \hookrightarrow X$.

Definition 3.3.4. We say that a stratification $X = \coprod X_\lambda$ of an ind-variety $X = \varinjlim X^{(N)}$ satisfies the Whitney condition, if all stratifications $X^{(N)} = \coprod_{\lambda\in\Lambda^{(N)}} X^{(N)}_\lambda$ satisfy the Whitney condition 3.2.20.

Definition 3.3.5. For a stratified ind-variety $X = \bigcup_N X^{(N)} = \coprod_\lambda X_\lambda$ we call a sheaf $\mathcal{F} \in \mathrm{Sh}(X)$ constructible with respect to Λ if $\mathcal{F} = i^{(N)}_*\mathcal{G}$ for a sheaf $\mathcal{G}$ which is constructible with respect to $\Lambda^{(N)}$. We write $D^b_\Lambda(X;k)$ for the full subcategory of $D^b(X;k)$ of complexes with constructible cohomology.

It follows from Lemma 3.3.1 that we have embeddings of categories

$$D^b_{\Lambda^{(1)}}(X^{(1)};k) \hookrightarrow D^b_{\Lambda^{(2)}}(X^{(2)};k) \hookrightarrow \cdots \hookrightarrow D^b_{\Lambda^{(N)}}(X^{(N)};k) \hookrightarrow \cdots \hookrightarrow D^b_{\Lambda}(X;k)$$

and that $D^b_\Lambda(X;k)$ is the direct limit of this system. In particular, we have that every object in $D^b_\Lambda(X;k)$ can be thought of as an element of $D^b_\Lambda(X^{(N)};k)$ for some N.

Corollary 3.3.6. *A sheaf $\mathcal{F}$ on X is constructible with respect to Λ if and only if $i^*_\lambda \mathcal{F}$ is a local system for all $\lambda \in \Lambda$ and zero for all but finitely many λ.*

Note also that for the inclusion $i_N \colon X^{(N)} \to X^{(N+1)}$ the Verdier duality functor satisfies

$$\mathbb{D}_{X^{(N+1)}} \circ (i_N)_* = (i_N)_* \circ \mathbb{D}_{X^{(N)}}.$$

We deduce that we also have Verdier duality for $D^b_{cb}(X;k)$ for an ind-variety X.

Now we verify that the decomposition $\mathrm{Gr}_G = \coprod_\lambda \mathrm{Gr}^\lambda_G$ is a Whitney stratification.

Proposition 3.3.7. *The Cartan Decomposition from Theorem 2.3.1*

$$\mathrm{Gr}_G(\mathbb{C}) = \coprod_{\lambda \in (X_*)_+} \mathrm{Gr}^\lambda_G$$

is a stratification satisfying the Whitney condition.

Proof. By Proposition 2.3.9 the Cartan Decomposition of the Affine Grassmannian restricts to a decomposition

$$\mathrm{Gr}^{(N)}_G(\mathbb{C}) = \coprod_{\lambda \in \Lambda^{(N)}} \mathrm{Gr}^\lambda_G$$

for some finite subsets $\Lambda^{(N)} \subseteq (X_*)_+$. By Theorem 2.3.15 all Schubert cells are smooth connected varieties and so $\mathrm{Gr}^{(N)}_G(\mathbb{C})$ is stratified.

Next we verify the Whitney condition. This argument is taken from [Sch11, Section 3.12]. Note that the set of points in $\mathrm{Gr}_G(\mathbb{C})$ satisfying the Whitney condition is $G(\mathbb{C}[[t]])$-invariant, because the multiplication by an element of $G(\mathbb{C}[[t]])$ is an automorphism of $\mathrm{Gr}_G(\mathbb{C})$ which restricts to all Gr^λ_G. Then the set of points that do not satisfy the Whitney condition is also $G(\mathbb{C}[[t]])$-invariant. However, by [Kal05, Theorem 2], the subset of points in Gr^λ_G, which do not satisfy the Whitney condition, is a locally closed subset of strictly smaller dimension. This subset is therefore a proper $G(\mathbb{C}[[t]])$-invariant subset of a single orbit and thus empty. $\square$

All of this allows us to define constructible sheaves on $\mathrm{Gr}_G(\mathbb{C})$.

3.4 t-Structures

In this section we introduce the necessary homological algebra, in the form of t-structures, in order to define perverse sheaves. These form a certain abelian subcategory of $D^b_{cb}(X;k)$. Hence, we need a systematic way to construct abelian subcategories of triangulated categories.

Definition 3.4.1. A t-structure (short for truncation structure) on a triangulated category $\mathcal{T}$ (such as $D^b(X;k)$ or $D^b_{cb}(X;k)$) is a pair of full additive subcategories $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ such that for $\mathcal{T}^{\leq n} := \mathcal{T}^{\leq 0}[-n]$ and $\mathcal{T}^{\geq n} := \mathcal{T}^{\geq 0}[-n]$ we have

- $\mathcal{T}^{\leq 0}$ is closed under taking [1] and $\mathcal{T}^{\geq 0}$ is closed under taking $[-1]$, or equivalently $\mathcal{T}^{\leq n} \subseteq \mathcal{T}^{\leq n+1}$ and $\mathcal{T}^{\geq n} \supseteq \mathcal{T}^{\geq n+1}$.
- If $T_1 \in \mathcal{T}^{\leq 0}$ and $T_2 \in \mathcal{T}^{\geq 1}$, we have $\mathrm{Hom}_{\mathcal{T}}(T_1, T_2) = 0$.
- For any $T \in \mathcal{T}$ there is a distinguished triangle

  $$T' \to T \to T'' \to T'[1]$$

  with $T' \in \mathcal{T}^{\leq 0}$ and $T'' \in \mathcal{T}^{\geq 1}$.

The heart $\mathcal{T}^{\heartsuit}$ of a t-structure is the intersection $\mathcal{T}^{\leq 0} \cap \mathcal{T}^{\geq 0} \subseteq \mathcal{T}$.

If the intersections $\bigcap_n \mathcal{T}^{\leq n}$ and $\bigcap_n \mathcal{T}^{\geq n}$ are trivial, we call the t-structure non-degenerate.

Example 3.4.2. If $\mathcal{T}$ is a triangulated category, we have two trivial t-structures, $(\mathcal{T}, 0)$ and $(0, \mathcal{T})$. For $(\mathcal{T}, 0)$ we see that $\mathcal{T}$ is closed under [1] and 0 is closed under $[-1]$, we have $\mathrm{Hom}_{\mathcal{T}}(T_1, T_2) = 0$ if $T_2 = 0$, and for every $T \in \mathcal{T}$ we have a distinguished triangle $T \to T \to 0 \to T[1]$. The argument for $(0, \mathcal{T})$ works the same. These t-structure are not non-degenerate and their heart is 0.

Example 3.4.3. Let $\mathcal{A}$ be an abelian category with weak Serre subcategory $\mathcal{B} \subseteq \mathcal{A}$ and $D_{\mathcal{B}}(\mathcal{A})$ its derived or bounded derived category with cohomology in $\mathcal{B}$. Recall that $D_{\mathcal{B}}(\mathcal{A})$ is triangulated by Proposition 3.2.16. Let

$$D^{\leq 0}_{\mathcal{B}}(\mathcal{A}) = \{\mathcal{F}^\bullet \in D_{\mathcal{B}}(\mathcal{A}) \mid H^i(\mathcal{F}^\bullet) = 0 \text{ for } i > 0\},$$
$$D^{\geq 0}_{\mathcal{B}}(\mathcal{A}) = \{\mathcal{F}^\bullet \in D_{\mathcal{B}}(\mathcal{A}) \mid H^i(\mathcal{F}^\bullet) = 0 \text{ for } i < 0\}.$$

This is a t-structure, called the standard t-structure. We have functors $\tau_{\leq 0}$ and $\tau_{\geq 0}$, which truncate our complexes given by

$$\begin{array}{rccccccccc}
\mathcal{F}^\bullet: & \cdots & \longrightarrow & \mathcal{F}^{-1} & \longrightarrow & \mathcal{F}^0 & \longrightarrow & \mathcal{F}^1 & \longrightarrow & \cdots \\
\tau_{\leq 0}\mathcal{F}^\bullet: & \cdots & \longrightarrow & \mathcal{F}^{-1} & \longrightarrow & \ker(\mathcal{F}^0 \to \mathcal{F}^1) & \longrightarrow & 0 & \longrightarrow & \cdots \\
\tau_{\geq 0}\mathcal{F}^\bullet: & \cdots & \longrightarrow & 0 & \longrightarrow & \mathcal{F}^0/\operatorname{im}(\mathcal{F}^{-1} \to \mathcal{F}^0) & \longrightarrow & \mathcal{F}^1 & \longrightarrow & \cdots
\end{array}$$

The heart of the standard t-structure is precisely $\mathcal{B}$. Indeed, let $\mathcal{F}^\bullet \in D^{\leq 0}(\mathcal{A}) \cap D^{\geq 0}(\mathcal{A})$.

Then $\mathcal{F}^\bullet$ has cohomology only in degree 0 and we have quasi-ismorphisms

$$\mathcal{F}^\bullet \xleftarrow{\sim} \tau_{\leq 0}\mathcal{F}^\bullet \xrightarrow{\sim} \tau_{\geq 0}\tau_{\leq 0}\mathcal{F}^\bullet$$

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] \\
\mathcal{F}^{-1} \arrow[d] & \mathcal{F}^{-1} \arrow[l] \arrow[r] \arrow[d] & 0 \arrow[d] \\
\mathcal{F}^{0} \arrow[d] & \ker\left(\mathcal{F}^0 \to \mathcal{F}^1\right) \arrow[l] \arrow[r] \arrow[d] & H^0(\mathcal{F}^\bullet) \arrow[d] \\
\mathcal{F}^{1} \arrow[d] & 0 \arrow[l] \arrow[r] \arrow[d] & 0 \arrow[d] \\
\vdots & \vdots & \vdots
\end{tikzcd}$$

In particular, the heart of the standard t-structure on $D(\mathcal{A}) = D_{\mathcal{A}}(\mathcal{A})$ is $\mathcal{A}$.

We could also have considered $(D^{\leq n}_{\mathcal{B}}(\mathcal{A}), D^{\geq n}_{\mathcal{B}}(\mathcal{A}))$ for $n \in \mathbb{Z}$. The heart of this t-structure is $\mathcal{B}[-n]$.

The truncation functors exist for general t-structures by the following proposition.

Proposition 3.4.4. *The embedding $\mathcal{T}^{\leq n} \to \mathcal{T}$ has a right adjoint, denoted $\tau_{\leq n}$, and $\mathcal{T}^{\geq n} \to \mathcal{T}$ has a left adoint, denoted $\tau_{\geq n}$. If $T \in \mathcal{T}$ and*

$$T' \to T \to T'' \to T'[1]$$

is a distinguished triangle with $T' \in \mathcal{T}^{\leq 0}$ and $T'' \in \mathcal{T}^{\geq 1}$, then we have $T' = \tau_{\leq 0}T$ and $T'' = \tau_{\geq 1}T$. In particular, the distinguished triangle from the definition of a t-structure is unique up to isomorphism. It follows that $T \in \mathcal{T}^{\leq 0}$ if and only if $\tau_{\geq 1}T = 0$, and $T \in \mathcal{T}^{\geq 1}$ if and only if $\tau_{\leq 0}T = 0$.

Proof. See [BBD82, Chapter 1.3]. □

Proposition 3.4.5. *The categories $\mathcal{T}^{\leq 0}$ and $\mathcal{T}^{\geq 0}$ are closed under extensions. This means that if $T' \to T \to T'' \to T'[1]$ is a distinguished triangle in $\mathcal{T}$ such that both $T', T'' \in \mathcal{T}^{\leq 0}$ lie in $\mathcal{T}^{\leq 0}$ respectively $\mathcal{T}^{\geq 0}$, then $T \in \mathcal{T}^{\leq 0}$, respectively $T \in \mathcal{T}^{\geq 0}$.*

Proof. See [BBD82, Chapter 1.3]. □

Definition 3.4.6. We call functor $\mathcal{T} \to \mathcal{T}^\heartsuit$ given by the composition of $\tau_{\leq 0}$ with $\tau_{\geq 0}$ the zeroth t-cohomology ${}^tH^0$.

In Example 3.4.3 the zeroth t-cohomology is given by the zeroth cohomology functor, which motivates the name.

Example 3.4.7. Let $\mathcal{A} = \mathrm{Vect}_k$. Every complex $V^\bullet \in D^b(\mathcal{A})$ is the direct sum of its cohomology groups by Proposition 3.2.3. Every such cohomology group is of the form k^n for some $n \geq 0$. Let $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ be some t-structure on $\mathcal{T} = D^b(\mathrm{Vect}_k)$. Each simple object $k[i]$ has to lie either in $\mathcal{T}^{\leq 0}$ or $\mathcal{T}^{\geq 1}$, since the categories $\mathcal{T}^{\leq 0}$ and $\mathcal{T}^{\geq 1}$ are closed under summands. It follows that the only non-degenerate t-structures on $D^b(\mathrm{Vect}_k)$ are shifts of the standard t-structure.

If $\mathcal{A}$ is semi-simple with more than one simple object, we find that the set of non-degenerate t-structures on $D^b(\mathcal{A})$ is given by functions from $\mathrm{Irr}(\mathcal{A})$ to $\mathbb{Z}$, since for any $X \in \mathrm{Irr}(\mathcal{A})$ we have to choose an integer $f(X)$ such that $X \in \mathcal{T}^{\leq f(X)} \setminus \mathcal{T}^{\leq f(X)-1}$. The heart of such a t-structure will be the category generated by $X[-f(X)]$ for $X \in \mathrm{Irr}(\mathcal{A})$.

Remark 3.4.8. If $\mathcal{A} \to \mathcal{B}$ is an equivalence of abelian categories, we obtain an equivalence of the derived categories $D(\mathcal{A}) \to D(\mathcal{B})$ which preserves the standard t-structure. If we have some other derived equivalence $D(\mathcal{A}) \to D(\mathcal{B})$, for example given by the right or left derived of some half-exact functor, we cannot expect this to preserve the standard t-structure.

In the case of the Riemann–Hilbert correspondence, we have an equivalence of derived categories and the heart of the standard t-structure on the side of $\mathcal{D}$-modules is identified with a category of perverse sheaves.

The following theorem concludes our study of t-structures in general.

Theorem 3.4.9. *For any t-structure $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ on a triangulated category $\mathcal{T}$, the heart $\mathcal{T}^\heartsuit := \mathcal{T}^{\leq 0} \cap \mathcal{T}^{\geq 0}$ is an abelian category and the functor $\tau_{\leq 0}\tau_{\geq 0} \colon \mathcal{T} \to \mathcal{T}^\heartsuit$ maps distinguished triangles*

$$T' \to T \to T'' \to T'[1]$$

to long exact sequences

$$\begin{tikzcd}
 & \cdots \arrow[r] & \tau_{\leq 0}\tau_{\geq 0}T[-1] \arrow[r] & \tau_{\leq 0}\tau_{\geq 0}T''[-1] \arrow[dll] \\
 & \tau_{\leq 0}\tau_{\geq 0}T' \arrow[r] & \tau_{\leq 0}\tau_{\geq 0}T \arrow[r] & \tau_{\leq 0}\tau_{\geq 0}T'' \arrow[dll] \\
 & \tau_{\leq 0}\tau_{\geq 0}T'[1] \arrow[r] & \tau_{\leq 0}\tau_{\geq 0}T[1] \arrow[r] & \cdots
\end{tikzcd}$$

Additionally, two maps $T' \to T$ and $T \to T''$ between the objects in the abelian category $\mathcal{T}^\heartsuit$ form an exact sequence

$$0 \to T' \to T \to T'' \to 0$$

if and only if there is a morphism $T'' \to T[1]$ such that

$$T' \to T \to T'' \to T'[1]$$

is a distinguished triangle in $\mathcal{T}$.

Proof. See [BBD82, Théorème 1.3.6]. □

3.5 Recollement and Perverse Sheaves

We return to geometry and sheaf-theory. Let X be an ind-variety. Consider a closed sub-ind-variety $Z \subseteq X$ with open complement $U \subseteq X$ and corresponding embeddings $i\colon Z \hookrightarrow X$ and $j\colon U \hookrightarrow X$.

Theorem 3.5.1 (Recollement)**.** *We have a diagram of functors*

$$\begin{tikzcd} D^b(Z;k) \arrow[r, "i_*=i_!"] & D^b(X;k) \arrow[l, bend right, "i^*"'] \arrow[l, bend left, "i^!"] \arrow[r, "j^*=j^!"] & D^b(U;k). \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "j_*"] \end{tikzcd}$$

These satisfy the following:

(1) $j^*i_* = 0$, $i^*j_! = 0$, *and* $i^!j_* = 0$.

(2) $(i^*, i_* = i_!, i^!)$ *and* $(j_!, j^! = j^*, j_*)$ *are adjoint triples.*

(3) $i_* = i_!$, j_*, *and* $j_!$ *are fully faithful and so the following maps are natural isomorphisms:* $i^*i_*\mathcal{F}^\bullet \to \mathcal{F}^\bullet \to i^!i_!\mathcal{F}^\bullet$ *for* $\mathcal{F}^\bullet \in D^b(Z;k)$ *as well as* $j^*j_*\mathcal{G}^\bullet \to \mathcal{G}^\bullet \to j^!j_!\mathcal{G}^\bullet$ *for* $\mathcal{G} \in D^b(U;k)$.

(4) The other adjunction maps define distinguished triangles in $D^b(X;k)$ *for any* $\mathcal{F}^\bullet \in D^b(X;k)$*:*

$$j_!j^!\mathcal{F}^\bullet \longrightarrow \mathcal{F}^\bullet \longrightarrow i_*i^*\mathcal{F}^\bullet \longrightarrow j_!j^!\mathcal{F}^\bullet[1],$$

$$i_!i^!\mathcal{F}^\bullet \longrightarrow \mathcal{F}^\bullet \longrightarrow j_*j^*\mathcal{F}^\bullet \longrightarrow i_!i^!\mathcal{F}^\bullet[1].$$

Everything also holds for the categories of constructible complexes.

We can use t-structure on $D^b(Z;k)$ and $D^b(U;k)$ to define a t-structure on $D^b(X;k)$.

Theorem 3.5.2. *Let* $(D_Z^{\leq 0}, D_Z^{\geq 0})$ *be a t-structure on* $D^b(Z;k)$ *and* $(D_U^{\leq 0}, D_U^{\geq 0})$ *a t-structure on* $D^b(U;k)$*. Then*

$$\begin{aligned} D_X^{\leq 0} &\coloneqq \{\mathcal{F}^\bullet \in D^b(X;k) \mid i^*\mathcal{F}^\bullet \in D_Z^{\leq 0} \text{ and } j^*\mathcal{F}^\bullet \in D_U^{\leq 0}\} \\ D_X^{\geq 0} &\coloneqq \{\mathcal{F}^\bullet \in D^b(X;k) \mid i^!\mathcal{F}^\bullet \in D_Z^{\geq 0} \text{ and } j^!\mathcal{F}^\bullet \in D_U^{\geq 0}\} \end{aligned}$$

define a t-structure on $D^b(X;k)$.

Proof. The proof is purely formal and can be deduced for any triple of triangulated categories satisfying the Recollement Theorem 3.5.1, see [BBD82, Théorème 1.4.10]. We only give the argument that $\mathrm{Hom}(\mathcal{F}_1^\bullet, \mathcal{F}_2^\bullet)$ vanishes for $\mathcal{F}_1^\bullet \in D_X^{\leq 0}$ and $\mathcal{F}_2^\bullet \in D_X^{\geq 1}$. We consider the distinguished triangle

$$j_!j^!\mathcal{F}_1^\bullet \to \mathcal{F}_1^\bullet \to i_*i^*\mathcal{F}_1^\bullet \to i_*i^*\mathcal{F}_1^\bullet[1]$$

from Theorem 3.5.1(4). Applying $\mathrm{Hom}(-, \mathcal{F}_2^\bullet)$ to this triangle, we obtain the exact sequence

$$\mathrm{Hom}(i_* i^* \mathcal{F}_1^\bullet, \mathcal{F}_2^\bullet) \to \mathrm{Hom}(\mathcal{F}_1^\bullet, \mathcal{F}_2^\bullet) \to \mathrm{Hom}(j_! j^! \mathcal{F}_1^\bullet, \mathcal{F}_2^\bullet).$$

But we have

$$\mathrm{Hom}(i_* i^* \mathcal{F}_1^\bullet, \mathcal{F}_2^\bullet) \overset{\text{Theorem 3.5.1(2)}}{=} \mathrm{Hom}(i^* \mathcal{F}_1^\bullet, i^! \mathcal{F}_2^\bullet) \overset{\text{t-structure on } Z}{=} 0$$

and

$$\mathrm{Hom}(j_! j^! \mathcal{F}_1^\bullet, \mathcal{F}_2^\bullet) \overset{\text{Theorem 3.5.1(2)}}{=} \mathrm{Hom}(j^! \mathcal{F}_1^\bullet, j^* \mathcal{F}_2^\bullet) \overset{\text{t-structure on } U}{=} 0.$$

We therefore have an exact sequence $0 \to \mathrm{Hom}(\mathcal{F}_1^\bullet, \mathcal{F}_2^\bullet) \to 0$ and so

$$\mathrm{Hom}(\mathcal{F}_1^\bullet, \mathcal{F}_2^\bullet) = 0.$$

□

Example 3.5.3. If $X = X_1 \amalg X_2$ is the disjoint union of two open and closed subsets, any complex of sheaves $\mathcal{F}^\bullet$ has a canonical decomposition $\mathcal{F}^\bullet = \mathcal{F}_1^\bullet \oplus \mathcal{F}_2^\bullet$ where $\mathcal{F}_1^\bullet$ is seen as a sheaf on X_1 and $\mathcal{F}_2^\bullet$ as a sheaf on X_2. We find that

$$\begin{aligned} D_X^{\leq 0} &= \{\mathcal{F}_1^\bullet \oplus \mathcal{F}_2^\bullet \mid \mathcal{F}_1^\bullet \in D_{X_1}^{\leq 0}, \mathcal{F}_2^\bullet \in D_{X_2}^{\leq 0}\}, \\ D_X^{\geq 0} &= \{\mathcal{F}_1^\bullet \oplus \mathcal{F}_2^\bullet \mid \mathcal{F}_1^\bullet \in D_{X_1}^{\geq 0}, \mathcal{F}_2^\bullet \in D_{X_2}^{\geq 0}\}. \end{aligned}$$

In particular, we see that the heart of this t-structure is just the direct sum of the hearts of the t-structures on X_1 and X_2.

We now come back to the situation of a stratified space X. We want to endow $D_\Lambda^b(X;k)$ with a t-structure whose heart is fixed by Verdier duality. If X has the trivial stratification $X = X$, the next proposition shows that a shifted version of the standard t-structure on $D_\Lambda^b(X;k)$ does the trick.

Proposition 3.5.4. *Let X be a smooth connected variety of dimension d. Consider the category $D_{const}^b(X;k)$ of complexes with locally constant cohomology (this is the same as $D_\Lambda^b(X;k)$ where $\#\Lambda = 1$). The heart of the standard t-structure shifted by d is fixed by Verdier duality.*

Proof. Verdier duality maps a local system $\mathcal{L}$ to $\mathcal{H}om(\mathcal{L}, k_X)[2d]$. And so

$$\mathbb{D}(\mathcal{L}[d]) = \mathcal{H}om(\mathcal{L}[d], k_X)[2d] = \mathcal{H}om(\mathcal{L}, k_X)[d],$$

where $\mathcal{H}om(\mathcal{L}, k_X)$ is a local system. □

Remark 3.5.5. It turns out that more is true: On a smooth connected variety, the Verdier duality functor switches the categories $D_X^{\leq -n}$ and $D_X^{\geq -n}$. This follows from computing $\mathcal{H}om(\mathcal{F}^\bullet, k_X)$ degreewise and then shifting for $\mathcal{F}^\bullet \in D_{const}^b(X;k)$.

We are now finally able to give the definition of perverse sheaves.

Definition 3.5.6. Let $X = \coprod_{\lambda \in \Lambda} X_\lambda$ be a stratified ind-variety and let $i_\lambda \colon X_\lambda \hookrightarrow X$ be the inclusions. Let $(D^{\leq 0}_{X_\lambda}, D^{\geq 0}_{X_\lambda})$ be the shifted standard t-structure on $D^b_{const}(X_\lambda; k)$ as in Proposition 3.5.4. Then we define the subcategories of $D^b_\Lambda(X; k)$

$$
\begin{aligned}
D^{\leq 0}_\Lambda &\coloneqq \{\mathcal{F}^\bullet \in D^b_\Lambda(X; k) \mid i^*_\lambda \mathcal{F}^\bullet \in D^{\leq 0}_{X_\lambda} \text{ for all } \lambda \in \Lambda\}, \\
D^{\geq 0}_\Lambda &\coloneqq \{\mathcal{F}^\bullet \in D^b_\Lambda(X; k) \mid i^!_\lambda \mathcal{F}^\bullet \in D^{\geq 0}_{X_\lambda} \text{ for all } \lambda \in \Lambda\}.
\end{aligned}
$$

We call a complex $\mathcal{F}^\bullet \in D^b_\Lambda(X; k)$ a **perverse sheaf** on X with respect to Λ if it lies in the intersection $D^{\leq 0}_\Lambda \cap D^{\geq 0}_\Lambda$. We denote the category of perverse sheaves on X with respect to Λ by $\mathsf{P}_\Lambda(X; k) \coloneqq D^{\leq 0}_\Lambda \cap D^{\geq 0}_\Lambda$. The zeroth t-cohomology with respect to this t-structure is denoted by ${}^pH^0$.

Proposition 3.5.7. *The pair $(D^{\leq 0}_\Lambda, D^{\geq 0}_\Lambda)$ defines a t-structure on $D^b_\Lambda(X; k)$ and so $\mathsf{P}_\Lambda(X; k)$ is an abelian category by Theorem 3.4.9.*

Proof. It suffices to consider the case where X is a (finite-dimensional) variety with a finite stratification, as any tuple of complexes will be supported on a finite-dimensional closed subvariety.

Now the statement follows by induction on the number of strata: If there is a single stratum, there is nothing. If there are more than one strata, we can pick an open stratum $X_{\lambda_0} \subseteq X$ and apply Theorem 3.5.2 to $X \setminus X_{\lambda_0} \subseteq X \supseteq X_{\lambda_0}$. □

Note that we obtain a canonical fully faithful functor $\mathsf{P}_\Lambda(X; k) \to \mathsf{P}_{\Lambda'}(X; k)$ whenever Λ' is a finer stratification than Λ. We can therefore define the category $\mathsf{P}(X; k)$ of all perverse sheaves as the union of all $\mathsf{P}_\Lambda(X; k)$.

Example 3.5.8. The category of perverse sheaves on a smooth connected variety X of dimension d with respect to the trivial stratification is given as the intersection of

$$D^{\leq 0} = \{\mathcal{F}^\bullet \in D^b_\Lambda(X; k) \mid \mathcal{H}^i(\mathcal{F}) = 0 \text{ for } i > -d\}$$

with

$$D^{\geq 0} = \{\mathcal{F}^\bullet \in D^b_\Lambda(X; k) \mid \mathcal{H}^i(\mathcal{F}) = 0 \text{ for } i < -d\}.$$

Note that in this case $D^b_\Lambda(X; k) = D^b_{const}(X; k)$ by Example 3.2.18 and so it follows from Example 3.4.3 that

$$\mathsf{P}_\Lambda(X; k) = \mathrm{Loc}_f(X; k)[-d] \subseteq D^b_{const}(X; k).$$

Recall that by Theorem 3.1.9 we have $\mathrm{Loc}_f(X; k) \simeq \mathrm{Rep}_k(\pi_1(X))$.

Example 3.5.9. There is a unique stratification on the point. The category of perverse sheaves on $\{pt\}$ is by the above example therefore just the category of local systems on the point. But $\mathrm{Loc}_x(\{pt\}; k) \cong \mathrm{Vect}_k$ by Corollary 3.1.10. If X is a finite disjoint union of points, there is also only one stratification, since the strata were assumed to be connected. It follows from Example 3.5.3 that $\mathsf{P}(X; k) = \oplus_{x \in X} \mathsf{P}(\{x\}; k) = \oplus_{x \in X} \mathrm{Vect}_k$.

Example 3.5.10. If X is a countably infinite disjoint union of points the conclusion of Example 3.5.9 holds as well. We consider some presentation of X with $n_1 < n_2 < \dots$

$$\{x_1, \dots, x_{n_1}\} \hookrightarrow \{x_1, \dots, x_{n_2}\} \hookrightarrow \{x_1, \dots, x_{n_3}\} \hookrightarrow \dots \hookrightarrow X = \{x_1, x_2, \dots\}.$$

Then we obtain fully faithful functors

$$\mathsf{P}(\{x_1 \dots, x_{n_1}\}; k) \hookrightarrow \mathsf{P}(\{x_1, \dots, x_{n_2}\}; k) \hookrightarrow \mathsf{P}(\{x_1, \dots, x_{n_3}\}; k) \hookrightarrow \cdots,$$

which under the identification $\mathsf{P}(\{x_1, \dots, x_{n_l}\}; k) = \bigoplus_{i=1}^{n_l} \mathrm{Vect}_k$ become the coordinate embeddings sending the n_l simple objects of $\bigoplus_{i=1}^{n_l} \mathrm{Vect}_k$ to the first n_l simple objects of $\bigoplus_{i=1}^{n_{l+1}} \mathrm{Vect}_k$.

It follows that for all (countable) X the abelian category $P(X; k)$ is equivalent to the category of X-graded vector spaces.

Proposition 3.5.11. *Verdier duality exchanges $D_\Lambda^{\leq 0}$ and $D_\Lambda^{\geq 0}$. Therefore $\mathbb{D}$ restricts to a duality on $\mathsf{P}_\Lambda(X; k)$.*

Proof. If $\mathcal{F}^\bullet \in D_\Lambda^{\leq 0}$, we have that $i_\lambda^* \mathcal{F} \in D_{X_\lambda}^{\leq 0}$. It follows that

$$i_\lambda^!(\mathbb{D}\mathcal{F}^\bullet) = \mathbb{D}(\underbrace{i_\lambda^* \mathcal{F}^\bullet}_{\in D_{X_\lambda}^{\leq 0}}) \in D_{X_\lambda}^{\geq 0}$$

by Remark 3.5.5. □

3.6 Formal Properties of $\mathsf{P}_\Lambda(X; k)$

In this section we collect some properties of the abelian category of perverse sheaves.

We saw in Example 3.5.8 that every perverse sheaf on a smooth connected variety of dimension d has cohomology only in degree $-d$. The next proposition generalizes this statement.

Proposition 3.6.1. *Let $\mathcal{F}^\bullet$ be a perverse sheaf on X. Then its cohomology sheaves $\mathcal{H}^i(\mathcal{F}^\bullet)$ vanish unless $-\dim X \leq i \leq 0$. Moreover, we have*

$$\dim \operatorname{supp} \mathcal{H}^i(\mathcal{F}^\bullet) \leq -i.$$

Proof. We first show that

$$\dim \operatorname{supp} \mathcal{H}^i(\mathcal{F}^\bullet) \leq -i$$

and $\mathcal{H}^i(\mathcal{F}^\bullet) = 0$ for $i > 0$ for all $\mathcal{F}^\bullet \in D_\Lambda^{\leq 0}(X)$. As $\mathcal{F}^\bullet$ has finite-dimensional support, we may assume that X is a variety with finite stratification. Take an open stratum $X_\lambda \subseteq X$ and let $i' \colon X' := X \setminus X_\lambda \hookrightarrow X$ be the closed immersion of the complement. By Theorem 3.5.1(4) there is a distinguished triangle

$$(i_\lambda)_! i_\lambda^! \mathcal{F}^\bullet \to \mathcal{F}^\bullet \to i'_* i'^* \mathcal{F}^\bullet \to (i_\lambda)_! i_\lambda^! \mathcal{F}^\bullet[1].$$

We obtain the exact sequence

$$\mathcal{H}^i((i_\lambda)_! i_\lambda^! \mathcal{F}^\bullet) \to \mathcal{H}^i(\mathcal{F}^\bullet) \to \mathcal{H}^i(i'_* i'^* \mathcal{F}^\bullet).$$

Therefore it suffices to verify the assertion for the left and right sheaf in this sequence.

Note that the non-derived functor $j_!$ and i_* are exact for j an open embedding and i_* a closed embedding. We therefore have

$$\mathcal{H}^i((i_\lambda)_! i_\lambda^! \mathcal{F}^\bullet) = (i_\lambda)_! \mathcal{H}^i(i_\lambda^! \mathcal{F}^\bullet) \quad \text{and} \quad \mathcal{H}^i(i'_* i'^* \mathcal{F}^\bullet) = i'_* \mathcal{H}^i(i'^* \mathcal{F}^\bullet).$$

Next note that $i_\lambda^! \mathcal{F}^\bullet = i_\lambda^* \mathcal{F}^\bullet$ lies in $D^{\leq 0}(X_\lambda)$ and $i'^* \mathcal{F}^\bullet$ lies in $D^{\leq 0}(X')$. Recall that $D^{\leq 0}(X_\lambda) = \{\mathcal{G} \in D^b_{const}(X_\lambda; k) \mid \mathcal{H}^i \mathcal{G}^\bullet = 0 \text{ for } i > -\dim X_\lambda\}$ and so $\mathcal{H}^i(i_\lambda^! \mathcal{F})$ contributes only in degrees $\leq -\dim X_\lambda$. The assertion now follows for $\mathcal{H}^i(i'^* \mathcal{F}^\bullet)$ by induction on the number of strata and we are done.

One can show in the same way that $\mathcal{H}^i \mathcal{F}^\bullet = 0$ for $i < -\dim X$ for $\mathcal{F}^\bullet \in D_\Lambda^{\geq 0}$. $\square$

We saw in Theorem 3.4.9 that $\mathsf{P}_\Lambda(X;k)$ is abelian, however, more is true.

Proposition 3.6.2. *The category $\mathsf{P}_\Lambda(X;k)$ is an abelian category of finite length.*

Proof. See [BBD82, Théorème 4.3.1]. $\square$

We now give the classification of the simple objects in $\mathsf{P}_\Lambda(X;k)$ following [CG97, Chapter 8.4]. For the proofs see [BBD82].

Proposition 3.6.3. *For any stratum X_λ of complex dimension d_λ and any local system $\mathcal{L}$ on X_λ, there is a unique object $\mathrm{IC}(X_\lambda, \mathcal{L})$ in $\mathsf{P}_\Lambda(X;k)$ such that*

(1) $i_\lambda^* \mathcal{H}^{-d_\lambda} \mathrm{IC}(X_\lambda, \mathcal{L}) = \mathcal{L}$,

(2) $\dim \operatorname{supp} \mathcal{H}^i \mathrm{IC}(X_\lambda, \mathcal{L}) < -i$, *for* $-d_\lambda < i$,

(3) $\dim \operatorname{supp} \mathcal{H}^i(\mathbb{D}_X(\mathrm{IC}(X_\lambda, \mathcal{L}))) < -i$, *for* $-d_\lambda < i$.

Proposition 3.6.4. *The object $\mathrm{IC}(X_\lambda, \mathcal{L})$ satisfies the following:*

(1) The cohomology sheaves $\mathcal{H}^i \mathrm{IC}(X_\lambda, \mathcal{L})$ vanish unless $-d_\lambda \leq i \leq 0$,

(2) $\mathcal{H}^{-d_\lambda} \mathrm{IC}(X_\lambda, \mathcal{L}) = \mathcal{H}^0((i_\lambda)_* \mathcal{L})$,

(3) $\mathrm{IC}(X_\lambda, \mathcal{L}^\vee) = \mathbb{D}_X(\mathrm{IC}(X_\lambda, \mathcal{L}))$.

Example 3.6.5. If X is smooth of dimension d with the trivial stratification, we have $\mathrm{IC}(X, \mathcal{L}) = \mathcal{L}[-\dim X]$.

Example 3.6.6. If X is stratified and there is a stratum $X_\lambda = \{x_0\}$, we have

$$\mathrm{IC}(\{x_0\}, k_{\{x_0\}}) : \quad \cdots \longrightarrow 0 \longrightarrow (i_{x_0})_* k_{\{x\}} \longrightarrow 0 \longrightarrow \cdots$$

Proposition 3.6.7. *The simple objects of $\mathsf{P}_\Lambda(X;k)$ are precisely the $\mathrm{IC}(X_\lambda, \mathcal{L})$ for simple local systems $\mathcal{L}$, where we run through all $\lambda \in \Lambda$.*

Corollary 3.6.8. *If $X = \coprod_{\lambda \in \Lambda} X_\lambda$ such that all X_λ are simply-connected, we have that the simple perverse sheaves are in 1:1-correspondence with Λ under*

$$\Lambda \to \mathrm{Irr}(\mathsf{P}_\Lambda(X;k)), \quad \lambda \mapsto \mathrm{IC}(X_\lambda, k_{X_\lambda}).$$

Proof. If X_λ is simply-connected, we have that $\mathrm{Loc}_f(X_\lambda;k)$ is equivalent to Vect_k by Corollary 3.1.10. Therefore, there is only one simple local system on every X_λ, namely the constant sheaf k_{X_λ}. We conclude by the previous Proposition 3.6.7. □

Example 3.6.9. If X is a discrete space, we have seen in Example 3.5.10 that the category of perverse sheaves on X is equivalent to the category of X-graded vector spaces. The simple objects of $\mathsf{P}(X;k)$ are in 1:1-correspondence with the points of X and are explicitly given as the degree zero stalk complex

$$\mathrm{IC}(\{x\}, k_{\{x\}}): \quad \cdots \longrightarrow 0 \longrightarrow (i_x)_* k_{\{x\}} \longrightarrow 0 \longrightarrow \cdots,$$

where x runs through all $x \in X$ and i_x denotes the inclusion $\{x\} \hookrightarrow X$.

We know now that that $\mathsf{P}_\Lambda(X;k)$ is a finite length abelian category with a specified set of simples, so we next are interested in extensions of perverse sheaves. Of course we have

$$\mathrm{Ext}^i_{\mathsf{P}_\Lambda(X;k)}(\mathcal{F}, \mathcal{G}) = \mathrm{Hom}_{D^b(\mathsf{P}_\Lambda(X;k)}(\mathcal{F}, \mathcal{G}[i]).$$

Note that the right hand side is in general not the same as $\mathrm{Hom}_{D^b(X;k)}(\mathcal{F}, \mathcal{G}[i])$. However, for $i = 0, 1$ there turns out to be an isomorphism. The inclusion $\mathsf{P}_\Lambda(X;k) \to D^b(X;k)$ induces a functor

$$D^b(\mathsf{P}_\Lambda(X;k) \to D^b(X;k),$$

which in turn yields a map

$$\mathrm{Hom}_{D^b(\mathsf{P}_\Lambda(X;k)}(\mathcal{F}, \mathcal{G}[i]) \to \mathrm{Hom}_{D^b(X;k)}(\mathcal{F}, \mathcal{G}[i]).$$

Proposition 3.6.10. *For $\mathcal{F}, \mathcal{G} \in \mathsf{P}_\Lambda(X;k)$ and we have that the map*

$$\mathrm{Ext}^i_{\mathsf{P}_\Lambda(X;k)}(\mathcal{F}, \mathcal{G}) = \mathrm{Hom}_{D^b(\mathsf{P}_\Lambda(X;k)}(\mathcal{F}, \mathcal{G}[i]) \longrightarrow \mathrm{Hom}_{D^b(X;k)}(\mathcal{F}, \mathcal{G}[i]),$$

is an isomorphism if $i = 0$ or $i = 1$.

Proof. The case $i = 0$ is clear and $i = 1$ follows from the description of short exact sequences in the heart of a t-structure in Theorem 3.4.9 and the fact that $\mathsf{P}_\Lambda(X;k)$ is closed under extensions in $D^b_\Lambda(X;k)$. □

The following theorem is a deep result whose original proof required techniques in positive characteristics and the Weil conjectures, see [BBD82, Théorème 6.2.5].

Theorem 3.6.11 (Decomposition Theorem)**.** *Let k be of characteristic zero. For a proper morphism $f \colon X \to Y$ of algebraic varieties, the complex of sheaves $f_*\mathrm{IC}(X_\lambda, \mathcal{L})$ is a finite direct sum of shifts of simple perverse sheaves on Y, i.e. a direct sum of shifts of $\mathrm{IC}(Y_\mu, \mathcal{L})$ for $Y_\mu \subseteq Y$ smooth locally closed subsets.*

Chapter 4

Geometric Satake Equivalence

In this chapter we finally state the Geometric Satake Equivalence as proven by Ginzburg [Gin95] and sketch the proof as presented in [BR18].

4.1 The Statement of Geometric Satake

Recall the Cartan Decomposition 2.3.1

$$\mathrm{Gr}_G(\mathbb{C}) = \coprod_{\lambda \in X_*(G,T)_+} \mathrm{Gr}_G^{\lambda}.$$

of the Affine Grassmannian $\mathrm{Gr}_G(\mathbb{C})$. Here we consider the space $\mathrm{Gr}_G(\mathbb{C})$ with the analytic topology.

In Proposition 3.3.7 we proved that this decomposition is a stratification satisfying the Whitney condition. This allows us to consider perverse sheaves on $\mathrm{Gr}_G(\mathbb{C})$, see Chapter 3.

Definition 4.1.1. We denote the category of perverse sheaves on Gr_G with respect to the stratification defined by the Cartan Decomposition by

$$\mathsf{Sat}_G \coloneqq \mathsf{P}_{X_*(G,T)_+}(\mathrm{Gr}_G(\mathbb{C}); \mathbb{C}).$$

The Geometric Satake Equivalence concerns this category of perverse sheaves and can be formulated as follows.

Theorem 4.1.2 (Geometric Satake Equivalence, [Gin95])**.** *The category* Sat_G *of perverse sheaves on* $\mathrm{Gr}_G(\mathbb{C})$ *with respect to the Cartan Decomposition is equivalent to the category* $\mathrm{Rep}_{\mathbb{C}}(G^{\vee})$ *of complex representations of the Langlands dual group of* G *as Tannakian categories.*

We recall the notion of Tannakian categories after Theorem 4.2.2. Essentially, a Tannakian category is an abelian category together with a monoidal structure and an underlying vector space for every object, such that the monoidal product behaves like the usual tensor product of representations and vector spaces.

Remark 4.1.3. Usually, the equivalence is stated as an equivalence between the $\mathrm{Rep}(G^\vee)$ and the category $\mathsf{P}_{G[[t]]}(\mathrm{Gr}_G(\mathbb{C});\mathbb{C})$ of $G[[t]]$-equivariant perverse sheaves. However, there is a Tannakian equivalence

$$\mathsf{P}_{G[[t]]}(\mathrm{Gr}_G(\mathbb{C}));\mathbb{C}) \longrightarrow \mathsf{P}_{X_*(G,T)_+}(\mathrm{Gr}_G(\mathbb{C});\mathbb{C}) = \mathsf{Sat}_G,$$

see [BR18, Corollary 4.8].

Example 4.1.4. By Corollary 2.3.18 the strata of $\mathrm{Gr}_G(\mathbb{C})$ are simply-connected and so by Corollary 3.6.8 the simple objects of Sat_G are of the form

$$\mathrm{IC}_\lambda \coloneqq \mathrm{IC}(\mathrm{Gr}_G^\lambda, \mathbb{C}_{\mathrm{Gr}_G^\lambda}), \quad \lambda \in X_*(G,T)_+.$$

On the other hand, the simples of $\mathrm{Rep}(G^\vee)$ are also labeled by $X^*(G^\vee, T^\vee)_+ = X_*(G,T)_+$ by Theorem 1.1.22. We see that we have the same labeling set for simples in both categories. We therefore recover the Cartan Decomposition 2.3.1 as a kind of "set-theoretic" version of the Geometric Satake Equivalence.

Example 4.1.5. The Affine Grassmannian of $\mathrm{GL}_1 = \mathbb{G}_m$ is just an infinite disjoint union of points $\mathrm{Gr}_{\mathrm{GL}_1}(\mathbb{C}) = \mathbb{Z}$ by Example 2.1.18. Therefore the category $\mathsf{Sat}_{\mathrm{GL}_1}$ is equivalent to the category of finite-dimensional $\mathbb{Z}$-graded vector spaces. This is also equivalent to $\mathrm{Rep}(\mathrm{GL}_1)$ and by 1.1.20 we indeed have $\mathrm{GL}_1 = \mathrm{GL}_1^\vee$.

Example 4.1.6. If $G = \mathrm{GL}_1 \times \mathrm{GL}_1$ we have by Proposition 2.2.11

$$\mathrm{Gr}_G(\mathbb{C}) = \mathrm{Gr}_{\mathrm{GL}_1}(\mathbb{C}) \times \mathrm{Gr}_{\mathrm{GL}_1}(\mathbb{C}) = \mathbb{Z}^2.$$

It follows that Sat_G is equivalent to the category of finite-dimensional $\mathbb{Z}^2$-graded vector spaces by Example 3.5.10 with simple objects $\mathrm{IC}_{(n,m)}$. Indeed,

$$(\mathrm{GL}_1 \times \mathrm{GL}_1)^\vee = \mathrm{GL}_1 \times \mathrm{GL}_1.$$

Example 4.1.7. By Corollary 2.2.16 and Proposition 2.3.6 we have closed-open embeddings

$$\mathrm{Gr}_{\mathrm{SL}_n}(\mathbb{C}) \hookrightarrow \mathrm{Gr}_{\mathrm{PGL}_n}(\mathbb{C}) \hookrightarrow \mathrm{Gr}_{\mathrm{GL}_n}(\mathbb{C}),$$

which are compatible with the stratifications. Therefore by Example 3.5.3 we obtain fully faithful exact functors

$$\mathsf{Sat}_{\mathrm{SL}_n} \hookrightarrow \mathsf{Sat}_{\mathrm{PGL}_n} \hookrightarrow \mathsf{Sat}_{\mathrm{GL}_n}.$$

Under Geometric Satake these correspond to functors

$$\mathrm{Rep}(\underbrace{\mathrm{SL}_n^\vee}_{\overset{1.1.21}{=}\mathrm{PGL}_n}) \hookrightarrow \mathrm{Rep}(\underbrace{\mathrm{PGL}_n^\vee}_{\overset{1.1.21}{=}\mathrm{SL}_n}) \hookrightarrow \mathrm{Rep}(\underbrace{\mathrm{GL}_n^\vee}_{\overset{1.1.20}{=}\mathrm{GL}_n}).$$

It turns out, the first functor is the one defined in Proposition 1.1.6 and the second is given by the identification

$$\mathrm{Rep}(\mathrm{SL}_n) = \{M \in \mathrm{Rep}(\mathrm{GL}_n) \mid \text{the matrix } A \text{ acts trivially on } M, \text{ if } A \in Z(\mathrm{GL}_n(\mathbb{C}))\}.$$

4.2 The Proof of Geometric Satake

We mainly follow the notes [BR18]. The proof of Geometric Satake 4.1.2 has two main steps. First one needs to define a monoidal structure on Sat_G and a fiber functor

$$\mathsf{Sat}_G \to \mathrm{Vect}_{\mathbb{C}},$$

which make Sat_G a Tannakian category.

However, before we investigate the tensor structure, we sketch the argument that Sat_G is a semi-simple. This will help us to prove that Sat_G is the category of representations of a reductive group.

Proposition 4.2.1. *The category* Sat_G *is semi-simple.*

Sketch of Proof. The simple objects of Sat_G are IC_λ with $\lambda \in X_*(G,T)_+$ by Corollary 3.6.8 and Corollary 2.3.18. Therefore it suffices to compute that

$$\mathrm{Ext}^1(\mathrm{IC}_\lambda, \mathrm{IC}_\mu) = 0$$

for all $\lambda, \mu \in X_*(G,T)_+$. This can be shown by using the theory of so-called parity sheaves, see [JMW14, Definition 2.4]. It turns out that IC_λ is an even, respectively an odd parity sheaf, if $\dim \mathrm{Gr}_G^\lambda = \langle 2\rho, \lambda\rangle$ is even, respectively odd, see [Ach21, Lema 9.3.8]. Notably, this step requires us to work in characteristic zero. This is related to Corollary 2.4.9 which states that the dimensions of all Schubert cells appearing in a Schubert variety have the same parity.

By [JMW14, Proposition 2.6], it follows that

$$\mathrm{Ext}^1(\mathrm{IC}_\lambda, \mathrm{IC}_\mu) = 0$$

if IC_λ and IC_μ are either both even or both odd.

The mixed case

$$\mathrm{Ext}^1(\mathrm{IC}_\lambda, \mathrm{IC}_\mu) = 0,$$

where either IC_λ is even and IC_μ is odd, or IC_λ is odd and IC_μ is even, follows from the fact that IC_λ and IC_μ are supported on different connected components of $\mathrm{Gr}_G(\mathbb{C})$. $\square$

We now come to the first main step of the proof of Geometric Satake.

4.2.1 Sat_G is Tannakian

This subsection is all about the proof of the following theorem.

Theorem 4.2.2. *The category* Sat_G *is Tannakian.*

We have seen in Theorem 3.4.9 that Sat_G is abelian. To show that Sat_G has the structure of a Tannakian category, one needs to do the following:

- Construct a monoidal structure, called convolution,
$$\star\colon \mathsf{Sat}_G \times \mathsf{Sat}_G \longrightarrow \mathsf{Sat}_G.$$
In particular, $\star$ needs to be associative with a monoidal unit $\mathbb{1} \in \mathsf{Sat}_G$.
- Show that Sat_G is symmetric monoidal.
- Define a faithful exact functor $F\colon \mathsf{Sat}_G \to \mathrm{Vect}_{\mathbb{C}}$, called the fiber functor, which satisfies
$$F(\mathcal{F} \star \mathcal{G}) = F(\mathcal{F}) \otimes_{\mathbb{C}} F(\mathcal{G}), \quad F(\mathbb{1}) \cong \mathbb{C}$$
and is compatible with the unity, associativity, and commuativity constraints.
- Show that for every $\mathcal{F}$ such that $\dim F(\mathcal{F}) = 1$ there is an element $\mathcal{F}^{-1}$ satisfying
$$\mathcal{F} \star \mathcal{F}^{-1} = \mathbb{1}.$$
This condition guarantees the existence of duals in Sat_G.

Construction of Monoidal Structure. For more details see [Gin95], or [BR18, Section I.6.2].

One defines the space
$$G(\mathbb{C}((t))) \times^{G(\mathbb{C}[[t]])} \mathrm{Gr}_G(\mathbb{C}) \coloneqq \big(G(\mathbb{C}((t))) \times \mathrm{Gr}_G(\mathbb{C})\big) \,/\, \sim,$$
where $(A, [B]) \sim (A', [B'])$ if and only if there is a $C \in G(\mathbb{C}[[t]])$ such that
$$(A, [B]) = (A'C, [C^{-1}B']).$$
Then one considers the natural maps
$$\begin{aligned} &G(\mathbb{C}((t))) \times \mathrm{Gr}_G(\mathbb{C}) \xrightarrow{p} \mathrm{Gr}_G(\mathbb{C}) \times \mathrm{Gr}_G(\mathbb{C}), \\ &G(\mathbb{C}((t))) \times \mathrm{Gr}_G(\mathbb{C}) \xrightarrow{q} G(\mathbb{C}((t))) \times^{G(\mathbb{C}[[t]])} \mathrm{Gr}_G(\mathbb{C}) \xrightarrow{m} \mathrm{Gr}_G(\mathbb{C}). \end{aligned}$$
Here p and q are the obvious quotient maps and m is the multiplication given by
$$m([A, [B]]) = [AB].$$
One easily checks that m is in fact well-defined.

For $\mathcal{F}, \mathcal{G} \in \mathsf{Sat}_G$ we can consider
$$\mathcal{F} \boxtimes \mathcal{G} \in D^b(\mathrm{Gr}_G(\mathbb{C}) \times \mathrm{Gr}_G(\mathbb{C}))$$
and show that there is a unique object
$$\mathcal{F} \mathbin{\widetilde{\boxtimes}} \mathcal{G} \in D^b\left(G(\mathbb{C}((t))) \times^{G(\mathbb{C}[[t]])} \mathrm{Gr}_G(\mathbb{C})\right)$$
such that
$$q^*(\mathcal{F} \mathbin{\widetilde{\boxtimes}} \mathcal{G}) = p^*(\mathcal{F} \boxtimes \mathcal{G}).$$
We then define the convolution product of $\mathcal{F}$ and $\mathcal{G}$ as
$$\mathcal{F} \star \mathcal{G} \coloneqq m_*(\mathcal{F} \mathbin{\widetilde{\boxtimes}} \mathcal{G}).$$

For a proof that $\star$ is associative and commutative, see [BR18, Chapter I.6 and 7].

Proposition 4.2.3. *The convolution* $\mathrm{IC}_\lambda \star \mathrm{IC}_\mu$ *of the simple perverse sheaves* IC_λ *and* IC_μ *is supported on* $\overline{\mathrm{Gr}_G^{\lambda+\mu}}$ *and the restriction to the open cell* $\mathrm{Gr}_G^{\lambda+\mu}$ *is* $\mathbb{C}_{\mathrm{Gr}_G^{\lambda+\mu}}[\dim \mathrm{Gr}_G^{\lambda+\mu}]$.

Sketch of Proof. The idea is to restrict and corestrict the maps p, q, m to closed subsets. We write $G(\mathbb{C}((t)))^{\leq\lambda}$ for the inverse image of $\overline{\mathrm{Gr}_G^\lambda} = \coprod_{\mu\leq\lambda} \mathrm{Gr}_G^\mu$ under the quotient map.

We can restrict and corestrict the map p to a function

$$p\colon G(\mathbb{C}((t)))^{\leq\lambda} \times \overline{\mathrm{Gr}_G^\mu} \longrightarrow \overline{\mathrm{Gr}_G^\lambda} \times \overline{\mathrm{Gr}_G^\mu}.$$

Note that applying $m \circ q$ to an element in $G(\mathbb{C}((t)))^{\leq\lambda} \times \overline{\mathrm{Gr}_G^\mu}$ gives an element in $\overline{\mathrm{Gr}_G^{\lambda+\mu}}$. So, restricting q to $G(\mathbb{C}((t)))^{\leq\lambda} \times \overline{\mathrm{Gr}_G^\mu}$, we can corestrict m to $\overline{\mathrm{Gr}_G^{\lambda+\mu}}$. For more details see [BR18, Chapter I.6]. □

Corollary 4.2.4. *The product* $\mathrm{IC}_\lambda \star \mathrm{IC}_\mu$ *has the "upper triangular" decomposition*

$$\mathrm{IC}_\lambda \star \mathrm{IC}_\mu \cong \mathrm{IC}_{\lambda+\mu} \oplus \bigoplus_{\nu<\lambda+\mu} \mathrm{IC}_\nu^{n_\nu^{\lambda,\mu}}$$

for some $n_\nu^{\lambda,\mu} \in \mathbb{N}_0$.

Proof. The perverse sheaf $\mathrm{IC}_\lambda \star \mathrm{IC}_\mu$ has a direct sum decomposition into IC-sheaves, because Sat_G is semi-simple by Proposition 4.2.1. By the first part of Proposition 4.2.3 we must have

$$\mathrm{IC}_\lambda \star \mathrm{IC}_\mu \cong \bigoplus_{\nu\leq\lambda+\mu} \mathrm{IC}_\nu^{n_\nu^{\lambda,\mu}},$$

because those are the only IC-sheaves supported on $\overline{\mathrm{Gr}_G^{\lambda+\mu}} = \coprod_{\nu\leq\lambda+\mu} \mathrm{Gr}_G^{\lambda+\mu}$. By the second part of Proposition 4.2.3 we must have $n_{\lambda+\mu}^{\lambda,\mu} = 1$, as $\mathrm{IC}_{\lambda+\mu}$ is the only appearing sheaf supported on $\mathrm{Gr}_G^{\lambda+\mu}$. □

Corollary 4.2.5. *If* Gr_G^λ *and* Gr_G^μ *are points (see Corollary 2.3.21) we have*

$$\mathrm{IC}_\lambda \star \mathrm{IC}_\mu \cong \mathrm{IC}_{\lambda+\mu}.$$

Proof. By Corollary 2.3.21 we deduce that $\mathrm{Gr}_G^{\lambda+\mu}$ is a singleton if Gr_G^λ and Gr_G^μ are. Therefore,

$$\{\nu \in (X_*)_+ \mid \nu < \lambda + \mu\} = \emptyset$$

and we conclude by Corollary 4.2.4. □

Remark 4.2.6. It follows from Geometric Satake that in fact

$$\mathrm{IC}_\lambda \star \mathrm{IC}_\mu \cong \mathrm{IC}_{\lambda+\mu}$$

even if only one of the two Gr_G^λ, Gr_G^μ is a singleton.

Example 4.2.7. In the case $G = \mathrm{GL}_1$ the simple objects are of the form IC_n for $n \in \mathbb{Z}$. These satisfy

$$\mathrm{IC}_n \star \mathrm{IC}_m = \mathrm{IC}_{n+m}.$$

This is precisely what we expect for representations of $\mathrm{GL}_1^\vee = \mathrm{GL}_1$.

Proposition 4.2.8. *The* IC*-sheaf* IC_0 *is the unit* $\mathbb{1}$ *of the monoidal functor* $\star$.

Proof. See [BR18, Chapter I.9]. □

We now define the fiber functor

$$F \colon \mathsf{Sat}_G \to \mathrm{Vect}_{\mathbb{C}}.$$

Definition 4.2.9. We set

$$F(\mathcal{F}) := \bigoplus_k H^k(\mathrm{Gr}_G(\mathbb{C}), \mathcal{F}),$$

where $H^k(X, -)$ is the sheaf cohomology functor.

This functor is additive and therefore exact, because Sat_G is semi-simple by Proposition 4.2.1.

Proposition 4.2.10. *The functor F preserves the monoidal unit, i.e.* $F(\mathrm{IC}_0) = \mathbb{C}$.

Proof. By Example 3.6.6 IC_0 is the complex

$$\mathrm{IC}_0 : \quad \cdots \longrightarrow 0 \longrightarrow (i_{\{pt\}})_* \mathbb{C}_{\{pt\}} \longrightarrow 0 \longrightarrow \cdots$$

Therefore, we have

$$\begin{aligned} F(\mathrm{IC}_0) &= \bigoplus_k H^k(\mathrm{Gr}_G, \mathrm{IC}_0) \\ &= \bigoplus_k H^k\left(\mathrm{Gr}_G, (i_{\{pt\}})_* \mathbb{C}_{\{pt\}}\right) \\ &= \bigoplus_k H^k(\{pt\}, \mathbb{C}_{\{pt\}}) \\ &= \bigoplus_k H^k(\{pt\}; \mathbb{C}) \\ &= \mathbb{C}. \end{aligned}$$ □

Proposition 4.2.11. *The functor F is faithful.*

Proof. Faithfulness follows if $F(\mathcal{F}) \neq 0$ for all simple objects in Sat_G by semi-simplicity. The simple objects are all of the form IC_λ and so it suffices to show that $F(\mathrm{IC}_\lambda) \neq 0$. This can be found in [BR18, Theorem 5.9]. □

Proposition 4.2.12. *The fiber functor F is compatible with the monoidal product* $\star$.

Proof. See [BR18, Chapter I.8]. □

We need the following lemma for the subsequent proposition.

Lemma 4.2.13. *We have* $\dim F(\mathrm{IC}_\lambda) = 1$ *if and only if* $\mathrm{Gr}_G^\lambda = \{pt\}$.

Proof. See [BR18, Theorem 5.13]. □

Proposition 4.2.14. *For every* $\mathcal{F} \in \mathsf{Sat}_G$ *with* $\dim F(\mathcal{F}) = 1$, *there is an* $\mathcal{F}^{-1} \in \mathsf{Sat}_G$ *satisfying*

$$\mathcal{F} \star \mathcal{F}^{-1} = \mathrm{IC}_0.$$

Proof. By Proposition 4.2.1 we know that $\mathcal{F}$ is a direct sum of IC-sheaves. But since F is faithful, we have that $\dim F(\mathcal{F}) = 1$ only if $\mathcal{F}$ is already a simple object IC_λ. By Proposition 4.2.13 we deduce that $\mathrm{Gr}_G^\lambda = \{pt\}$. Setting $\mathcal{F}^{-1} := \mathrm{IC}_{-\lambda}$, we obtain by Corollary 4.2.5

$$\mathcal{F} \star \mathcal{F}^{-1} = \mathrm{IC}_\lambda \star \mathrm{IC}_{-\lambda} = \mathrm{IC}_{\lambda-\lambda} = \mathrm{IC}_0.$$

This is the monoidal unit of Sat_G. □

All of these propositions and constructions together show that Sat_G is a Tannakian category.

4.2.2 Reconstructing $G^\vee$

In this subsection we use the Tannakian Reconstruction Theorem, see [Saa72], to proof Geometric Satake.

Theorem 4.2.15 (Tannakian Reconstruction)**.** *If the category* $\mathcal{C}$ *together with a monoidal functor* $\otimes$ *and a fiber functor* $F\colon \mathcal{C} \to \mathrm{Vect}$ *is Tannakian, there is an equivalence of Tannakian categories*

$$\mathcal{C} \simeq \mathrm{Rep}(\widetilde{G})$$

for some affine group scheme $\widetilde{G}$.

The following corollary is Tannakian Reconstruction applied to the Tannakian category Sat_G.

Corollary 4.2.16. *The category* Sat_G *is equivalent to* $\mathrm{Rep}(\widetilde{G})$ *as Tannakian categories for some affine group scheme* $\widetilde{G}$.

The second step of the proof of Geometric Satake is to show that $\widetilde{G} \cong G^\vee$. To do this we first show that $\widetilde{G}$ is reductive and then compute the root datum. Showing that the root datum of the reductive group $\widetilde{G}$ is the dual root datum of G implies that $\widetilde{G} \cong G^\vee$ by Definition 1.1.18 and concludes the proof.

The next three propositions are all statements about the group scheme $\widetilde{G}$ which have equivalent formulations as properties about the category $\mathrm{Rep}(\widetilde{G})$. So we learn about the group $\widetilde{G}$ by investigating its representation category $\mathrm{Rep}(\widetilde{G})$.

Proposition 4.2.17. *The group scheme $\widetilde{G}$ is algebraic.*

Proof. This proof is taken from [BR18, Lemma 9.2].

The group $\widetilde{G}$ is algebraic if and only if there exists a representation $M \in \operatorname{Rep}(\widetilde{G})$ such that M generates $\operatorname{Rep}(\widetilde{G})$ by taking direct sums,tensor products, duals, and subquotients, see [BR18, Proposition 2.11.1].

Take a finite set of elements $\lambda_1, \ldots, \lambda_l \in (X_*)_+$ such that every element $\lambda \in (X_*)_+$ can be written as a finite sum of λ_i's. Set $\mathcal{F} := \mathrm{IC}_{\lambda_1} \oplus \cdots \oplus \mathrm{IC}_{\lambda_l}$. This $\mathcal{F}$ generates $\mathsf{Sat}_G = \operatorname{Rep}(\widetilde{G})$. Indeed let $\lambda \in (X_*)_+$. By assumption on the λ_i's, we may write

$$\lambda = k_1\lambda_1 + \cdots + k_l\lambda_l.$$

Then $\mathcal{F}^{\star(k_1+\cdots+k_l)}$ contains IC_λ as a direct summand: Indeed, it contains

$$\mathrm{IC}_{\lambda_1}^{\star k_1} \star \cdots \star \mathrm{IC}_{\lambda_l}^{\star k_l}$$

as a direct summand. But this contains IC_λ as a direct summand by Proposition 4.2.3 and induction.

Therefore every simple object of the semi-simple category Sat_G is contained in the subcategory generated by $\mathcal{F}$. Hence, the representation $\mathcal{F} \in \operatorname{Rep}(\widetilde{G})$ generates and we conclude that $\widetilde{G}$ is algebraic. □

Proposition 4.2.18. *The affine algebraic group $\widetilde{G}$ is connected.*

Proof. This proof is taken from [BR18, Lemma 9.3].

The affine algebraic group $\widetilde{G}$ is not connected if and only if there is a non-trivial representation M in $\operatorname{Rep}(\widetilde{G})$ such that the category generated from M by taking direct sums and subquotients is stable under $\star$, see [BR18, Proposition 2.11.2].

If we take some non-trivial $\mathcal{F} \in \mathsf{Sat}_G$, then every object arising from $\mathcal{F}$ by taking subquotients and direct sums will be supported only on the support of $\mathcal{F}$. However, the support of $\mathcal{F}$ is a finite union of strata. Now take a simple direct summand IC_λ of $\mathcal{F}$, with $\lambda \neq 0$. The powers $\mathrm{IC}_\lambda^{\star n}$ satisfy

$$\operatorname{supp} \mathrm{IC}_\lambda^{\star n} \supseteq \mathrm{Gr}_G^{n \cdot \lambda}$$

by Proposition 4.2.3. We have $n\lambda \neq n'\lambda$ for $n \neq n'$, since $\lambda \neq 0$. Therefore the combined support of tensor powers of IC_λ contains infinitely many strata. We conclude that the category generated by $\mathcal{F}$ cannot be $\star$-stable and therefore $\widetilde{G}$ must be connected. □

Proposition 4.2.19. *The connected affine algebraic group $\widetilde{G}$ is reductive.*

Proof. We have seen in Proposition 4.2.1 that $\operatorname{Rep}(\widetilde{G}) = \mathsf{Sat}_G$ is semi-simple. But a connected affine algebraic group with semi-simple representation category is reductive by definition. □

We now know that $\mathsf{Sat}_G \simeq \operatorname{Rep}(\widetilde{G})$ for a reductive algebraic group. What is left to show is that $\widetilde{G}$ has the same root datum as $G^\vee$.

Proposition 4.2.20. *The root datum of the reductive group $\widetilde{G}$ is dual to the root datum of G.*

Proof. To do this one first needs to construct a torus and then compute the roots and coroots. This is done in [BR18, Chapter 9.2 and 9.3]. □

With this proposition it follows that $\widetilde{G} \cong G^\vee$ by definition of the Langlands dual Definiton 1.1.18. This finishes our sketch of the proof of the Geometric Satake Equivalence.

4.3 Geometric Satake with General Coefficients

We have discussed the Geometric Satake equivalence as proven by Ginzburg, [Gin95]. However, Mirković and Vilonen have given a generalization of Ginzburg's Geometric Satake to admit more general coefficients.

They still consider the Affine Grassmannian $\mathrm{Gr}_G(\mathbb{C})$ for a complex reductive group G with the analytic topology. Next they look at $\mathsf{P}_{(X_*)_+}(\mathrm{Gr}_G(\mathbb{C}); k)$ where k is a Noetherian ring of finite global dimension. Ginzburg's result is the case $k = \mathbb{C}$.

Theorem 4.3.1 (Geometric Satake Equivalence, Mirković–Vilonen)**.** *There is an equivalence of Tannakian categories*

$$\mathsf{P}_{(X_*)_+}(\mathrm{Gr}_G(\mathbb{C}); k) \longrightarrow \mathrm{Rep}_k(G_k^\vee).$$

Here $G_k^\vee$ denotes the split-reductive group defined over k with the dual root datum of the complex reductive group G. This group was defined by Demazure, see [DG11].

The proof of Mirković–Vilonen is quite similar to the one sketched above. They, too, begin by constructing the monoidal functor $\star$ on $\mathsf{P}_{(X_*)_+}(\mathrm{Gr}_G(\mathbb{C}); k)$ and show that this is indeed symmetric. However, this category is no longer semi-simple and therefore more work needed to be put into constructing $\widetilde{G}$. To remedy this, they introduced a weight decomposition on the fiber functor

$$F \colon \mathsf{P}_{(X_*)_+}(\mathrm{Gr}_G(\mathbb{C}); k) \to k\text{-mod}$$

which models the weight decomposition one expects to have on representations. For an exposition of this proof see [BR18, Part II].

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