A brief review of abelian categorifications

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Abstract

This article contains a review of categorifications of semisimple representations of various rings via abelian categories and exact endofunctors on them. A simple definition of an abelian categorification is presented and illustrated with several examples, including categorifications of various representations of the symmetric group and its Hecke algebra via highest weight categories of modules over the Lie algebra $\mathfrak{sl}_n$. The review is intended to give non-experts in representation theory who are familiar with the topological aspects of categorification (lifting quantum link invariants to homology theories) an idea for the sort of categories that appear when link homology is extended to tangles.

1 Introduction

The idea of categorification goes back to Crane and Frenkel [28] and became more and more popular in recent years. In the present review we want to give an idea what abelian categorifications should be and present some examples naturally arising in representation theory. We give references to related categorifications arising in knot theory as well as in symplectic geometry.

In the following we will give a definition of abelian categorifications. To get a feeling for this concept it might be useful to view it as an "inverse process" of *decategorification.* Here by decategorification we mean passing from an abelian category $\mathcal{C}$ to its (complexified) Grothendieck group, i.e., the complex vector space generated by the isomorphism classes of objects of $\mathcal{C}$, modulo short exact sequences. Decategorification forgets enormous amount of structure, in particular, it does not directly remember about morphisms in the category.

Categorification starts with a vector space M and tries to find an (interesting) abelian category $\mathcal{C}$ with decategorification M. Assuming that M is in addition a module or an algebra, one would also like to lift this extra structure. The idea of categorification is not completely new in the context of representation theory, but the focus changed in recent years from trying to understand the combinatorics, decomposition numbers and multiplicities of representations, to the opposite—given certain combinatorics one would like to enrich the structure by categorification. This idea was, for instance, successfully applied to constructing functorial link and knot invariants which on the one hand side categorify well-known invariants (such as the Jones-polynomial) ([5], [45], [47], [81], [82]) and on the other side extend to invariants of cobordisms. At this point it becomes clear that one should pass from abelian categories to 2-categories. An axiomatic definition of such a categorification is not yet available, but this review can be viewed as a modest naive step in this direction.

We start by giving the definition of a weak abelian categorification, then proceed with several well-known examples of weak abelian categorifications of modules over the symmetric group and Lie algebras. In Section 2 we give detailed explicit constructions of irreducible representations of the symmetric group. The last section indicates relations to knot theory and combinatorial representation theory.

Chuang and Rouquier, in the recent work [27], introduced the notion of a strong $\mathfrak{sl}_2$ categorification which is the data of an abelian category together with endofunctors E and F (corresponding to the usual Chevalley generators of $\mathfrak{sl}_2$), and elements X and T in the ring of natural transformations of E respectively E^2 with certain compatibility conditions. Using this machinery of strong $\mathfrak{sl}_2$-categorification they proved the so-called Broué conjecture for the representation theory of the symmetric group in positive characteristic: two blocks with the same defect are derived equivalent.

Our list of examples of abelian categorifications is very far from complete. Many great results in the geometric representation theory can be interpreted as categorifications via abelian or triangulated categories. This includes the early foundational work of Beilinson-Bernstein and Brylinsky-Kashiwara on localization [6], [19], [67], the work of Kazhdan and Lusztig on geometric realization of representations of affine Hecke algebras [43], [26], Lusztig's geometric construction of the Borel subalgebras of quantum groups [60], Nakajima's realization of irreducible Kac-Moody algebra representations as middle cohomology groups of quiver varieties [68], and various constructions related to Hilbert schemes of surfaces [32], [69], quantum groups at roots of unity [4], geometric Langlands correspondence [30], etc.

Since the first version of the paper appeared, a significant further progress has been made in developing the notion of categorification, especially for categorifications of Lie algebra and quantum groups representations in [17], [20]-[22], [54]-[56], [58], [73] and other papers.

2 A simple framework for categorification

Categorification. The Grothendieck group $K(\mathcal{B})$ of an abelian category $\mathcal{B}$ has as generators the symbols $[M]$, where M runs over all the objects of $\mathcal{B}$, and defining relations $[M_2] = [M_1] + [M_3]$, whenever there is a short exact sequence

$$0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0.$$

An exact functor F between abelian categories induces a homomorphism $[F]$ between their Grothendieck groups.

Let A be a ring which is free as an abelian group, and $\mathbf{a} = \{a_i\}_{i \in I}$ a basis of A such that the multiplication has nonnegative integer coefficients in this basis:

$$a_i a_j = \sum_k c_{ij}^k a_k, \quad c_{ij}^k \in \mathbb{Z}_{\geq 0}. \tag{1}$$

Let B be a (left) A-module.

Definition 1 *A (weak) abelian categorification of* $(A, \mathbf{a}, B)$ *consists of an abelian category* $\mathcal{B}$, *an isomorphism* $\varphi : K(\mathcal{B}) \xrightarrow{\sim} B$ *and exact endofunctors* $F_i : \mathcal{B} \longrightarrow \mathcal{B}$ *such that the following holds:*

(C-I) The functor F_i *lifts the action of* a_i *on the module* B, *i.e. the action of* $[F_i]$ *on the Grothendieck group of* $\mathcal{B}$ *corresponds to the action of* a_i *on the module* B *so that the diagram below is commutative.*

$$\begin{array}{ccc} K(\mathcal{B}) & \xrightarrow{[F_i]} & K(\mathcal{B}) \\ \varphi \downarrow & & \downarrow \varphi \\ B & \xrightarrow{a_i} & B \end{array}$$

(C-II) There are isomorphisms

$$F_iF_j \cong \bigoplus_k F_k^{c_{ij}^k},$$

i.e., the composition F_iF_j decomposes as the direct sum of functors F_k with multiplicities c_{ij}^k as in (1).

The word *weak* refers here to the absence of any compatibility constraints on the isomorphisms appearing in (C-II). A stronger version of categorification keeping track of this extra information has to use the notion of 2-categories and is only available so far for special cases (se e.g. [71] for such an example).

If there is a categorification as above we say *the action of the functors F_i on the category $\mathcal{B}$ categorifies the A-module B.*

In all our examples, the objects of $\mathcal{B}$ will have finite length (finite Jordan-Hölder series). Consequently, if $\{L_j\}_{j\in J}$ is a collection of simple objects of $\mathcal{B}$, one for each isomorphism class, the Grothendieck group $K(\mathcal{B})$ is free abelian with basis elements $[L_j]$. The image of any object $M \in \mathcal{B}$ in the Grothendieck group is

$$[M] = \sum_j m_j(M)[L_j]$$

where $m_j(M)$ is the multiplicity of L_j in some (and hence in any) composition series of M.

The free group $K(\mathcal{B})$ has therefore a distinguished basis $[L_j]_{j\in J}$, and the action of $[F_i]$ in this basis has integer non-negative coefficients:

$$[F_i(L_j)] = \sum d_{ij}^k [L_k],$$

with d_{ij}^k being the multiplicity $m_k(F_i(L_j))$. Via the isomorphism φ we obtain a distinguished basis $\mathbf{b} = \{b_j\}_{j\in J}$ of B, and

$$a_ib_j = \sum d_{ij}^k b_k. \tag{2}$$

Conversely, we could fix a basis $\mathbf{b}$ of B with a positivity constraint for the action of A. as in (2). Then our definition of a categorification of $(A, \mathbf{a}, B)$ can be amended to a similar definition of a categorification of $(A, \mathbf{a}, B, \mathbf{b})$, with the additional data being the fixed basis $\mathbf{b}$. Ideally the basis $\mathbf{b}$ corresponds then via the isomorphism φ to a basis $[M_j]_{j\in J}$ for certain objects

$M_j \in \mathcal{B}$. Varying the choice of basis might give rise to an interesting combinatorial interplay between several, maybe less prominent than $[L_j]_{j\in J}$ but more interesting, families $\{M_j\}_{j\in J}$ of objects in $\mathcal{B}$. Typical examples of such an interplay can be found in [13], [31, Section 5].

Of course, any such data $(A, \mathbf{a}, B, \mathbf{b})$ admits a rather trivial categorification, via a semisimple category $\mathcal{B}$. Namely, choose a field $\Bbbk$ and denote by $\Bbbk-$vect the category of finite-dimensional $\Bbbk$-vector spaces. Let

$$\mathcal{B} = \bigoplus_{j\in J} \Bbbk-\mathrm{vect}$$

be the direct sum of categories $\Bbbk-$vect, one for each basis vector of B. The category $\mathcal{B}$ is semisimple, with simple objects L_j enumerated by elements of J, and

$$\mathrm{Hom}_{\mathcal{B}}(L_j, L_k) = \begin{cases} \Bbbk & \text{if } \ j = k, \\ 0 & \text{otherwise.} \end{cases}$$

We identify $K(\mathcal{B})$ with B by mapping $[L_j]$ to b_j The functors F_i are determined by their action on simple objects, hence, given (2), we can define

$$F_i(L_j) = \bigoplus_{k\in J} L_k^{d_{ij}^k}$$

and obtain a categorification of $(A, \mathbf{a}, B, \mathbf{b})$. With few exceptions, semisimple categorifications bring little or no new structure into play, and we will ignore them. More interesting instances of categorifications occur for non-semisimple categories $\mathcal{B}$. Here is a sample list.

The Weyl algebra

1 Let $\mathcal{A}_1$ be the first *Weyl algebra* (the algebra of polynomial differential operators in one variable) with integer coefficients,

$$\mathcal{A}_1 = \mathbb{Z}\langle x, \partial\rangle / (\partial x - x\partial - 1).$$

One takes $\{x^i\partial^j\}_{i,j\geq 0}$ as the basis $\mathbf{a}$ of $\mathcal{A}_1$. The $\mathbb{Z}$-lattice $B \subset \mathbb{Q}[x]$ with the basis $\mathbf{b} = \{\frac{x^n}{n!}\}_{n\geq 0}$ is an $\mathcal{A}_1$-module.

To categorify this data we consider the category $\mathcal{B} = \bigoplus_{n\geq 0} R_n-$mod, i.e. the direct sum of the categories of finite-dimensional modules over the nilCoxeter

$\Bbbk$-algebra R_n. The latter has generators $Y_1, \dots, Y_{n-1}$ subject to relations

$$\begin{aligned} Y_i^2 &= 0, \\ Y_i Y_j &= Y_j Y_i \quad \text{if} \quad |i-j| > 1, \\ Y_i Y_{i+1} Y_i &= Y_{i+1} Y_i Y_{i+1}. \end{aligned}$$

The algebra R_n has a unique, up to isomorphism, finite dimensional simple module L_n, and $K(R_n\text{−mod}) \cong \mathbb{Z}$. The Grothendieck group $K(\mathcal{B})$ is naturally isomorphic to the $\mathcal{A}_1$-module B, via the isomorphism φ which maps $[L_n]$ to $\frac{x^n}{n!}$. The endofunctors X, D in $\mathcal{B}$ that lift the action of x and ∂ on B are the induction and restriction functors for the inclusion of algebras $R_n \subset R_{n+1}$. Basis elements lift to functors $X^i D^j$, and the isomorphisms (C-II) of definition 1 are induced by an isomorphism of functors $DX \cong XD \oplus \mathrm{Id}$ which lifts the defining relation $\partial x = x\partial + 1$ in $\mathcal{A}_1$. A detailed analysis of this categorification can be found in [46].

The symmetric group

2. The *regular representation* of the group ring $\mathbb{Z}[S_n]$ of the symmetric group S_n has a categorification via projective functors acting on a regular block of the highest weight BGG category $\mathcal{O}$ from [12] for $\mathfrak{sl}_n$. (For an introduction to the representation theory of semisimple Lie algebras we refer to [36]).

To define category $\mathcal{O}$, start with the standard triangular decomposition $\mathfrak{sl}_n = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$, where the first and the last terms are the Lie algebras of strictly upper-triangular (resp. lower-triangular) matrices, while $\mathfrak{h}$ is the algebra of traceless diagonal matrices. The highest weight category $\mathcal{O}$ of $\mathfrak{sl}_n$ is the full subcategory of the category of finitely-generated $\mathfrak{sl}_n$-modules consisting of $\mathfrak{h}$-diagonalize (possibly infinite dimensional) modules on which $U(\mathfrak{n}_+)$ acts locally-nilpotently. Thus, any $M \in \mathcal{O}$ decomposes as

$$M = \bigoplus_{\lambda \in \mathfrak{h}^*} M_\lambda,$$

where $hx = \lambda(h)x$ for any $h \in \mathfrak{h}$ and $x \in M_\lambda$. Here $\mathfrak{h}^*$ is the dual vector space of $\mathfrak{h}$, its elements are called weights.

The one-dimensional modules $\underline{\mathbb{C}}_\lambda = \mathbb{C}v_\lambda$ over the positive Borel subalgebra $\mathfrak{b}_+ = \mathfrak{n}_+ \oplus \mathfrak{h}$ are classified by elements λ of $\mathfrak{h}^*$. The subalgebra $\mathfrak{n}_+$ acts trivially on v_λ, while $hv_\lambda = \lambda(h)v_\lambda$ for $h \in \mathfrak{h}$.

The *Verma module* $M(\lambda)$ is the $\mathfrak{sl}_n$-module induced from the $\mathfrak{b}_+$-module $\underline{\mathbb{C}}_\lambda$,

$$M(\lambda) = U(\mathfrak{sl}_n) \otimes_{U(\mathfrak{b}_+)} \underline{\mathbb{C}}_\lambda.$$

The Verma module $M(\lambda)$ has a unique simple quotient, denoted $L(\lambda)$, and any simple object of $\mathcal{O}$ is isomorphic to $L(\lambda)$ for some $\lambda \in \mathfrak{h}^*$.

We call a weight λ *positive integral* if $\langle \lambda, \alpha \rangle \in \mathbb{Z}_{\geq 0}$ for any positive simple root $\alpha \in \mathfrak{h}^*$. The representation $L(\lambda)$ is finite-dimensional if and only if λ is a positive integral weight.

Although most of the objects in $\mathcal{O}$ are infinite dimensional vector spaces, every object M of $\mathcal{O}$ has finite length, i.e. there is an increasing filtration by subobjects $0 = M^0 \subset M^1 \subset \cdots \subset M^m = M$ such that the subsequent quotients M^{i+1}/M^i are isomorphic to simple objects, hence have the form $L(\lambda)$ (where λ may vary). The Grothendieck group of $\mathcal{O}$ is thus a free abelian group with generators $[L(\lambda)]$ for $\lambda \in \mathfrak{h}^*$.

It turns out that $\mathcal{O}$ has enough projective objects: given M there exists a surjection $P \twoheadrightarrow M$ with a projective $P \in \mathcal{O}$. Moreover, isomorphism classes of indecomposable projective objects are enumerated by elements of $\mathfrak{h}^*$. The indecomposable projective object $P(\lambda)$ is determined by the property of being projective and

$$\mathrm{Hom}_{\mathcal{O}}(P(\lambda), L(\mu)) = \begin{cases} \mathbb{C} & \text{if } \lambda = \mu, \\ 0 & \text{otherwise.} \end{cases}$$

We should warn the reader that the $P(\lambda)$'s are not projective when viewed as objects of the category of *all* $\mathfrak{sl}_n$-modules, while the $L(\lambda)$'s remain simple in the latter category.

The symmetric group S_n, the Weyl group of $\mathfrak{sl}_n$, acts naturally on $\mathfrak{h}$ by permuting the diagonal entries and then also on $\mathfrak{h}^*$. Let $\rho \in \mathfrak{h}^*$ be the half-sum of positive roots. In the study of the category $\mathcal{O}$ an important role is played by the shifted (dot) action of S_n,

$$w \cdot \lambda = w(\lambda + \rho) - \rho.$$

Two simple modules $L(\lambda), L(\lambda')$ have the same central character (i.e. are annihilated by the same maximal ideal of the center of the universal enveloping algebra) if and only if λ and λ' belong to the same S_n-orbit under the shifted action. Consequently, $\mathcal{O}$ decomposes into a direct sum of categories

$$\mathcal{O} = \bigoplus_{\nu \in \mathfrak{h}^*/S_n} \mathcal{O}_\nu \tag{3}$$

indexed by orbits ν of the shifted action of S_n on $\mathfrak{h}^*$. Here, $\mathcal{O}_\nu$ consists of all modules with composition series having only simple subquotients isomorphic to $L(\lambda)$ for $\lambda \in \nu$. There is no interaction between $\mathcal{O}_\nu$ and $\mathcal{O}_{\nu'}$ for different orbits ν, ν'. More accurately, if $\nu \neq \nu'$ then $\mathrm{Ext}^i_{\mathcal{O}}(M, M') = 0$ for any $i \geq 0$, $M \in \mathcal{O}_\nu$ and $M' \in \mathcal{O}_{\nu'}$.

Furthermore, each $\mathcal{O}_\nu$ is equivalent to the category of finite-dimensional modules over some finite-dimensional $\mathbb{C}$-algebra A_ν. Here's the catch, though: *explicitly* describing A_ν for $n > 3$ and interesting ν is very hard, see [79]. For an *implicit* description we just form $P = \bigoplus_{\lambda \in \nu} P(\lambda)$, the direct sum of all indecomposable projectives over $\lambda \in \nu$. Then $A_\nu \cong \mathrm{Hom}_{\mathcal{O}}(P, P)^{op}$.

An orbit ν (for the shifted action) is called *generic* if $w \cdot \lambda - \lambda$ is never integral, for $\lambda \in \nu$ and $w \in S_n, w \neq 1$. For a generic orbit ν, the category $\mathcal{O}_\nu$ is boring and equivalent to the direct sum of $n!$ copies of the category of finite-dimensional $\mathbb{C}$-vector spaces, one for each $\lambda \in \nu$. For such λ we have $P(\lambda) = M(\lambda) = L(\lambda)$, i.e. the Verma module with the highest weight λ is simple as well as projective in $\mathcal{O}$.

We call an orbit *integral* if it is a subset of the integral weight lattice in $\mathfrak{h}^*$. In [76] it is shown that $\mathcal{O}_\nu$ for non-integral ν reduces to those for integral weights. From now on we therefore assume that ν is integral. Then the category $\mathcal{O}_\nu$ is indecomposable (unlike in the generic case). Moreover, the complexity of $\mathcal{O}_\nu$ only depends on the type of the orbit. If two orbits ν and ν' contain points $\lambda \in \nu$, $\lambda' \in \nu'$ with identical stabilizers, then the categories $\mathcal{O}_\nu$ and $\mathcal{O}_{\nu'}$ are equivalent, see [11], [76]. If the stabilizer of ν under the shifted action is trivial, the category $\mathcal{O}_\nu$ is called a *regular block*. Regular blocks are the most complicated indecomposable direct summands of $\mathcal{O}$, for instance in the sense of having the maximal number of isomorphism classes of simple modules.

There is a natural bijection between the following three sets: positive integral weights, isomorphism classes of irreducible finite-dimensional representations of $\mathfrak{sl}_n$, and regular blocks of $\mathcal{O}$ for $\mathfrak{sl}_n$. A positive integral weight λ is the highest weight of an irreducible finite-dimensional representation $L(\lambda)$, determined by the weight uniquely up to isomorphism. In turn, $L(\lambda)$ belongs to the regular block $\mathcal{O}_\nu$, where $\nu = S_n \cdot \lambda$ is the orbit of λ.

Any two regular blocks of $\mathcal{O}$ are equivalent as categories, as shown in [40]. For this reason, we can restrict our discussion to the uniquely defined regular block which contains the one-dimensional trivial representation $L(0)$ of $\mathfrak{sl}_n$. We denote this block by $\mathcal{O}_0$. It has $n!$ simple modules $L(w) = L(w \cdot 0)$, enumerated by all permutations $w \in S_n$ (with the identity element e of S_n corresponding to $L(0)$ which is the only finite dimensional simple module in $\mathcal{O}_0$). Thus, $K(\mathcal{O}_0)$ is free abelian of rank $n!$ with basis $\{[L(w)]\}_{w\in S_n}$. Other notable objects in $\mathcal{O}_0$ are the Verma modules $M(w) = M(w \cdot 0)$ and the indecomposable projective modules $P(w) = P(w \cdot 0)$, over all $w \in S_n$. The sets $\{[M(w)]\}_{w\in S_n}$ and $\{[P(w)]\}_{w\in S_n}$ form two other prominent bases in $K(\mathcal{O}_0)$. For the set $\{[M(w)]\}_{w\in S_n}$ this is easy to see, because the transformation matrix between Verma modules and simple modules is upper triangular with ones on the diagonal. For the set $\{[P(w)]\}_{w\in S_n}$ this claim is not obvious and relates to the fact that $\mathcal{O}_0$ has finite homological dimension, see [12].

Equivalences between regular blocks are established by means of translation functors. First note that we can tensor two $U(\mathfrak{sl}_n)$-modules over the ground field. If V is a *finite-dimensional* $\mathfrak{sl}_n$-module it follows from the definitions that $V \otimes M$ lies in $\mathcal{O}$ whenever M is in $\mathcal{O}$. Hence, tensoring with V defines an endofunctor $V \otimes \_$ of the category $\mathcal{O}$. Taking direct summands of the functors $V \otimes \_$ provides a bewildering collection of different functors and allows one to analyze $\mathcal{O}$ quite deeply. By definition, a *projective functor* is any endofunctor of $\mathcal{O}$ isomorphic to a direct summand of $V \otimes \_$ for some finite-dimensional $\mathfrak{sl}_n$-module V. Projective functors were classified by J. Bernstein and S. Gelfand [11]. *Translation functors* are special cases of projective functors—they are direct summands of projective functors obtained by first restricting to a certain block and afterwards also projecting onto a fixed block.

Let us restrict our discussion to projective endofunctors in the regular block $\mathcal{O}_0$. Each projective endofunctor $\mathcal{O}_0 \longrightarrow \mathcal{O}_0$ decomposes into a finite direct sum of indecomposable functors θ_w, enumerated by permutations w and determined by the property $\theta_w(M(e)) \cong P(w)$. We have $P(e) = M(e)$ and the functor θ_e is the identity functor. The composition or the direct sum of two projective functors are again projective functors. With respect to these two operations, projective endofunctors on $\mathcal{O}_0$ are (up to isomorphism) generated by the projective functors $\theta_i := \theta_{s_i}$ corresponding to the simple transpositions/reflections $s_i = (i, i + 1)$. The functor θ_i is called *the translation through the i-th wall.* The functor θ_w is a direct summand of $\theta_{i_k} \dots \theta_{i_1}$, for any reduced decomposition $w = s_{i_1} \dots s_{i_k}$. The induced endomorphism $[\theta_i]$ of the Grothendieck group acts (in the basis given by Verma modules) by

$$[\theta_i][(M(w))] = [\theta_i(M(w))] = [M(w)] + [M(ws_i)].$$

Now we are prepared to explain the categorification. We first fix the unique isomorphism φ of groups

$$\begin{array}{rrcl} \varphi: & K(\mathcal{O}_0) & \longrightarrow & \mathbb{Z}[S_n] \\ & [M(w)] & \longmapsto & w \end{array}$$

and define $C_w := \varphi([P(w)])$. Then the C_w, $w \in W$, form a basis $\mathbf{a}$ of $\mathbb{Z}[S_n]$.

The action of $[\theta_i]$ corresponds under φ to the endomorphism of $\mathbb{Z}[S_n]$ given by right multiplication with $C_{s_i} := 1 + s_i$.

The defining relations of the generators $1+s_i$ in $\mathbb{Z}[S_n]$ lift to isomorphisms of functors as follows

$$\begin{aligned} \theta_i^2 &\cong \theta_i \oplus \theta_i, \\ \theta_i\theta_j &\cong \theta_j\theta_i \text{ if } |i-j| > 1, \\ \theta_i\theta_{i+1}\theta_i \oplus \theta_{i+1} &\cong \theta_{i+1}\theta_i\theta_{i+1} \oplus \theta_i. \end{aligned}$$

Here, the last isomorphism follows from the existence of decompositions of functors

$$\begin{aligned} \theta_i\theta_{i+1}\theta_i &\cong \theta_{w_1} \oplus \theta_i, \\ \theta_{i+1}\theta_i\theta_{i+1} &\cong \theta_{w_1} \oplus \theta_{i+1}, \end{aligned}$$

where $w_1 = s_is_{i+1}s_i = s_{i+1}s_is_{i+1}$. In particular, $[\theta_{w_1}]$ corresponds under φ to the right multiplication with $1 + s_i + s_{i+1} + s_is_{i+1} + s_{i+1}s_i + s_is_{i+1}s_i$.

By the classification theorem of projective functors, the endomorphism $[\theta_w]$, $w \in W$, corresponds then to right multiplication with the element C_w. From this one can then actually deduce that the multiplication in the basis $\mathbf{a}$ has non-negative integral coefficients

$$C_wC_{w'} = \sum_{w''} c_{w,w'}^{w''} C_{w''}, \qquad c_{w,w'}^{w''} \in \mathbb{Z}_{\geq 0}. \tag{4}$$

Hence we are in the situation of (1) and are looking for an abelian categorification of $(\mathbb{Z}[S_n], \mathbf{a}, \mathbb{Z}[S_n])$. We already have the isomorphism φ and the exact endofunctor θ_w corresponding to the generator C_w satisfying condition (C-I).

The composition of two projective functors decomposed as a direct sum of indecomposable functors $[\theta_w]$, $w \in W$, has nonnegative integral coefficients, and the equations (4) turn into isomorphisms of functors

$$\theta_w\theta_{w'} \cong \bigoplus_{w''} (\theta_{w''})^{c_{ww'}^{w''}}, \qquad c_{ww'}^{w''} \in \mathbb{Z}_{\geq 0}, \tag{5}$$

It turns out that each $[\theta_w]$ acts by a multiplication with a linear combination of y's for $y \leq w$. Moreover, all coefficients are nonnegative integers. For instance, if $w \in S_4$ then $[\theta_w] \doteq \sum_{y \leq w} y$, with two exceptions:

$$\begin{array}{lcll} [\theta_w] & \doteq & \sum_{y\leq w} y + 1 + s_2, & w = s_2 s_1 s_3 s_2, \\ [\theta_w] & \doteq & \sum_{y\leq w} y + 1 + s_1 + s_3 + s_1 s_3, & w = s_1 s_3 s_2 s_1 s_3. \end{array}$$

We can summarize the above results into a theorem.

Theorem 2 *The action of the indecomposable projective functors $\theta_w, w \in S_n$, on the block $\mathcal{O}_0$ for $\mathfrak{sl}_n$ categorifies the right regular representation of the integral group ring of the symmetric group S_n (in the basis* **a** *of the elements C_w, $w \in S_n$).*

This theorem is due to Bernstein and Gelfand, see [11], where it was stated in different terms, since the word "categorification" was not in the mathematician's vocabulary back then. In fact, Bernstein and Gelfand obtained a more general result by considering any simple Lie algebra $\mathfrak{g}$ instead of $\mathfrak{sl}_n$ and its Weyl group W in place of S_n.

In the explanation to the theorem we did not give a very explicit description of the basis **a** due to the fact that there is no explicit (closed) formula for the elements C_w available. However, the elements C_w can be obtained by induction (on the length of w) using the Kazhdan-Lusztig theory [41], [42]. The Kazhdan-Lusztig theory explains precisely the complicated interplay between the basis **a** and the standard basis of $\mathbb{Z}[S_n]$.

3. Parabolic blocks of $\mathcal{O}$ categorify representations of the symmetric group S_n *induced from the sign representation of parabolic subgroups.*

Let $\mu = (\mu_1, \dots, \mu_k), \mu_1 + \cdots + \mu_k = n$, be a composition of n and $\lambda = (\lambda_1, \dots, \lambda_k)$ the corresponding partition. In other words, λ is a permutation of the sequence μ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$. Denote by p_μ the subalgebra of $\mathfrak{sl}_n$ consisting of μ-block upper-triangular matrices. Consider the full subcategory $\mathcal{O}^\mu$ of $\mathcal{O}$ which consists of all modules M on which the action of $U(p_\mu)$ is locally finite. The category $\mathcal{O}^\mu$ is an example of a *parabolic* subcategory of $\mathcal{O}$, introduced in [70]. A simple object $L(\lambda)$ of $\mathcal{O}$ belongs to $\mathcal{O}^\mu$ if and only if the weight λ is positive integral with respect to all roots of the Lie algebra p_μ. The two extreme cases are $\mu = (1, 1, \dots, 1)$, in which case

$\mathcal{O}^\mu$ is all of $\mathcal{O}$, and $\mu = (n)$, for $\mathcal{O}^{(n)}$ is the semisimple category consisting exactly of all finite-dimensional $\mathfrak{sl}_n$-modules.

The direct sum decomposition (3) induces a similar decomposition of the parabolic category:

$$\mathcal{O}^\mu \cong \bigoplus_{\nu \in \mathfrak{h}^*/S_n} \mathcal{O}^\mu_\nu.$$

Each category $\mathcal{O}^\mu_\nu$ is either trivial (i.e. contains only the zero module) or equivalent to the category of finite-dimensional modules over some finite-dimensional $\mathbb{C}$-algebra (but describing this algebra explicitly for interesting μ and ν is a hard problem, see [18]). Unless $\mu = (1^n)$, for generic ν the summand $\mathcal{O}^\mu_\nu$ is trivial. Again, the most complicated summands are the $\mathcal{O}^\mu_\nu$ where the orbit ν contains a dominant regular integral weight. Translation functors establish equivalences between such summands for various such ν, and allow us to restrict our consideration to the block $\mathcal{O}^\mu_0$ corresponding to the (shifted) orbit through 0. The inclusion

$$\mathcal{O}^\mu_0 \subset \mathcal{O}_0$$

is an exact functor and induces an inclusion of Grothendieck groups

$$K(\mathcal{O}^\mu_0) \subset K(\mathcal{O}_0). \tag{6}$$

Indeed, the Grothendieck group of $\mathcal{O}_0$ is free abelian with generators $[L(w)]$, $w \in S_n$. A simple module $L(w)$ lies in $\mathcal{O}^\mu_0$ if and only if w is a minimal left coset representative for the subgroup S_μ of S_n (we informally write $w \in (S_\mu \backslash S_n)_{short}$). The Grothendieck group of $\mathcal{O}^\mu_0$ is then the subgroup of $K(\mathcal{O}_0)$ generated by such $L(w)$.

The analogues of the Verma modules in the parabolic case are the so-called parabolic Verma modules

$$M(p_\mu, V) = U(\mathfrak{sl}_n) \otimes_{U(p_\mu)} V,$$

where V is a finite-dimensional simple p_μ-module. The module $M(p_\mu, V)$ is a homomorphic image of some ordinary Verma module from $\mathcal{O}$, in particular, it has a unique simple quotient isomorphic to some $L(w)$ for some unique $w \in S_n$. In this way we get a canonical bijection between parabolic Verma modules in $\mathcal{O}^\mu_0$ and the set $(S_\mu \backslash S_n)_{short}$ of shortest coset representatives. Hence it is convenient to denote the parabolic Verma module with simple quotient $L(w)$, $w \in (S_\mu \backslash S_n)_{short}$, simply by $M^\mu(w)$.

Generalized Verma modules provide a basis for the Grothendieck group of $\mathcal{O}_0^\mu$. Under the inclusion (6) of Grothendieck groups the image of the generalized Verma module $M^\mu(w)$ is the alternating sum of Verma modules, see [70] and [59]:

$$[M^\mu(w)] = \sum_{u\in S_\mu} (-1)^{l(u)}[M(uw)]. \tag{7}$$

Since the projective endofunctors θ_w preserve $\mathcal{O}_0^\mu$, the inclusion (6) is actually an inclusion of S_n-modules, and, in view of the formula (7), we can identify $K(\mathcal{O}_0^\mu)$ with the submodule I_μ^- of the regular representation of S_n isomorphic to the representation induced from the sign representation of S_μ,

$$I_\mu^- \cong \mathrm{Ind}_{S_\mu}^{S_n}\mathbb{Z}v,$$

where we denoted by $\mathbb{Z}v$ the sign representation, so that $wv = (-1)^{l(w)}v$ for $w \in S_\mu$.

To summaries, we have:

Theorem 3 *The action of the projective functors θ_w, $w \in W$, on the parabolic subcategory $\mathcal{O}_0^\mu$ of $\mathcal{O}$ categorifies the induced representation I_μ^- of the integral group ring of the symmetric group S_n (with basis $\mathbf{a} = \{C_w\}_{w\in S_n}$).*

As in the previous example the Grothendieck group $K(\mathcal{O}_0^\mu)$ has three distinguished basis, given by simple objects, projective objects, and parabolic Verma modules respectively.

Remark: If we choose a pair μ, μ' of decompositions giving rise to the same partition λ of n, then the modules I_μ^- and $I_{\mu'}^-$ are isomorphic, and will be denoted I_λ^-. However, the categories $\mathcal{O}^\mu$ and $\mathcal{O}^{\mu'}$ are not equivalent in general, which means the two categorifications of the induced representation I_λ^- are also not equivalent. This problem disappears if we leave the world of abelian categorifications, since the derived categories $D^b(\mathcal{O}^\mu)$ and $D^b(\mathcal{O}^{\mu'})$ are equivalent [48]. The equivalence is based on the geometric description of $\mathcal{O}^\mu$ and $\mathcal{O}^{\mu'}$ in terms of complexes of sheaves on partial flag varieties.

4. Self-dual projectives in a parabolic block categorify *irreducible representations* of the symmetric group.

Let I_μ be the representation of $\mathbb{Z}[S_n]$ induced from the trivial representation of the subgroup S_μ. Up to isomorphism, it only depends on the partition

λ associated with μ. Partitions of n naturally index the isomorphism classes of irreducible representations of S_n over any field of characteristic zero (we use $\mathbb{Q}$ here). Denote by $\mathcal{S}_{\mathbb{Q}}(\lambda)$ the irreducible (Specht) module associated with λ. It is an irreducible representation defined as the unique common irreducible summand of $I_\lambda \otimes \mathbb{Q}$ and $I^-_{\lambda^*} \otimes \mathbb{Q}$, where λ^* is the dual partition of λ. Passing to duals, we see that $\mathcal{S}_{\mathbb{Q}}(\lambda^*)$ is the unique common irreducible summand of $I_{\lambda^*} \otimes \mathbb{Q}$ and $I^-_\lambda \otimes \mathbb{Q}$.

We have already categorified the representation I^-_λ (in several ways) via the parabolic categories $\mathcal{O}^\mu_0$, where μ is any decomposition for λ. It's natural to try to realize a categorification of some integral lift $\mathcal{S}(\lambda^*)$ of the irreducible representation $\mathcal{S}_{\mathbb{Q}}(\lambda^*)$ via a suitable subcategory of some $\mathcal{O}^\mu_0$ stable under the action of projective endofunctors.

The correct answer, presented in [50], is to pass to a subcategory generated by those projective objects in $\mathcal{O}^\mu_0$ which are also injective. Note that these modules are neither projective nor injective in $\mathcal{O}_0$ (unless if $\mathcal{O}^\mu_o = \mathcal{O}_0$).

Any projective object in $\mathcal{O}^\mu_0$ is isomorphic to a direct sum of indecomposable projective modules $P^\mu(w)$, for $w \in (S_\mu \backslash S_n)_{short}$. Let $J \subset (S_\mu \backslash S_n)_{short}$ be the subset indexing indecomposable projectives modules that are also injective: $w \in J$ if and only if $P^\mu(w)$ is injective. Projective endofunctors θ_w, $w \in S_n$, take projectives to projectives and injectives to injectives. Therefore, they take projective-injective modules (modules that are both projective and injective, also called self-dual projective, for instance, in Irving [37]) to projective-injective modules.

The category of projective-injective modules is additive, not abelian. To remedy this, consider the full subcategory $\mathcal{C}^\mu$ of $\mathcal{O}^\mu_0$ consisting of modules M admitting a resolution

$$P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0 \tag{8}$$

with projective-injective P_1 and P_0. The category $\mathcal{C}^\mu$ is abelian and stable under all endofunctors θ_w for $w \in S_n$, see [50].

Irving [37] classified projective-injective modules in $\mathcal{O}^\mu_0$. His results were interpreted in [50] in the language of categorification:

Theorem 4 *The action of the projective endofunctors θ_w, $w \in S_n$, on the abelian category $\mathcal{C}^\mu$ categorifies (after tensoring the Grothendieck group with $\mathbb{Q}$ over $\mathbb{Z}$) the irreducible representation $\mathcal{S}_{\mathbb{Q}}(\lambda^*)$ of the symmetric group S_n.*

The Grothendieck group $K(\mathcal{C}^\mu)$ is a module over the integral group ring of S_n, with s_i acting by $[\theta_i] - \mathrm{Id}$, and the theorem says that $K(\mathcal{C}^\mu) \otimes_{\mathbb{Z}} \mathbb{Q}$ is an irreducible representation of the symmetric group corresponding to the partition λ^*. Several explicit examples of categorifications via $\mathcal{C}^\mu$ will be given in Section 3.

Remark: Suppose μ and ν are two decompositions of the same partition λ. It's shown in [65] (Theorem 5.4.(2)) that the categories $\mathcal{C}^\mu$ and $\mathcal{C}^\nu$ are equivalent, through an equivalence which commutes with the action of the projective functors θ_u on these categories (the equivalence is given by a non-trivial composition of derived Zuckerman functors). Therefore, the categorification of $\mathcal{S}(\lambda^*)$ does not depend on the choice of the decomposition μ that represents λ, and we can denote the category $\mathcal{C}^\mu$ by $\mathcal{C}^\lambda$. (This should be compared with the remark after Theorem 3.)

Remark: Theorem 3 and Theorem 4 can be generalized to arbitrary semisimple complex finite-dimensional Lie algebras, see [66]. However, in the general case Theorem 4 does not categorify simple modules for the corresponding Weyl group but rather the Kazhdan-Lusztig cell modules from [41]. This can be used to describe the so-called "rough" structure of generalized Verma modules, which shows that "categorification theoretic" ideas can lead to new results in representation theory.

Remark: The inclusion of categories $\mathcal{C}^\mu \subset \mathcal{O}_0^\mu$ is not an exact functor; however, it is a part of a very natural filtration of the category $\mathcal{O}_0^\mu$ which can be defined using the Gelfand-Kirillov dimension of modules, see [66, 6.9]. To get the inclusion of Grothendieck groups analogous to the inclusion of representations from the irreducible Specht module into the induced sign representation, we pass to the subgroup $K'(\mathcal{C}^\mu)$ of $K(\mathcal{C}^\mu)$ generated by the images of projective modules in $\mathcal{C}^\mu$. This additional technicality is necessary as the category $\mathcal{C}^\mu$ does not have finite homological dimension in general. The subgroup $K'(\mathcal{C}^\mu)$ is always a finite index subgroup, stable under the action of the $[\theta_w]$'s. We denote this subgroup by $\mathcal{S}'(\lambda^*)$:

$$\mathcal{S}'(\lambda^*) \stackrel{\text{def}}{=} K'(\mathcal{C}^\mu) \subset K(\mathcal{C}^\mu) \cong \mathcal{S}(\lambda^*).$$

The inclusion of categories $\mathcal{C}^\mu \subset \mathcal{O}_0^\mu$ induces the inclusion

$$\mathcal{S}'(\lambda^*) \subset K(\mathcal{O}_0^\mu) \cong I_\mu^-$$

of $\mathbb{Z}[S_n]$-modules, hence realising the integral lift $\mathcal{S}'(\lambda^*)$ of the Specht module as a subrepresentation of I_μ^-.

5. Categorification of the *induced representations* I_μ via projectively presentable modules.

Let $\mathcal{P}^\mu$ denote the category of all modules M admitting a resolution (8) in which each indecomposable direct summand of both P_0 and P_1 has the form $P(w)$, where w is a longest left coset representative for S_μ in S_n (we will write $w \in (S_\mu \backslash S_n)_{long}$). Such modules are called p_μ-presentable modules, see [62]. As in the previous example, the category $\mathcal{P}^\mu$ is stable under all endofunctors θ_w, $w \in S_n$.

By definition, $\mathcal{P}^\mu$ is a subcategory of $\mathcal{O}_0$, but just as in the example above, the natural inclusion functor is not exact. The category $\mathcal{P}^\mu$ does not have finite homological dimension in general, so we again pass to the subgroup $K'(\mathcal{P}^\mu)$ of $K(\mathcal{P}^\mu)$, generated by the images of indecomposable projective modules in $\mathcal{P}^\mu$. The latter are (up to isomorphism) the $P(u)$, $u \in (S_\mu \backslash S_n)_{long}$. This is a finite index subgroup, stable under the action of the $[\theta_w]$'s and we have the following statement proved in [62]:

Theorem 5 *The action of projective endofunctors on the abelian category $\mathcal{P}^\mu$ categorifies (after tensoring with $\mathbb{Q}$ over $\mathbb{Z}$) the induced representation $(I_\mu)_\mathbb{Q}$ of the group algebra of the symmetric group S_n (with the basis $\mathbf{a} = \{C_w\}_{w \in S_n}$).*

Consider the diagram of $\mathbb{Q}[S_n]$-modules

$$(I_\lambda)_\mathbb{Q} \xrightarrow{\iota_1} \mathbb{Q}[S_n] \xrightarrow{p_1} (I^-_{\lambda^*})_\mathbb{Q}.$$

The map ι_1 is the symmetrization inclusion map, while p_1 is the antisymmetrization quotient map. We have

$$\mathcal{S}_\mathbb{Q}(\lambda) \stackrel{\text{def}}{=} p_1 \iota_1((I_\lambda)_\mathbb{Q}).$$

The map ι_1 is categorified as the inclusion of $\mathcal{P}^\mu$ to $\mathcal{O}_0$. The map p_1 is categorified as the projection of $\mathcal{O}_0$ onto $\mathcal{O}_0^{\mu^*}$, where μ^* is some decomposition corresponding to λ^*. Unfortunately, the composition of the two functors categorifying these two maps will be trivial in general. To repair the situation we first project $\mathcal{P}^\mu$ onto the full subcategory of $\mathcal{P}^\mu$ given by simple objects of minimal possible Gelfand-Kirillov dimension. It is easy to see that the image category contains enough projective modules, and using the equivalence constructed in [66, Theorem I], these projective modules can be functorially mapped to projective modules in $\mathcal{C}^{\mu^*}$, where μ^* is a (good choice of) composition with associated partition λ^*. The latter category embeds into $\mathcal{O}_0^{\mu^*}$ as was explained in the previous example.

6. Categorification of the *group algebra* The representation theory of groups like $GL(n, \mathbb{C})$, considered as a real Lie group, naturally leads to the notion of Harish-Chandra bimodules. A Harish-Chandra bimodule over $\mathfrak{sl}_n$ is a finitely-generated module over the universal enveloping algebra $U(\mathfrak{sl}_n \times \mathfrak{sl}_n)$ which decomposes into a direct sum of finite-dimensional $U(\mathfrak{sl}_n)$-modules with respect to the diagonal copy $\{(X, -X)|X \in \mathfrak{sl}_n\}$ of $\mathfrak{sl}_n$. Let $\mathcal{HC}_{0,0}$ be the category of Harish-Chandra bimodules which are annihilated, on both sides, by some power of the maximal ideal I_0 of the center Z of $U(\mathfrak{sl}_n)$. Here I_0 is the annihilator of the trivial $U(\mathfrak{sl}_n)$-module considered as a Z-module. Thus, $M \in \mathcal{HC}_{0,0}$ if and only if $xM = 0 = Mx$ for all $x \in I_0^N$ for N large enough.

By [11, Section 5] there exists an exact and fully faithful functor

$$\mathcal{O}_0 \longrightarrow \mathcal{HC}_{0,0}.$$

Moreover, this functor induces an isomorphism of Grothendieck groups

$$K(\mathcal{O}_0) \cong K(\mathcal{HC}_{0,0}).$$

Since the former group is isomorphic to $\mathbb{Z}[S_n]$, we can identify the Grothendieck group of $\mathcal{HC}_{0,0}$ with $\mathbb{Z}[S_n]$ as well.

The advantage of bimodules is that we now have two sides and can tensor with a finite-dimensional $\mathfrak{sl}_n$-module both on the left and on the right. In either case, we preserve the category of Harish-Chandra bimodules. Taking all possible direct summands of these functors and restricting to endofunctors on the subcategory $\mathcal{HC}_{0,0}$ leads to two sets of commuting projective functors, $\{\theta_{r,w}\}_{w\in S_n}$ and $\{\theta_{l,w}\}_{w\in S_n}$ which induce endomorphisms on the Grothendieck group $K(\mathcal{HC}_{0,0}) \cong \mathbb{Z}[S_n]$ given by left and right multiplication with $\{C_w\}_{w\in S_n}$ respectively. Summarizing, we have

Theorem 6 *The action of the functors $\{\theta_{r,w}\}_{w\in S_n}$ and $\{\theta_{l,w}\}_{w\in S_n}$ on the category $\mathcal{HC}_{0,0}$ of Harish-Chandra bimodules for $\mathfrak{sl}_n$ with generalized trivial character on both sides categorifies $\mathbb{Z}[S_n]$, viewed as a bimodule over itself. The functors $\{\theta_{r,w}\}_{w\in S_n}$ induces the left multiplication with C_w on the*

Grothendieck group, whereas the functors $\{\theta_{l,w}\}_{w\in S_n}$ induces the right multiplication with C_w on the Grothendieck group,

The first half of the theorem follows at once from [11], the second half from [77]. The category of Harish-Chandra bimodules is more complicated than the category $\mathcal{O}$. For example, $\mathcal{HC}_{0,0}$ does not have enough projectives, and is not Koszul with respect to the natural grading, in contrast to $\mathcal{O}_0$ (for the Koszulity of $\mathcal{O}_0$ see [9]). The study of translation functors on Harish-Chandra modules goes back to Zuckerman [85].

Lie algebras

7. In the following we will mention several instances of categorifications of *modules over Lie algebras*. Our definition of categorification required an associative algebra rather than a Lie algebra, so one should think of this construction as a categorification of representations of the associated universal enveloping algebra.

Let V be the fundamental two-dimensional representation of the complex Lie algebra $\mathfrak{sl}_2$. Denote by $\{e, f, h\}$ the standard basis of $\mathfrak{sl}_2$. The n-th tensor power of V decomposes into a direct sum of weight spaces:

$$V^{\otimes n} = \bigoplus_{k=0}^{n} V^{\otimes n}(k),$$

where $hx = (2k-n)x$ for $x \in V^{\otimes n}(k)$.

A categorification of $V^{\otimes n}$ was constructed in [10]. The authors considered certain singular blocks $\mathcal{O}_{k,n-k}$ of the category $\mathcal{O}$ for $\mathfrak{sl}_n$. More precisely $\mathcal{O}_{k,n-k}$ is the choice of an integral block $\mathcal{O}_\nu$, where the stabilizer of ν is isomorphic to $S_k \times S_{n-k}$. Note that different choices give equivalent blocks. The Grothendieck group of this block has rank $\binom{n}{k}$ equal to the dimension of the weight space $V^{\otimes n}(k)$, and there are natural isomorphisms

$$K(\mathcal{O}_{k,n-k}) \otimes_{\mathbb{Z}} \mathbb{C} \cong V^{\otimes n}(k).$$

The Grothendieck group of the direct sum

$$\mathcal{O}_n = \bigoplus_{k=0}^{n} \mathcal{O}_{k,n-k}$$

is isomorphic to $V^{\otimes n}$ (after tensoring with $\mathbb{C}$ over $\mathbb{Z}$). Suitable translation functors $\mathcal{E}, \mathcal{F}$ in $\mathcal{O}_n$ lift the action of the generators e, f of $\mathfrak{sl}_2$ on $V^{\otimes n}$.

To make this construction compatible with Definition 1 one should switch to Lusztig's version $\dot{\mathbf{U}}$ of the universal enveloping algebra $U(\mathfrak{sl}_2)$ (see [60], [10]) and set $q = 1$. Instead of the unit element, the ring $\dot{\mathbf{U}}$ contains idempotents $1_n, n \in \mathbb{Z}$, which can be viewed as projectors onto integral weights. The Lusztig basis $\dot{\mathbb{B}}$ in $\dot{\mathbf{U}}$ has the positivity property required by Definition 1, and comes along with an integral version $V_{\mathbb{Z}}^{\otimes n}$ of the tensor power representation. The triple $(\dot{\mathbf{U}}, \dot{\mathbb{B}}, V_{\mathbb{Z}}^{\otimes n})$ is categorified using the above-mentioned category $\mathcal{O}_n$ and projective endofunctors of it. In fact, each element of $\dot{\mathbb{B}}$ either corresponds to an indecomposable projective endofunctor on $\mathcal{O}_n$ or acts by 0 on $V_{\mathbb{Z}}^{\otimes n}$. We refer the reader to [10] for details, to [27] for an axiomatic development of $\mathfrak{sl}_2$ categorifications, and to [10] and [81] for a categorification of the Temperley-Lieb algebra action on $V_{\mathbb{Z}}^{\otimes n}$ via projective endofunctors on the category Koszul dual to $\mathcal{O}_n$ (see [9] and [61] for details on Koszul duality).

The Lie algebra $\mathfrak{sl}_2$ has one irreducible $(n+1)$-dimensional representation V_n, for each $n \geq 0$ ($V_1 \cong V$, of course). A categorification of arbitrary tensor products $V_{n_1} \otimes \cdots \otimes V_{n_m}$ is described in [31]. This tensor product is a submodule of $V^{\otimes n}$, where $n = n_1 + \cdots + n_m$. Knowing that $\mathcal{O}_n$ categorifies $V^{\otimes n}$, we find a "subcategorification," a subcategory of $\mathcal{O}_n$ stable under the action of projective functors, with the Grothendieck group naturally isomorphic to $V_{n_1} \otimes \cdots \otimes V_{n_m}$. The subcategory has an intrinsic description via Harish-Chandra modules similar to the one from Example **5**.

8. A categorification of *arbitrary tensor products of fundamental representations* $\Lambda^i V$, where V is the k-dimensional $\mathfrak{sl}_k$-representation and $1 \leq i \leq k-1$ was found by J. Sussan [84], see also [64] for a more combinatorial approach. A tensor product $\Lambda^{i_1} V \otimes \cdots \otimes \Lambda^{i_r} V$ decomposes into weight spaces $\Lambda^{i_1} V \otimes \cdots \otimes \Lambda^{i_r} V(\nu)$, over various integral weights ν of $\mathfrak{sl}_k$. Each weight space becomes the Grothendieck group of a parabolic-singular block of the highest weight category for $\mathfrak{sl}_N$, where $N = i_1 + \cdots + i_r$. For the parabolic subalgebra one takes the Lie algebra of traceless $N \times N$ matrices which are $(i_1, \ldots, i_r)$ block upper-triangular. The choice of the singular block is determined by ν. For the precise dictionary how to determine ν we refer to [64].

Translation functors between singular blocks, restricted to the parabolic category, provide an action of the generators $\mathcal{E}_j$ and $\mathcal{F}_j$ of the Lie algebra

$\mathfrak{sl}_k$. Relations in the universal enveloping algebra lift to functor isomorphisms. Conjecturally, Sussan's categorification satisfies the framework of Definition 1 above, with respect to Lusztig's completion $\dot{\mathbf{U}}$ of the universal enveloping algebra of $\mathfrak{sl}_k$ and Lusztig's canonical basis there.

9. Ariki, in a remarkable paper [2], categorified all *finite-dimensional irreducible representations* of $\mathfrak{sl}_m$, for all m, as well as integrable irreducible representations of affine Lie algebras $\widehat{\mathfrak{sl}}_r$. Ariki considered certain finite-dimensional quotient algebras of the affine Hecke algebra $\widehat{H}_{n,q}$, known as Ariki-Koike cyclotomic Hecke algebras, which depend on a number of discrete parameters. He identified the Grothendieck groups of blocks of these algebras, for generic values of $q \in \mathbb{C}$, with the weight spaces $V_\lambda(\mu)$ of finite-dimensional irreducible representations

$$V_\lambda = \bigoplus_\mu V_\lambda(\mu)$$

of $\mathfrak{sl}_m$. Direct summands of the induction and restriction functors between cyclotomic Hecke algebras for n and $n+1$ act on the Grothendieck group as generators e_i and f_i of $\mathfrak{sl}_m$.

Specializing q to a primitive r-th root of unity, Ariki obtained a categorification of integrable irreducible representations of the affine Lie algebra $\widehat{\mathfrak{sl}}_r$.

We conjecture that direct summands of arbitrary compositions of Ariki's induction and restriction functors correspond to elements of the Lusztig canonical basis $\dot{\mathbb{B}}$ of Lusztig's completions $\dot{\mathbf{U}}$ of these universal enveloping algebras. This conjecture would imply that Ariki's categorifications satisfy the conditions of Definition 1.

Lascoux, Leclerc and Thibon, in an earlier paper [57], categorified *level-one irreducible* $\widehat{\mathfrak{sl}}_r$*-representations*, by identifying them with the direct sum of Grothendieck groups of finite-dimensional Hecke algebras $H_{n,q}$, over all $n \geq 0$, with q a primitive r-th root of unity. Their construction is a special case of Ariki's. We also refer the reader to related works [3], [33]. Categorifications of the adjoint representation and of irreducible $\mathfrak{sl}_m$-representations with highest weight $\omega_j + \omega_k$ are described explicitly in [34], [35] and [24].

Another way to categorify all irreducible finite-dimensional representations of $\mathfrak{sl}_m$, for all m, was found by Brundan and Kleshchev [15], via the representation theory of W-algebras. There is a good chance that their categorification is equivalent to that of Ariki, and that an equivalence of two categorifications can be constructed along the lines of Arakawa-Suzuki [1] and Brundan-Kleshchev [16].

Biadjointness. Definition 1 of (weak) categorifications was minimalistic. Categorifications in the above examples share extra properties, the most prominent of which is biadjointness: there exists an involution $a_i \to a_{i'}$ on the basis $\mathbf{a}$ of A such that the functor $F_{i'}$ is both left and right adjoint to F_i. This is the case in the examples **2** through **9**, while in example **1** the functors are almost biadjoint. Namely, the induction functor F_x lifting the action of x is left adjoint to the restriction functor F_∂ (which lifts the action of ∂) and right adjoint to F_∂ conjugated by an involution.

A conceptual explanation for the pervasiveness of biadjointness in categorifications is given by the presence of the Hom bifunctor in any abelian category. The Hom bifunctor in $\mathcal{B}$ descends to a bilinear form on the Grothendieck group B of $\mathcal{B}$, via

$$([M],[N]) \stackrel{\text{def}}{=} \dim \mathrm{Hom}_{\mathcal{B}}(M,N),$$

where M is projective or N is injective, and some standard technical conditions are satisfied. When a representation naturally comes with a bilinear form, the form is usually compatible with the action of A: there exists an involution $a \to a'$ on A such that $(ax,y) = (x,a'y)$ for $x,y \in B$. A categorification of this equality should be an isomorphism

$$\mathrm{Hom}(F_a M, N) \cong \mathrm{Hom}(M, F_{a'} N)$$

saying that the functor lifting the action of a' is right adjoint to the functor lifting the action of a. If the bilinear form is symmetric, we should have the adjointness property in the other direction as well, leading to biadjointness of F_a and $F_{a'}$.

A beautiful approach to $\mathfrak{sl}_2$ categorifications via biadjointness was developed by Chuang and Rouquier [27]. The role of biadjointness in TQFTs and their categorifications is clarified in [47, Section 6.3]. An example how the existence of a categorification with a bilinear form can be used to determine dimensions of hom-spaces can be found in [82].

Grading and q-deformation. In all of the above examples, the data $(A, \mathbf{a}, B)$ that is being categorified admits a natural q-deformation $(A_q, \mathbf{a}_q, B_q)$. Here A_q is a $\mathbb{Z}[q,q^{-1}]$-algebra, B_q an A_q-module, and $\mathbf{a}_q$ a basis of A_q. We assume that both A_q and B_q are free $\mathbb{Z}[q,q^{-1}]$-modules, that the multiplication in A_q in the basis $\mathbf{a}_q$ has all coefficients in $\mathbb{N}[q,q^{-1}]$, and that taking the quotient by the ideal $(q-1)$ brings us back to the original data:

$$A = A_q/(q-1)A_q, \quad B = B_q/(q-1)B_q, \quad \mathbf{a}_q \stackrel{q=1}{\longrightarrow} \mathbf{a}.$$

An automorphism τ of an abelian category $\mathcal{B}$ (more accurately, an invertible endofunctor on $\mathcal{B}$) induces a $\mathbb{Z}[q,q^{-1}]$-module structure on the Grothendieck group $K(\mathcal{B})$. Multiplication by q corresponds to the action of τ:

$$[\tau(M)] = q[M], \quad [\tau^{-1}(M)] = q^{-1}[M].$$

In many of the examples, $\mathcal{B}$ will be the category of graded modules over a graded algebra, and τ is just the functor which shifts the grading. To emphasize this, we denote τ by $\{1\}$ and its n-th power by $\{n\}$.

Definition 7 *A (weak) abelian categorification of $(A_q, \mathbf{a}_q, B_q)$ consists of an abelian category $\mathcal{B}$ equipped with an invertible endofunctor $\{1\}$, an isomorphism of $\mathbb{Z}[q,q^{-1}]$-modules $\varphi : K(\mathcal{B}) \stackrel{\sim}{\longrightarrow} B_q$ and exact endofunctors $F_i : \mathcal{B} \longrightarrow \mathcal{B}$ that commute with $\{1\}$ and such that the following hold*

(qC-I) F_i lifts the action of a_i on the module B_q, i.e. the action of $[F_i]$ on the Grothendieck group corresponds to the action of a_i on B_q, under the isomorphism ϕ, in the sense that the diagram below is commutative.

$$\begin{array}{ccc} K(\mathcal{B}) & \xrightarrow{[F_i]} & K(\mathcal{B}) \\ \varphi \downarrow & & \downarrow \varphi \\ B_q & \xrightarrow{a_i} & B_q \end{array}$$

(qC-II) There are isomorphisms of functors

$$F_i F_j \cong \bigoplus_k F_k^{c_{ij}^k},$$

i.e., the composition $F_i F_j$ decomposes as the direct sum of functors F_k with multiplicities $c_{ij}^k \in \mathbb{N}[q,q^{-1}]$

The graded versions are well-known in all of the examples above up to Example **8**. In Example **1** the nilCoxeter algebra R_n is naturally graded with $\deg(Y_i) = 1$. The inclusion $R_n \subset R_{n+1}$ induces induction and restriction functors between categories of graded R_n and R_{n+1}-modules. In the graded case, induction and restriction functors satisfy the isomorphism

$$DX \cong XD\{1\} \oplus \mathrm{Id}$$

which lifts the defining relation $\partial x = qx\partial + 1$ of the q-Weyl algebra (see [46] for more detail).

An accurate framework for graded versions of examples **2**–**8** is a rather complicated affair. To construct a canonical grading on a regular block of the highest weight category [9] requires étale cohomology, perverse sheaves [7], and the Beilinson-Bernstein-Brylinski-Kashiwara localization theorem [6], [19]. Soergel's approach to this grading is more elementary [76], [78], [77], but still relies on these hard results. Extra work is needed to show that translation or projective functors can be lifted to endofunctors in the graded category [80].

Ariki's categorification of irreducible integrable representations (Example **9** above) should admit a graded version as well.

3 Four examples of categorifications of irreducible representations

In the example **4** above we categorified an integral lift of the irreducible representation $\mathcal{S}_{\mathbb{Q}}(\lambda^*)$ of the symmetric group via the abelian category $\mathcal{C}^\lambda$ built out of projective-injective modules in a parabolic block of $\mathcal{O}$. The category $\mathcal{C}^\lambda$ is equivalent to the category of finite-dimensional representations over a finite-dimensional algebra A^λ, the algebra of endomorphisms of the direct sum of indecomposable projective-injective modules $P^\mu(w)$. Under this equivalence, projective functors θ_i turn into the functors of tensoring with certain A^λ-bimodules. It's not known how to describe A^λ and these bimodules explicitly, except in a few cases, four of which are discussed below.

a. The sign representation. The sign representation of the symmetric group (over $\mathbb{Z}$) is a free abelian group $\mathbb{Z}v$ on one generator v, with $s_i v = -v$

for all i. It corresponds to the partition (1^n) of n, which in our notation is λ^* for $\lambda = (n)$. The parabolic category $\mathcal{O}_0^{(n)}$ has as objects exactly the finite-dimensional modules from $\mathcal{O}_0$ since the parabolic subalgebra in this case is all of $\mathfrak{sl}_n$.

Actually, $\mathcal{O}_0$ has only one simple module with this property, the one-dimensional trivial representation $\mathbb{C}$. In our notation, this is the module $L(e)$, the simple quotient of the Verma module $M(e)$ assigned to the unit element of the symmetric group.

Consequently, any object of $\mathcal{O}_0^{(n)}$ is isomorphic to a direct sum of copies of $L(e)$, and the category is semisimple. Furthermore, the category $\mathcal{C}^{(n)}$ is all of $\mathcal{O}_0^{(n)}$. Thus, $\mathcal{C}^{(n)}$ is equivalent to the category of finite-dimensional $\mathbb{C}$-vector spaces. Projective functors θ_w act by zero on $\mathcal{C}^{(n)}$ for all $w \in S_n, w \neq e$, while θ_e is the identity functor.

The graded version $\mathcal{C}_{gr}^{(n)}$ is equivalent to the category of graded finite-dimensional $\mathbb{C}$-vector spaces. Again, projective functors $\theta_w, w \neq e$, act by zero, and θ_e is the identity functor.

Thus, our categorification of the sign representation is rather trivial.

b. The trivial representation. The trivial representation $\mathbb{Z}z$ of $\mathbb{Z}[S_n]$ is a free abelian group on one generator z, with the action $wz = z$, $w \in S_n$. The corresponding partition is (n), with the dual partition $\lambda = (1^n)$. Only one decomposition $\mu = (1^n)$ corresponds to the dual partition; the parabolic subalgebra associated with (1^n) is the positive Borel subalgebra, and the parabolic category $\mathcal{O}_0^{(1^n)}$ is all of $\mathcal{O}_0$.

The unique self-dual indecomposable projective P in $\mathcal{O}_0$ is usually called the big projective module. Its endomorphism algebra $\mathrm{End}_{\mathcal{O}}(P)$ is isomorphic to the cohomology ring H_n of the full flag variety Fl of $\mathbb{C}^n$, see [76].

The category $\mathcal{C}^{(1^n)}$ is equivalent to the category of finite-dimensional H_n-modules. The unique (up to isomorphism) simple H_n-module generates the Grothendieck group $K_0(\mathrm{H}-\mathrm{mod}) \cong \mathbb{Z}$.

To describe how the functors θ_i act on $\mathcal{C}^\lambda$ consider generalized flag varieties

$$\begin{aligned}\mathrm{Fl}_i &= \{0 \subset L_1 \subset L_2 \subset \cdots \subset L_{n-1} \subset \mathbb{C}^n, L'_i | \\ &\quad \dim(L_j) = j, \dim(L'_i) = i, L_{i-1} \subset L'_i \subset L_{i+1}\}.\end{aligned}$$

This variety is a $\mathbb{P}^1$-bundle over the full flag variety Fl in two possible ways, corresponding to forgetting L_i, respectively L'_i. These two maps from Fl_i onto

F induce two ring homomorphisms

$$\mathrm{H}_n = \mathrm{H}(\mathrm{Fl}, \mathbb{C}) \longrightarrow \mathrm{H}(\mathrm{Fl}_i, \mathbb{C})$$

which turn $\mathrm{H}(F_i, \mathbb{C})$ into an H_n-bimodule. The functor $\theta_i : \mathcal{C}^{(1^n)} \longrightarrow \mathcal{C}^{(1^n)}$ is given by tensoring with this H_n-bimodule (under the equivalence $\mathcal{C}^{(1^n)} \cong \mathrm{H}_n-\mathrm{mod}$).

To describe functors θ_w for an arbitrary $w \in S_n$, we recall that $\mathrm{Fl} = G/B$ where $G = SL(n, \mathbb{C})$ and B the Borel subgroup of G. The orbits of the natural left action of G on $\mathrm{Fl} \times \mathrm{Fl}$ are in natural bijection with elements of the symmetric group. Denote by O_w the orbit associated with w and by $\mathrm{IC}(\overline{O}_w)$ the simple perverse sheaf on the closure of this orbit. The cohomology of $\mathrm{IC}(\overline{O}_w)$ is an H_n-bimodule, and the functor

$$\theta_w : \mathrm{H}_n-\mathrm{mod} \longrightarrow \mathrm{H}_n-\mathrm{mod}$$

takes a module M to the tensor product

$$\mathrm{H}(\mathrm{IC}(\overline{O}_w), \mathbb{C}) \otimes_{\mathrm{H}_n} M.$$

Notice that all cohomology rings above have a canonical grading (by cohomological degree). The graded version of $\mathcal{C}^{(1^n)}$ is the category of finite-dimensional graded H_n-modules and the graded version of θ_w tensors a graded module with the graded H_n-bimodule $\mathrm{H}(\mathrm{IC}(\overline{O}_w), \mathbb{C})$.

It is surprising how sophisticated the categorification of the trivial representation is, especially when compared with the categorification of the sign representation. Both the trivial and the sign representation are one-dimensional, but their categorifications have amazingly different complexities. All of the complexity is lost when we pass to the Grothendieck group, which has rank one.

c. Categorification of the Burau representation. Consider the partition $\lambda^* = (2, 1^{n-2})$ and the dual partition $\lambda = (n-1, 1)$. The category $\mathcal{C}^{(n-1,1)}$ admits an explicit description, as follows. For $n > 3$ let A_{n-1} be the quotient of the path algebra of the graph from Figure 1 by the relations

$$\begin{aligned} (i|i+1|i+2) &= 0, \\ (i|i-1|i-2) &= 0, \\ (i|i-1|i) &= (i|i+1|i) \end{aligned}$$

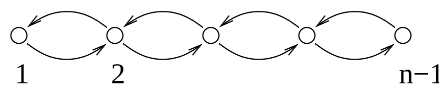


Figure 1: Quiver diagram of A_{n-1}

Also, let A_1 be the exterior algebra on one generator, and A_2 be the quotient of the path algebra of the graph from Figure 1 (for $n = 2$) by the relations $(1|2|1|2) = 0 = (2|1|2|1)$. The $\mathbb{C}$-algebra A_{n-1} is finite-dimensional.

Proposition 8 *The category $\mathcal{C}^{(n-1,1)}$ is equivalent to the category of finite-dimensional left A_{n-1}-modules.*

This is a well-known result, see e.g. [79] for $n = 2$ and [83] for the general case.

Denote by P_i the indecomposable left projective A_{n-1}-module $A_{n-1}(i)$. This module is spanned by all paths that end in vertex i. Likewise, let ${}_iP$ stand for the indecomposable right projective A_{n-1}-module $(i)A_{n-1}$. Under the equivalence between $\mathcal{C}^{(n-1,1)}$ and the category $A_{n-1}-$mod of finite-dimensional A_{n-1}-modules, the functor θ_i becomes the functor of tensoring with the bimodule

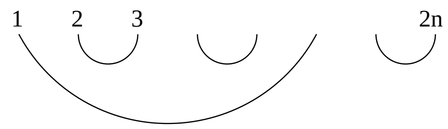
$$P_i \otimes {}_iP.$$

The functors θ_w are zero for most $w \in S_n$. They are nonzero only when the corresponding composition of θ_i's is nonzero (which rarely happens, note that already $\theta_i\theta_j = 0$ for $|i - j| > 1$).

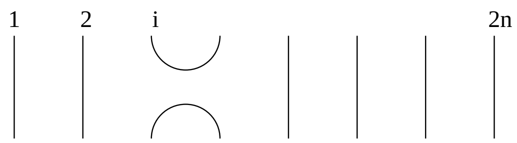
The algebras A_{n-1}, as well as the modules P_i, ${}_iP$ are naturally graded by the length of paths. The categories of finite dimensional graded modules over these algebras provide a categorification of the reduced Burau representation of each of the corresponding braid groups. For more information about the algebras A_{n-1} and their uses we refer the reader to the papers [52], [72], [75], [82], [34].

d. Categorification of the 2-column irreducible representation (partition (2^n)).

Let $\lambda^* = (2^n)$ and $\lambda = (n, n)$. The irreducible representation $S_{\mathbb{Q}}(\lambda^*)$ has the following explicit description. The basis of the representation consists of crossingless matchings of $2n$ points positioned on the x-axis by n arcs lying in the lower half-plane, as depicted below.



The element $1 + s_i$ acts on a basis element by concatenating it with the diagram



If the concatenation contains a circle, we remove it and multiply the result by 2, see figure 2.

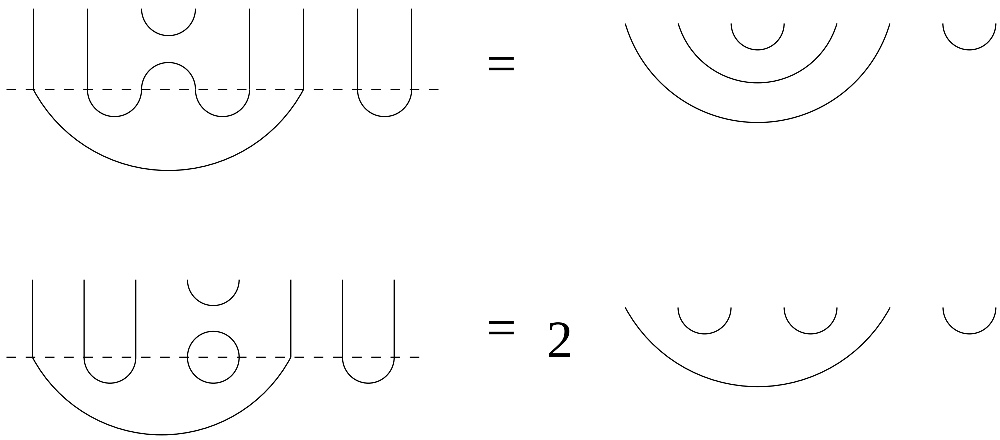


Figure 2: The product $(1 + s_i)b$, for a basis element b, is either another basis element (top diagram) or the same basis element times 2.

A categorification of this representation and of its quantum deformation was described in [47], in the context of extending a categorification of the Jones polynomial to tangles. The basis elements b corresponding to crossingless matchings become indecomposable projective modules P_b over a certain finite-dimensional algebra H^n. The space of homs $\mathrm{Hom}_{H^n}(P_a, P_b)$ between projective modules is given by gluing crossingless matchings a and b along their endpoints and applying a 2-dimensional TQFT to the resulting 1-manifold. The TQFT is determined by a commutative Frobenius algebra, which is just the cohomology of the 2-sphere. Spaces of these homs together with compositions

$$\mathrm{Hom}_{H^n}(P_a, P_b) \times \mathrm{Hom}_{H^n}(P_b, P_c) \longrightarrow \mathrm{Hom}_{H^n}(P_a, P_c)$$

determine H^n uniquely. The above geometric action of $1 + s_i$ lifts to the action on the category of H^n-mod of finite-dimensional H^n-modules given by tensoring with a certain H^n-module. This results in a very explicit categorification of the 2-column irreducible representation of S_n (and of the corresponding representation of the Hecke algebra) via the category of H^n-modules.

It was shown in [83] that H^n-mod is equivalent to the category $\mathcal{C}^{(n,n)}$ generated by projective-injective modules in the parabolic block $\mathcal{O}_0^{(n,n)}$. Subquotient algebras of H^n considered in [83], [25], [24] can be used to categorify other 2-column representations of the Hecke algebra and the symmetric group. These subquotient algebras provide also a graphical description of the whole category $\mathcal{O}_0^{\mu}$ for any composition $\mu_1 + \mu_2 = n$ of n ([83]).

4 Miscellaneous

Braid group actions. Graded versions of projective functors θ_w categorify the action of the Hecke algebra $H_{n,q}$ on its various representations. There is a homomorphism from the braid group on n strands to the group of invertible elements in $H_{n,q}$. This homomorphism, too, admits a categorification. The categorification should be at least an action of the braid group on a category, and this action is indeed well-known. To define it we need to pass to one of the triangulated extensions of the highest weight category: there does not seem to exist any interesting braid group actions on abelian categories, due to the positivity imposed by the abelian structure (see discussion in Section 6 of [47]).

The translation through the wall functors θ_i, $i \leq 1 \leq n-1$, for the regular block $\mathcal{O}_0$ are compositions of two projective endofunctors (*on* and *off the wall*) of $\mathcal{O}$, which are biadjoint to each other. This results in natural transformations $\theta_i \longrightarrow \mathrm{Id}$ and $\mathrm{Id} \longrightarrow \theta_i$. Let $D(\mathcal{O}_0)$ be the bounded derived category of $\mathcal{O}_0$. The complexes R_i and R_i' of functors $0 \longrightarrow \theta_i \longrightarrow \mathrm{Id} \longrightarrow 0$ and $0 \longrightarrow \mathrm{Id} \longrightarrow \theta_i \longrightarrow 0$ can be viewed as endofunctors of $D(\mathcal{O}_0)$ (we normalize the above functors so that Id sits in cohomological degree 0).

Proposition 9 *The functors R_i define a braid group action on $D(\mathcal{O}_0)$. The functor R_i' is the inverse of R_i.*

We distinguish between *weak* and *genuine* group actions; the terminology can be found in [52], [71], [53]. That R_i' and R_i are inverses of each other follows from a more general result of J. Rickard. That R_i define a weak braid group action follows from [62], [63, Proposition 10.1].

The Koszul duals of the functor R_i and its inverse are described in [61] in terms of the so-called twisting and completion functors on $\mathcal{O}_0$. A geometric description of these functors can be found in [8] and [71].

The functors θ_i restrict to exact endofunctors of the parabolic categories $\mathcal{O}_0^\mu$ and of the categories $\mathcal{C}^\mu$. Hence, the functors R_i and R_i' first induce endofunctors on $D^b(\mathcal{O}_0^\mu)$, and define braid group actions there and then restrict to endofunctors of the subcategory given by complexes of projective-injective modules in $\mathcal{O}_0^\mu$.

The braid group acts by functors respecting the triangulated structure of the involved categories, resulting in a categorification of parabolic braid group modules as well as those irreducible representations of the braid group that factor through the Hecke algebra. The two commuting actions of projective functors on the category of Harish-Chandra bimodules as described in example **6** of Section 2 give rise to two commuting actions on the braid group on the derived category of the category of Harish-Chandra bimodules. For more examples of braid group actions on triangulated categories and a possible framework for these actions see [53].

Invariants of tangle cobordisms. In several cases, braid group actions on triangulated categories can be extended to representations of the 2-category of tangle cobordisms. The objects of this 2-category (when 2-tangles are not decorated) are non-negative integers, morphisms from n to m are tangles with n bottom and m top boundary components, and 2-morphisms are isotopy classes of tangle cobordisms. A representation of the 2-category of tangle cobordisms associated a triangulated category $\mathcal{K}_n$ to the object n, an exact functor $\mathcal{K}_n \longrightarrow \mathcal{K}_m$ to a tangle, and a natural transformation of functors to a tangle cobordism. Such representations can be derived from examples **7** and **8** of Section 2 (see [82], [84], [64]) and from example **d** of Section 3 (see [47]). Example **6** is related to at least braid cobordisms (if not tangle cobordisms) via the construction of [49]. We expect that a categorification of tensor products of representations of quantum $\mathfrak{sl}_2$, mentioned at the end of example **7** extends (after passing to derived categories, suitable functors, and natural transformations) to a representation of the 2-category of tangle cobordisms colored by irreducible representations of quantum $\mathfrak{sl}_2$. Such an extension would give a categorification of the colored Jones polynomial.

The Cautis-Kamnitzer invariant of tangle cobordisms [23] is based on a similar framework, but their version of the category $\mathcal{K}_n$ is the derived category of coherent sheaves on a certain iterated $\mathbb{P}^1$-bundle. The Grothendieck group of their category is isomorphic to $V^{\otimes n}$, where V is the fundamental representation of quantum $\mathfrak{sl}_2$, just like in the example **7**, but these two categorifications of $V^{\otimes n}$ are noticeably different. For instance, in the example **7** the category decomposes into the direct sum matching the weight decomposition of the tensor product, while the category in [23] is indecomposable. When the parameter is even, the two categorifications of $V^{\otimes 2n}$ appear to have a common "core" subcategory, a categorification of the invariants in $V^{\otimes 2n}$ (the latter isomorphic to $\mathcal{S}((2^n))$) briefly reviewed in the example **d** above.

In the matrix factorization invariant of tangle cobordisms [51], the abelian category remains hidden inside the triangulated category of matrix factorizations.

Determinant of the Cartan matrix. With λ and μ as in example **4**, let $\{P_a\}_{a\in I}$ be a collection of indecomposable projectives in $\mathcal{C}^\mu$, one for each isomorphism class. The Cartan matrix of $\mathcal{C}^\mu$ is an $I \times I$ matrix C with the (a, b)-entry being the dimension of $\mathrm{Hom}(P_a, P_b)$, the space of homomorphisms between projective modules P_a and P_b. Since $\mathrm{End}(P, P)$ is a symmetric algebra by [65], where $P = \bigoplus_{a\in I} P_a$, the Cartan matrix is symmetric, $c_{a,b} = c_{b,a}$. These algebras are not commutative, but the center has a nice geometrical description as the cohomology of some Springer fibre ([48], and more general [14], [83]).

What is the determinant of this Cartan matrix? Since $\mathcal{C}^\mu$ depends (up to equivalence) on the partition λ only ([65]), so does the determinant. The answer to the question is obvious in each of the first three cases considered in the previous section: the determinant is equal to 1 for $\lambda = (n)$, to $n!$ for $\lambda = (1^n)$ and to n for $\lambda = (n-1, 1)$. The fourth case, when $\lambda = (2^n)$, requires more work, and follows from the results of [29] and [44]. The determinant equals

$$\prod_{i=1}^{n} (i+1)^{r_{n,i}}, \quad r_{n,i} = \binom{2n}{n-i} - 2\binom{2n}{n-i-1} + \binom{2n}{n-i-2}, \tag{9}$$

with the convention $\binom{j}{s} = 0$ if $s < 0$. The answer for an arbitrary λ is more complicated. However, we want to point out that this determinant of the Cartan matrix is the determinant of the Shapovalov form ([74]) on a certain weight space of some irreducible $\mathfrak{sl}_n$-module, as can be obtained from instance from [16]. It can be computed using the so-called Jantzen-Schaper formula [39, Satz 2].

The absolute value of the determinant has an interesting categorical interpretation. $\mathcal{C}^\mu$ is equivalent to the category of finite-dimensional modules over some symmetric $\mathbb{C}$-algebra A^μ. Given any symmetric $\mathbb{C}$-algebra A (an algebra with a nondegenerate symmetric trace $A \longrightarrow \mathbb{C}$), the stable category $A-\underline{\mathrm{mod}}$ is triangulated. Objects of $A-\underline{\mathrm{mod}}$ are finite-dimensional A-modules and the set of morphisms from M to N is the quotient vector space of all module maps modulo those that factor through a projective module. If $\det(C) \neq 0$ then the Grothendieck group of the stable category is finite abelian of cardinality equal to the absolute value of the determinant.

The graded version of this problem makes sense as well. Modules P_a are naturally graded, and to a pair (a, b) we can assign the Laurent polynomial in q which is the graded dimension of the graded vector space $\mathrm{Hom}(P_a, P_b)$. Arrange these polynomials into an $I \times I$ matrix (the graded Cartan matrix of $\mathcal{C}^\mu_{gr}$).

Problem: Find the determinant of the graded Cartan matrix of $\mathcal{C}^\mu$.

The determinant depends only on λ. Again, the answer is known in the above four cases. In the last case, the determinant of the graded Cartan matrix is given by formula (9), with the quantum integer $[i+1] = 1 + q^2 + \cdots + q^{2i}$ in place of $(i+1)$ in the product (the proof follows by combining results of [29] and [44]).

The determinant is algorithmically computable, since the entries of the graded Cartan matrix can be computed from the Kazhdan-Lusztig polynomials of the symmetric group. We are almost tempted to conjecture that, for any λ, the determinant (up to a power of q) is a product of quantum integers $[j] = q^{j-1} + q^{j-3} + \cdots + q^{1-j}$, for small j, with some multiplicities.

In [38], a q-analogue of the Jantzen-Schaper formula is obtained. Generalizing [14] by working out a graded or q-version, should imply that the determinant is equal to the determinant of the q-analogue of the Shapovalov form on a suitable weight space of an irreducible $\mathfrak{sl}_m$-module.

Acknowledgement M.K. was partially supported by the NSF grant DMS-0407784. V. M. was supported by STINT, the Royal Swedish Academy of Sciences, and the Swedish Research Council, C. S. was partially supported by EPSRC grant 32199. We thank all these institutions. We also thank Kenneth Brown for useful discussions and remarks and Jonathan Brundan for remarks on an earlier version of this paper.

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