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BLOCKS OF THE CATEGORY OF CUSPIDAL $\mathfrak{sp}_{2n}$-MODULES

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In this paper we show that every block of the category of cuspidal generalized weight modules with finite dimensional generalized weight spaces over the Lie algebra $\mathfrak{sp}_{2n}(\mathbb{C})$ is equivalent to the category of finite dimensional $\mathbb{C}[[t_1, t_2, \dots, t_n]]$-modules.

1. Introduction and description of the results

Fix the ground field to be the complex numbers. Fix $n \in \{2, 3, \dots\}$ and consider the symplectic Lie algebra $\mathfrak{sp}_{2n} =: \mathfrak{g}$ with a fixed Cartan subalgebra $\mathfrak{h}$ and root space decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Delta} \mathfrak{g}_\alpha,$$

where Δ denotes the corresponding root system. For a $\mathfrak{g}$-module V and $\lambda \in \mathfrak{h}^*$ set

$$V_\lambda := \{v \in V : h \cdot v = \lambda(h)v \text{ for any } h \in \mathfrak{h}\},$$
$$V^\lambda := \{v \in V : (h - \lambda(h))^k \cdot v = 0 \text{ for any } h \in \mathfrak{h} \text{ and } k \gg 0\}.$$

A $\mathfrak{g}$-module V is called

- a *weight module* provided that $V = \bigoplus_{\lambda\in\mathfrak{h}^*} V_\lambda$;
- a *generalized weight module* provided that $V = \bigoplus_{\lambda\in\mathfrak{h}^*} V^\lambda$;
- a *cuspidal module* provided that for any $\alpha \in \Delta$ the action of any nonzero element from $\mathfrak{g}_\alpha$ on V is bijective.

If V is a generalized weight module, then the set $\{\lambda \in \mathfrak{h}^* : V_\lambda \neq 0\}$ is called the *support* of V and is denoted by $\mathrm{supp}(V)$.

Denote by $\hat{\mathscr{C}}$ the full subcategory in $\mathfrak{g}$-mod that consists of all cuspidal generalized weight modules with finite dimensional generalized weight spaces, and by $\mathscr{C}$ the full subcategory of $\hat{\mathscr{C}}$ consisting of all weight modules. Understanding the categories $\mathscr{C}$ and $\hat{\mathscr{C}}$ is a classical problem in the representation theory of Lie algebras. The first major step towards the solution of this problem was made in [Mathieu 2000], where all simple objects in $\hat{\mathscr{C}}$ were classified. Britten et al. [2004]

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showed that the category $\mathscr{C}$ is semisimple, hence completely understood. The aim of the present note is to describe the category $\hat{\mathscr{C}}$.

Apart from $\mathfrak{sp}_{2n}$, cuspidal weight modules with finite dimensional weight spaces exist only for the Lie algebra $\mathfrak{sl}_n$ [Fernando 1990]. In the latter case, simple objects in the corresponding category $\hat{\mathscr{C}}$ are classified in [Mathieu 2000], the category $\mathscr{C}$ is described in [Grantcharov and Serganova 2010] (see also [Mazorchuk and Stroppel 2011]), and the category $\hat{\mathscr{C}}$ is described in [Mazorchuk and Stroppel 2011]. Taking all these results into account, the present paper completes the study of cuspidal generalized weight modules with finite dimensional generalized weight spaces over semisimple finite dimensional Lie algebras.

Let $U(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$ and $Z(\mathfrak{g})$ the center of $U(\mathfrak{g})$. The action of $Z(\mathfrak{g})$ on any object from $\hat{\mathscr{C}}$ is locally finite. Using this and the standard support arguments gives the following *block decomposition* of $\hat{\mathscr{C}}$:

$$\hat{\mathscr{C}} \cong \bigoplus_{\substack{\chi:Z(\mathfrak{g})\to\mathbb{C}\\ \xi\in\mathfrak{h}^*/\mathbb{Z}\Delta}} \hat{\mathscr{C}}_{\chi,\xi},$$

where $\hat{\mathscr{C}}_{\chi,\xi}$ consists of all V such that $\operatorname{supp}(V)\subset\xi$ and $(z-\chi(z))^k\cdot v=0$ for all $v\in V$, $z\in Z(\mathfrak{g})$ and $k\gg 0$. Set

$$\mathscr{C}_{\chi,\xi} := \mathscr{C}\cap\hat{\mathscr{C}}_{\chi,\xi}.$$

From [Mathieu 2000, Section 9] it follows that each nontrivial $\hat{\mathscr{C}}_{\chi,\xi}$ contains a unique (up to isomorphism) simple object. In particular, $\hat{\mathscr{C}}_{\chi,\xi}$ is indecomposable, hence a block. From this and [Britten et al. 2004] we thus get that every nontrivial block $\mathscr{C}_{\chi,\xi}$ is equivalent to the category of finite dimensional $\mathbb{C}$-modules. Our main result is the following:

Theorem 1. *Every nontrivial block $\hat{\mathscr{C}}_{\chi,\xi}$ is equivalent to the category of finite dimensional $\mathbb{C}[[t_1, t_2, \dots, t_n]]$-modules.*

To prove Theorem 1 we use and further develop the technique of extension of the module structure from a Lie subalgebra, originally developed in [Mazorchuk and Stroppel 2011] for the study of categories of singular and nonintegral cuspidal generalized weight $\mathfrak{sl}_n$-modules. The proof of Theorem 1 is given in Section 4. In Section 2 we recall the standard reduction to the case of the so-called simple *completely pointed* modules (that is, simple weight cuspidal modules for which *all* nontrivial weight spaces are one-dimensional) and a realization of such modules using differential operators. In Section 3 we define a functor from the category of finite dimensional $\mathbb{C}[[t_1, t_2, \dots, t_n]]$-modules to any block $\hat{\mathscr{C}}_{\chi,\xi}$ containing a simple completely pointed module. In Section 4 we prove that this functor is an equivalence of categories. In Section 5 we present some consequences of our main result. In particular, we recover the main result of [Britten et al. 2004] stated above.

2. Completely pointed simple cuspidal weight modules

A weight $\mathfrak{g}$-module V is called *pointed* provided that $\dim V_\lambda = 1$ for some $\lambda \in \mathfrak{h}^*$. If V is a pointed simple cuspidal weight $\mathfrak{g}$-module, then, obviously, all nontrivial weight spaces of V are one-dimensional, in which case one says that V is *completely pointed* (see [Britten et al. 2004]). It is enough to consider blocks with completely pointed simple modules because of the following:

Lemma 2. *All nontrivial blocks of $\hat{\mathscr{C}}$ are equivalent.*

Proof. In the case of the category $\mathscr{C}$, this is proved in [Britten et al. 2004, Lemma 2]. The same argument works in the case of the category $\hat{\mathscr{C}}$. □

We recall the explicit realization of completely pointed simple cuspidal modules from [Britten and Lemire 1987]. Denote by W_n the n-th *Weyl algebra*, that is, the algebra of differential operators with polynomial coefficients in variables $x_1, x_2, \dots, x_n$. The algebra W_n is generated by x_i and $\partial/\partial x_i$, $i = 1, \dots, n$, which satisfy the relations $[\partial/\partial x_i, x_j] = \delta_{i,j}$. Let $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n$ be the vectors of the standard basis in $\mathbb{C}^n$. Identify $\mathbb{C}^n$ with $\mathfrak{h}^*$ such that Δ becomes the following *standard root system* of type C_n:

$$\{\pm(\varepsilon_i \pm \varepsilon_j) : 1 \le i < j \le n\} \cup \{\pm 2\varepsilon_i : 1 \le i \le n\}.$$

Then

$$\boldsymbol{H} = \boldsymbol{H}_n = \{2\varepsilon_1, \varepsilon_2 - \varepsilon_1, \varepsilon_3 - \varepsilon_2, \dots, \varepsilon_n - \varepsilon_{n-1}\}$$

is a basis of Δ. Fix a basis of $\mathfrak{g}$ of the form

$$\boldsymbol{C} := \{X_{\pm\varepsilon_i \pm \varepsilon_j} : 1 \le i < j \le n\} \cup \{X_{\pm 2\varepsilon_i} : i = 1, 2, \dots, n\} \cup \{H_\alpha : \alpha \in \boldsymbol{H}\}$$

such that the following map defines an injective Lie algebra homomorphism from $\mathfrak{g}$ to the Lie algebra associated with W_n:

$$\begin{aligned}
&X_{\varepsilon_i - \varepsilon_j} \mapsto x_i \frac{\partial}{\partial x_j}, && 1 \le i \ne j \le n,\\
&X_{\varepsilon_i + \varepsilon_j} \mapsto x_i x_j, && i, j = 1, 2, \dots, n,\\
&X_{-\varepsilon_i - \varepsilon_j} \mapsto \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}, && i, j = 1, 2, \dots, n,\\
&H_{\varepsilon_{i+1} - \varepsilon_i} \mapsto x_{i+1}\frac{\partial}{\partial x_{i+1}} - x_i \frac{\partial}{\partial x_i}, && i = 1, 2, \dots, n-1,\\
&H_{2\varepsilon_1} \mapsto \frac{1}{2}\Big(x_1 \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_1} x_1\Big).
\end{aligned} \tag{1}$$

Set

$$\boldsymbol{B} := \{(b_1, b_2, \dots, b_n) \in \mathbb{Z}^n : b_1 + b_2 + \cdots + b_n \in 2\mathbb{Z}\}.$$

For $\boldsymbol{a} = (a_1, a_2, \dots, a_n) \in \mathbb{C}^n$ define $N(\boldsymbol{a})$ to be the linear span of

$$\{\boldsymbol{x}^{\boldsymbol{b}} := x_1^{a_1+b_1} x_2^{a_2+b_2} \cdots x_n^{a_n+b_n} : \boldsymbol{b} \in \boldsymbol{B}\}.$$

First define an action of the elements from $\boldsymbol{C}$ on $N(\boldsymbol{a})$ using the formulae from (1) as follows:

$$
\begin{array}{rll}
& X_{\varepsilon_i-\varepsilon_j}\boldsymbol{x}^{\boldsymbol{b}} = (a_j+b_j)\boldsymbol{x}^{\boldsymbol{b}+\varepsilon_i-\varepsilon_j} & 1 \le i \neq j \le n, \\
& X_{\varepsilon_i+\varepsilon_j}\boldsymbol{x}^{\boldsymbol{b}} = \boldsymbol{x}^{\boldsymbol{b}+\varepsilon_i+\varepsilon_j} & i, j = 1, 2, \dots, n, \\
& X_{-\varepsilon_i-\varepsilon_j}\boldsymbol{x}^{\boldsymbol{b}} = (a_i+b_i)(a_j+b_j)\boldsymbol{x}^{\boldsymbol{b}-\varepsilon_i-\varepsilon_j} & 1 \le i \neq j \le n, \\
(2) & X_{-2\varepsilon_i}\boldsymbol{x}^{\boldsymbol{b}} = (a_i+b_i)(a_i+b_i-1)\boldsymbol{x}^{\boldsymbol{b}-2\varepsilon_i} & i = 1, 2, \dots, n, \\
& H_{\varepsilon_{i+1}-\varepsilon_i}\boldsymbol{x}^{\boldsymbol{b}} = (a_{i+1}+b_{i+1}-a_i-b_i)\boldsymbol{x}^{\boldsymbol{b}} & i = 1, 2, \dots, n-1, \\
& H_{2\varepsilon_1}\boldsymbol{x}^{\boldsymbol{b}} = \tfrac{1}{2}(2a_1+2b_1+1)\boldsymbol{x}^{\boldsymbol{b}}. &
\end{array}
$$

Theorem 3 [Britten and Lemire 1987]. (i) *For every $\boldsymbol{a} \in \mathbb{C}^n$ the formulae in (2) define on $N(\boldsymbol{a})$ the structure of a completely pointed weight $\mathfrak{g}$-module.*

(ii) *If $a_i \notin \mathbb{Z}$ for all $i = 1, \dots, n$, then the module $N(\boldsymbol{a})$ is simple and cuspidal.*

(iii) *Every completely pointed simple cuspidal $\mathfrak{g}$-module is isomorphic to $N(\boldsymbol{a})$ for some $\boldsymbol{a} \in \mathbb{C}^n$ such that $a_i \notin \mathbb{Z}$, $i = 1, \dots, n$.*

3. The functor F

This section is similar to [Mazorchuk and Stroppel 2011, Section 3.1]. Fix $\boldsymbol{a} \in \mathbb{C}^n$ such that $a_i \notin \mathbb{Z}$, $i = 1, \dots, n$. Let $\hat{\mathscr{C}}_{\boldsymbol{a}}$ denote the block of $\hat{\mathscr{C}}$ containing $N(\boldsymbol{a})$. The category $\hat{\mathscr{C}}_{\boldsymbol{a}}$ is closed under extensions. Denote the category of finite dimensional $\mathbb{C}[[t_1, t_2, \dots, t_n]]$-modules by $\mathbb{C}[[t_1, t_2, \dots, t_n]]$-mod. For $V \in \mathbb{C}[[t_1, t_2, \dots, t_n]]$-mod denote by T_i the linear operator describing the action of t_i on V. Set $\mathbf{0} = (0, 0, \dots, 0) \in \boldsymbol{B}$.

For $\boldsymbol{b} \in \boldsymbol{B}$ consider a copy $V^{\boldsymbol{b}}$ of V. Define

$$
\mathrm{F}V := \bigoplus_{\boldsymbol{b}\in\boldsymbol{B}} V^{\boldsymbol{b}}.
$$

Define the action of elements from $\boldsymbol{C}$ on the vector space $\mathrm{F}V$ in the following way: for $v \in V^{\boldsymbol{b}}$ set

$$
(3)\quad
\begin{cases}
X_{\varepsilon_i-\varepsilon_j}v = (T_j + (a_j+b_j)\,\mathrm{Id}_V)v & \in V^{\boldsymbol{b}+\varepsilon_i-\varepsilon_j}, \\
X_{\varepsilon_i+\varepsilon_j}v = v & \in V^{\boldsymbol{b}+\varepsilon_i+\varepsilon_j}, \\
X_{-\varepsilon_i-\varepsilon_j}v = (T_i + (a_i+b_i)\,\mathrm{Id}_V)(T_j + (a_j+b_j)\,\mathrm{Id}_V)v & \in V^{\boldsymbol{b}-\varepsilon_i-\varepsilon_j}, \\
X_{2\varepsilon_i}v = (T_i + (a_i+b_i)\,\mathrm{Id}_V)(T_i + (a_i+b_i-1)\,\mathrm{Id}_V)v & \in V^{\boldsymbol{b}-2\varepsilon_i}, \\
H_{\varepsilon_{i+1}-\varepsilon_i}v = (T_{i+1} - T_i + (a_{i+1}+b_{i+1}-a_i-b_i)\,\mathrm{Id}_V)v & \in V^{\boldsymbol{b}}, \\
H_{2\varepsilon_1}v = \tfrac{1}{2}(2T_1 + (2a_1+2b_1+1)\,\mathrm{Id}_V)v & \in V^{\boldsymbol{b}},
\end{cases}
$$

where i and j are as in the respective row of (2). For a homomorphism $f : V \to W$ of $\mathbb{C}[\![t_1, t_2, \dots, t_n]\!]$-modules denote by $\mathrm{F}f$ the diagonally extended linear map from $\mathrm{F}V$ to $\mathrm{F}W$, that is, for every $\boldsymbol{b} \in \boldsymbol{B}$ and $v \in V^{\boldsymbol{b}}$, set

$$\mathrm{F}f(v) = f(v) \in W^{\boldsymbol{b}}. \tag{4}$$

Proposition 4. (i) *The formulae of (3) define on* $\mathrm{F}V$ *the structure of a* $\mathfrak{g}$*-module.*

(ii) *Every* $V^{\boldsymbol{b}}$ *is a generalized weight space of* $\mathrm{F}V$*. Moreover, for* $\boldsymbol{b} \neq \boldsymbol{b}'$ *the weights of* $V^{\boldsymbol{b}}$ *and* $V^{\boldsymbol{b}'}$ *are different.*

(iii) *The module* $\mathrm{F}V$ *belongs to* $\hat{\mathscr{C}}_{\boldsymbol{a}}$*.*

(iv) *Formulae (3) and (4) turn* F *into a functor*

$$\mathrm{F} : \mathbb{C}[\![t_1, t_2, \dots, t_n]\!]\text{-mod} \to \hat{\mathscr{C}}_{\boldsymbol{a}}.$$

(v) *The functor* F *is exact, faithful and full.*

Proof. Consider the $\mathfrak{g}$-module $N(\boldsymbol{a})$ for $\boldsymbol{a}$ as above. Then, for every $\boldsymbol{b}$, the defining relations of $\mathfrak{g}$ (in terms of elements from $\boldsymbol{C}$) applied to $\boldsymbol{x}^{\boldsymbol{b}}$ can be written as some polynomial equations in the a_i. Since (2) defines a $\mathfrak{g}$-module for any $\boldsymbol{a}$ by Theorem 3(i), these equations hold for any $\boldsymbol{a}$, that is, they are actually formal identities in the a_i. Now write

$$T_j + (a_j + b_j)\,\mathrm{Id}_V = A_j + B_j,$$

a sum of matrices, where $A_j = T_j + a_j\,\mathrm{Id}_V$ and $B_j = b_j\,\mathrm{Id}_V$. All A_i and B_j commute with each other and with all the T_l. For a fixed $\boldsymbol{b}$, the defining relations for $\mathfrak{g}$ on $\mathrm{F}V$ reduce to our formal identities (in the A_i) and hence are satisfied. This proves claim (i). Claim (ii) follows from the last two lines in (3) and the fact that all the T_i are nilpotent (hence zero is the only eigenvalue).

As f commutes with all T_i, the map $\mathrm{F}f$ commutes with the action of all elements from $\boldsymbol{C}$ and hence defines a homomorphism of $\mathfrak{g}$-modules. By construction we also have $\mathrm{F}(f \circ f') = \mathrm{F}f \circ \mathrm{F}f'$, which implies claim (iv).

By construction, F is exact and faithful. It sends the simple one-dimensional $\mathbb{C}[\![t_1, t_2, \dots, t_n]\!]$-module to $N(\boldsymbol{a})$ (as in this case all $T_i = 0$ and hence (3) gives (2)), which is an object of the category $\hat{\mathscr{C}}_{\boldsymbol{a}}$ closed under extensions. Claim (iii) follows.

To complete the proof of claim (v) we are left to show that F is full. Let $\varphi : \mathrm{F}V \to \mathrm{F}W$ be a $\mathfrak{g}$-homomorphism. Then φ commutes with the action of all elements from $\mathfrak{h}$. Using claim (ii), we get that φ induces, by restriction, a linear map $f : V = V^{\boldsymbol{0}} \to W^{\boldsymbol{0}} = W$. As φ commutes with all $H_{\varepsilon_{i+1}-\varepsilon_i}$, the map f commutes with all operators $T_{i+1} - T_i$. As φ commutes with $H_{2\varepsilon_1}$, the map f commutes with T_1. It follows that f is a homomorphism of $\mathbb{C}[\![t_1, t_2, \dots, t_n]\!]$-modules. This yields $\varphi = \mathrm{F}f$ and thus the functor F is full. This completes the proof of claim (v) and of the whole proposition. □

4. Proof of Theorem 1

Because of Lemma 2 it is enough to fix one particular block and show there that F is an equivalence. Thus, we may assume that $a_i + a_j \notin \mathbb{Z}$ for all i, j (in particular, $a_i \notin \mathbb{Z}$ for all i). According to Proposition 4, we are only left to show that F is dense (that is, essentially surjective). We establish the density of F by induction on n. We first prove the induction step and then the basis of the induction, which is the case $n = 2$.

Denote by λ the weight of $\boldsymbol{x}^{\mathbf{0}} \in N(\boldsymbol{a})$ (see Proposition 4(ii)). Let $M \in \hat{\mathscr{C}}_{\boldsymbol{a}}$. Set $V := M_\lambda$ and denote by M' the $\mathfrak{a}$-module $U(\mathfrak{a})V$.

4.1. *Reduction to the case n = 2.*

The main result of this section is the following:

Proposition 5. *If the functor* F *is dense for $n = 2$, then it is dense for any $n \geq 2$.*

Proof. Assume that $n > 2$ and that the functor F is dense in the case of the algebra $\mathfrak{sp}_{2n-2}$. Realize $\mathfrak{sp}_{2n-2}$ as the subalgebra $\mathfrak{a}$ of $\mathfrak{g}$ corresponding to the subset $\boldsymbol{H}_{n-1} \subset \boldsymbol{H}$ of simple roots.

Let $Y_1, Y_2, \dots, Y_n$ be the linear operators representing the action of the elements $H_{2\varepsilon_1}$, $H_{\varepsilon_2-\varepsilon_1}$, $H_{\varepsilon_3-\varepsilon_2}, \dots, H_{\varepsilon_n-\varepsilon_{n-1}}$ on V, respectively. Set

$$\begin{aligned} T_1 &:= Y_1 - \tfrac{1}{2}(2a_1+1)\,\mathrm{Id}_V, \\ T_2 &:= Y_2 + T_1 - (a_2 - a_1)\,\mathrm{Id}_V, \\ T_3 &:= Y_3 + T_2 - (a_3 - a_2)\,\mathrm{Id}_V, \\ &\vdots \\ T_n &:= Y_n + T_{n-1} - (a_n - a_{n-1})\,\mathrm{Id}_V . \end{aligned} \tag{5}$$

The T_i are obviously pairwise commuting nilpotent linear operators.

The module M' is a cuspidal generalized weight $\mathfrak{a}$-module with finite dimensional weight spaces. Moreover, as all composition subquotients of M are of the form $N(\boldsymbol{a})$, all composition subquotients of M' are of the form $N(\boldsymbol{a})'$, the latter being a completely pointed simple cuspidal $\mathfrak{a}$-module. By our inductive assumption, the functor F is dense in the case of the algebra $\mathfrak{a}$. Hence $M' \cong N' := \bigoplus_{\boldsymbol{b}} V^{\boldsymbol{b}}$, where $\boldsymbol{b} \in \boldsymbol{B}$ is such that $b_n = 0$, and the action of $\mathfrak{a}$ on N' is given by (3).

Lemma 6. *There is a unique* (*up to isomorphism*) $\mathfrak{g}$-*module* $Q \in \hat{\mathscr{C}}_{\boldsymbol{a}}$ *such that* $Q' = N'$ *and which gives the linear operator* T_n *when computed using* (5).

Proof. The existence statement is clear, so we need only to show uniqueness. Assume that $Q \in \hat{\mathscr{C}}_{\boldsymbol{a}}$ is such that $Q' = N'$ and the formulae in (5) applied to Q produce the linear operator T_n. Since $a_n \notin \mathbb{Z}$, the endomorphism $T_n + (a_n + b_n)\,\mathrm{Id}_V$ is invertible for all $b_n \in \mathbb{Z}$. As the action of $X_{\varepsilon_n-\varepsilon_{n-1}}$ on Q is bijective, we can fix a weight basis in Q such that both the $\mathfrak{a}$-action on $Q' = N'$ and the action of $X_{\varepsilon_n-\varepsilon_{n-1}}$ on the whole Q is given by (3). As $n > 2$, the elements $X_{\pm 2\varepsilon_1}$ commute with $X_{\varepsilon_n-\varepsilon_{n-1}}$ and hence their action extends uniquely to the whole of Q using this commutativity. This holds similarly for all elements $X_{\pm(\varepsilon_i-\varepsilon_{i-1})}$, $i<n-1$, and for the element $X_{\varepsilon_{n-2}-\varepsilon_{n-1}}$. This leaves us with the elements $X_{\varepsilon_{n-1}-\varepsilon_{n-2}}$ and $X_{\varepsilon_{n-1}-\varepsilon_n}$. The simple roots $\varepsilon_{n-1}-\varepsilon_{n-2}$ and $\varepsilon_n-\varepsilon_{n-1}$ corresponding to the elements $X_{\varepsilon_{n-1}-\varepsilon_{n-2}}$ and $X_{\varepsilon_n-\varepsilon_{n-1}}$ generate a root system of type A_2 (this corresponds to the algebra $\mathfrak{sl}_3$). Lemmas 21 and 22 of [Mazorchuk and Stroppel 2011] prove that the actions of $X_{\varepsilon_{n-1}-\varepsilon_{n-2}}$ and $X_{\varepsilon_{n-1}-\varepsilon_n}$ extend uniquely to Q. This completes the proof of Lemma 6. ☐

The module $\mathrm{F}V$ obviously satisfies $(\mathrm{F}V)'=N'$ and defines the linear operator T_n when computed using (5). Hence Lemma 6 implies $M\cong \mathrm{F}V$. Since $M\in\hat{\mathscr{C}}_{\boldsymbol{a}}$ was arbitrary, the functor F is dense, completing the proof of Proposition 5. ☐

4.2. ***Base of the induction: some $\mathfrak{sl}_2$-theory as preparation.*** In this section we will recall (and slightly improve) some classical $\mathfrak{sl}_2$-theory. For details see [Mazorchuk 2010]. Consider the Lie algebra $\mathfrak{sl}_2=\mathfrak{sl}_2(\mathbb{C})$ with standard basis

$$\boldsymbol{e}:=\begin{pmatrix}0&1\\0&0\end{pmatrix},\quad \boldsymbol{f}:=\begin{pmatrix}0&0\\1&0\end{pmatrix},\quad \boldsymbol{h}:=\begin{pmatrix}1&0\\0&-1\end{pmatrix}.$$

Let V be a finite dimensional vector space and A and B be two commuting linear operators on V. For $i\in\mathbb{Z}$ denote by $V^{(i)}$ a copy of V and consider the vector space $\overline{V}:=\bigoplus_{i\in\mathbb{Z}}V^{(i)}$ (a direct sum of copies of V indexed by i). Define the actions of $\boldsymbol{e}$, $\boldsymbol{f}$ and $\boldsymbol{h}$ on $\overline{V}$ as follows: for $v\in V^{(i)}$ set

$$\begin{aligned}\mathbf{v}&:=(P-i\,\mathrm{Id}_V)v &&\in V^{(i+1)},\\ \mathbf{v}&:=(Q+i\,\mathrm{Id}_V)v &&\in V^{(i-1)},\\ \mathbf{v}&:=(Q-P+2i\,\mathrm{Id}_V)v &&\in V^{(i)}.\end{aligned}\tag{6}$$

This can be depicted as follows (here right arrows represent the action of $\boldsymbol{e}$, left arrows represent the action of $\boldsymbol{f}$ and loops represent the action of $\boldsymbol{h}$):

$$\begin{tikzcd}
\cdots \arrow[r, bend left, "P+2\,\mathrm{Id}_V"] & V^{(-1)} \arrow[l, bend left, "Q-\mathrm{Id}_V"] \arrow[loop below, "Q-P-2\,\mathrm{Id}_V"] \arrow[r, bend left, "P+\mathrm{Id}_V"] & V^{(0)} \arrow[l, bend left, "Q"] \arrow[loop below, "Q-P"] \arrow[r, bend left, "P"] & V^{(1)} \arrow[l, bend left, "Q+\mathrm{Id}_V"] \arrow[loop below, "Q-P+2\,\mathrm{Id}_V"] \arrow[r, bend left, "P-\mathrm{Id}_V"] & \cdots \arrow[l, bend left, "Q+2\,\mathrm{Id}_V"]
\end{tikzcd}$$

Proposition 7. (i) *The formulae in (6) define on $\overline{V}$ the structure of a generalized weight $\mathfrak{sl}_2$-module with finite dimensional generalized weight spaces.*

(ii) *Every cuspidal generalized weight $\mathfrak{sl}_2$-module with finite dimensional generalized weight spaces is isomorphic to $\overline{V}$ for some V with P and Q as above.*

(iii) *The action of the Casimir element $\boldsymbol{c}:=(\boldsymbol{h}+1)^2+4\boldsymbol{f}\boldsymbol{e}$ on $\overline{V}$ is given by the linear operator $(P+Q+\mathrm{Id}_V)^2$.*

(iv) *Let $\mathbb{C}^2$ denote the natural $\mathfrak{sl}_2$-module (the unique two-dimensional simple $\mathfrak{sl}_2$-module). Then the linear operator $(\boldsymbol{c}-(P+Q+2\,\mathrm{Id}_V)^2)(\boldsymbol{c}-(P+Q)^2)$ annihilates the $\mathfrak{sl}_2$-module $\mathbb{C}^2\otimes\overline{V}$.*

(v) *Let $\mathbb{C}^3$ denote the unique three-dimensional simple $\mathfrak{sl}_2$-module. Then the linear operator $(\boldsymbol{c}-(P+Q+3\,\mathrm{Id}_V)^2)(\boldsymbol{c}-(P+Q+\mathrm{Id}_V)^2)(\boldsymbol{c}-(P+Q-\mathrm{Id}_V)^2)$ annihilates the $\mathfrak{sl}_2$-module $\mathbb{C}^3\otimes\overline{V}$.*

Proof. The fact that $\overline{V}$ is an $\mathfrak{sl}_2$-module is checked by a direct computation. That $\overline{V}$ is a generalized weight module follows from the fact that the action of $\boldsymbol{h}$ on $\overline{V}$ preserves (by (6)) each V^i and hence is locally finite. Since the category of generalized weight modules is closed under extensions, to prove that $\overline{V}$ has finite dimensional generalized weight spaces it is enough to consider the case when $\boldsymbol{h}$ has a unique eigenvalue on $V^{(0)}$, say λ. However, in this case $\boldsymbol{h}$ has a unique eigenvalue on V^i, namely $\lambda+2i$, which implies that $\overline{V}^\lambda=V$ is finite dimensional. Claim (i) follows. To prove Claim (iii) we observe that the action of $\boldsymbol{c}$ on V^i is given by

$$(Q-P+(2i+1)\,\mathrm{Id}_V)^2+4(Q+(i+1)\,\mathrm{Id}_V)(P-i\,\mathrm{Id}_V)=(P+Q+\mathrm{Id}_V)^2.$$

Claim (ii) can be found with all details in [Mazorchuk 2010, Chapter 3].

To prove claim (iv) choose a basis $\{v_1,\dots,v_k\}$ in V, which gives rise to a basis $\{v_1^{(i)},\dots,v_k^{(i)},i\in\mathbb{Z}\}$ in $\overline{V}$. Choose the standard basis $\{e_1,e_2\}$ in $\mathbb{C}^2$. Since $\boldsymbol{h}e_1=e_1$, $\boldsymbol{h}e_2=-e_2$ and $\boldsymbol{h}$ acts by $Q-P+2i\,\mathrm{Id}_V$ on $V^{(i)}$, we obtain that $\boldsymbol{h}$ acts by $Q-P+(2i+1)\,\mathrm{Id}_V$ on the vector space $W^{(i)}$ with basis

$$\{e_1\otimes v_1^{(i)},\dots,e_1\otimes v_1^{(i)},e_2\otimes v_1^{(i+1)},\dots,e_2\otimes v_1^{(i+1)}\}.$$

We have $\mathbb{C}^2\otimes\overline{V}\cong\bigoplus_{i\in\mathbb{Z}}W^{(i)}$ and one easily computes that in the above basis the actions of $\boldsymbol{e}$ and $\boldsymbol{f}$ on $\mathbb{C}^2\otimes\overline{V}$ are given by the following picture:

$$\begin{tikzcd}
\cdots \arrow[r, bend left] & W^{(-1)} \arrow[l, bend left] \arrow[r, bend left, "\begin{pmatrix} P+\mathrm{Id} & \mathrm{Id} \\ 0 & P \end{pmatrix}"] & W^{(0)} \arrow[l, bend left, "\begin{pmatrix} Q & 0 \\ \mathrm{Id} & Q+\mathrm{Id} \end{pmatrix}"] \arrow[r, bend left, "\begin{pmatrix} P & \mathrm{Id} \\ 0 & P-\mathrm{Id} \end{pmatrix}"] & W^{(1)} \arrow[l, bend left, "\begin{pmatrix} Q+\mathrm{Id} & 0 \\ \mathrm{Id} & Q+2\,\mathrm{Id} \end{pmatrix}"] \arrow[r, bend left] & \cdots \arrow[l, bend left]
\end{tikzcd}$$

The action of $\boldsymbol{c}$ on $W^{(0)}$ is now easily computed to be given by the linear operator

$$G:=\begin{pmatrix} (Q-P+2\,\mathrm{Id})^2+4(Q+\mathrm{Id})P & 4(Q+\mathrm{Id}) \\ 4P & (Q-P+2\,\mathrm{Id})^2+4(Q+2\,\mathrm{Id})(P-\mathrm{Id})+4\,\mathrm{Id} \end{pmatrix}.$$

The characteristic polynomial of G is

$$\chi_G(\lambda)=(\lambda-(P+Q+2\,\mathrm{Id})^2)(\lambda-(P+Q)^2).$$

Claim (iv) now follows from the Cayley–Hamilton theorem.

We have an isomorphism of $\mathfrak{sl}_2$-modules as follows: $\mathbb{C}^2\otimes\mathbb{C}^2\cong\mathbb{C}^3\oplus\mathbb{C}$ (here $\mathbb{C}$ is the trivial module), and hence claim (v) follows applying claim (iv) twice.

Alternatively, one could do a direct calculation, similar to the proof of (iii). The proposition follows. ☐

The statement of Proposition 7(ii) is a special case of a more general result of Gabriel and Drozd describing blocks of the category of (generalized) weight $\mathfrak{sl}_2$-modules, in particular, simple weight $\mathfrak{sl}_2$-modules (see [Drozd 1983; Dixmier 1996, 7.8.16]). The statements of Proposition 7(iv) and (v) are $\mathfrak{sl}_2$-refinements of a theorem of Kostant [1975, Theorem 5.1] describing possible (generalized) central characters of the tensor product of a finite dimensional module with an infinite dimensional module.

4.3. ***The case $n = 2$.*** Assume now that $n = 2$. We have $a_1, a_2, a_1 + a_2 \notin \mathbb{Z}$. Let $\mathfrak{a}$ denote the Lie subalgebra of $\mathfrak{g}$ generated by $X_{\pm(\varepsilon_2-\varepsilon_1)}$. The algebra $\mathfrak{a}$ is isomorphic to $\mathfrak{sl}_2$.

Let $M \in \hat{\mathscr{C}}_{\boldsymbol{a}}$. Denote by λ the weight of $\boldsymbol{x}^{\mathbf{0}} \in N(\boldsymbol{a})$ and set $V := M_\lambda$. Let Y_1 and Y_2 be the linear operators representing the actions of the elements $H_{\varepsilon_2-\varepsilon_1}$ and $C := (H_{\varepsilon_2-\varepsilon_1} + 1)^2 + 4X_{\varepsilon_1-\varepsilon_2}X_{\varepsilon_2-\varepsilon_1}$ on V. The element C is a Casimir element for $\mathfrak{a}$. In particular, the operators Y_1 and Y_2 commute. Our first observation is the following:

Lemma 8. *The action of C on V is invertible and hence has a square root.*

Proof. From (2) we have that C acts on $\boldsymbol{x}^{\mathbf{0}}$ by

$$(a_2 - a_1 + 1)^2 + 4(a_2 + 1)a_1 = (a_1 + a_2 + 1)^2.$$

Since $a_1 + a_2 \notin \mathbb{Z}$ by our assumptions, $\boldsymbol{x}^{\mathbf{0}}$ is an eigenvector of C with a nonzero eigenvalue. As the module M has a composition series with subquotients isomorphic to $N(\boldsymbol{a})$, the complex number $(a_1 + a_2 + 1)^2 \neq 0$ is the only eigenvalue of C on V. The claim follows. ☐

Consider the $\mathfrak{a}$-module $M' := U(\mathfrak{a})M_\lambda$. Let Y_2' denote any square root of Y_2, which is a polynomial in Y_2 (it exists by Lemma 8). So Y_2' commutes with Y_1. Set

$$T_1 := \frac{Y_2' - Y_1 - \mathrm{Id}_V}{2} - a_1 \,\mathrm{Id}_V, \qquad T_2 := \frac{Y_2' + Y_1 - \mathrm{Id}_V}{2} - a_2 \,\mathrm{Id}_V\,.$$

Then T_1 and T_2 are two commuting nilpotent linear operators (it is easy to check that 0 is the unique eigenvalue for both T_1 and T_2), hence define on V the structure of a $\mathbb{C}[[t_1, t_2]]$-module. The aim of this section is to establish an isomorphism $\mathrm{F}V \cong M$, which would complete the proof of Theorem 1.

Set $R' := U(\mathfrak{a})(\mathrm{F}V)_\lambda$. A direct computation using (3) shows that $H_{\varepsilon_2-\varepsilon_1}$ and C act on $(\mathrm{F}V)_\lambda = V^{\mathbf{0}}$ as the linear operators Y_1 and Y_2, respectively. As any cuspidal generalized weight $\mathfrak{a}$-module is uniquely determined by the actions of $H_{\varepsilon_2-\varepsilon_1}$ and C (see [Drozd 1983; Mazorchuk 2010, 3.7] for full details), it follows that $M' \cong R'$. The isomorphism $\mathrm{F}V \cong M$ now follows from the next proposition:

Proposition 9. *There is at most one* (*up to isomorphism*) $\mathfrak{g}$-*module* $R \in \hat{\mathcal{C}}_{\boldsymbol{a}}$ *such that* $U(\mathfrak{a})R_\lambda = R'$.

Proof. Let $R \in \hat{\mathcal{C}}_{\boldsymbol{a}}$ be such that $U(\mathfrak{a})R_\lambda = R'$. Choose a weight basis in R such that the action of $\mathfrak{a}$ on R' and the action of $X_{2\varepsilon_1}$ on R is given by (3) (in other words these actions coincide with the corresponding actions on $\mathrm{F}V$). Since $X_{\varepsilon_1-\varepsilon_2}$ commutes with $X_{2\varepsilon_1}$, it follows that the action of $X_{\varepsilon_1-\varepsilon_2}$ on R is also given by (3).

It is left to show that the action of $X_{\varepsilon_2-\varepsilon_1}$ extends uniquely from R' to R and then that there is a unique way to define the action of $X_{-2\varepsilon_1}$. This will be done in the Lemmata 10 and 11 below. □

Lemma 10. *There is a unique way to extend the action of* $X_{\varepsilon_2-\varepsilon_1}$ *from* R' *to* R.

Proof. We first show that for every $k \in \{1, 2, \dots\}$, the action of $X_{\varepsilon_2-\varepsilon_1}$ extends uniquely from $X_{2\varepsilon_1}^{k-1}R'$ to $X_{2\varepsilon_1}^{k}R'$ (here $X_{2\varepsilon_1}^{0}R' = R'$).

Consider the following picture:

(7)

$$\begin{tikzcd}
\bullet \arrow[r, dashed, bend left, "X"] & \bullet \arrow[l, bend left, "Q"] & \\
\bullet \arrow[u, bend left, "1"] \arrow[r, bend left, "P+1"] & \bullet \arrow[u, bend left, "1"] \arrow[l, bend left, "Q"] \arrow[r, bend left, "P"] & \bullet \arrow[l, bend left, "Q+1"]
\end{tikzcd}$$

Here bullets are weight spaces with some fixed bases. The lower row is a part of $X_{2\varepsilon_1}^{k-1}R'$ where the $\mathfrak{a}$-action is already known by induction. The bases in the weight spaces in the lower row are chosen such that the action of $\mathfrak{a}$ in the lower row is given by (3). The upper row is a part of $X_{2\varepsilon_1}^{k}R'$ where the $\mathfrak{a}$-action is to be determined. Arrows pointing up indicate the action of $X_{2\varepsilon_1}$. The bases of the weight spaces in the upper row are chosen such that the action of $X_{2\varepsilon_1}$ is given by the operator Id_V (as in (3)). Left arrows indicate the action of $X_{\varepsilon_1-\varepsilon_2}$. The latter commutes with the action of $X_{2\varepsilon_1}$ and hence is given by the same linear operator in each column. Right arrows indicate the action of $X_{\varepsilon_2-\varepsilon_1}$ (which is known for $X_{2\varepsilon_1}^{k-1}R'$ and is to be determined for $X_{2\varepsilon_1}^{k}R'$). The part to be determined is given by the dashed arrow. Labels P and Q represent coefficients (which are linear operators on V) appearing in the corresponding parts of formulae (3). Note that P and Q commute. The action of $X_{\varepsilon_2-\varepsilon_1}$ on $X_{2\varepsilon_1}^{k}R'$ which is to be determined is given by some unknown linear operator X.

From $H_{\varepsilon_2-\varepsilon_1} = [X_{\varepsilon_2-\varepsilon_1}, X_{\varepsilon_1-\varepsilon_2}]$ we see that the action of $H_{\varepsilon_2-\varepsilon_1}$ on the middle weight space in the lower row is given by $Q-P$. Using $[H_{\varepsilon_2-\varepsilon_1}, X_{2\varepsilon_1}] = -2X_{2\varepsilon_1}$ we get that $H_{\varepsilon_2-\varepsilon_1}$ acts on the right dot of the upper row via $Q-P-2$. Using $[H_{\varepsilon_2-\varepsilon_1}, X_{\varepsilon_1-\varepsilon_2}] = -2X_{\varepsilon_1-\varepsilon_2}$ we get that $H_{\varepsilon_2-\varepsilon_1}$ acts on the left dot of the upper row via $Q-P-4$. So the action of C on the upper row is given by $(Q-P-3)^2+4XQ$.

The action of C on the lower row is given by $(Q-P-1)^2+4(P+1)Q=(Q+P+1)^2$.

The elements $X_{2\varepsilon_1}$, $X_{2\varepsilon_2}$ and $X_{\varepsilon_1+\varepsilon_1}$ form a weight basis of a simple three-dimensional $\mathfrak{a}$-module $\mathbb{C}^3$ with respect to the adjoint action of $\mathfrak{a}$. Hence the upper row of our picture is a subquotient of the tensor product of the lower row and $\mathbb{C}^3$. Therefore, from Proposition 7(v) we obtain that the linear operator

$$(C-(Q+P-1)^2)(C-(Q+P+1)^2)(C-(Q+P+3)^2)$$

annihilates the upper row. A direct computation using (3) shows that the action of the operators $C-(Q+P-1)^2$ and $C-(Q+P+1)^2$ on the part $X_{2\varepsilon_1}^k N(\boldsymbol{a})'$ of the module $N(\boldsymbol{a})$ is invertible. As the $\mathfrak{g}$-module we are working with must have a composition series with subquotients $N(\boldsymbol{a})$, it follows that the action of both $C-(Q+P-1)^2$ and $C-(Q+P+1)^2$ on $X_{2\varepsilon_1}^k R'$ is invertible. Hence $C-(Q+P+3)^2$ annihilates $X_{2\varepsilon_1}^k R'$, which gives us the equation

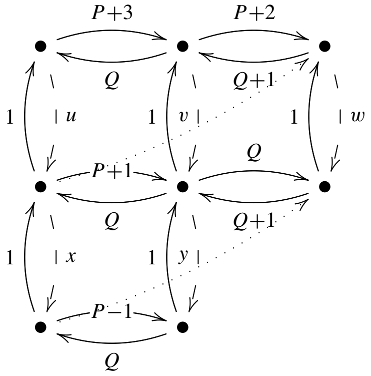
$$(Q-P-3)^2+4XQ=(Q+P+3)^2.$$

This equation has a unique solution, namely $X=Q+3$, which gives the required extension.

Similarly one shows that for $k\in\{-1,-2,\dots\}$, the action of $X_{\varepsilon_2-\varepsilon_1}$ extends uniquely from $X_{2\varepsilon_1}^{k+1}R'$ to $X_{2\varepsilon_1}^k R'$ (here again $X_{2\varepsilon_1}^0 R'=R'$). ☐

Lemma 11. *There is a unique way to define the action of $X_{-2\varepsilon_1}$ on N.*

Proof. To determine this action of $X_{-2\varepsilon_1}$ on N we consider the following extension of the picture (7) with the same notation as in the proof of Lemma 10:



Here all right arrows, representing the action of $X_{\varepsilon_2-\varepsilon_1}$, are now determined by Lemma 10 and we have to figure out the down arrows, representing the action of $X_{-2\varepsilon_1}$. The two dotted arrows will be used later on in the proof.

Consider the $\mathfrak{sl}_2$-subalgebra $\mathfrak{c}$ of $\mathfrak{g}$ generated by $e:=X_{2\varepsilon_1}$ and $f:=X_{-2\varepsilon_1}$. Set $h:=[e,f]$. Denote by Z the action of h in the leftmost weight space of the middle row. Then $Z=x-u$. The element h commutes with both h and $H_{\varepsilon_2-\varepsilon_1}$. Therefore, by (3), the operator Z commutes with both T_1 and T_2 and hence with both P and Q.

The algebra $\mathfrak{c}$ has the quadratic Casimir element $C_{\mathfrak{c}}$, whose action on the $\mathfrak{c}$-module given by the leftmost column of our picture is given by $x+f(Z)$, where f is some polynomial of degree two. From (3) it follows that the unique eigenvalue of this action is nonzero, in particular, $x+f(Z)$ is invertible. Let x' be a fixed square root $x+f(Z)$, which is a polynomial in $x+f(Z)$.

The elements $X_{\varepsilon_2-\varepsilon_1}$ and $X_{\varepsilon_2+\varepsilon_1}$ form a basis of a simple two-dimensional $\mathfrak{c}$-module with respect to the adjoint action. Using Proposition 7(iv) and arguments similar to those used in the proof of Lemma 10, we get that $C_{\mathfrak{c}}-(x'+1)^2$ or $C_{\mathfrak{c}}-(x'-1)^2$ annihilates the middle column (the sign depends on the original choice of x'). The middle column equals $X_{\varepsilon_2-\varepsilon_1}$ applied to the leftmost column.

Similarly, the elements $X_{\varepsilon_1-\varepsilon_2}$ and $X_{-\varepsilon_2-\varepsilon_1}$ form a basis of a simple two-dimensional $\mathfrak{c}$-module with respect to the adjoint action. Applying the same arguments as in the previous paragraph we get that $C_{\mathfrak{c}}-(x')^2$ annihilates any vector of the form $X_{\varepsilon_1-\varepsilon_2}X_{\varepsilon_2-\varepsilon_1}\mathrm{v}$, where v is from the leftmost column. This implies that the actions of $C_{\mathfrak{c}}$ and $X_{\varepsilon_1-\varepsilon_2}X_{\varepsilon_2-\varepsilon_1}$ and thus the actions of $C_{\mathfrak{c}}$ and C on the leftmost column commute. As the action of H commutes with the action of C, we thus obtain that x commutes with the action of C. This implies that x commutes with T_1+T_2. As it obviously commutes with T_1-T_2, we get that x commutes with both T_1 and T_2 and hence with both P and Q.

Similarly one shows that y, u, v and w commute with both P and Q. From the commutativity of $X_{\varepsilon_2-\varepsilon_1}$ and $X_{-2\varepsilon_1}$ we get the conditions

$$y(P+1)=(P-1)x,\quad V(P+3)=(P+1)u,\quad w(P+2)(P+3)=P(P+1)u.$$

Here everything commutes by the above and $P+1$, $P+2$ and $P+3$ are invertible (as $X_{\varepsilon_2-\varepsilon_1}$ acts bijectively). Therefore

$$y=(P-1)(P+1)^{-1}x,\quad v=(P+1)(P+3)^{-1}u,\quad w=P(P+1)(P+3)^{-1}(P+2)^{-1}u.$$

This implies that y, v and w are uniquely determined by x and u.

Since the actions of both $X_{\varepsilon_2-\varepsilon_1}$ and $X_{2\varepsilon_1}$ are completely determined, we can compute the action of $X_{2\varepsilon_2}$ and see that it is given (similarly to the action of $X_{2\varepsilon_1}$) by Id_V (this is depicted by the dotted arrows in the picture). As $X_{-2\varepsilon_2}$ and $X_{2\varepsilon_2}$ commute, we obtain that $w=x$, that is,

$$x=P(P+1)(P+3)^{-1}(P+2)^{-1}u. \tag{8}$$

Therefore the only parameter left for now is u.

On the one hand, the action of the element h on the middle dot of the second row is given by $y-v=(P-1)(P+1)^{-1}x-(P+1)(P+3)^{-1}u$. On the other hand, from $[h, X_{\varepsilon_2-\varepsilon_1}]=4X_{\varepsilon_2-\varepsilon_1}$ we have that this action equals $Z+4=x-u+4$.

This gives us the equation

$$(P-1)(P+1)^{-1}x-(P+1)(P+3)^{-1}u=x-u+4. \tag{9}$$

Using (9) and (8) we get the equation

$$\frac{P(P-1)}{(P+2)(P+3)}u+\frac{P+1}{P+3}u=\frac{P(P+1)}{(P+2)(P+3)}u-u+4.$$

This is a linear equation with nonzero coefficients and thus it has a unique solution, namely $u=(P+3)(P+2)$. Hence u is uniquely defined. The claim of the lemma follows. □

5. Consequences

Corollary 12. *Let $\boldsymbol{a}\in\mathbb{C}^n$ be such that $a_i\notin\mathbb{Z}$ and $a_i+a_j\notin\mathbb{Z}$ for all i and j. Let $M\in\hat{\mathscr{C}}$ and $\lambda\in\operatorname{supp}(M)$. Denote by U_0 the centralizer of $\mathfrak{h}$ in $U(\mathfrak{g})$. Then for any $A, B\in U_0$ the actions of A and B on M_λ commute.*

Proof. By Proposition 4, we may assume that $M\cong \mathrm{F}V$. For the module $\mathrm{F}V$ the claim follows from the formulae in (3). □

Corollary 13. *For any simple weight cuspidal $\mathfrak{g}$-module L with finite dimensional weight spaces we have* $\dim\operatorname{Ext}^1_{\mathfrak{g}}(L,L)=n$.

Proof. This follows from Theorem 1 and the observation that a similar equality is true for the unique simple $\mathbb{C}[[t_1,t_2,\dots,t_n]]$-module. □

We also recover the main result of [Britten et al. 2004]:

Corollary 14. *The category of all weight cuspidal $\mathfrak{g}$-modules is semisimple.*

Proof. By [Britten et al. 2004, Lemma 2], all blocks of the category of weight cuspidal $\mathfrak{g}$-modules are equivalent. Hence it is enough to prove the claim for the block containing $N(\boldsymbol{a})$ for some $\boldsymbol{a}\in\mathbb{C}^n$ such that $a_i+a_j\notin\mathbb{Z}$ for all i, j. From (3) it follows that the module $\mathrm{F}V$ is weight if and only if all operators T_i are semisimple, hence zero. Therefore from Theorem 1 we get that the block of the category of weight cuspidal modules is equivalent to the category of finite dimensional modules over $\mathbb{C}[[t_1,t_2,\dots,t_n]]/(t_1-0,t_2-0,\dots,t_n-0)\cong\mathbb{C}$. The claim follows. □

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