Tensor powers of the natural representation of $\mathrm{OSp}(r|2n)$

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Introduction

Lie algebras have been widely studied and are well-understood. Via highest weight theory one can e.g. classify finite dimensional irreducible modules, which arise as a quotient of Verma modules and the category of their finite dimensional representations is semisimple. Lie superalgebras provide a generalization of Lie algebras, which include a $\mathbb{Z}/2\mathbb{Z}$-grading. One can similarly define Cartan subalgebras, roots and in some settings apply tools from highest weight theory as well. However, it turns out that the category of finite dimensional representations is almost never semisimple. One can associate to every block an integer called the *atypicality*, which describes how far away this block is from being semisimple.

The toy example for a Lie superalgebra is the $\mathbb{Z}/2\mathbb{Z}$-graded analogue of $\mathfrak{gl}(n)$. Given a $\mathbb{Z}/2\mathbb{Z}$-graded vector space $V = V_0 \oplus V_1$, we can look at its linear endomorphisms $\mathfrak{gl}(V)$. This admits a $\mathbb{Z}/2\mathbb{Z}$-grading as well and a linear endomorphism f has degree $i \in \mathbb{Z}/2\mathbb{Z}$ if $f(V_j) \subseteq V_{i+j}$ for $j \in \mathbb{Z}$. By setting $[x, y] = x \circ y - (-1)^{|x||y|} y \circ x$ for homogeneous x and $y \in \mathfrak{gl}(V)$ we obtain the *general linear Lie superalgebra of* V.

The *orthosymplectic Lie superalgebra* $\mathfrak{g} = \mathfrak{osp}(r|2n)$ can be thought of as the super analogue of $\mathfrak{so}(r)$ and $\mathfrak{sp}(2n)$ simultaneously. It is the Lie subsuperalgebra of $\mathfrak{gl}(V)$ (for V of superdimension $(r|2n)$) leaving invariant a fixed nondegenerate supersymmetric bilinear form on V (i.e. a form of degree 0, which is symmetric on V_0 and skew-symmetric on V_1, see Definition 1.5). The extreme cases for $r = 0$ respectively $n = 0$ give the classical Lie algebras $\mathfrak{so}(r)$ respectively $\mathfrak{sp}(2n)$.

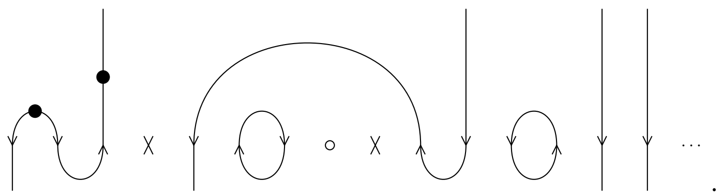
In Section 1.3, we will pass to the supergroup $\mathrm{OSp}(r|2n)$ due to better combinatorics. This is the supergroup of automorphisms preserving the fixed nondegenerate supersymmetric bilinear form on V. If r is now odd, the category of finite dimensional representations $\mathcal{F}$ of $\mathrm{OSp}(r|2n)$ decomposes as a direct sum $\mathcal{F}' \oplus \mathcal{F}'$, where $\mathcal{F}'$ describes the category of finite dimensional representations of $\mathfrak{osp}(r|2n)$. The difference between the two summands is given by whether $-\mathrm{id} \in \mathrm{OSp}(r|2n)$ acts by 1 or -1 on a module. In particular every finite dimensional irreducible module is given by a finite dimensional irreducible module for $\mathfrak{osp}(r|2n)$ and the action of $-\mathrm{id}$ by ± 1 (see Proposition 1.18).

The vector superspace V turns via matrix multiplication into a representation of $\mathrm{OSp}(r|2n)$. The main goal of this thesis is then to describe $V^{\otimes d}$ for $d \in \mathbb{N}$. As the category of finite dimensional representations is in general semisimple, $V^{\otimes d}$ does not decompose into a direct sum of irreducible modules (unlike the classical case), so we rather try to characterize the indecomposable summands. For small d and $\mathfrak{osp}(r|2n)$ Benkart, Shader and Ram described the direct summands combinatorially and gave character formulas for them in [BSR98]. Using new tools, we will give a description

of the indecomposable summands for all d. In particular, we can describe explicitly the socle and radical filtrations of these direct summands.
Classifying the indecomposable summands has already been achieved using Schur–Weyl duality for $\mathrm{OSp}(r|2n)$. There exists a surjection of the Brauer algebra (see (1.19) and [LZ17, Theorem 5.6]) onto the endomorphisms of $V^{\otimes d}$,

$$\mathrm{Br}_d(r-2n) \twoheadrightarrow \mathrm{End}_{\mathrm{OSp}(r|2n)}(V^{\otimes d}).$$

Using this in conjunction with the knowledge of idempotents in the Brauer algebra, Comes and Heidersdorf obtained a classification of indecomposable summands in [CH17]. Unfortunately, this characterization provides no information about the socle or head of these summands, much less on the socle and radical filtration. Even the Loewy length cannot be read off. This is also the approach Benkart, Shader and Ram [BSR98] used to obtain their combinatorial description of the direct summands for $\mathfrak{osp}(r|2n)$.
In Chapter 1 we are going to recall the definition of the Lie superalgebra $\mathfrak{osp}(r|2n)$ and summarize the basics on their representation theory and in particular the characterization of finite dimensional irreducible modules in terms of integral dominant weights (see Lemma 1.7). Via a reminder on Harish-Chandra pairs we recall the definition of $\mathrm{OSp}(r|2n)$ as well as the connections between its representation theory and the one of $\mathfrak{osp}(r|2n)$ (see Section 1.3).
After that we will recall the Brauer category, which is a categorical way to talk about the Brauer algebra, and in Theorem 1.32 present the classification of indecomposable summands in $V^{\otimes d}$ via certain partitions from [CH17, Thm. 7.3].
To provide more information on the structure of these indecomposable summands, we are going to use the Khovanov algebra of type B, first introduced by [ES16a]. The name Khovanov algebra originates from Khovanov homology, which gives a categorification of the Jones polynomial (at least for knots). The Khovanov algebra (of type A) extends this concept to categorify the Jones polynomial for all tangles. In type A the approach of using the Khovanov algebra to analyze indecomposable summands in $V^{\otimes d} \otimes (V^{\circledast})^{\otimes d'}$ for the general linear superalgebra $\mathfrak{gl}(m|n)$ was already pursued by [BS12] and in this thesis we will consider its type B analogue.
A basis for the Khovanov algebra K of type B is given by *oriented circle diagrams*, which look something like



The multiplication procedure (see Section 3.1) is based on a topological procedure (a TQFT), indicating again the connections to Khovanov homology. This algebra can be endowed with a grading and is locally unital. The category of locally finite dimensional modules is then an upper finite highest weight category in the sense of [BS21] with

standard modules $V(\lambda)$, irreducible modules $L(\lambda)$, and the indecomposable projective modules $P(\lambda)$ indexed by partitions λ (see Theorem 3.6).
This algebra can be thought of as another diagrammatical description of (a limit of) $\mathrm{Br}_d(r-2n)$. In contrast to the Brauer algebra, which is defined by generators and relations, the Khovanov algebra of type B is not given by generators and relations, but rather it has a description of the primitive idempotents built in.
In [ES21, Theorem 10.5], Ehrig and Stroppel proved that a subquotient (here called $e\tilde{K}e$) of this Khovanov algebra of type B is in fact isomorphic to a projective generator for $\mathrm{OSp}(r|2n)$ and thus gives rise to an equivalence

$$\Psi\colon (e\tilde{K}e)\text{-mod} \to \mathrm{OSp}(r|2n)\text{-mod} \tag{0.1}$$

of categories between the finite dimensional representations of $\mathrm{OSp}(r|2n)$ and finite dimensional $e\tilde{K}e$-modules. Both algebras K and $e\tilde{K}e$ can be endowed with a nonnegative grading and this actually induces a grading on $\mathrm{OSp}(r|2n)$-mod.
In Chapter 2 we present results from [ES21, Sections 7 + 8] to go back and forth between the two sides. Via the Brauer algebra and Schur–Weyl duality, we are able to classify indecomposable summands of $V^{\otimes d}$ via certain partitions. We will present in Definition 2.18 how one can translate these partitions into weight diagrams which are sequences of $\times$, $\circ$, $\vee$ and $\wedge$. On the other hand we can characterize irreducible finite dimensional modules L via an integral dominant weight λ and specifying the action of $-\mathrm{id}$ by ± 1 on L. We will also recall in Lemma 2.25 how one associates a weight diagram to this. A further very important result from [ES21] is an explicit map $\dagger$, which, using weight diagrams, provides the highest weight of the head of an indecomposable summand (Theorem 2.31).
Unfortunately, the equivalence Ψ above is not monoidal, so we do not have a direct interpretation of the tensor product of $\mathrm{OSp}(r|2n)$-modules on the Khovanov algebra side. In [BS12], Brundan and Stroppel encountered the same problem for the general linear superalgebra, when they tried to relate its finite dimensional representations to modules over the Khovanov algebra of type A. They solved it by considering the decomposition of the endofunctor $\_ \otimes V = \bigoplus_{i\in\mathbb{Z}} \theta_i$ and finding an analogue on the Khovanov algebra side. Namely they defined certain *geometric bimodules* $K^t_{\Lambda\Gamma}$ such that tensoring with these coincides with θ_i under the equivalence relating $\mathfrak{gl}(m|n)$-modules with modules over the Khovanov algebra of type A.
In Chapter 3, after recalling the definition of the Khovanov algebra of type B from [ES16a], we define type B equivalents of the geometric bimodules from [BS12]. After their definition we explicitly characterize their effect on the projective, standard and irreducible modules for K and we later extend this picture and define an analogue of these bimodules also for the case of $e\tilde{K}e$ in Section 3.4. There we also analyze their effect on projective and irreducible modules (observe that $e\tilde{K}e$-mod is not a highest weight category anymore).
Following up on that in Chapter 4 we will define the functors $\tilde{\Theta}_i\colon e\tilde{K}e\text{-mod} \to e\tilde{K}e\text{-mod}$, which are essentially given by tensoring with geometric bimodules and subsequently prove the following theorem.

Theorem A. *We have an equivalence of categories* $\Psi\colon (e\tilde{K}e)$*-mod* $\to \mathrm{OSp}(r|2n)$*-mod such that* $\theta_i \circ \Psi \cong \Psi \circ \tilde{\Theta}_i$*.*

We will, however, only sketch how one could prove some preparatory result relating the Brauer algebra with the Khovanov algebra of type B, and using this, prove the claim.
With the results so far, we are now able to analyze indecomposable summands of $V^{\otimes d}$ via tensoring the irreducible K-module corresponding to the trivial representation with geometric bimodules. Using our explicit descriptions of this from Chapter 3 we are able to prove the following results in Chapter 5 about these indecomposable summands $\mathbb{F}\mathrm{R}_\delta(\lambda)$ associated to a partition λ:

- $\mathbb{F}\mathrm{R}_\delta(\lambda)$ is self-dual and has simple head and socle (Theorem 5.1).
- The Loewy length is given by $2d(\lambda)+1$, where $d(\lambda)$ denotes the number of caps in the cap diagram of the weight diagram associated to λ (Proposition 5.2).
- The highest weight constituent sits in the middle Loewy layer with multiplicity 1 (Proposition 5.4).
- Every block has a unique irreducible $\mathbb{F}\mathrm{R}_\delta(\lambda)$ (Proposition 5.5).

Ultimately we will in Proposition 5.2 see that the grading filtration of every $\mathbb{F}\mathrm{R}_\delta(\lambda)$ agrees with its radical and its socle filtration (in particular they are rigid).
Given this, we investigate the question which irreducible $\mathrm{OSp}(r|2n)$-modules appear as direct summands in $V^{\otimes d}$. We will look at this question from two different angles. First we give in Corollary 5.14 different characterizations, when an indecomposable summand $\mathbb{F}\mathrm{R}_\delta(\lambda)$ is irreducible and after that we try to classify the irreducibles $L(\lambda,\varepsilon)$ appearing as a direct summand in Corollary 5.19.
For this we will recall the notion of a Kostant module in Section 5.1 (for further motivation see e.g. [BH09]). The definition originates from Kostant's theorem about the Lie algebra cohomology of a semisimple complex Lie algebra, which proves that the Lie algebra cohomology is multiplicity free as an $\mathfrak{h}$-module.

Theorem. *Let* $\mathfrak{g}$ *be a semisimple complex Lie algebra and* $L(\lambda)$ *a finite dimensional irreducible module of highest weight* λ*. Then there exists an* $\mathfrak{h}$*-module isomorphism*

$$H^n(\mathfrak{n}^+, L(\lambda)) \cong \bigoplus_{\substack{w\in W\\ l(w)=n}} \mathbb{C}_{w.\lambda}.$$

Using [Sch81, Lemma 5.13], we have $\mathrm{Ext}^n_{\mathcal{O}}(M(\mu), L(\lambda)) \cong H^n(\mathfrak{n}^+, L(\lambda))_\mu$ and under this isomorphism the Lie algebra cohomology being multiplicity free translates to

$$\sum_{n\geq 0} \dim \mathrm{Ext}^n_{\mathcal{O}}(M(\mu), L(\lambda)) \leq 1$$

for all μ.

We will also revisit Kazhdan–Lusztig polynomials in this setting, which were introduced to give a combinatorial way of describing multiplicities of irreducible modules inside Verma modules.
In type A there is the following proposition characterizing irreducible direct summands of $V^{\otimes d} \otimes (V^{\circledast})^{\otimes d'}$.

Proposition B. *Let λ be an integral dominant weight for $\mathfrak{gl}(m|n)$. Then the following are equivalent:*

- *For some Berezin twist μ of λ, we have that $L(\mu)$ is a direct summand of $V^{\otimes d} \otimes (V^{\circledast})^{\otimes d'}$.*
- *$L(\lambda)$ is a Kostant module.*
- *$L(\lambda)$ has a BGG-resolution.*
- *The Kazhdan–Lusztig polynomials $p_{\mu,\lambda}(q)$ are monomials for all $\mu \leq \lambda$.*
- *The character of $L(\lambda)$ is given by the Kac–Wakimoto formula.*

The equivalence of the middle three statements can be found in [BS12], for the first two see [Hei17] and the equivalence of the last two can be found in [CHR15].
In fact, we will find a very similar statement to hold true in our setting. Namely the following statements are equivalent for an indecomposable direct summand $\mathrm{R}(\lambda)$ in $V^{\otimes d}$ associated to a partition λ due to Corollary 5.14.

- $\mathrm{R}(\lambda)$ is irreducible.
- λ is a Kostant weight.
- The Kazhdan–Lusztig polynomials $p_{\mu,\lambda}(q)$ are monomials for all $\mu \leq \lambda$.
- The weight diagram associated to λ is $\vee\wedge$- and $\wedge\wedge$-avoiding.
- The cap diagram associated to λ is cap-free.
- The weight diagram associated to λ is maximal in the Bruhat order.

In order to try and classify the irreducible $\mathrm{OSp}(r|2n)$-modules, which appear as a direct summand in some $V^{\otimes d}$, we introduce an automorphism of order 2 on the category of finite dimensional $\mathrm{OSp}(r|2n)$-modules. It is defined via some manipulation on the Khovanov algebra side and e.g. interchanges the trivial with the natural representation for $r = 2n+1$. It maps $L(\lambda, \varepsilon)$ to $L(\lambda^{\square}, -\varepsilon)$ for some combinatorially defined weight $\lambda^{\square}$.
Corollary 5.19 then proves that the following statements are equivalent.

- $L(\lambda, \varepsilon)$ is a direct summand of some $V^{\otimes d}$ for some ε.
- λ or $\lambda^{\square}$ is a Kostant weight in the sense of [GH21].

And if r is odd or $\mathrm{at}(\lambda) > 1$ these are equivalent to

- $L(\lambda)$ or $L(\lambda^{\square})$ satisfies the Kac–Wakimoto conditions (considered as $\mathfrak{osp}(r|2n)$-modules).

After that in Chapter 6 we prove

Theorem C. *The Khovanov algebra K of type B is a locally unital Koszul algebra.*

A Koszul algebra A_0 is a positively graded algebra with semisimple A_0, which is as "close" to being semisimple as a graded algebra possibly can be, i.e.

$$\operatorname{Ext}^n_A(A_0, A_0\langle k\rangle) = 0 \quad \text{unless} \quad n = k.$$

The general idea is to reduce to finite weight diagrams, where the statement is known (see [Sey17]) and then do a limit argument. We provide (under some mild assumptions) a very general argument, which probably can also be used for other algebras arising as a limit.
Observe that this theorem does not deal with the subquotient $e\tilde{K}e$ which is related to $\mathrm{OSp}(r|2n)$, but rather K.
For $e\tilde{K}e$ Ehrig and Stroppel conjectured in [ES21] that the algebra $e\tilde{K}e$ (they call it $A_{(r|2n)}$) is Koszul as well and we will repeat the conjecture here.

Conjecture. *We conjecture that the algebra $e\tilde{K}e$ is a locally finite dimensional Koszul algebra.*

Unfortunately the tools provided here and also in [BS12] do not seem to suffice to conclude this from the existing theory. It would be very interesting for further research to develop new tools and provide results how Koszulity behaves under idempotent truncation and taking quotients by ideals.
Finally we provide some explicit examples. Namely we take a look at $\mathrm{OSp}(1|2)$, $\mathrm{OSp}(3|2)$ and $\mathrm{OSp}(2|2)$, classify and describe every indecomposable summand R of $V^{\otimes d}$ and compute explicitly $\mathrm{R} \otimes V$ for every indecomposable summand. For $\mathrm{OSp}(1|2)$, the category $\mathrm{OSp}(1|2)$-mod is semisimple and we even provide a closed formula for the multiplicities of $L(\lambda, \varepsilon)$ in $V^{\otimes d}$. Although these results have already been known (see e.g. [RS82] or [ES21]), they illustrate the power of the Khovanov algebra and we included them anyways and present the results explicitly.

Acknowledgments

First of all I would like to thank Dr. Heidersdorf for introducing me to the topic of Lie superalgebras. It has developed ever since into a real passion of mine and I really enjoyed working on this topic. Therefore I would like to thank Prof. Stroppel and Dr. Heidersdorf for supervising my thesis and giving me the opportunity to explore the world of Lie superalgebras further in my thesis. I am deeply grateful for every discussion, be it over some minor detail or the general picture. I appreciate you providing and sharing your mathematical insights and experience with me and making always time for my questions. I would like to thank you for proofreading earlier drafts of this thesis and providing lots of helpful comments. Furthermore, I would like to thank Lukas Bonfert, Timothy Meyer and Timm Peerenboom for reading and commenting my thesis and correcting my mistakes. Without you many orthographic and grammatical as well as mathematical errors would have stayed undetected. Finally I would like to thank my parents and friends for supporting me throughout my thesis. I appreciate their continued interest, even though I had difficulties explaining them the contents. It helped me very much to really focus on the core of the problem and get not distracted by other details.

1 Representation theory of $\mathfrak{osp}(r|2n)$ and $\mathrm{OSp}(r|2n)$

The ground field is always assumed to be $\mathbb{C}$. In this chapter we are going to introduce Lie superalgebras in particular $\mathfrak{osp}(r|2n)$ and look at the representation theory of the latter. After this we are going to present the connections between the representation theory of $\mathrm{OSp}(r|2n)$ and $\mathfrak{osp}(r|2n)$. Section 1.1 loosely follows [CW12a, Chapter 1] and subsequently we follow closely [ES17, Section 2] presenting the representation theory of $\mathrm{OSp}(r|2n)$.

1.1 Basic definitions

Definition 1.1. A *vector superspace* is a vector space equipped with a $\mathbb{Z}/2\mathbb{Z}$-grading $V = V_0 \oplus V_1$. An element of V_0 is called *even* and one of V_1 *odd.* For a homogeneous element $x \in V$ we denote its parity by $|x| \in \mathbb{Z}/2\mathbb{Z}$. In the following we will assume that $x \in V$ is homogeneous whenever $|x|$ occurs.
A morphism of vector superspaces is a linear morphism $f : V \to W$ such that $f(V_i) \subseteq W_i$.
The tensor product of super vector spaces V and W is the one of vector spaces graded by

$$(V \otimes W)_k = \bigoplus_{i+j=k} V_i \otimes W_j \tag{1.1}$$

for $i, j, k \in \mathbb{Z}/2\mathbb{Z}$. Together with the unit object $\mathbb{C}$ (in even degree) this turns the category of vector superspaces into a monoidal category. It is even symmetric with respect to the symmetry $c_{V,W} : V \otimes W \xrightarrow{\cong} W \otimes V$ given by

$$c_{V,W}(v \otimes w) = (-1)^{|v||w|} w \otimes v. \tag{1.2}$$

Definition 1.2. A *Lie superalgebra* is a vector superspace $\mathfrak{g}$ together with a bilinear map $[\_,\_] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ satisfying for $x, y, z \in \mathfrak{g}$

(i) $|[x, y]| = |x| + |y|$ ($\mathbb{Z}/2\mathbb{Z}$-*grading*),

(ii) $[x, y] = -(-1)^{|x||y|}[y, x]$ (*super skew symmetry*),

(iii) $(-1)^{|x||z|}[x, [y, z]] + (-1)^{|y||x|}[y, [z, x]] + (-1)^{|z||y|}[z, [x, y]] = 0$ (*super Jacobi identity*).

Definition 1.3. Given a Lie superalgebra $\mathfrak{g}$, a $\mathfrak{g}$-*module* is a vector superspace V together with a bilinear map $\mathfrak{g} \times V \to V, (x, v) \mapsto xv$ such that

$$x(yv) - (-1)^{|x||y|}y(xv) = [x, y]v$$

for all $x, y \in \mathfrak{g}$ and $v \in V$.
The vector superspace tensor product of two $\mathfrak{g}$-modules V and W again forms a $\mathfrak{g}$-module via

$$x(v \otimes w) = x(v) \otimes w + (-1)^{|x||v|}v \otimes x(w) \text{ (for } v \in V,\ w \in W).$$

Example 1.4. Given a vector superspace V, we can define the Lie superalgebra $\mathfrak{gl}(V) = \mathrm{End}_{\mathbb{C}}(V)$ given by all linear endomorphisms of V. An element $x \in \mathfrak{gl}(V)$ is even if $x(V_i) \subseteq V_i$ and odd if $x(V_i) \subseteq V_{i+1}$ for $i \in \mathbb{Z}/2\mathbb{Z}$. The bilinear bracket is given by

$$[x, y] = x \circ y - (-1)^{|x||y|}y \circ x$$

for $x, y \in \mathfrak{gl}(V)$.
The vector superspace V is a representation for $\mathfrak{gl}(V)$ by setting $xv = x(v)$ for all $x \in \mathfrak{gl}(V)$ and $v \in V$.
If $V = \mathbb{C}^{m|n}$ (i.e. $V_0 = \mathbb{C}^m$ and $V_1 = \mathbb{C}^n$), we denote $\mathfrak{gl}(V)$ by $\mathfrak{gl}(m|n)$.

Definition 1.5. Let V be a vector superspace and $\beta\colon V \times V \to \mathbb{C}$ a nondegenerate supersymmetric bilinear form (i.e. a nondegenerate bilinear form that is symmetric on V_0, skewsymmetric on V_1 and 0 on mixed products). Then $\mathfrak{osp}(V)$ is the Lie subsuperalgebra of $\mathfrak{gl}(V)$ given by

$$\mathfrak{osp}(V)_i := \{x \in \mathfrak{gl}(V)_i \mid \beta(x(a), b) = -(-1)^{|x||a|}\beta(a, x(b)) \text{ for all } a, b \in V\}.$$

Again if $V = \mathbb{C}^{r|2n}$ we write $\mathfrak{osp}(r|2n)$ for $\mathfrak{osp}(V)$.

Remark 1.6. If $V = \mathbb{C}^{r|2n}$ for odd r, an explicit choice for β is given by the bilinear form induced by the matrix

$$B := \left(\begin{array}{ccc|cc} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathcal{I}_m & 0 & 0 \\ 0 & \mathcal{I}_m & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & \mathcal{I}_n \\ 0 & 0 & 0 & -\mathcal{I}_n & 0 \end{array}\right).$$

If r is even, we just delete the first row and column.
Then $\mathfrak{osp}(r|2n)$ is given by all $x \in \mathfrak{gl}(r|2n)$ such that $x^{st}B + Bx = 0$, where x^{st} is the supertranspose of x which is defined by

$$\left(\begin{array}{c|c} a & b \\ \hline c & d \end{array}\right)^{st} = \left(\begin{array}{c|c} a^t & c^t \\ \hline -b^t & d^t \end{array}\right).$$

Explicitly $\mathfrak{osp}(r|2n)$ for odd r is given by all matrices of the following form

$$\left(\begin{array}{ccc|cc} 0 & -u^t & -v^t & x & x_1 \\ v & a & b & y & y_1 \\ u & c & -a^t & z & z_1 \\ \hline -x_1^t & -z_1^t & -y_1^t & d & e \\ x^t & z^t & y^t & f & -d^t \end{array}\right)$$

where a is any $(m\times m)$-matrix; b and c are skew-symmetric $(m\times m)$-matrices; d is any $(n\times n)$-matrix; e and f are symmetric $(n\times n)$-matrices; u and v are $(m\times 1)$-matrices; y, y_1, z and z_1 are $(m\times n)$-matrices; and x as well as x_1 are $(1\times n)$-matrices. In case that r is even, we again have to delete the first row and column.

1.2 Integral dominance for $\mathfrak{osp}(r|2n)$ and (n,m)-hook partitions

In this section we are going to classify the integral dominant weights for $\mathfrak{osp}(r|2n)$, where $r=2m$ or $r=2m+1$ and provide a different labelling set for these weights via (n,m)-hook partitions. A Cartan algebra $\mathfrak{h}$ for $\mathfrak{osp}(r|2n)$ is given by all diagonal matrices. These are of the form

$$\begin{cases} \operatorname{diag}(0,h_1,\ldots,h_m,-h_1,\ldots,-h_m|h_1',\ldots h_n',-h_1',\cdots-h_n') & \text{if } r \text{ is odd,}\\ \operatorname{diag}(h_1,\ldots,h_m,-h_1,\ldots,-h_m|h_1',\ldots h_n',-h_1',\cdots-h_n') & \text{if } r \text{ is even.}\end{cases}$$

We let ε_i for $1\leq i\leq m$ be the standard basis vectors of $\mathfrak{h}^*$, which pick out the $(i+1)$th (resp. ith) diagonal entry if r is odd (resp. even), and δ_j for $1\leq j\leq n$ be the ones picking out the $(r+j)$th diagonal entry. Now $\mathfrak{osp}(r|2n)$ decomposes into root spaces with respect to the adjoint action of $\mathfrak{h}$

$$\mathfrak{osp}(r|2n)=\mathfrak{h}\oplus\bigoplus_{\alpha\in\Phi}\mathfrak{osp}(r|2n)^{\alpha}, \tag{1.3}$$

where the set of roots Φ is given by (for details and the corresponding root vectors we refer to [CW12a, Section 1.2.4 + Section 1.2.5]):

- If r is odd
$$\Phi=\{\pm\varepsilon_i,\pm\varepsilon_i\pm\varepsilon_{i'},\pm\delta_j\pm\delta_{j'}\mid i\neq i'\}\cup\{\pm\delta_j,\pm\varepsilon_i\pm\delta_j\},$$
- and if r is even
$$\Phi=\{\pm\varepsilon_i\pm\varepsilon_{i'},\pm\delta_j\pm\delta_{j'}\mid i\neq i'\}\cup\{\pm\varepsilon_i\pm\delta_j\}$$
for $1\leq i\leq m$ and $1\leq j\leq n$.

The first set denotes the even and the second one the odd roots.

We will denote by

$$X(\mathfrak{osp}(r|2n)) := \bigoplus_{i=1}^{m} \mathbb{Z}\varepsilon_i \oplus \bigoplus_{j=1}^{n} \mathbb{Z}\delta_j$$

the integral weight lattice. When referring to a weight we will always mean an *integral weight*, i.e. an element of $X(\mathfrak{osp}(r|2n))$. The parity shift Π, which interchanges the even and odd part of a vector superspace, gives rise to a decomposition of $\mathfrak{osp}(r|2n)$-mod $= \mathcal{F}' \oplus \Pi\mathcal{F}'$, where $\mathcal{F}'$ contains all objects such that the parity of the weight space agrees with the parity of the corresponding weight (the parity of a weight is given by extending the assignment $\varepsilon_i \mapsto 0, \delta_j \mapsto 1$ $\mathbb{Z}$-linearly, see also [ES17, Section 2]). By [Ser11, Theorem 9.9] the finite dimensional irreducible $\mathfrak{osp}(r|2n)$-modules are all highest weight modules and the finite dimensional irreducible modules are up to isomorphism and parity shift uniquely determined by their highest weight.
In the following we will restrict ourselves to $\mathcal{F}'$ an thus its irreducible objects are uniquely determined by their highest weight. Next we describe those weight (called *integral dominant*) which appear as a highest weight of a finite dimensional irreducible module. For this we will follow [GS10, Section 5] and fix a certain choice of simple roots. This gives then rise to a set of positive roots Φ^+ and the corresponding ρ is given by $\frac{1}{2}(\sum_{\alpha\in\Phi_0^+}\alpha - \sum_{\beta\in\Phi_1^+}\beta)$. For this we let $\delta = r - 2n$.

- If $\mathfrak{g} = \mathfrak{osp}(2m+1|2n)$ and $m \geq n$ the simple roots are

$$\begin{gathered}\varepsilon_1-\varepsilon_2, \varepsilon_2-\varepsilon_3, \dots, \varepsilon_{m-n}-\varepsilon_{m-n+1},\\ \varepsilon_{m-n+1}-\delta_1, \delta_1-\varepsilon_{m-n+2}, \varepsilon_{m-n+2}-\delta_2, \dots, \varepsilon_m-\delta_n, \delta_n\end{gathered} \tag{1.4}$$

  and $\rho = (\frac{\delta}{2}-1, \frac{\delta}{2}-2, \dots, \frac{1}{2}, -\frac{1}{2}, \dots, -\frac{1}{2}|\frac{1}{2}, \dots, \frac{1}{2})$.

- If $\mathfrak{g} = \mathfrak{osp}(2m+1|2n)$ and $m < n$ the simple roots are

$$\begin{gathered}\delta_1-\delta_2, \delta_2-\delta_3, \dots, \delta_{n-m-1}-\delta_{n-m},\\ \delta_{n-m}-\varepsilon_1, \varepsilon_1-\delta_{n-m+2}, \delta_{n-m+2}-\varepsilon_2, \dots, \varepsilon_m-\delta_n, \delta_n\end{gathered} \tag{1.5}$$

  and $\rho = (-\frac{1}{2}, \dots, -\frac{1}{2}|-\frac{\delta}{2}, -\frac{\delta}{2}-1, \dots, \frac{1}{2}, \dots, -\frac{1}{2})$.

- If $\mathfrak{g} = \mathfrak{osp}(2m|2n)$ and $m > n$ the simple roots are

$$\begin{gathered}\varepsilon_1-\varepsilon_2, \varepsilon_2-\varepsilon_3, \dots, \varepsilon_{m-n-1}-\varepsilon_{m-n},\\ \varepsilon_{m-n}-\delta_1, \delta_1-\varepsilon_{m-n+1}, \varepsilon_{m-n+1}-\delta_2 \dots, \delta_n-\varepsilon_m, \delta_n+\varepsilon_m\end{gathered} \tag{1.6}$$

  and $\rho = (\frac{\delta}{2}-1, \frac{\delta}{2}-2, \dots, 1, 0, \dots, 0|0, \dots, 0)$.

- If $\mathfrak{g} = \mathfrak{osp}(2m|2n)$ and $m \leq n$ the simple roots are

$$\begin{gathered}\delta_1-\delta_2, \delta_2-\delta_3, \dots, \delta_{n-m}-\delta_{n-m+1},\\ \delta_{n-m+1}-\varepsilon_1, \varepsilon_1-\delta_{n-m+2}, \delta_{n-m+1}-\varepsilon 2, \dots, \delta_n-\varepsilon_m, \delta_n+\varepsilon_m\end{gathered} \tag{1.7}$$

and $\rho = (0, \ldots, 0| - \frac{\delta}{2}, -\frac{\delta}{2} - 1, \ldots, 1, 0 \ldots, 0)$.

The following lemma is due to [GS10, Cor. 3]:

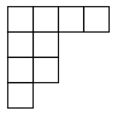
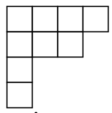
Lemma 1.7. *Let $\lambda \in X(\mathfrak{osp}(r|2n))$ and write $\lambda + \rho = \sum_{i=1}^m a_i\varepsilon_i + \sum_{j=1}^n b_j\delta_j$. Then λ is integral dominant if and only if $\lambda \in \bigoplus_{i=1}^m \mathbb{Z}\varepsilon_i \oplus \bigoplus_{j=1}^n \mathbb{Z}\delta_j$ and the following conditions hold:*

- *If $\mathfrak{g} = \mathfrak{osp}(2m+1|2n)$*
  - (i) *either $a_1 > a_2 > \cdots > a_m \geq \frac{1}{2}$ and $b_1 > b_2 > \cdots > b_n \geq \frac{1}{2}$,*
  - (ii) *or $a_1 > a_2 > \cdots > a_{m-l-1} > a_{m-l} = \cdots = a_m = -\frac{1}{2}$ and $b_1 > b_2 > \cdots > b_{n-l-1} \geq b_{n-l} = \cdots = b_n = \frac{1}{2}$.*
- *If $\mathfrak{g} = \mathfrak{osp}(2m|2n)$*
  - (i) *either $a_1 > a_2 > \cdots > |a_m|$ and $b_1 > b_2 > \cdots > b_n > 0$,*
  - (ii) *or $a_1 > a_2 > \cdots > a_{m-l-1} \geq a_{m-l} = \cdots = a_m = 0$ and $b_1 > b_2 > \cdots > b_{n-l-1} > b_{n-l} = \cdots = b_n = 0$.*

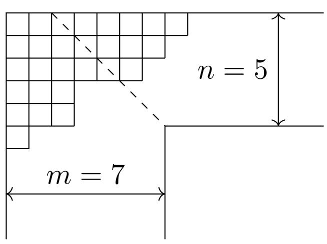
Definition 1.8. We denote the set of integral dominant weights for $\mathfrak{osp}(r|2n)$ by $X^+(\mathfrak{osp}(r|2n))$. We write $L^{\mathfrak{g}}(\lambda)$ for the finite dimensional irreducible module in $\mathcal{F}'$ with highest weight $\lambda \in X^+(\mathfrak{osp}(r|2n))$.

There also exists another commonly used labelling set of the integral dominant weights for $\mathfrak{osp}(r|2n)$ by so called (n,m)-*hook partitions* (for $r = 2m+1$ or $r = 2m$).

Definition 1.9. A *partition* is a sequence $\lambda = (\lambda_1, \lambda_2, \ldots)$ such that $\lambda_i \geq \lambda_{i+1}$ for all $i \geq 1$ and $\lambda_i \neq 0$ only for finitely many λ_i. If $\lambda_j = 0$ for all $j > k$, we will also write $\lambda = (\lambda_1, \ldots, \lambda_k)$. We define $|\lambda| = \sum_i \lambda_i$. If $|\lambda| = d$, we say that λ is a *partition of d*. A partition $\ulcorner\lambda$ is called (n,m)-*hook* if $\ulcorner\lambda_{n+1} \leq m$. By $\emptyset$ we denote the partition given by $\emptyset = (0, 0, \ldots)$ and by Λ we denote the set of all partitions.

In the following we will identify partitions with their *Young diagram*, i.e. a diagram that consists out of λ_i boxes in row i, e.g.  corresponds to the partition $(4, 2, 2, 1)$. The *transpose* λ^t of a partition λ is given by reflecting the Young diagram along the antidiagonal. In other words it is the Young diagram which has λ_i boxes in column i, e.g the transpose of the partition before would be  or $(4, 3, 1, 1)$.

Using the language of Young diagrams the following example gives an interpretation of being (n,m)-hook.



The following definition from [ES17, Definition 2.19] relates integral dominant weights and (m,n)-hook partitions.

Definition 1.10. We associate to an (n,m)-hook partition $^\ulcorner\lambda$ the weight $\mathrm{wt}(^\ulcorner\lambda) \in X^+(\mathfrak{osp}(r|2n))$ via $\mathrm{wt}(^\ulcorner\lambda) = (a_1,\dots,a_m|b_1,\dots,b_n) - \rho$, where a_i and b_j are defined as follows:

- If r is odd:
$$b_j = \max\Big(^\ulcorner\lambda_j - j - \frac{\delta}{2} + 1, \frac{1}{2}\Big) \quad \text{and} \quad a_i = \max\Big(^\ulcorner\lambda_i^t - i + \frac{\delta}{2}, -\frac{1}{2}\Big).$$
- If r is even:
$$b_j = \max\Big(^\ulcorner\lambda_j - j - \frac{\delta}{2} + 1, 0\Big) \quad \text{and} \quad a_i = \max\Big(^\ulcorner\lambda_i^t - i + \frac{\delta}{2}, 0\Big).$$

This almost defines an identification of (n,m)-hook partitions with $X^+(\mathfrak{osp}(r|2n))$. Only the integral dominant weights for $\mathfrak{osp}(2m|2n)$ with $a_m < 0$ do not correspond to an (n,m)-hook partition. As we will later concentrate on the algebraic supergroup $\mathrm{OSp}(r|2n)$ instead of the Lie superalgebra, we will first introduce the theory of finite dimensional representations of $\mathrm{OSp}(r|2n)$ and then revisit and upgrade this correspondence (we will see that the two representations with $a_m > 0$ and $-a_m$ form together one representation of the supergroup). Example 1.11 illustrates this almost bijection. For the actual bijection for $\mathrm{OSp}(r|2n)$ we refer to Proposition 1.18 and Proposition 1.22. This is also a reason why it is in this setup more convenient to work with the algebraic supergroup $\mathrm{OSp}(r|2n)$ instead of the Lie superalgebra $\mathfrak{osp}(r|2n)$ (see also Remark 1.33).

Example 1.11. For $\mathfrak{osp}(3|2)$ we have $m = n = 1$ and $\delta = 1$. We choose the simple roots to be $\varepsilon_1 - \delta_1$ and δ_1, and then ρ is given by $(-\frac{1}{2}|\frac{1}{2})$. A weight $\lambda \in \mathfrak{h}^*$ is then integral dominant by Lemma 1.7 (we write $\lambda + \rho = a'\epsilon_1 + b'\delta_1$ for some half-integers a' and b') if and only if $a', b' \geq \frac{1}{2}$ or $a' = -\frac{1}{2}$ and $b' = \frac{1}{2}$. This means that $\lambda = a\varepsilon_1 + b\delta_1$ is integral dominant if and only if $a = b = 0$ (corresponding to the trivial representation) or $a \geq 1$ and $b \geq 0$ for integers a and b. Via $(a|b) \mapsto (a, 1^b)^t$ we can identify integral dominant weights with $(1,1)$-hook partitions. This assignment is the inverse to Definition 1.10.
For $\mathfrak{osp}(2|2)$ we choose the simple roots $\delta_1 - \varepsilon_1$ and $\delta_1 + \varepsilon_1$ together with $\rho = (0|0)$. By Lemma 1.7 the integral dominant weights are given by $(a|b)$ for integers a and $b \in \mathbb{Z}$ such that either $b > 0$ or $a = b = 0$. The map defined in Definition 1.10 associates the integral dominant weight $(a|b)$ to a $(1,1)$-hook partition $(b, 1^a)$. Note that in this case the integral dominant weights $(a|b)$ with negative a have no $(1,1)$-hook partition counterpart.

Similar to the classical setting and category $\mathcal{O}$, the category $\mathcal{F}'$ decomposes into a direct sum $\mathcal{F}' = \bigoplus_\chi \mathcal{F}'_\chi$ for certain central characters χ. We define an equivalence relation on $X(\mathfrak{osp}(r|2n))$ via $\lambda \sim \mu$ if and only if $L^{\mathfrak{g}}(\lambda)$ and $L^{\mathfrak{g}}(\mu)$ belong to the

same $\mathcal{F}'_\chi$. Equivalence classes of this relation are called *blocks*. We will not go into detail here and refer to [Mus12, Section 8.2.4] as we are going to give a combinatorial description of the blocks in Definition 2.27 using the language of weight diagrams which is much more accessible.
We will later use another important concept called atypicality. We will define the degree of atypicality (see also [CW12a, Section 1.2 + Definition 2.29]), which indicates how far a block is from being semisimple. The following definitions can be found in [CW12a, Section 1.2.2 + (1.18) + Definition 2.29].

Definition 1.12. On $\mathfrak{h}^*$ we have the standard symmetric bilinear form $(\_,\_)$ which is given by $(\varepsilon_i, \varepsilon_j) = \delta_{i,j}$, $(\varepsilon_i, \delta_j) = 0$ and $(\delta_i, \delta_j) = -\delta_{i,j}$.
A root $\alpha \in \Phi$ is called *isotropic* if $(\alpha, \alpha) = 0$.
The *degree of atypicality* $\mathrm{at}(\lambda)$ of a weight $\lambda \in \mathfrak{h}^*$ is then the maximum number of mutually orthogonal odd isotropic roots $\alpha \in \Phi_1^+$ such that $(\lambda + \rho, \alpha) = 0$. An element $\lambda \in \mathfrak{h}^*$ is called *typical* if $\mathrm{at}(\lambda) = 0$ and *atypical* otherwise.

By [CW12a, Theorem 2.30] any two weights lying in the same block have the same atypicality.

Example 1.13. For $\mathfrak{osp}(3|2)$ the positive odd isotropic roots are given by $\varepsilon_1 - \delta_1$ and $\varepsilon_1 + \delta_1$, which are not orthogonal. Therefore the degree of atypicality is either 0 or 1. If λ is integral dominant and $\lambda + \rho = a\varepsilon_1 + b\delta_1$, we have $(\lambda + \rho, \varepsilon_1 - \delta_1) = a + b$ (this can only be 0 if $a = -\frac{1}{2}$ and $b = \frac{1}{2}$) and $(\lambda + \rho, \varepsilon_1 + \delta_1) = a - b$. Getting rid of $\rho = (-\frac{1}{2}|\frac{1}{2})$, the atypical weights for $\mathfrak{osp}(3|2)$ are given by $(0|0)$ and $(a|a-1)$ for $a \in \mathbb{Z}_{>0}$.

Definition 1.14. We define a partial order on $X^+(\mathfrak{osp}(r|2n))$ by saying that $\lambda \geq \mu$ for λ, $\mu \in X^+(\mathfrak{osp}(r|2n))$ if $\lambda - \mu \in \sum_{\alpha \in \Phi^+} \mathbb{N}_0 \alpha$.

1.3 The algebraic supergroups $\mathrm{SOSp}(r|2n)$ and $\mathrm{OSp}(r|2n)$

1.3.1 Algebraic supergroups and Harish-Chandra pairs

Definition 1.15. A *super Harish-Chandra pair* is a triple $(\mathfrak{g}, G_0, a)$, where $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is a Lie superalgebra, G_0 is an algebraic group with corresponding Lie algebra $\mathfrak{g}_0$ and a is a G_0-module structure on $\mathfrak{g}$ whose differential agrees with the adjoint action of $\mathfrak{g}_0$.
A *Harish-Chandra module* or a $(\mathfrak{g}, G_0, a)$*-module* is a $\mathfrak{g}$-module with a G_0-action such that the derivative of the G_0-action agrees with the action of $\mathfrak{g}_0$.
We denote by $(\mathfrak{g}, G_0, a)$-mod the category of finite dimensional $(\mathfrak{g}, G_0, a)$-modules.

We call the Harish-Chandra pair $(\mathfrak{osp}(r|2n), \mathrm{SO}(r) \times \mathrm{Sp}(2n), a)$ the *special orthosymplectic supergroup* $\mathrm{SOSp}(r|2n)$ and the pair $(\mathfrak{osp}(r|2n), \mathrm{O}(r) \times \mathrm{Sp}(2n), a)$ the *orthosymplectic supergroup* $\mathrm{OSp}(r|2n)$. In both cases a denotes the conjugation action of $\mathrm{SO}(r) \times \mathrm{Sp}(2n)$ (resp. $\mathrm{O}(r) \times \mathrm{Sp}(2n)$) on $\mathfrak{osp}(r|2n)$.

1.3.2 Finite dimensional representations of $\mathrm{SOSp}(r|2n)$

The Lie algebra $\mathfrak{osp}(r|2n)_0 = \mathfrak{so}(r) \oplus \mathfrak{sp}(2n)$ is semisimple, $\mathrm{char}(\mathbb{C}) = 0$ and $\mathrm{SO}(r) \times \mathrm{Sp}(2n)$ is simply connected, so every finite dimensional representation of $\mathfrak{so}(r) \oplus \mathfrak{sp}(2n)$ admits a $\mathrm{SO}(r) \times \mathrm{Sp}(2n)$-module structure by classical theory (see e.g. [Jan87, p. 194]).
Any finite dimensional $\mathfrak{osp}(r|2n)$-module is a $\mathfrak{so}(r) \times \mathfrak{sp}(2n)$-module via restriction and hence gives rise to a compatible $\mathrm{SO}(r) \times \mathrm{Sp}(2n)$-module structure. Thus every finite dimensional representation is also a Harish-Chandra module and hence a representation of $\mathrm{SOSp}(r|2n)$. This gives rise to a monoidal isomorphism of categories

$$\mathfrak{osp}(r|2n)\text{-mod} \cong \mathrm{SOSp}(r|2n)\text{-mod}. \tag{1.8}$$

1.3.3 Finite dimensional representations of $\mathrm{OSp}(r|2n)$

For combinatorial purposes we will look at $\mathrm{OSp}(r|2n)$ rather than $\mathrm{SOSp}(r|2n)$. This is very similar to $\mathfrak{gl}(n)$ admitting better combinatorics than $\mathfrak{sl}(n)$. The main advantage is that for $\mathrm{OSp}(r|2n)$ there are only morphisms $V^{\otimes d} \to V^{\otimes d'}$ if $d \equiv d' \mod 2$ (see also Remark 1.33).
Similarly to the decomposition $\mathfrak{osp}(r|2n)\,\mathrm{mod} = \mathcal{F}' \oplus \Pi\mathcal{F}'$, the category of finite dimensional $\mathrm{OSp}(r|2n)$-modules decomposes into $\mathcal{F} \oplus \Pi\mathcal{F}$ where $\mathcal{F}$ contains all objects such that the restriction to $\mathfrak{osp}(r|2n)$-mod lies in $\mathcal{F}'$. In the following we will restrict ourselves to $\mathcal{F}$.
To construct the irreducible representations of $\mathrm{OSp}(r|2n)$ we introduce a *(Harish-Chandra) induction functor*. So suppose we have a Harish-Chandra pair $(\mathfrak{g}, H, a)$ and a subgroup H' of H such that $(\mathfrak{g}, H', a|_{H'})$ is also a Harish-Chandra pair. Then we define the induction functor

$$\mathrm{Ind}_{\mathfrak{g},H'}^{\mathfrak{g},H} \colon (\mathfrak{g}, H', a|_{H'})\text{-mod} \to (\mathfrak{g}, H, a)\text{-mod} \tag{1.9}$$

by $\mathrm{Ind}_{\mathfrak{g},H'}^{\mathfrak{g},H} N = \{f \colon H \to N \mid f(xh) = xf(h)\ \forall h \in H, x \in H'\}$, just the usual induction for algebraic groups. The H-action is given by the right regular action and the $\mathfrak{g}$ action is the usual one on N. This functor is left exact, preserves injective objects and is right adjoint to the restriction functor (see [Jan87, Section 3]).

Remark 1.16. The category $\mathcal{F}$ contains enough projectives and injectives and these two actually agree. For a proof of this we refer the reader to [BKN11, Proposition 2.2.2].

In order to explicitly describe the irreducible objects we need to distinguish whether $r = 2m + 1$ is odd or $r = 2m$ is even. We will use $L^{\mathfrak{g}}(\lambda)$ to refer to the irreducible $\mathrm{SOSp}(r|2n)$-module of highest weight λ for an integral dominant weight $\lambda \in X^+(\mathfrak{osp}(r|2n))$.

Remark 1.17. The case distinction between $r = 2m+1$ and $r = 2m$ also happens in the classical case. Namely $\mathrm{O}(2m+1)$ is a direct product of $\mathrm{SO}(2m+1)$ with $\mathbb{Z}/2\mathbb{Z}$, where the nontrivial element in $\mathbb{Z}/2\mathbb{Z}$ corresponds to $-\,\mathrm{id}$. However, $\mathrm{O}(2m)$ is only given by a semidirect product $\mathrm{SO}(2m) \rtimes \mathbb{Z}/2\mathbb{Z}$ and the story is more involved. For a detailed analysis we refer to [GW09, Section 5.5.5]. The approach they use to classify irreducible $\mathrm{O}(r)$-representations from ones for $\mathrm{SO}(r)$ is very similar to the one in [ES17, Section 2.2] for $\mathrm{OSp}(r|2n)$, which we are following here.

The odd case: $\mathrm{OSp}(2m+1|2n)$

This case is fairly simple as

$$\mathrm{OSp}(2m+1|2n) \cong \mathrm{SOSp}(2m+1|2n) \times \mathbb{Z}/2\mathbb{Z} \tag{1.10}$$

where the generator of $\mathbb{Z}/2\mathbb{Z}$ corresponds to $-\,\mathrm{id}$ and we get the following classification (see [ES17, (1.4) + Proposition 2.6]):

Proposition 1.18. *For* $G = \mathrm{OSp}(2m+1|2n)$ *the set*

$$X^+(G) = X^+(\mathfrak{g}) \times \mathbb{Z}/2\mathbb{Z} = \{(\lambda, \varepsilon) \mid \lambda \in X^+(\mathfrak{g}), \varepsilon \in \{\pm\}\} \tag{1.11}$$

is a labelling set for the isomorphism classes of finite dimensional irreducible G*-modules in* $\mathcal{F}$*. The irreducible module* $L(\lambda, \varepsilon)$ *is just the irreducible module* $L^{\mathfrak{g}}(\lambda)$ *extended to a* G*-module by letting* $-\,\mathrm{id}$ *act by* ± 1*. Moreover the map*

$$\begin{aligned} \Psi\colon \{(m,n)\text{-}\textit{hook partitions}\} \times \mathbb{Z}/2\mathbb{Z} &\to X^+(G) \\ (\ulcorner\lambda, \varepsilon) &\mapsto (\mathrm{wt}(\ulcorner\lambda), \varepsilon) \end{aligned}$$

is a bijection.

Proof. The first part follows directly as $\mathrm{OSp}(2m+1|2n)$ is a direct product of $\mathrm{SOSp}(2m+1|2n)$ and $\mathbb{Z}/2\mathbb{Z}$. That Ψ is a bijection is [ES17, Lemma 2.21]. ☐

Remark 1.19. For the indecomposable projectives and injectives we have

$$\begin{aligned} \mathrm{Hom}_{\mathcal{F}}(I(\lambda,\varepsilon), I(\lambda,-\varepsilon)) &= \{0\}, \\ \mathrm{Hom}_{\mathcal{F}}(P(\lambda,\varepsilon), P(\lambda,-\varepsilon)) &= \{0\}, \\ \mathrm{Hom}_{\mathcal{F}}(P(\lambda,\varepsilon), P(\lambda,\varepsilon)) &= \mathrm{Hom}_{\mathcal{F}'}(P^{\mathfrak{g}}(\lambda), P^{\mathfrak{g}}(\lambda)). \end{aligned}$$

In particular, we have that $L(\lambda, \varepsilon)$ and $L(\mu, \varepsilon')$ lie in the same block if and only if $\varepsilon = \varepsilon'$ and $L^{\mathfrak{g}}(\lambda)$ and $L^{\mathfrak{g}}(\mu)$ belong to the same block in $\mathrm{SOSp}(2m+1|2n)$-mod.
For details see [ES17, Remark 2.8 + Corollary 2.9].

The even case: $\mathrm{OSp}(2m|2n)$

In this case we only have

$$\mathrm{OSp}(2m|2n) \cong \mathrm{SOSp}(2m|2n) \rtimes \mathbb{Z}/2\mathbb{Z} \tag{1.12}$$

and the situation is rather involved (see also [ES17, (1.5)]).

Definition 1.20. For $G = \mathrm{OSp}(2m|2n)$ and σ corresponding to the nontrivial element of $\mathbb{Z}/2\mathbb{Z}$ we introduce the set

$$X^+(G) := \{(\lambda, \varepsilon) \mid \lambda \in X^+(\mathfrak{g})/\sigma \text{ and } \varepsilon \in \mathrm{Stab}_\sigma(\lambda)\} \tag{1.13}$$

where $\mathrm{Stab}_\sigma(\lambda)$ is the stabilizer of λ for the group generated by σ.
Every $\lambda \in X^+(\mathfrak{g})$ is contained in a unique orbit consisting of either one or two elements. In the former case we denote the orbit by λ. The stabilizer has two elements and we will write $(\lambda, +)$ for (λ, e) and $(\lambda, -)$ for (λ, σ). In the latter case the stabilizer is trivial and we will denote the orbit by λ^G and abbreviate (λ^G, e) by λ^G.

Definition 1.21. A *signed (n, m)-hook partition* is an (n, m)-hook partition $^\ulcorner\lambda$ with $^\ulcorner\lambda_{n+1} = m$, i.e. the "hook" is completely filled (see Definition 1.9), or a pair $(^\ulcorner\lambda, \varepsilon)$ of an (n, m)-hook partition with $^\ulcorner\lambda_{n+1} < m$ (i.e. the hook is not completely filled) and a sign $\varepsilon \in \{\pm\}$.

The following Proposition constructs and classifies the irreducible G-modules and relate them to (m, n)-signed hook partitions.

Proposition 1.22. *Let $G = \mathrm{OSp}(2m|2n)$, $\mathfrak{g} = \mathfrak{osp}(2m|2n)$ and $G' = \mathrm{SOSp}(2m|2n)$. Assume that*

$$\lambda = \sum_{i=1}^{m} a_i \varepsilon_i + \sum_{j=1}^{n} b_j \delta_j - \rho \in X^+(\mathfrak{g}) \tag{1.14}$$

is an integral dominant weight and denote by $L^{\mathfrak{g}}(\lambda)$ the corresponding irreducible G'-module with highest weight λ. Then we have the following

(i) *If $a_m \neq 0$, then the $\mathrm{OSp}(2m|2n)$-module*

$$L(\lambda^G) = L(\lambda^G, e) = \mathrm{Ind}_{\mathfrak{g},G'}^{\mathfrak{g},G} L^{\mathfrak{g}}(\lambda) \tag{1.15}$$

is irreducible and moreover we have

$$\mathrm{Ind}_{\mathfrak{g},G'}^{\mathfrak{g},G} L^{\mathfrak{g}}(\lambda) \cong \mathrm{Ind}_{\mathfrak{g},G'}^{\mathfrak{g},G} L^{\mathfrak{g}}(\sigma(\lambda)). \tag{1.16}$$

If $a_m = 0$ then

$$\mathrm{Ind}_{\mathfrak{g},G'}^{\mathfrak{g},G} L^{\mathfrak{g}}(\lambda) = L(\lambda, +) \oplus L(\lambda, -) \tag{1.17}$$

decomposes into a direct sum of $L(\lambda, +)$ and $L(\lambda, -)$, which are two nonisomorphic irreducible $\mathrm{OSp}(2m|2n)$-modules. But they are isomorphic to $L^{\mathfrak{g}}(\lambda)$ when restricted to G'.

(ii) *The $L(\lambda, \varepsilon)$ for $(\lambda, \varepsilon) \in X^+(G)$ form a complete list of pairwise nonisomorphic irreducible $\mathrm{OSp}(2m|2n)$-modules in $\mathcal{F}$.*

(iii) *We have a bijection*

$$\Psi\colon \{\textit{signed } (n,m)\textit{-hook partitions}\} \to X^+(G), \quad \ulcorner\lambda \mapsto \mathrm{wt}(\lambda),$$
$$(\ulcorner\lambda, \pm) \mapsto (\mathrm{wt}(\lambda), \pm,).$$

Proof. This can be found in [ES17, Proposition 2.12+2.13, Lemma 2.21]. □

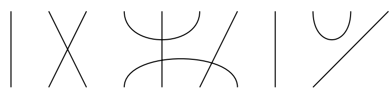
Example 1.23. For $\mathfrak{g} = \mathfrak{osp}(2|2)$ we had the irreducible representations $L^{\mathfrak{g}}(a|b)$, where either $a = b = 0$ or $b > 0$ for integers a and b. For $G = \mathrm{OSp}(2|2)$, we have irreducible representations of the form $L(0|b,+)$ and $L(0|b,-)$ which agree with $L^{\mathfrak{g}}(0|b)$ when restricting to $\mathfrak{g}$. For $a \neq 0$, we obtain a new (bigger) irreducible module $L((a|b)^G)$ which agrees with $L((-a|b)^G)$. We could also say that the irreducible $\mathrm{OSp}(2|2)$-representations are parametrized by $(a|b)$ with $a = b = 0$ or $a \geq 0$ and $b > 0$ integers. Now recall Example 1.11. For those integral dominant weights with $a \geq 0$ we could give a bijection to $(1,1)$-hook partitions. Using the additional signs in signed $(1,1)$-hook partitions and that for $\mathrm{OSp}(2|2)$ the irreducible modules for $(a|b)$ and $(-a|b)$ agree, we actually get the bijection between $X^+(G)$ and signed $(1,1)$-hook partitions.

Remark 1.24. There are also descriptions for the homomorphism spaces between indecomposable projectives and injectives, but these are more involved and we refer the reader to [ES17, Proposition 2.16 + Lemma. 2.17].

1.4 Brauer algebra

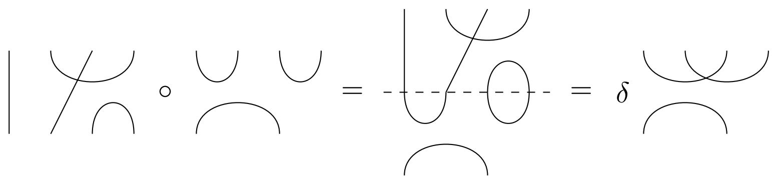
Definition 1.25. Let $\delta \in \mathbb{C}$. A *Brauer diagram of type* (r,s) is a partitioning of the set $\{1,\ldots,r+s\}$ into subsets of cardinality 2. This can be represented diagrammatically by identifying $p \in \{1,\ldots,r+s\}$ with the point $(p,0)$ if $1 \leq p \leq r$ and $(p-r,1)$ if $r < p \leq r+s$ in the plane and connecting the points in each subset by an arc inside the rectangle spanned by these points.

Example 1.26. The following is an example for a Brauer diagram of type $(9,11)$:



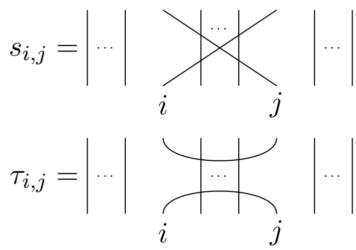
Definition 1.27. The *Brauer category* $\mathrm{Br}(\delta)$ for $\delta \in \mathbb{C}$ is the category with objects $d \in \mathbb{Z}_{\geq 0}$ and $\mathrm{Hom}_{\mathrm{Br}(\delta)}(r,s)$ is the $\mathbb{C}$-vector space with basis given by all (r,s)-Brauer diagrams. The multiplication is given by stacking diagrams vertically and evaluating a circle to δ. The Brauer category admits a monoidal structure, given by $m \otimes n = m+n$ on objects and on morphisms it is giving by stacking diagrams horizontally. The *Deligne category* Rep_δ is the additive Karoubian envelope of the $\mathrm{Br}(\delta)$. The *Brauer algebra* $\mathrm{Br}_d(\delta)$ is the endomorphism algebra $\mathrm{End}_{\mathrm{Br}(\delta)}(d)$ of d.

Example 1.28. The following illustrates the composition rule for Brauer diagrams.



The Brauer category $\mathrm{Br}(\delta)$ can also be described by generating morphisms with relations, for this we refer the reader to [ES21, Prop. 1.2].

We will define now certain special elements in $\mathrm{Br}_d(\delta)$ which will be of use later. By definition the group algebra $\mathbb{C}[\mathfrak{S}_d]$ of the symmetric group $\mathfrak{S}_d$ is a subalgebra of the Brauer algebra $\mathrm{Br}_d(\delta)$ (given by all permutation diagrams, i.e. every point on the bottom is connected to one on the top). We denote by $s_{i,j}$ the image of the transposition (i,j) $(1 \le i < j \le d)$ in $\mathrm{Br}_d(\delta)$. This is the diagram, which connects $(i,0)$ with $(j,1)$ and $(i,1)$ with $(j,0)$ as well as $(k,0)$ with $(k,1)$ for $1 \le k \le d, k \ne i,j$. Quite similarly we define the elements $\tau_{i,j}$ for $1 \le i < j \le d$ as the diagram, which connects $(k,0)$ with $(k,1)$ for $1 \le k \le d, k \ne i,j$ and $(i,0)$ with $(j,0)$ and $(i,1)$ with $(j,1)$. Graphically they look like



In the following we try to understand indecomposable objects in Rep_δ. For this we need to understand idempotents in $\mathrm{Br}_d(\delta)$ and we achieve this via primitive idempotents for $\mathbb{C}[\mathfrak{S}_d]$.

In $\mathrm{Br}_d(\delta)$ the span of all diagrams with at least one nonpropagating strand forms a two sided ideal J. The quotient $\mathrm{Br}_d(\delta)/J$ is canonically isomorphic to $\mathbb{C}[\mathfrak{S}_d]$ and we denote the quotient map by π. Observe that for a permutation diagram σ we have $\pi(\sigma) = \sigma$.

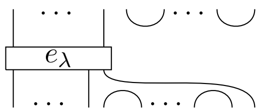
It is well-known that the primitive idempotents of $\mathbb{C}[\mathfrak{S}_d]$ are parametrized by partitions of d. For each partition λ of d we denote by z_λ the corresponding primitive idempotent in $\mathbb{C}[\mathfrak{S}_d]$. If we consider z_λ as an element of $\mathrm{Br}_d(\delta)$, we can decompose it into primitive idempotents $z_\lambda = e_1 + \cdots + e_k$. Then $\pi(e_1), \ldots, \pi(e_k)$ are idempotents summing to z_λ. But z_λ was primitive, hence there exists a unique $1 \le i \le k$ such that $\pi(e_i) \ne 0$. Define $e_\lambda \coloneqq e_i$. This definition depends a priori on the choice of a decomposition of z_λ into mutually orthogonal idempotents, but the conjugacy classes of the appearing idempotents in any decomposition of z_λ are unique, so e_λ is a primitive idempotent in $\mathrm{Br}_d(\delta)$, which is defined uniquely up to conjugacy.

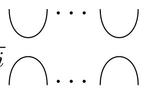
This gives a way of constructing primitive idempotents in $\mathrm{Br}_d(\delta)$ for a partition λ with $|\lambda| = d$. Next we are going to look at a way to construct idempotents in $\mathrm{Br}_d(\delta)$

corresponding to partitions with $|\lambda| < d$. For this let $\cup$ (resp. $\cap$) be the unique $(0,2)$-Brauer diagram (resp. $(2,0)$). We define

$$\psi_{r,i} = \mathrm{id}_r \otimes \cup^{\otimes i},$$
$$\phi_{r,i} = \begin{cases} \mathrm{id}_{r-1} \otimes \cap^{\otimes i} \otimes \mathrm{id}_1 & \text{if } r > 0, \\ \frac{1}{\delta^i} \cap^{\otimes i} & \text{if } r = 0 \text{ and } \delta \neq 0. \end{cases}$$

and set $e_\lambda^{(i)} = \psi_{|\lambda|,i} e_\lambda \phi_{|\lambda|,i}$. Pictorially speaking $e_\lambda^{(i)}$ is defined as follows

$e_\lambda^{(i)} =$  whenever $\lambda \neq 0$,

$e_\emptyset^{(i)} = \frac{1}{\delta^i}$  whenever $\delta \neq 0$,

where we have i cups and i caps in both pictures. Note that $e_\lambda^{(i)} \in \mathrm{Br}_{|\lambda|+2i}(\delta)$ and that it is defined for all partitions λ and all integers $i \geq 0$ except for $\lambda = \emptyset$ and $i = 0$. One can easily check that the $e_\lambda^{(i)}$ are all idempotents. The following theorem which can be found in [CH17, Thm 3.4] finishes our study of idempotents in the Brauer algebra.

Theorem 1.29. *The set $\{e_\lambda^{(i)} \mid \lambda \in \Lambda_d(\delta)\}$ is a complete set of pairwise nonconjugate primitive idempotents in $\mathrm{Br}_d(\delta)$, where $\Lambda_d(\delta)$ denotes the set*

$$\Lambda_d(\delta) := \begin{cases} \{\lambda \in \Lambda \mid |\lambda| = d - 2i, 0 \leq i \leq \frac{d}{2}\} & \text{if } \delta \neq 0 \text{ or } d \text{ is odd or } d = 0, \\ \{\lambda \in \Lambda \mid |\lambda| = d - 2i, 0 \leq i < \frac{d}{2}\} & \text{if } \delta = 0 \text{ and } d > 0 \text{ is even.} \end{cases} \tag{1.18}$$

Example 1.30. For the Brauer algebra $\mathrm{Br}_2(\delta)$ we have the two idempotents coming from $\mathfrak{S}_2$ given by $\frac{1-s_{1,2}}{2}$ and $\frac{1+s_{1,2}}{2}$. If $\delta = 0$ these are also the primitive idempotents, but if $\delta \neq 0$ the second one is not primitive anymore. Namely it decomposes as a sum of $\frac{1}{\delta}\tau_{1,2}$ and $\frac{1+s_{1,2}}{2} - \frac{1}{\delta}\tau_{1,2}$.

Definition 1.31. In Rep_δ every idempotent has an image and we set $\mathrm{R}_\delta(\lambda) := \operatorname{im} e_\lambda^0$.

The following theorem can be found in [CH17, Thm. 3.5] and classifies the indecomposable objects in $\mathrm{Br}(\delta)$.

Theorem 1.32. *The assignment $\lambda \mapsto \mathrm{R}_\delta(\lambda)$ defines a bijection between the set Λ of all partitions and isomorphism classes of nonzero indecomposable objects in Rep_δ.*

1.4.1 From Rep_δ to $\mathrm{OSp}(r|2n)$-mod

As the natural representation V of $\mathrm{OSp}(r|2n)$ has superdimension $\delta = r - 2n$ there exists a monoidal functor $\mathbb{F} = \mathbb{F}_{(r|2n)} \colon \mathrm{Rep}_\delta \to \mathcal{F}$ by the universal property of the

Deligne category Rep_δ, which is given by sending 1 to V (see [Del07, Proposition 9.4]). Then we have $\mathbb{F}(d) = V^{\otimes d}$ and we get an action of $\mathrm{Br}_d(\delta)$ on $V^{\otimes d}$. This functor is actually full (see [LZ17, Thm. 5.6]) and thus we have in particular a surjective algebra homomorphism

$$\Phi_{d,\delta}\colon \mathrm{Br}_d(\delta) \to \mathrm{End}_{\mathcal{F}}(V^{\otimes d}). \tag{1.19}$$

Remark 1.33. Note that for this functor $\mathbb{F}$ to be full it is crucial to work over $\mathrm{OSp}(r|2n)$ instead of $\mathfrak{osp}(r|2n)$. For $\mathfrak{osp}(r|2n)$ there exist morphisms $V^{\otimes d} \to V^{\otimes d'}$ for $d \not\equiv d' \mod 2$, whereas in this case $\mathrm{Hom}_{\mathrm{Br}(\delta)}(d, d') = 0$ (see also [ES16c, Remark 5.35] or [LZ17]).

Using the full functor $\mathbb{F}$ and some abstract categorical results (see [CW12b, Prop. 2.7.4]) we can extend the classification of indecomposable objects in Rep_δ to the indecomposable summands in $V^{\otimes d}$. The next result can be found in [CH17, Thm. 7.3].

Theorem 1.34. *The assignment $\lambda \mapsto \mathbb{F}\,\mathrm{R}_\delta(\lambda)$ defines a bijection between the set $\Lambda(d, r, n) := \{\lambda \in \Lambda_d(\delta) \mid \mathbb{F}\,\mathrm{R}_\delta(\lambda) \neq 0\}$ and a set of representatives of isomorphism classes of nonzero indecomposable summands in $V^{\otimes d}$.*

In Chapter 2 we are going to describe the set $\Lambda(d, r, n)$ in terms of characteristics of weight diagrams, but first we are going to take a look at the endofunctor $\_ \otimes V$, which is going to be crucial for the structure analysis of the indecomposable summands in $V^{\otimes d}$.

In the category Rep_δ, we have the endofunctor $\mathrm{ind} = \_ \boxtimes \mathrm{R}_\delta(\square)$ which is given by tensoring with $\mathrm{R}_\delta(\square)$. Note that diagrammatically it adds to each basis morphism one strand to the right. On the other hand we can consider the endofunctor $\_ \otimes V$ in the category $\mathrm{OSp}(r|2n)$-mod, and as the functor $\mathbb{F}$ is monoidal, we also have

$$(\_ \otimes V) \circ \mathbb{F} \cong \mathbb{F} \circ \mathrm{ind}\,.$$

In the following we would like to refine this isomorphism by decomposing $\_ \otimes V$ and ind into a direct sum of functors. For this we are going to introduce the so called *Jucys–Murphy elements*, originally defined by Nazarov in [Naz96].

Definition 1.35. The *Jucys–Murphy elements* $\xi_k \in \mathrm{Br}_d(\delta)$ for $1 \leq k \leq d$ are the elements

$$\xi_k := \frac{\delta - 1}{2} + \sum_{1 \leq i < k} (s_{i,k} - \tau_{i,k}).$$

Furthermore we define $\Omega_d := 2(\xi_1 + \cdots + \xi_d)$.

Lemma 1.36. *The Jucys–Murphy elements generate a commutative subalgebra $\mathrm{GZ}_d(\delta)$ of $\mathrm{Br}_d(\delta)$ and the element Ω_d is central in $\mathrm{Br}_d(\delta)$.*

Proof. This is [Naz96, Cor. 2.2]. □

This leads us to the following refinement of the induction functor for Rep_δ. For the well-definedness, we invite the reader to consult [ES21, Lemma 2.15].

Definition 1.37. For $i \in \mathbb{Z} + \frac{\delta}{2}$ we define the *i-induction functor*

$$\begin{aligned} i\text{-ind}\colon \mathrm{Rep}_\delta &\to \mathrm{Rep}_\delta, \\ M &\mapsto \mathrm{proj}_i(M \boxtimes \mathrm{R}_\delta(\lambda)). \end{aligned}$$

Here, for an indecomposable object $\mathrm{R}_\delta(\lambda)$, proj_i is the projection onto the generalized i-eigenspace of $\xi_{|\lambda|+1}$ viewed as an element in $\mathrm{End}_{\mathrm{Rep}_\delta}(\mathrm{R}_\delta(\lambda) \boxtimes \mathrm{R}_\delta(\square))$, which is then extended to arbitrary objects M.

We clearly have $\bigoplus_{i \in \mathbb{Z}+\frac{\delta}{2}} i\text{-ind} = \mathrm{ind}$.
On the other hand we have the Casimir element C in the universal enveloping algebra $U(\mathfrak{osp}(r|2n))$ (see [Mus12, Lemma 8.5.1]). This is central and thus multiplication by C denotes an endomorphism of every $\mathrm{OSp}(r|2n)$-module and we can look at the eigenvalue of this endomorphism.

Definition 1.38. The endofunctor $\_ \otimes V$ of $\mathcal{F}$ decomposes as $\_ \otimes V = \bigoplus_{i \in \mathbb{Z}+\frac{\delta}{2}} \theta_i$, where θ_i denotes the projection onto the summand, which changes the generalized eigenvalue of C by $2i$. We call θ_i the *i-translation functor*.

The following theorem from [ES21, Thm. 8.10] relates the notions of i-induction and i-translation.

Theorem 1.39. *The functor $\mathbb{F}_{(r|2n)}$ intertwines i-induction with i-translation for any $i \in \mathbb{Z} + \frac{\delta}{2}$, i.e.*

$$\mathbb{F}_{(r|2n)} \circ i\text{-ind} \cong \theta_i \circ \mathbb{F}_{(r|2n)}.$$

Example 1.40. The endofunctor 0-ind maps $\mathrm{R}_\delta(\emptyset)$ to $\mathrm{R}_\delta(\square)$. On $\mathcal{F}$ this corresponds to the fact that 0-translation applied to the trivial representation gives the natural representation.

2 Weight diagrams

The main reason to look at weight diagrams instead of partitions is the following. In Theorem 1.32 we have seen that the indecomposable objects in Rep_δ are given by $\mathrm{R}_\delta(\lambda)$ for a partition λ and the indecomposable summands in $V^{\otimes d}$ then by $\mathbb{F}\,\mathrm{R}_\delta(\lambda)$ (see Theorem 1.34). It turns out that these $\mathrm{OSp}(r|2n)$-modules have irreducible head (which would then be described via (n,m)-hook partitions). One really would like to determine the head of $\mathbb{F}\,\mathrm{R}_\delta(\lambda)$ by using a combinatorial procedure to construct the correct (n,m)-hook partition from λ. So far there is no way known (yet), to relate these concepts by only using the language of partitions. We also have been unsuccessful. Ehrig and Stroppel used instead the language of weight diagrams in [ES21, Theorem 7.8] to overcome this problem.

We will begin by recalling from [ES17] the notion of weight diagrams. These are certain sequences of the symbols $\circ$, $\times$, $\wedge$, $\vee$ and $\diamond$. To this one can associate cup and cap diagrams and a (compatible) triple of a cup, a cap and a weight diagram will give a basis vector in the Khovanov algebra K (see Chapter 3 for a definition). We will also define certain numbers associated to a weight diagram, which will later be connected to the grading of K, the allowed weight diagrams for $e\tilde{K}e$ and the Kazhdan–Lusztig polynomials defined in Chapter 5.

The rest of this chapter we will mainly follow [ES21, Sections 7 + 8] and present the explicit translation between weight diagrams and partitions.

Throughout this chapter we fix natural numbers r and n and set $\delta = r - 2n$, $m = \lfloor \frac{\delta}{2} \rfloor$ and denote by $L = \left(\mathbb{Z} + \frac{\delta}{2}\right) \cap \mathbb{R}_{\geq 0}$ the nonnegative (half) integer line. Furthermore we call elements of L *vertices*.

For every $\mathrm{R}_\delta(\lambda)$ we will later get an irreducible K-module. These irreducible modules will be labeled by *Deligne weight diagrams* and we will present the correspondence between partitions and Deligne weight diagrams. On the other hand the highest weights of irreducible $\mathrm{OSp}(r|2n)$-modules are parametrized by (n,m)-hook partitions (together with a sign ε) and we will also recall the associated weight diagram to this. These are called *hook weight diagrams*.

Ehrig and Stroppel provided in [ES21, Definition 7.7] a combinatorially defined map $\dagger$ which associates to the Deligne weight diagram the hook weight diagram of the head of the corresponding $\mathbb{F}\,\mathrm{R}_\delta(\lambda)$ (see Theorem 2.31).

Ehrig and Stroppel provided a third set of weight diagrams, the so called *super weight diagrams*. For proving a Morita equivalence between $\mathrm{OSp}(r|2n)$-mod and a subquotient $e\tilde{K}e$ of the Khovanov algebra, they identified $e\tilde{K}e$ with a projective generator for $\mathrm{OSp}(r|2n)$ (see [ES21, Theorem 10.5] and also the proof of Theorem 4.4). These are mainly the Deligne weight diagrams which are associated to partitions which give rise to projective $\mathbb{F}\mathrm{R}_\delta(\lambda)$. We will also give a reminder on those and explain how one can translate from hook weight diagrams to super weight diagrams.

2.1 Basic definitions

Definition 2.1. A *weight diagram* μ is a map $\mu\colon L \to \{\times, \circ, \vee, \wedge, \diamond\}$ such that $\diamond$ can only occur as image of 0 and conversely the image of 0 can only be $\circ$ or $\diamond$. Furthermore for $? \in \{\circ, \times, \vee, \wedge, \diamond\}$, we denote by $\#?(\mu) \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$ the number of ?'s appearing in μ.
The symbols $\circ$, $\times$, $\vee$, $\wedge$ and $\diamond$ are called *nought*, *cross*, *down*, *up* and *diamond*, respectively.

Definition 2.2. We call a weight diagram μ *admissible* if $\#\circ(\mu)+\#\times(\mu)+\#\wedge(\mu) < \infty$ and *flipped* if $\#\circ(\mu) + \#\times(\mu) + \#\vee(\mu) < \infty$.

Definition 2.3. Two admissible weight diagrams λ and μ belong to the same block Λ if the position of $\circ$'s and $\times$'s agree and either $\#\wedge(\lambda) \equiv \#\wedge(\mu) (\mathrm{mod}\ 2)$ or they both start with $\diamond$.

We usually draw a weight diagram as a sequence together with the lowest number of L (sometimes we also omit this number), i.e.

$$\begin{array}{ccccccccc} 0 & & & & & & & & \\ \diamond & \vee & \vee & \wedge & \circ & \times & \vee & \vee & \cdots \end{array}$$

or

$$\begin{array}{ccccccccc} \frac{1}{2} & & & & & & & & \\ \wedge & \wedge & \times & \times & \circ & \vee & \circ & \vee & \cdots \end{array}$$

By turning every symbol upside down (i.e. exchanging $\vee$'s and $\wedge$'s) we obtain a bijection between admissible and flipped weight diagrams.

Definition 2.4. We call two symbols neighbored if they are only separated by $\circ$'s and $\times$'s. For the following a $\diamond$ can be interpreted either as $\vee$ or $\wedge$. For two admissible weight diagrams μ, λ we say that μ is obtained from λ by a *Bruhat move*, if one of the following holds:

- λ has a pair of neighboring labels $\vee\wedge$ (say at positions i,j) and μ is obtained by replacing these by $\wedge\vee$. This is called a *type A move* applied at positions i and j.
- λ starts (up to some $\circ$'s and $\times$'s) with neighboring labels $\wedge\wedge$ at positions i and j and μ is obtained by replacing these with $\vee\vee$. This is called a *type D move* applied at positions i and j.

We define a partial order on the set of admissible weight diagrams by saying $\lambda \leq \mu$ if μ can be obtained from λ by a sequence of Bruhat moves. Note that $\lambda \leq \mu$ implies that λ and μ lie in the same block.

Remark 2.5. Observe that for even δ and a weight diagram starting at positions 0 and 1 with $\diamond\wedge$, we can apply a type A move and a type D move at these positions, both produce $\diamond\vee$.

Example 2.6. For $\delta = 1$, $\lambda = \wedge\times\wedge\wedge\circ\vee\dots$ and $\mu = \wedge\times\vee\vee\circ\vee\dots$ we have $\lambda \leq \mu$ because we can apply the following sequences of moves

$$\wedge\times\wedge\wedge\circ\vee\dots \xrightarrow{D} \vee\times\vee\wedge\circ\vee\dots \xrightarrow{A} \vee\times\wedge\vee\circ\vee\dots \xrightarrow{A} \wedge\times\vee\vee\circ\vee\dots.$$

Definition 2.7. Let λ and μ be two admissible weight diagrams belonging to the same block. Suppose that μ has m symbols $\wedge$ and λ has $m+2k$ symbols $\wedge$. We define then $l_0(\lambda,\mu) := 2k$ and for $i \in L$ we set $l_i(\lambda,\mu) = 0$ if $\lambda(i) \in \{\times,\circ\}$ and otherwise

$$l_i(\lambda,\mu) := 2k + \#\{i \geq j \in L \mid \lambda(j) = \vee\} - \#\{i \geq j \in L \mid \mu(j) = \vee\}.$$

Note that, as λ and μ are admissible, we have $l_n(\lambda,\mu) = 0$ for big enough n. Therefore $l(\lambda,\mu) = \sum_{i\geq 1} l_i(\lambda,\mu)$ is well-defined and finite.

Lemma 2.8. *Let λ and μ be admissible weight diagrams. Then $\lambda \leq \mu$ if and only if $l_i(\lambda,\mu) \geq 0$ for all $i \geq 0$.*

Proof. This is [Sey17, Prop. 1.1.8] after observing that we can restrict to the finite case as $\lambda(i) = \mu(i) = \vee$ for all $i \in L$ big enough. □

Example 2.9. Let λ and μ be as in Example 2.6. Then the integers $l_i(\lambda,\mu)$ are given by

i	0	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{5}{2}$	$\geq \frac{7}{2}$
$l_i(\lambda, mu)$	2	2	0	1	0

This gives then $l(\lambda,\mu) = 3$. We also see that $l_i(\lambda,\mu) \geq 0$ for all i and in Example 2.6 we also have seen that $\lambda \leq \mu$.

Definition 2.10. The *cup diagram* $\underline{\mu}$ associated to an admissible or a flipped weight diagram μ is obtained by applying the following steps.

(C-1) First connect neighbored vertices labeled $\vee\wedge$ successively by a cup, i.e. we connect the vertices by an arc forming a cup below. Repeat this step as long as possible, ignoring already joint vertices. Note that the result is independent of the order in which the connections are made

(C-2) Attach a vertical ray to each remaining $\vee$.

(C-3) Connect pairs of neighbored $\wedge$'s from left to right by cups (we interpret $\diamond$ for this as a $\wedge$). It might be necessary to attach infinitely many cups in this step.

(C-4) If a single $\wedge$ or $\diamond$ remains, attach a vertical ray.

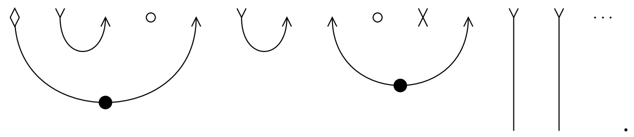
(C-5) Put a marker $\bullet$ on each cup created in (C-3) and each ray created in (C-4).

(C-6) We erase the marker from the component that contains the $\diamond$ if the number of placed markers in (C-5) is finite and odd.

(C-7) Finally delete all $\vee$ and $\wedge$ labels at vertices.

The cups and rays are always drawn without intersections, and two cup diagrams are said to be the same if there is a bijection between the cups and rays, respecting the connected vertices and the markers $\bullet$. We call cups and rays with a marker *dotted* and those without $\bullet$ *undotted*.
A *cap diagram* is just the horizontal mirror image of a cup diagram, for a cup diagram a we denote by a^* the cap diagram obtained by horizontal mirroring and vice versa.

Example 2.11. The associated cup diagram to $\diamond \vee \wedge \circ \wedge \vee \wedge \wedge \circ \times \wedge \vee \vee \cdots$ looks like



We observe that admissible diagrams produce infinitely many undotted vertical rays, whereas flipped diagrams have infinitely many dotted cups.

Definition 2.12. Given a weight diagram μ, we call the total number of cups (dotted as well as undotted) in its weight diagram $\underline{\mu}$ the *defect* $\operatorname{def}(\mu)$ of μ. The *rank* of μ is defined to be $\operatorname{rk}(\mu) \coloneqq \min(\#\circ(\mu), \#\times(\mu))$. The *layer number* of μ is $\kappa(\mu) \coloneqq \operatorname{def}(\mu) + \operatorname{rk}(\mu)$.

Definition 2.13. We associate to a subset $S \subseteq \mathbb{Z} + \frac{\delta}{2}$ the *weight diagram* λ_S, which is given at position $i \in L$ by $\diamond$ if $i = 0 \in S$, and otherwise

$$\begin{cases} \wedge & \text{if } i \in S \text{ but } -i \notin S, \\ \vee & \text{if } -i \in S \text{ but } i \notin S, \\ \times & \text{if } i \in S \text{ and } -i \in S, \\ \circ & \text{if } i \notin S \text{ and } -i \notin S. \end{cases}$$

Definition 2.14. An *oriented cup diagram* $a\lambda$ is a cup diagram a together with a weight diagram λ such that the positions of the appearing $\circ$'s (resp. $\times$'s) agree and every cup (resp. ray) is oriented as in Figure 2.1. An oriented cap diagram λb is just a cap diagram b together with a Deligne weight diagram λ such that $b^*\lambda$ is an oriented cup diagram.

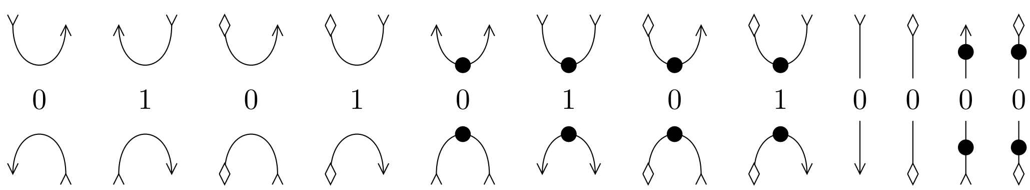
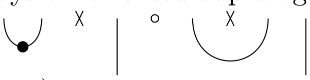
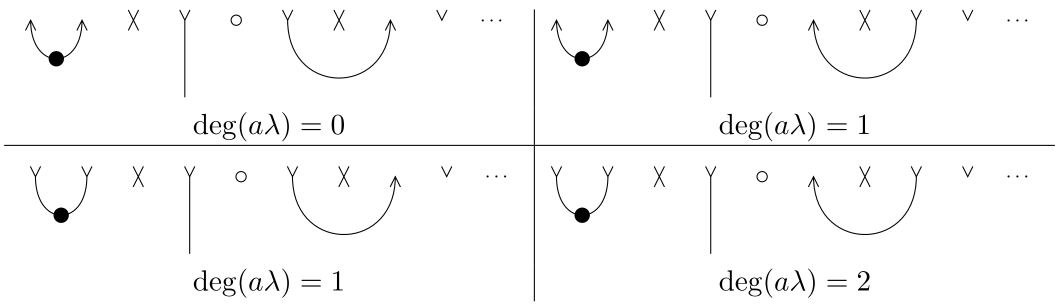


Figure 2.1: Orientations and degrees

Definition 2.15. A *circle diagram* ab is a cup diagram a put beneath a cap diagram b, such that the positions of the appearing $\circ$'s (resp. $\times$'s) agree. An *oriented circle diagram* $a\lambda b$ is a circle diagram ab together with a Deligne weight diagram λ such that $a\lambda$ and λb are oriented cup (resp. cap) diagrams.

Definition 2.16. Given an oriented cup diagram $a\lambda$ each cup and ray has an associated integer according to Figure 2.1. The sum of all these integers is called the *degree* $\deg(a\lambda)$ of the oriented cup diagram $a\lambda$. The degree of an oriented cap diagram λb is defined as $\deg(\lambda b) := \deg(b^*\lambda)$. For an oriented circle diagram $a\lambda b$, we define $\deg(a\lambda b) = \deg(a\lambda) + \deg(\lambda b)$.
The cups and caps in Figure 2.1 with a 1 are called *clockwise* and those with a 0 *anticlockwise*.

Example 2.17. If we take any weight diagram λ, by the definition of the associated cup diagram $\underline{\lambda}\lambda$ is always an oriented cup diagram.
The cup diagram $a =$  ... admits exactly four orientations, which (and whose degrees) are given by



2.2 Deligne weight diagrams

Definition 2.18. Given a partition $\lambda \in \Lambda$, we associate to it the set

$$X(\lambda) := \{\lambda_i^t - i + 1 - \frac{\delta}{2} \mid i \geq 1\} \subset \mathbb{Z} + \frac{\delta}{2}.$$

Using Definition 2.13 we can associate a weight diagram to $X(\lambda)$. We denote it by λ_δ and call it *Deligne weight diagram*. Furthermore we denote the set of all Deligne weight diagrams by Λ_δ.

Example 2.19. For odd δ, the partition $\emptyset$ corresponds to

$$\emptyset_\delta = \begin{cases} \overset{\frac{1}{2}}{\circ} \quad \cdots \quad \circ \quad \overset{\frac{\delta}{2}}{\vee} \quad \vee \quad \vee \quad \vee \quad \cdots & \text{if } \delta > 0, \\ \overset{\frac{1}{2}}{\times} \quad \cdots \quad \times \quad \overset{-\frac{\delta}{2}}{\times} \quad \vee \quad \vee \quad \vee \quad \cdots & \text{if } \delta < 0. \end{cases}$$

For even δ we have

$$\emptyset_\delta = \begin{cases} \begin{array}{cccccccc} 0 & & & \frac{\delta}{2} & & & & \\ \circ & \cdots & \circ & \vee & \vee & \vee & \vee & \cdots \end{array} & \text{if } \delta > 0, \\ \begin{array}{cccccccc} 0 & & & -\frac{\delta}{2} & & & & \\ \Diamond & \times & \cdots & \times & \vee & \vee & \vee & \cdots \end{array} & \text{if } \delta \leq 0. \end{cases}$$

Remark 2.20. Suppose we have the Deligne weight diagram $\emptyset_\delta$ associated to the empty partition $\emptyset$. Then we can deduce the Deligne weight diagram for any other partition using

$$\begin{array}{cccccccccc} \vee & \vee & \vee & \vee & \vee & \vee & \vee & \vee & \vee & \cdots \\ \wedge & \wedge & \wedge & \wedge & \wedge & & & & & \end{array}$$

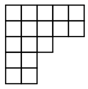
and the following construction: The above picture denotes the Deligne weight diagram $\emptyset_\delta$ for $\delta = -9$.

For this remark we think of $\times$ as a $\vee$ and $\wedge$ together at the same position and of $\circ$ as the absence of these two symbols. Furthermore, we think of $\wedge$ as an upside down $\vee$. Now in the picture above we indicated an arrow. We enumerate the $\vee$'s (and the upside down $\vee$'s) by their position on the arrow. Namely the rightmost $\wedge$ is at the first position, the $\wedge$ to the left at the second and so on. The leftmost $\vee$ is at position six in this example and the $\vee$ to the right of it at position seven. We can then move the i-th symbol in this enumeration λ_i^t steps along the arrow. For example, if we move the first symbol (i.e. the rightmost $\wedge$) four steps along the arrow, it will end up under the last $\vee$ which is drawn in the picture. A $\vee$ moves first to the left until it reaches the left boundary of the picture, then turns upside down into a $\wedge$ and moves further to the right.

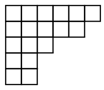
Doing this for a partition λ the resulting weight diagram is λ_δ. The following example presents this procedure in detail.

Example 2.21. Let $\delta = -5$. We want to deduce the Deligne weight diagram for $\lambda = \ydiagram{6,5,3,2,2}$ as described in the previous remark.

$$\begin{array}{cccccccccccc}
\emptyset & \rightsquigarrow & \begin{smallmatrix}\vee\\ \wedge\end{smallmatrix} & \begin{smallmatrix}\vee\\ \wedge\end{smallmatrix} & \begin{smallmatrix}\vee\\ \wedge\end{smallmatrix} & \vee & \vee & \vee & \vee & \vee & \vee & \cdots \\
\ydiagram{1,1,1,1,1} & \rightsquigarrow & \begin{smallmatrix}\vee\\ \wedge\end{smallmatrix} & \begin{smallmatrix}\vee\\ \wedge\end{smallmatrix} & \vee & \vee & \vee & \vee & \vee & \begin{smallmatrix}\vee\\ \wedge\end{smallmatrix} & \vee & \cdots \\
\ydiagram{2,2,2,2,2} & \rightsquigarrow & \begin{smallmatrix}\vee\\ \wedge\end{smallmatrix} & \vee & \vee & \vee & \vee & \vee & \begin{smallmatrix}\vee\\ \wedge\end{smallmatrix} & \begin{smallmatrix}\vee\\ \wedge\end{smallmatrix} & \vee & \cdots \\
\ydiagram{3,3,3,2,2} & \rightsquigarrow & \vee & \vee & \vee & \begin{smallmatrix}\vee\\ \wedge\end{smallmatrix} & \vee & \vee & \begin{smallmatrix}\vee\\ \wedge\end{smallmatrix} & \begin{smallmatrix}\vee\\ \wedge\end{smallmatrix} & \vee & \cdots \\
\ydiagram{4,4,3,2,2} & \rightsquigarrow & & \begin{smallmatrix}\vee\\ \wedge\end{smallmatrix} & \vee & \begin{smallmatrix}\vee\\ \wedge\end{smallmatrix} & \vee & \vee & \begin{smallmatrix}\vee\\ \wedge\end{smallmatrix} & \begin{smallmatrix}\vee\\ \wedge\end{smallmatrix} & \vee & \cdots
\end{array}$$



$$\rightsquigarrow \begin{array}{cccccccccc} & & \vee & \vee & \vee & \vee & \vee & \vee & \vee & \dots \\ \wedge & \wedge & & \wedge & & & \wedge & \wedge & & \end{array}$$



$$\rightsquigarrow \begin{array}{cccccccccc} & \vee & & \vee & \vee & \vee & \vee & \vee & \vee & \dots \\ \wedge & \wedge & & \wedge & & & \wedge & \wedge & & \end{array}$$

Therefore we have $\lambda_\delta = \wedge \times \circ \times \vee \vee \times \times \vee \vee \dots$, which also agrees with Definition 2.18.

Remark 2.22. We observe the following facts for Deligne weight diagrams (for fixed δ):

- Every Deligne weight diagram has only finitely many positions which are different from $\vee$ and hence is admissible.
- The number $\#\circ(\lambda) - \#\times(\lambda)$ is fixed for all Deligne weight diagrams λ and is given by $\lfloor \frac{\delta}{2} \rfloor$.

The above remark actually gives us enough information to classify all weight diagrams, which arise from partitions.

Lemma 2.23. *The assignment $\lambda \mapsto \lambda_\delta$ defines a bijection*

$$\{\textit{partitions}\} \to \{\textit{admissible weight diagrams } \mu \textit{ such that } \#\circ(\mu) - \#\times(\mu) = \lfloor \frac{\delta}{2} \rfloor\}$$

Proof. This is [ES21, Lemma. 7.1] □

With the description of partitions in terms of Deligne weight diagrams, we are now able to classify the set $\Lambda(d, r, n)$ using the following theorem which can be found in [CH17, Cor. 7.14].

Theorem 2.24. *There is an equality of sets $\Lambda(d, r, n) = \{\lambda \in \Lambda_d(\delta) \mid \kappa(\lambda_\delta) \leq \min(m, n)\}$.*

We will call these diagrams *tensor weight diagrams*.
Furthermore we have that $\mathbb{F}\mathrm{R}_\delta(\lambda)$ is projective if and only if $\kappa(\lambda_\delta) = \min(n, m)$ (see [CH17, Lemma 7.16]).
So far we have no idea what the head of the indecomposable summands $\mathbb{F}\mathrm{R}_\delta(\lambda)$ looks like. The next section is going to address this and for this purpose introduces *hook weight diagrams*.

2.3 Hook weight diagrams

In [GS13], Gruson and Serganova introduced some combinatorially defined weight diagrams for $\mathfrak{osp}(r|2n)$ using symbols $<$, $>$, $\times$ and $\circ$. The combinatorics of hook weight diagrams in the sense of [ES21] (which were first introduced in [ES17]), which we recall here, are very similar to those, except that the symbols of Gruson and Serganova correspond to $\times$, $\circ$, $\vee$ and $\wedge$ respectively. For a detailed translation between

those two pictures we refer the reader to [ES17, Section 6]. The main advantage of the approach by Ehrig and Stroppel is that they could endow their combinatorics with a multiplicative structure, thus turning it into an algebraic object.
Given a partition λ, we denote by λ^∞ the weight diagram, which is obtained from λ_δ by turning all symbols upside down (or equivalently swapping $\vee$'s and $\wedge$'s). Note that these diagrams are then *flipped*.
The bijection in Lemma 2.23 clearly induces a bijection

$$\{\text{partitions}\} \to \{\text{flipped weight diagrams } \mu \text{ such that } \#\circ(\mu) - \#\times(\mu) = \lfloor \frac{\delta}{2} \rfloor\} \tag{2.1}$$

which is given by $\lambda \mapsto \lambda^\infty$. Integral dominant weights for $\mathfrak{osp}(r|2n)$ are characterized by (n,m)-hook partitions and thus we restrict the bijection (2.1) to (n,m)-hook partitions. It turns out that we can explicitly describe the image of (n,m)-hook partitions under this map.

Lemma 2.25. *The map $\lambda \mapsto \lambda^\infty$ gives rise to a bijections of sets*

$$\{(n,m)\text{-hook partitions}\} \to \left\{ \begin{matrix} \text{flipped weight diagrams } \mu \text{ such that} \\ \#\circ(\mu) - \#\times(\mu) = \lfloor \frac{\delta}{2} \rfloor \text{ and} \\ \#\vee(\mu) \leq \min(m,n) - \mathrm{rk}(\mu) \end{matrix} \right\} =: \Gamma_\delta(n,m).$$

We can transport the equivalence relation on $X^+(\mathfrak{g}) \times \{\pm\}$ (see Proposition 1.18 and Proposition 1.22) to $\Gamma_\delta(n,m) \times \{\pm\}$. We denote the set of equivalence classes by $s\Gamma_\delta(n,m)$ and call such equivalence classes *signed hook weight diagrams*. Hence we have a bijection between $X^+(G)$ and $s\Gamma_\delta(n,m)$. We will abuse notation and write (λ,ε) for the equivalence class of (λ,ε) in $s\Gamma_\delta(n,m)$.

Example 2.26. For $\mathfrak{osp}(3|2)$ the highest weight $(0|0)$ corresponds to the partition $\emptyset$ and $\emptyset^\infty$ is given by $\wedge\wedge\wedge\ldots$. The standard representation has highest weight $(1,0)$ which corresponds to the partition $\square$. The associated flipped weight diagram is then $\vee\wedge\wedge\ldots$.

Definition 2.27. Two signed hook weight diagrams (λ,ε), (μ,ε') belong to the same block if the positions of $\circ$ and $\times$ in λ and μ agree and if $\varepsilon = \varepsilon'$ for some representatives of the respective equivalence classes.
For two signed hook weight diagrams (λ,ε) and (μ,ε') belonging to the same block, we have $(\lambda,\varepsilon) \leq (\mu,\varepsilon')$ if μ can be obtained from λ via changing some $\wedge$'s into $\vee$'s or by changing $\vee\wedge$'s into $\wedge\vee$'s.

Remark 2.28. The notion of blocks according to Definition 2.27 agrees with the one given before Definition 1.12 by [GS10, Section 6] (see also [GS13, Section 4.5]) after translating their combinatorics to the one of Ehrig and Stroppel using [ES17, Section 6].
For the degree of atypicality from Definition 1.12 for a weight $\lambda \in X^+(\mathfrak{g})$ (with $a_m \geq 0$ in the notation of Lemma 1.7 if $r = 2m$), we have $\mathrm{at}(\lambda) = \min(m,n) - \mathrm{rk}(\lambda^\infty)$. The condition $a_m \geq 0$ in the even case is necessary because for those weights with

$a_m < 0$ we did not define an associated (n, m)-hook partition. This follows from [GS13, Section 4.5] by translating their combinatorics to our setting.
Additionally we can see that if such a weight is typical, λ^∞ is actually $\vee$-avoiding (i.e. no $\vee$ occurs) as $\min(m, n) = \mathrm{rk}(\lambda^\infty)$ and thus $\#\vee(\mu) = 0$ by Lemma 2.25.
Furthermore Definition 2.27 agrees with Definition 1.14 under the identifications in Definition 1.10 and Lemma 2.25 for two weights of the same block.

Example 2.29. For $\mathrm{OSp}(3|2)$ the weights $(0|0, \varepsilon)$ and $(a|a-1, \varepsilon)$ are atypical by Example 1.13. Under the identification from Lemma 2.25, these correspond to the signed hook weight diagrams $(\wedge\wedge\ldots, \varepsilon)$ for $(0|0, \varepsilon)$ and $(\wedge\cdots\wedge\vee\wedge\ldots, \varepsilon)$ for $(a|a-1, \varepsilon)$ (where the $\vee$ is at position $a - \frac{1}{2}$).
As we have the condition $\#\vee(\mu) \leq \min(m, n) - \mathrm{rk}(\mu)$ and $\#\circ(\mu) = \#\times(\mu)$ (as $\delta = 1$) in Lemma 2.25, these are in fact all possible signed hook weight diagrams without $\circ$'s and $\times$'s. So the notion of atypicality for weights for $\mathfrak{osp}(r|2n)$ and hook weight diagrams agree in this example.
Furthermore Definition 1.14 gives us an ordering on the atypical weights by $(0|0, \varepsilon) \leq (1|0, \varepsilon) \leq (2|1, \varepsilon) \ldots$. Considering the associated hook weight diagrams, this agrees with Definition 2.27.

Given a tensor weight diagram λ_δ (i.e. a Deligne weight diagram with $\kappa(\lambda_\delta) \leq \min(n, m)$), we would like to determine the head of the associated indecomposable $\mathrm{OSp}(r|2n)$-module $\mathbb{F}\,\mathrm{R}_\delta(\lambda)$. It will turn out that the head is actually irreducible, and its highest weight can be obtained via the map $\dagger$ defined below.

Definition 2.30. The map $\dagger\colon \{\text{tensor weight diagrams}\} \to s\Gamma_\delta(n, m)$ is defined as follows. For a Deligne weight diagram λ_δ with $\kappa(\lambda_\delta) = \min(m, n)$ (i.e. $\mathbb{F}\,\mathrm{R}_\delta(\lambda)$ is projective), we define $\lambda_\delta^\dagger := (\Phi(\lambda), \varepsilon)$, where $\Phi(\lambda)$ is the weight diagram obtained from λ_δ by turning all symbols $\vee$ corresponding to rays in $\underline{\lambda_\delta}$ into $\wedge$'s. In case that δ is odd, the sign ε is given by $+$ (resp. $-$) if the parity of the partition λ (under the bijection from Lemma 2.23) is even (resp. odd). In case that δ is even, the sign ε is $+$ (resp. $-$) if the leftmost ray of $\underline{\lambda_\delta}$ is undotted (resp. dotted) and not at position zero and $\varepsilon = \pm$ is the leftmost ray is at position zero. For a tensor weight diagram λ_δ with $\kappa(\lambda_\delta) < \min(n, m)$, we define $\lambda_\delta^\dagger := (\Phi(\lambda), \varepsilon)$, where $\Phi(\lambda)$ is given by turning <u>all</u> symbols corresponding to rays in $\underline{\lambda_\delta}$ upside down. The sign is defined in the same way as for projective tensor weight diagrams if δ is odd. In case that δ is even, we always set $\varepsilon = +$.

The main result of this section is the classification theorem from [ES21, Thm. 7.8].

Theorem 2.31. *Let $\lambda \in \Lambda(d, r, n)$, then:*

(i) *The indecomposable summand $\mathbb{F}\,\mathrm{R}_\delta(\lambda)$ of the $\mathrm{OSp}(r|2n)$-module $V^{\otimes d}$ has irreducible head isomorphic to $L(\lambda_\delta^\dagger)$.*

(ii) *In particular, if $\mathbb{F}\,\mathrm{R}_\delta(\lambda)$ is projective, then $\mathbb{F}\,\mathrm{R}_\delta(\lambda) \cong P(\lambda_\delta^\dagger)$.*

(iii) *Any indecomposable projective in* $\mathrm{OSp}(r|2n)$*-mod is obtained in this way for some* λ *and* d*.*

Example 2.32. For $\mathrm{OSp}(3|2)$ we know that $\mathbb{F}\,\mathrm{R}_\delta(\emptyset) \cong L(0|0,+)$ and $\mathbb{F}\,\mathrm{R}_\delta(\square) \cong V \cong L(1|0,-)$. The Deligne weight diagram $\emptyset_\delta$ with its associated cup diagram is given by (see also Example 2.19). It has $\kappa(\emptyset_\delta) = 0 < \min(m,n)$. Thus we have to turn all symbols upside down and hence we see that $\emptyset_\delta^\dagger = (\wedge\wedge\wedge\ldots,+)$, which is exactly the signed hook weight diagram associated to $(0|0,+)$ (see Example 2.29). For $\square$ we get the Deligne weight diagram . For this we have $\kappa(\square_\delta) = 0$ and so $\square_\delta^\dagger = (\vee\wedge\wedge\ldots,-)$, which is the signed hook weight diagram associated to $(1|0,-)$.

2.4 Super weight diagrams

The "problem" with hook weight diagrams is that the associated cup diagrams always have infinitely many dotted cups. Ultimately we want to endow the vector space with basis given by certain circle diagrams with a multiplication. The general approach for the multiplication in diagram algebras is given by stacking diagrams on top of each other (see e.g. Definition 1.27). But when stacking two circle diagrams on top of each other, the result is not a circle diagram anymore. There may be some cups and caps in the middle which one wants to remove (similar to the loop in the Brauer algebra). This is done by certain procedures called *surgeries* (for details see Chapter 3). Thus this approach can only be well-defined for circle diagrams with a finite number of cups and caps. On the other hand if we are working with Deligne weight diagrams, we have seen that they correspond to indecomposable objects in Rep_δ and not every one of those gives a nonzero object in $\mathcal{F}$.

Furthermore, one normally establishes a Morita equivalence via a projective generator. And for a projective generator in $\mathcal{F}$ we "only need" the Deligne weight diagrams λ such that $\mathbb{F}\,\mathrm{R}_\delta(\lambda)$ is projective. Up to some technicalities these are the *super weight diagrams.*

In this section we are going to make this introduction precise and provide an algorithm to compute the corresponding super weight diagram given a hook weight diagram.

Definition 2.33. Given a signed hook weight diagram $(\lambda,\varepsilon) \in s\Gamma_\delta(n,m)$, we define the associated *super weight diagram* $\lambda_\varepsilon^{\oslash}$ as the unique admissible weight diagram μ with $\kappa(\mu) = \mathrm{rk}(\mu) + \mathrm{def}(\mu) = \min(m,n)$ such that

- $\underline{\mu}$ is obtained from $\underline{\lambda}$ by replacing (infinitely many) dotted cups by two vertical rays each

- and possibly a dot on the resulting leftmost ray depending on ε according to the following rule:

  - If δ is even, we put a dot on the leftmost ray if $\varepsilon = +$ and no dot if $\varepsilon = -$.

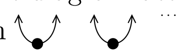
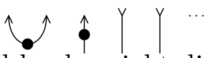
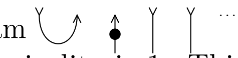
– If δ is odd, we do the following: For each symbol $\circ$ or $\times$ we count the number of endpoints of rays and cups in λ to the left of this symbol (this is the same as the number of $\vee$'s and $\wedge$'s to the left), and take the sum plus the total number of undotted cups in λ (this equals the number of $\vee$'s). Let this be s. If s is even, we put a dot on the first ray if $\varepsilon = +$ and no dot if $\varepsilon = -$. If s is odd, we put a dot if $\varepsilon = -$ and no dot if $\varepsilon = +$.

If one follows the explicit construction steps, one sees the following (or consult [ES21, Proposition 8.4])

Proposition 2.34. *Let λ_δ be a Deligne weight diagram associated to a projective $\mathbb{F}\,\mathrm{R}_\delta(\lambda)$. We denote the super weight diagram $(\lambda_\delta^\dagger)_\varepsilon^{\oslash}$ by μ. Then we have that $\underline{\mu}$ and $\underline{\lambda_\delta}$ agree up to a dot on the leftmost ray, and additionally a dot on the cup attached to $\diamond$ in case there is such a cup.*

Remark 2.35. We would like to emphasize here that the rule whether or not to put a dot, can be altered. We could have also chosen the reverse association, but we decided to stick with the convention of [ES21, Definition 8.1]. In case of the reverse association, the analogue of Proposition 2.34 would say that $(\lambda_\delta^\dagger)_\varepsilon^{\oslash} = \lambda_\delta$ for a Deligne weight diagram associated to a projective $\mathbb{F}\,\mathrm{R}_\delta(\lambda)$.

Remark 2.36. The atypicality of a signed hook weight diagram (λ, ε) was given by $\min(m,n) - \mathrm{rk}(\lambda)$ (see Remark 2.28). Observe that by Definition 2.33 this is also given by $\mathrm{def}(\lambda_\varepsilon^{\oslash})$.

Example 2.37. For OSp(3|2) the signed hook weight diagram attached to $(0|0,+)$ is given by $(\wedge\wedge\wedge\ldots,+)$ and its associated cup diagram  (see Example 2.29 and Definition 2.10). Using Definition 2.33 this then translates to the super weight diagram $\wedge\wedge\wedge\vee\vee\ldots$ with associated cup diagram .
For the standard representation the associated signed hook weight diagram was given by $(\vee\wedge\wedge\ldots,-)$ and the associated cup diagram is . The super weight diagram associated to it is $\vee\wedge\wedge\vee\vee\ldots$ with associated cup diagram .
We know by Example 1.13 that in both cases the degree of atypicality is 1. This agrees exactly with the number of cups in the cup diagram of the attached super weight diagram (as Remark 2.36 claims).

3 Khovanov's arc algebra of type B

Throughout this chapter we fix $\delta \in \mathbb{Z}$. By a weight diagram we mean a Deligne weight diagram corresponding to this δ and by a cup or cap diagram, we mean a cup or cap diagram associated to some Deligne weight diagram for δ.

The equivalence Ψ of categories between $\mathcal{F}$ and $e\tilde{K}e$-mod from [ES21, Theorem 10.5] (which will be refined in Theorem 4.4) is not monoidal. That means that we have no direct analogue of $V^{\otimes d}$ on the Khovanov algebra side. The key idea to overcome this problem is to look at the endofunctor $\_ \otimes V = \bigoplus_{i \in \mathbb{Z}+\frac{\delta}{2}} \theta_i$ and find an endofunctor on the Khovanov side, which identifies with $\_ \otimes V$ under Ψ.

This approach was also successfully taken by Brundan and Stroppel for $\mathfrak{gl}(m|n)$ and the Khovanov algebra of type A in [BS10] and [BS12]. They defined certain *geometric bimodules* $K^t_{\Lambda\Gamma}$ and proved that tensoring with these actually corresponds to $\_ \otimes V$ for $\mathfrak{gl}(m|n)$.

We follow their ideas and adapt the definitions to the type B setting. We will look at two different versions of geometric bimodules. First we are going to introduce those for the Khovanov algebra K. The theory of these is parallel to [BS10] (statements as well as proofs), although some proofs are a bit more complex due to the existence of $\bullet$ in circle diagrams.

Ehrig and Stroppel proved in [ES21, Theorem 6.22] that K is related to Brauer algebras and we will see that tensoring with this geometric bimodules then corresponds to the i-induction from Definition 1.37 on the Brauer category, for the precise statement consider Theorem 4.3. However, $\mathcal{F}$ is equivalent to a subquotient of K, called $e\tilde{K}e$ here (see [ES21, Theorem 10.5] or Theorem 4.4). So we also define geometric bimodules for $e\tilde{K}e$. But in this case, even though the statements are very similar to [BS10, Sections 3–4], the proofs differ markedly.

In Theorem 4.4 we will see that tensoring with these geometric bimodules translates to i-translation from Definition 1.38.

Explicitly we will prove the effect of tensoring with geometric bimodules on irreducible modules, indecomposable projective modules and in case of K the effect on standard modules in the sense of [BS21] (or [GL96]).

Definition 3.1. The Khovanov algebra K is the graded associative algebra with underlying basis given by all oriented circle diagrams $a\lambda b$, where $a\lambda b$ is homogeneous of degree $\deg(a\lambda b)$. The multiplication $(a\lambda b)(c\mu d)$ is defined to be 0 whenever $b^* \neq c$ and if $b^* = c$ we draw the circle diagram $(a\lambda b)$ under the circle diagram $(b^*\mu d)$, where we connect the rays of b and b^* and apply a certain surgery procedure defined in Section 3.1.

For this we choose a cap in b. Then we have at the same position a cup in b^*. In every step of the surgery procedure, we take such a cup-cap pair or and replace it in a certain way (which we specify later) by . We have to choose an order such that all intermediate results are again admissible (i.e. every dot can be connected to the left boundary without intersecting other components). This can for example be achieved, when one first resolves all dotted cups from right to left and then all the undotted ones.
After resolving all pairs of cups and caps, we are left with a middle section consisting only of straight rays. Removing this and identifying the two weight diagrams (which necessarily agree now), we obtain a linear combination of oriented circle diagram and we set this to be the product of $(a\lambda b)$ and $(b^*\lambda a)$.

Remark 3.2. Note that in the definition of the Khovanov algebra we claimed that this is an associative algebra. This is highly nontrivial, when looking at the definition of the surgery procedures below. For this to be associative it is crucial that all intermediate results are admissible, otherwise the associativity would fail. Details for this can be found in [ES16a, Section 5].

Drawing $b^*\mu d$ on top of $a\lambda b$ gives a so called *oriented stacked circle diagram of height* 2. This can be generalized to arbitrary height by stacking more compatible diagrams (for details we refer to [ES16a, Section 5.1]). We give the vertices the coordinate $(x, l-1)$ if it appears in the l-th diagram at position x for $l \in \mathbb{Z}_{>0}$ and $x \in L$. Note that in an oriented stacked circle diagram the positions of $\circ$ and $\times$ in each of the weight diagrams agree.
A *tag* of a stacked circle diagram associates to each circle C a rightmost vertex $t(C)$, i.e. a vertex such that the horizontal coordinate is maximal among all vertices C. Given an orientable stacked circle diagram D, a tag t and a coordinate (x, l) such that the connected component of (x, l) in D is a circle, we define

$$\operatorname{sign}_D(i, l) = (-1)^{\#\{j \mid \gamma_j \text{ is a dotted cup/cap}\}}, \tag{3.1}$$

where $\gamma_1, \ldots, \gamma_t$ is a sequence of arcs in D such that their concatenation is a path from (i, l) to $t(C)$. This sign is actually independent of the chosen tag t and the sequence of arcs. A proof of this can be found in [ES16a, Lemma 5.7], but it is very similar to Lemma 3.7.

Definition 3.3. A circle C in an oriented stacked circle diagram is oriented *clockwise* if the symbol at $t(C)$ is $\vee$ and *anticlockwise* if it is $\wedge$. A line is always oriented *anticlockwise* by convention.

3.1 The surgery procedure

Now suppose we are given an oriented stacked circle diagram $D = \underline{\lambda}(\boldsymbol{a}, \boldsymbol{\nu})\overline{\mu}$ and a cup-cap pair inside. We assume that the coordinates of the cup are (i, l) and (j, l)

with $i < j$ and $(i, l-1)$ and $(j, l-1)$ for the cap. We denote the stacked circle diagram, where we replace this pair by two undotted rays, by $D' = \underline{\lambda}(\boldsymbol{b})\overline{\mu}$. We distinguish between three different cases *Merge*, *Split* and *Reconnect*. A *Merge* occurs if the cup and cap belong to two different components, but after replacing the cup and cap by two rays, these components get merged. A *Split* happens, when the cup and cap belong to the same component (after replacing them we necessarily have two components) and a *Reconnect* occurs if the cup and cap belong to two different lines and after the replacement we have still two lines.

3.1.1 Merge

Denote the component of D containing $(i, l-1)$ by C_{l-1}, the component containing (i, l) by C_l and the component in D' containing $(i, l-1)$ by C. Then the surgery is given by

$$\underline{\lambda}(\boldsymbol{a}, \boldsymbol{\nu}) \mapsto \begin{cases} \underline{\lambda}(\boldsymbol{b}, \boldsymbol{\nu}') & \text{if } C_{l-1} \text{ and } C_l \text{ are both anticlockwise,} \\ \sigma_1 \underline{\lambda}(\boldsymbol{b}, \boldsymbol{\nu}'') & \text{if } C_{l-1} \text{ is clockwise and } C_l \text{ anticlockwise,} \\ \sigma_2 \underline{\lambda}(\boldsymbol{b}, \boldsymbol{\nu}'') & \text{if } C_{l-1} \text{ is anticlowise and } C_l \text{ clockwise,} \\ 0 & \text{if } C_{l-1} \text{ and } C_l \text{ are both clockwise,} \end{cases}$$

where $\boldsymbol{\nu}'$ (resp. $\boldsymbol{\nu}''$) are obtained by changing $\boldsymbol{\nu}$ such that the component C is oriented anticlockwise (resp. clockwise). If C cannot be oriented clockwise (i.e. it is a line) then the corresponding term is defined to be zero. Furthermore the signs σ_1, $\sigma_2 \in \{\pm 1\}$ are given by

$$\begin{aligned} \sigma_1 &= \operatorname{sign}_D(i, l-1)\operatorname{sign}_{D'}(i, l-1), \\ \sigma_2 &= \operatorname{sign}_D(i, l)\operatorname{sign}_{D'}(i, l). \end{aligned}$$

This *Merge* is based on the *multiplication* of the algebra $\mathbb{C}[X]/(X^2)$ and its action on the trivial module $\mathbb{C} = \mathbb{C}y$ with basis y,

$\mathbb{C}[X]/(X^2)$	$1 \otimes 1 \mapsto 1,$	$1 \otimes x \mapsto x,$	$x \otimes 1 \mapsto x,$	$x \otimes x \mapsto 0,$
$\mathbb{C}y$	$1 \otimes y \mapsto y,$		$x \otimes y \mapsto 0$	

where 1 is interpreted as an anticlockwise circle, x as a clockwise one and y as a line.

Example 3.4. Consider the left part of Figure 3.1. If we choose the cup-cap pair to be the undotted one, replacing these two by two rays results in a *Merge* and the result is presented in the middle of Figure 3.1. Note that the coordinate at the left ends of the chosen cup is connected via an undotted arc to a tag of its circle before and after the *Merge* respectively, i.e. no sign appears. After that we apply a *Merge* for the dotted cup-cap pair. Again we are in the situation that all involved signs cancel, in this case because the vertex at the left end of the dotted cap is connected before and after the surgery to a tag via a dotted arc.

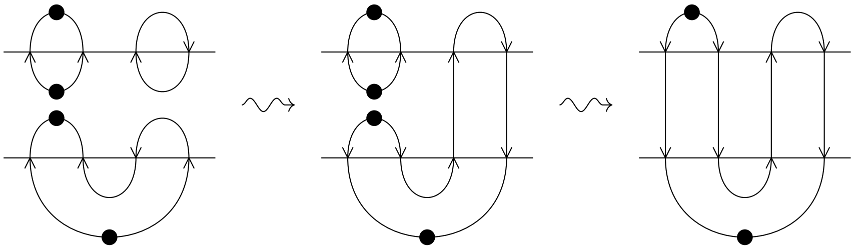


Figure 3.1: Illustration of the surgery procedure *Merge*

3.1.2 Split

Denote by C and C_a respectively the components in D containing (i, l) and in D' containing (a, l) for $a = i, j$. Then the surgery is given by

$$\underline{\lambda}(\boldsymbol{a}, \boldsymbol{\nu})\overline{\mu} \mapsto \begin{cases} (-1)^{\mathrm{p}(i)}(\sigma_1 \underline{\lambda}(\boldsymbol{b}, \boldsymbol{\nu}')\overline{\mu} + \sigma_2 \underline{\lambda}(\boldsymbol{b}, \boldsymbol{\nu}'')\overline{\mu}) & \text{if } C \text{ is anticlockwise and the} \\ & \text{cup-cap pair is dotted in } D, \\ (-1)^{\mathrm{p}(i)}(\sigma_1 \underline{\lambda}(\boldsymbol{b}, \boldsymbol{\nu}')\overline{\mu} - \sigma_2 \underline{\lambda}(\boldsymbol{b}, \boldsymbol{\nu}'')\overline{\mu}) & \text{if } C \text{ is anticlockwise and the} \\ & \text{cup-cap pair is undotted in } D, \\ (-1)^{\mathrm{p}(i)} \sigma_3 \underline{\lambda}(\boldsymbol{b}, \boldsymbol{\nu}''')\overline{\mu} & \text{if } C \text{ is clockwise}, \\ 0 & \text{if } D' \text{ is not orientable}, \end{cases}$$

where $\boldsymbol{\nu}'$ (resp. $\boldsymbol{\nu}''$) are obtained by changing $\boldsymbol{\nu}$ such that the C_j is oriented clockwise and C_i is oriented anticlockwise (resp. C_j anticlockwise and C_i clockwise). We obtain $\boldsymbol{\nu}'''$ by changing $\boldsymbol{\nu}$ such that C_i and C_j both are oriented clockwise. Furthermore the signs σ_k are given by

$$\begin{aligned} \sigma_1 &= \mathrm{sign}_{D'}(j, l), \\ \sigma_2 &= \mathrm{sign}_{D'}(i, l), \\ \sigma_3 &= \mathrm{sign}_D(i, l)\mathrm{sign}_{D'}(i, l)\mathrm{sign}_{D'}(j, l) \end{aligned}$$

and

$$\mathrm{p}(i) = \#\{k \in L \mid k \leq i \text{ and } \nu(k) \notin \{\circ, \times\} \text{ for one of the weight diagrams } \nu \text{ of } \boldsymbol{\nu}'\}.$$

In this case the surgery is based on the *comultiplication* of the algebra $\mathbb{C}[X]/(X^2)$ and its trivial comodule $\mathbb{C} = \mathbb{C}y$ with basis y

$$\begin{array}{c|c} \mathbb{C}[X]/(X^2) & 1 \mapsto 1 \otimes x + x \otimes 1, \quad x \mapsto x \otimes x, \\ \hline \mathbb{C}y & y \mapsto y \otimes x, \end{array}$$

where we use the same interpretation as in the *Merge* case.

Example 3.5. Consider the left part of Figure 3.2. The surgery procedure that

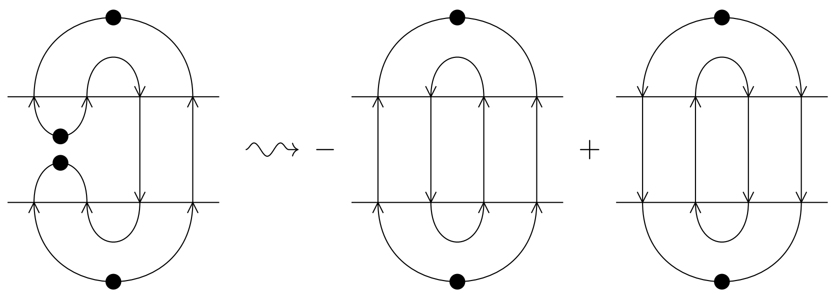


Figure 3.2: Illustration of the surgery procedure *Split*

we have to apply to the dotted cup-cap pair is a *Split*. As the circle is oriented clockwise we are in the first case of the definition. We observe that $\mathrm{p}(i) = 1$, as we are considering the leftmost vertex. We then have to compute σ_1 and σ_2. The sign σ_1 has to be 1 as one considers the inner circle, which has no dotted parts. As the leftmost vertex in the bottom weight diagram is connected to a tag via a dotted arc, σ_2 has to be -1.

3.1.3 Reconnect

This situation can only occur if the cup and cap lie on two distinct lines. In this case the surgery is given by

$$\underline{\lambda}(\boldsymbol{a}, \boldsymbol{\nu})\overline{\mu} = \begin{cases} \underline{\lambda}(\boldsymbol{b}, \boldsymbol{\nu})\overline{\mu} & \text{if } \boldsymbol{\nu} \text{ is an orientation for } D' \text{ and the} \\ & \text{two lines in } D \text{ were propagating,} \\ 0 & \text{otherwise.} \end{cases}$$

In the notation of the algebra $\mathbb{C}[X]/(X^2)$ and the trivial module $\mathbb{C}y$ this is the rule

$$y \otimes y \mapsto \begin{cases} y \otimes y & \text{if both lines propagate and reconnecting} \\ & \text{gives an oriented diagram} \\ 0 & \text{otherwise.} \end{cases} \tag{3.2}$$

For any Deligne weight diagram λ the circle diagram $e_\lambda := \underline{\lambda}\lambda\overline{\lambda}$ is an idempotent in K and $e_\lambda e_\mu = 0$ whenever $\lambda \neq \mu$. This gives the algebra $K = \bigoplus_{\lambda,\mu \in \Lambda_\delta} e_\lambda K e_\mu$ the structure of a locally unital algebra. By $\mathrm{mod}_{lf}(K)$ we refer to locally finite dimensional graded modules over K, i.e. graded modules M such that $\dim e_\lambda M < \infty$ for all $\lambda \in \Lambda_\delta$.

The irreducible locally finite dimensional K-modules are in bijection with Λ_δ. Given $\lambda \in \Lambda_\delta$ we construct a one dimensional irreducible K-module $L(\lambda)$ as follows. As a vector space it is just $\mathbb{C}$ and e_μ acts by 1 if $\lambda = \mu$ and 0 otherwise. The indecomposable projective objects in $\mathrm{mod}_{lf}(K)$ are given by $P(\lambda) := Ke_\lambda$ for $\lambda \in \Lambda_\delta$.

We have an anti-involution $*$ on K which is given by sending $a\lambda b$ to $b^*\lambda a^*$. And this gives rise to a duality (also denoted $*$) on $\operatorname{mod}_{lf}(K)$. For a locally finite dimensional graded K-module M, we define the graded piece $(M^\circledast)_j := \operatorname{Hom}_{\mathbb{C}}(M_{-j}, \mathbb{C})$ and $x \in K$ acts on $f \in M^\circledast$ by $(xf)(m) := f(x^*m)$. We also easily see that $L(\lambda)^\circledast = L(\lambda)$.
The indecomposable injective objects are then $I_\delta(\lambda) := (P(\lambda))^\circledast$ for $\lambda \in \Lambda_\delta$.
Furthermore, we define *standard modules* $V(\mu)$ for $\mu \in \Lambda_\delta$. These are the cell modules associated to the cellular structure (in the sense of [GL96]) of K in [ES16a, Theorem 7.1]. As a vector space it has a basis given by formal symbols $(\underline{\gamma}\mu|$ for all $\gamma \in \Lambda_\delta$ such that $\underline{\gamma}\mu$ is oriented. The multiplication is defined as

$$(a\lambda b)(\underline{\gamma}\mu| = \begin{cases} s_{a\lambda b}(\mu)(a\mu| & \text{if } b \neq \overline{\gamma} \text{ and } a\mu \text{ is oriented,} \\ 0 & \text{otherwise,} \end{cases} \tag{3.3}$$

where $s_{a\lambda b}(\mu)$ is either the coefficient from [ES16a, Thm. 7.1] or Theorem 3.19. The standard module $V(\mu)$ is also the quotient of $P(\mu)$ and the K-submodule generated by all oriented circle diagrams $a\lambda\overline{\mu}$ with $\lambda \neq \mu$ (then we necessarily have $\lambda > \mu$). The irreducible module $L(\mu)$ is the quotient of $V(\mu)$ and the K-submodule generated by all $(\underline{\gamma}\mu|$ with $\gamma \neq \mu$ (and hence $\gamma > \mu$).
With these definitions we can conclude this section with the following theorem:

Theorem 3.6. *The category* $\operatorname{mod}_{lf}(K)$ *is an upper finite highest weight category in the sense of* [BS21] *with standard objects* $V(\lambda)$*,* $\lambda \in \Lambda_\delta$*.*

Proof. This is just [ES21, Cor 2.11] after identifying their category $\mathcal{D}(\delta)$ (which consists of contravariant functors from $\operatorname{Rep}_\delta$ to Vect, the category of finite dimensional complex vector spaces) with $\operatorname{mod}_{lf}(K)$ using [ES21, Thm. 6.22] or Theorem 4.3. □

Brundan and Stroppel define an upper finite highest weight category in [BS21] as follows. Let $\mathcal{R}$ be an abelian category which is equivalent to the category of locally finite dimensional A-modules for a locally finite dimensional locally unital algebra A. An *upper finite stratification* for $\mathcal{R}$ is a set $\boldsymbol{B}$ labeling a full set $\{L(b) \mid b \in \boldsymbol{B}\}$ of pairwise nonisomorphic irreducible objects in $\mathcal{R}$ together with a function $\rho \colon \boldsymbol{B} \to \Lambda$ such that all fibres are finite, where $(\Lambda, \leqslant)$ is an upper finite poset. For every $\lambda \in \Lambda$ we can consider the sets $\boldsymbol{B}_{<\lambda} := \bigcup_{\mu<\lambda} \rho^{-1}(\mu)$ and $\boldsymbol{B}_{\leqslant\lambda} := \bigcup_{\mu\leqslant\lambda} \rho^{-1}(\mu)$. We can then look at the Serre subcategories $\mathcal{R}_{<\lambda}$ and $\mathcal{R}_{\leqslant\lambda}$ given by $\boldsymbol{B}_{<\lambda}$ and $\boldsymbol{B}_{\leqslant\lambda}$.
Defining the *stratum* $\mathcal{R}_\lambda$ to be the Serre quotient category of $\mathcal{R}_{\leq\lambda}$ by $\mathcal{R}_{<\lambda}$ we are in a recollement situation (for details see [BS21, Section 2.5]):

$$\begin{tikzcd} \mathcal{R}_{<\lambda} \arrow[r, "i_{<\lambda}"] & \mathcal{R}_{\leqslant\lambda} \arrow[l, bend right, "i^!_{<\lambda}"'] \arrow[l, bend left, "i^*_{<\lambda}"] \arrow[r, "j^\lambda"] & \mathcal{R}_\lambda \arrow[l, bend right, "j^\lambda_*"'] \arrow[l, bend left, "j^\lambda_!"] \end{tikzcd}$$

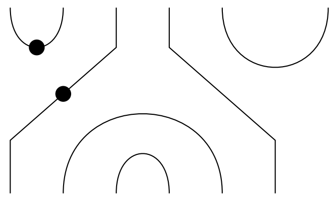
Define then $\Delta(b)$ for $b \in \boldsymbol{B}_\lambda := \rho^{-1}(\lambda)$ as $j^\lambda_* P_\lambda(b)$ where $P_\lambda(b)$ denotes the projective cover of the irreducible object $j^\lambda L(b)$ in $\mathcal{R}_\lambda$.

Then $\mathcal{R}$ is called an *upper finite highest weight category* if all the strata are simple (i.e. equivalent to the category of finite dimensional vector spaces) and for every $b \in \boldsymbol{B}$ there exists a projective object P_b which admits a Δ-flag with $\Delta(b)$ at the top and other sections $\Delta(c)$ with $\rho(c) \geq \rho(b)$. Note that ρ is necessarily a bijection for an upper finite highest weight category.

3.2 Geometric bimodules

In this section we generalize the diagrammatics of Khovanov's arc algebra by incorporating *crossingless matchings* (of type B). This section proves furthermore analogous results to [BS10, Sections 2–4] and many ideas from the proofs there can be directly applied to our setting.

A *crossingless matching* is a diagram t, which is obtained by drawing an admissible cap diagram c underneath an admissible cup diagram d and connecting the rays in c to the rays in d from left to right. This means that we allow dotted cups, caps and lines but each dot necessarily needs to be able to be connected to the left boundary without crossing anything, just as in the case of admissible circle diagrams (see [ES16a, Def. 3.5]). Furthermore we delete pairs of dots on each segment, such that each line segment contains at most one dot. Any crossingless matching is a union of (dotted) cups, caps and line segments, for example:



We denote by $\text{cups}(t)$ respectively $\text{caps}(t)$ the number of cups respectively caps in t. Furthermore let t^* be the horizontally reflected image of t.

We say that t is a $\Lambda\Gamma$-*matching* if the bottom and top number lines of t agree with the number lines of Λ respectively Γ. More generally, given a sequence of blocks $\boldsymbol{\Lambda} = \Lambda_k \ldots \Lambda_0$, we define a $\boldsymbol{\Lambda}$-*matching* to be a diagram $\boldsymbol{t} = t_k \ldots t_1$ obtained by glueing a sequence $t_1, \ldots, t_k$ of crossingless matchings together from top to bottom such that

- each t_i is a $\Lambda_i\Lambda_{i-1}$-matching for each $i = 1, \ldots, k$,
- the free vertices at the bottom of t_i are in the same position with the free vertices at the top of t_{i+1} for $i = 1, \ldots, k-1$.

Given additionally a cup diagram a and a cap diagram b such that their number lines agree with the bottom number lines of t_k respectively the top number line of t_1, we can glue them together and obtain a Λ-*circle diagram* $a\boldsymbol{t}b = at_k \ldots t_1 b$.

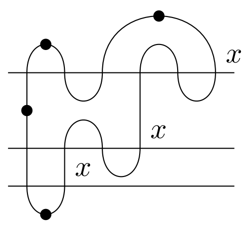
Let Λ and Γ be blocks and let t be a $\Lambda\Gamma$-matching. Given weights $\lambda \in \Lambda$ and $\mu \in \Gamma$ we can glue these together from bottom to top to obtain a new diagram $\lambda t \mu$. We call this an *oriented* $\Lambda\Gamma$-*matching* if

- each pair of vertices lying on the same dotted cup or the same undotted line segment is labeled such that both are either $\vee$ or both are $\wedge$,
- each pair of vertices lying on the same undotted cup or the same dotted line segment is labeled such that one is $\vee$ and one is $\wedge$,
- all other vertices are labeled $\circ$ or $\times$.

A diamond $\diamond$ can be interpreted as either $\vee$ or $\wedge$.
More generally an *oriented* $\mathbf{\Lambda}$*-matching* for a sequence of blocks $\mathbf{\Lambda} = \Lambda_k \dots \Lambda_0$ is a composite diagram of the form

$$\boldsymbol{t}[\boldsymbol{\lambda}] = \lambda_k t_k \lambda_{k-1} \dots \lambda_1 t_1 \lambda_0$$

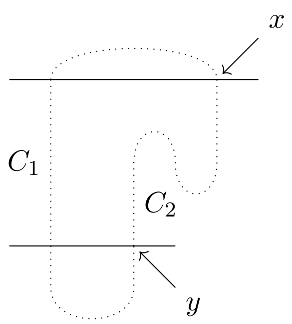
where $\boldsymbol{\lambda} = \lambda_k \dots \lambda_0$ is a sequence of weights such that $\lambda_i t_i \lambda_{i-1}$ is an oriented $\Lambda_i \Lambda_{i-1}$-matching for each $i = 1, \dots k$.
Finally given an oriented $\mathbf{\Lambda}$-matching and cap and cup diagrams a and b such that $a\lambda_k$ (resp. $\lambda_0 b$) is an oriented cup (resp cap) diagram we can glue these together to obtain an *oriented* $\mathbf{\Lambda}$*-circle diagram* $a\boldsymbol{t}[\boldsymbol{\lambda}]b$.
We call a $\mathbf{\Lambda}$-matching $\boldsymbol{t}$ *proper* if there exists at least one oriented $\mathbf{\Lambda}$-matching for $\boldsymbol{t}$. By a rightmost vertex x on a circle C we mean a vertex lying on C such that on this numberline, there is no vertex to the right of x. In the bottom picture every rightmost vertex is marked by x.



We refer to a circle in an oriented $\mathbf{\Lambda}$-diagram as *clockwise* respectively *anticlockwise* if a rightmost vertex on the circle is labeled $\vee$ respectively $\wedge$. This notion is well-defined by the following lemma.

Lemma 3.7. *Let $a\boldsymbol{t}[\boldsymbol{\lambda}]b$ be an oriented $\mathbf{\Lambda}$-circle diagram and let C be a closed component of this diagram. Then the rightmost vertices of C all have the same orientation.*

Proof. Take two rightmost vertices x and y in a circle c and assume that $x \neq y$. Then there are exactly two paths connecting x with y in C. The crucial observation is that the "right" one of them is cut off by the other one from the left boundary of the diagram and thus cannot contain any dots. Without loss of generality assume that y appears on a lower number line as the picture indicates.



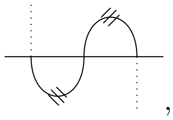
One of the paths leaves the vertex y to the top (C_2) and one to the bottom (C_1). Note that C_1 has to cross the number line of y again but by our assumption this happens to the left of y. Then this paths always stays to the left of C_2, hence C_2 is "cut off" by C_1. So C_2 cannot contain any dots as otherwise those could not be connected to the left boundary (contradicting the admissiblity assumption in the definition of crossingless matching). By a similar reasoning C_2 is also the path which enters x from the bottom and hence C_2 has to contain an even number of cups which are all undotted. So the symbol ($\vee$ or $\wedge$) gets changed an even number of times, when moving from y to x along C_2 and thus the orientations agree. $\square$

Definition 3.8. The *degree* of a circle or a line in an oriented $\mathbf{\Lambda}$-circle diagram is the total number of clockwise cups or caps that it contains. The *degree* of an oriented $\mathbf{\Lambda}$-circle diagram is the sum of the degrees of each of its circles and lines. We call a circle only consisting of one cup and one cap a *small circle.*

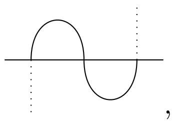
Lemma 3.9. *The degree of an anticlockwise circle in an oriented $\mathbf{\Lambda}$-circle diagram is one less than the total number of caps (equivalently, cups) that it contains. The degree of a clockwise circle is one more than the total number of caps (equivalently, cups) that it contains. The degree of a line is equal to the number of caps or the number of cups that it contains, whichever is greater.*

Proof. We will prove the statement via induction on the number of cups and caps. Our base cases are all small circles and lines with at most one cup or cap.
First of all, note that a cup (resp. cap) is oriented clockwise if the right vertex is oriented $\vee$ and anticlockwise if it is oriented $\wedge$, regardless of any dots. In the base case of a line without any cup and cap it clearly has degree 0, which is the maximum of the number of cups respectively caps. For a line with one cup (resp. one cap), there are two rays and by admissibility the right one cannot be dotted, as otherwise it could not be connected to the left boundary. But this means that the cup (resp. the cap) is oriented $\vee$ at its right endpoint and thus has anticlockwise orientation. Hence the degree of the line is 1 which equals the maximum of the number of cups respectively caps.
For a circle consisting only of one cup and one cap the right endpoint of the cap is connected by a straight line without dots (by admissibility) to the right endpoint of the cup. So either both are oriented clockwise or both are oriented anticlockwise. In

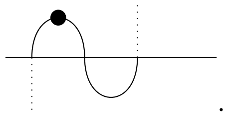
the former case, the circle is oriented clockwise and has degree 2, which is one more than the number of cups (resp. caps) and in the latter case, the circle is oriented anticlockwise and has degree 0 which is one less than the number of cups (resp. caps). If we are given a component, which is not of the above form, we necessarily have a subpicture (or its horizontal mirror image) looking like



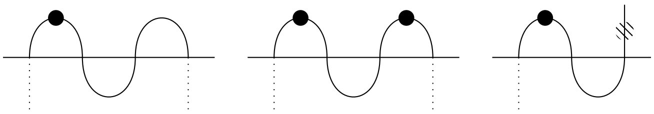
where a dashed dot means that a dot can be present or not. We may assume that we choose this subpicture such that the horizontal distance between its endpoints is minimal, i.e. if a cup or cap is attached to one of the endpoints, the other one is either to the left or to the right of the picture. Furthermore we cannot have two dots present by admissibility, as otherwise one of the dots would necessarily be cut off the left boundary. In the case that we have no dots, i.e. a picture of the following form



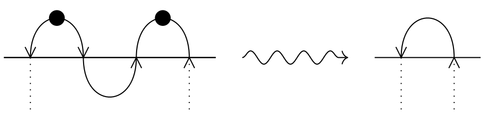
one of the cap and cup is oriented clockwise and the other one is oriented anticlockwise. Thus by replacing this cap and cup by a straight line we reduce the degree of the diagram by 1 and the number of caps respectively cups also by 1, and hence we are done by the induction hypothesis.
The only other case (up to symmetry) is, that there is exactly one dot, i.e.

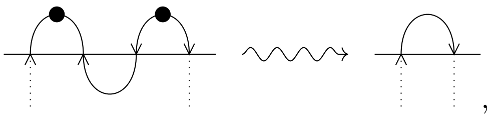


We make a case distinction by looking at the part of the circle connected to the right endpoint. By admissibility and assumptions we have to consider the following cases (note that the third case cannot happen for the vertical mirror image):

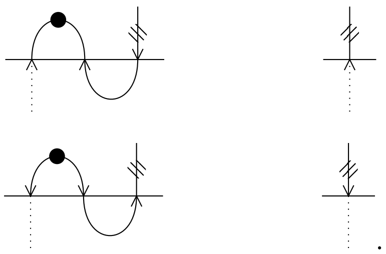


For the first case we can apply the case before with no dots, so we are done by induction. For the second case we can apply (depending on the orientation) one of the following two cases:





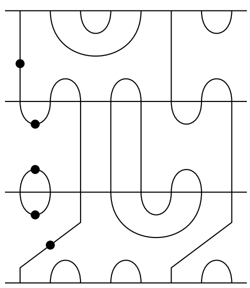
In each of these cases the number of cups and caps decreases by 1 each and the degree of the component also decreases by 1, hence we are done by induction.
Lastly for the third case we can apply one of the reductions



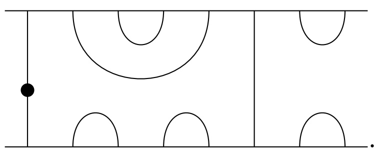
Here different dashed patterns correspond to exactly opposite choices of whether a dot is present, i.e. if a dot is present before the reduction, there is none after and vice versa. Again we reduce the number of cups respectively caps by 1 and the degree also by 1, and we can apply the induction hypothesis, finishing this case and thus the proof. ☐

Definition 3.10. Suppose we have a $\mathbf{\Lambda}$-matching $\boldsymbol{t} = t_k \dots t_1$ for some sequence $\mathbf{\Lambda} = \Lambda_k \dots \Lambda_0$ of blocks. We refer to circles in $\boldsymbol{t}$ not meeting the top or bottom number line as *internal* circles. The *reduction* of $\boldsymbol{t}$ is the $\Lambda_k\Lambda_0$-matching which is obtained by removing all internal circles, all but the top and bottom number line and maintaining the parity of dots on each component.

Example 3.11. The reduction fo the $\mathbf{\Lambda}$-matching

$\boldsymbol{t} =$ 

is given by



Informally speaking, reduction is given by removing all circles and straightening lines.

Lemma 3.12. *Assume that $a\boldsymbol{t}[\boldsymbol{\lambda}]b$ is an oriented $\boldsymbol{\Lambda}$-circle diagram for some sequence $\boldsymbol{\lambda} = \lambda_k \dots \lambda_0$ of weights. Let u be the reduction of $\boldsymbol{t}$. Then $a\lambda_k u\lambda_0 b$ is an oriented $\Lambda_k\Lambda_0$-circle diagram and*

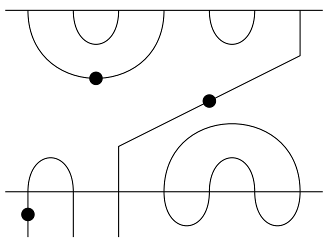
$$\begin{aligned}\deg(a\boldsymbol{t}[\boldsymbol{\lambda}]b) &= \deg(a\lambda_k u\lambda_0 b) + \mathrm{caps}(t_1) + \dots + \mathrm{caps}(t_k) - \mathrm{caps}(u) + p - q \\ &= \deg(a\lambda_k u\lambda_0 b) + \mathrm{cups}(t_1) + \dots + \mathrm{cups}(t_k) - \mathrm{cups}(u) + p - q,\end{aligned}$$

where p (resp. q) denotes the number of internal circles of $\boldsymbol{t}$ that are clockwise (resp. anticlockwise) in the diagram $a\boldsymbol{t}[\boldsymbol{\lambda}]b$.

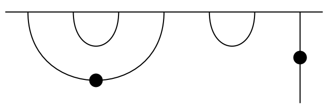
Proof. When passing from $\boldsymbol{t}$ to u we remove all internal circles, which obviously have the same number of caps as cups, and we remove an equal number of cups and caps from every other component of the circle diagram. Moreover the total number of caps removed is $\mathrm{caps}(t_1) + \dots + \mathrm{caps}(t_k) - \mathrm{caps}(u)$. Then the statement follows directly from Lemma 3.9. □

Definition 3.13. Let t be a $\Lambda\Gamma$-matching for some blocks Λ and Γ. Let a be a cup diagram such that its number line agrees with the bottom one of t. We refer to circles or lines not meeting the top number line in at as *lower* circles or lines. The *lower reduction* of at refers to the cup diagram which is obtained by removing all lower circles and lines as well as the bottom number line.
Similarly if b is a cap diagram whose number line agrees with the top one of t, we call each circle or line not meeting the bottom number line *upper* circle or line. Similarly the *upper reduction* of bt means removing all upper lines or circles and the top number line.

Example 3.14. Suppose at as in Definition 3.13 looks like:



Then the lower reduction is:



Lemma 3.15. *If $a\lambda t\mu b$ is an oriented $\Lambda\Gamma$-circle diagram and c is the lower reduction of at, then $c\mu b$ is an oriented circle diagram and*

$$\deg(a\lambda t\mu b) = \deg(c\mu b) + \mathrm{caps}(t) + p - q,$$

where p (resp. q) is the number of lower circles that are clockwise (resp. anticlockwise) in the diagram $a\lambda t\mu b$. For the dual statement about upper reduction one needs to replace $\operatorname{caps}(t)$ *by* $\operatorname{cups}(t)$.

Proof. We prove only the statement about lower reduction as the one about upper reduction is similar. When passing from at to c, we remove all lower circles, which obviously have the same number of cups and caps, and lower lines, which have one more cap than cup. From every other component we remove an equal number of cups and caps. The total number of caps removed is $\operatorname{caps}(t)$. The statement then follows from Lemma 3.9. □

Definition 3.16. Let $\boldsymbol{\Lambda} = \Lambda_k \dots \Lambda_0$ be a sequence of blocks, and let $\boldsymbol{t} = t_k \dots t_1$ be a $\boldsymbol{\Lambda}$-matching. Define $K^{\boldsymbol{t}}_{\boldsymbol{\Lambda}}$ to be the graded vector space with homogeneous basis

$$\{(a\boldsymbol{t}[\boldsymbol{\lambda}]b) \mid \text{ for all closed oriented } \boldsymbol{\Lambda}\text{-circle diagrams } a\boldsymbol{t}[\boldsymbol{\lambda}]b\}.$$

Define a degree preserving linear map

$$*\colon K^{\boldsymbol{t}}_{\boldsymbol{\Lambda}} \to K^{\boldsymbol{t}^*}_{\boldsymbol{\Lambda}^*}, \quad (a\boldsymbol{t}[\boldsymbol{\lambda}]b) \mapsto (b^*\boldsymbol{t}^*[\boldsymbol{\lambda}^*]a^*), \tag{3.4}$$

where $\boldsymbol{\Lambda}^* = \Lambda_0 \dots \Lambda_k$, $\boldsymbol{\lambda}^* = \lambda_0 \dots \lambda_k$, $t^* = t_1^* \dots t_k^*$ and t_i^*, a^* and b^* denote the mirror images of t_i, a, b in the horizontal axis.

Remark 3.17. Note that $K^{\boldsymbol{t}}_{\boldsymbol{\Lambda}}$ is nonzero if and only if $\boldsymbol{t}$ is a proper $\boldsymbol{\Lambda}$-matching. If we assume $k = 0$ in the above definition, then $\boldsymbol{\Lambda}$ consists of a single block Λ, $\boldsymbol{t}$ is empty and $K^{\boldsymbol{t}}_{\boldsymbol{\Lambda}}$ is the vector space underlying K_Λ. The map $*$ reduces in this case to the anti-involution of K_Λ (see the paragraph before Theorem 3.6).

Let $\boldsymbol{\Gamma} = \Gamma_l \dots \Gamma_0$ be another sequence of blocks with $\Lambda_0 = \Gamma_l$. We denote by $\boldsymbol{\Lambda} \wr \boldsymbol{\Gamma}$ the block sequence $\Lambda_k \dots \Lambda_1 \Gamma_l \dots \Gamma_0$. Observe that one copy of Λ_0 is left out in comparison to the concatenation of the block sequences. Furthermore note that if $\boldsymbol{u} = u_l \dots u_1$ is a $\boldsymbol{\Gamma}$-matching the concatenation $\boldsymbol{tu} = t_k \dots t_1 u_l \dots u_1$ is a $\boldsymbol{\Lambda} \wr \boldsymbol{\Gamma}$-matching. We then define a degree preserving linear multiplication

$$m\colon K^{\boldsymbol{t}}_{\boldsymbol{\Lambda}} \otimes K^{\boldsymbol{u}}_{\boldsymbol{\Gamma}} \to K^{\boldsymbol{tu}}_{\boldsymbol{\Lambda}\wr\boldsymbol{\Gamma}} \tag{3.5}$$

as follows. The product $(a\boldsymbol{t}[\boldsymbol{\lambda}]b)(c\boldsymbol{u}[\boldsymbol{\mu}]d)$ is defined to be 0 whenever $b \neq c^*$. In the case $b = c^*$ we draw $(a\boldsymbol{t}[\boldsymbol{\lambda}]b)$ underneath $(c\boldsymbol{u}[\boldsymbol{\mu}]d)$ and we then smooth out the symmetric middle section using surgery procedures exactly as in the Khovanov algebra K of type B. Then we collapse the middle section by identifying the number lines adjacent to the middle section and declaring the product to be this sum of oriented $\boldsymbol{\Lambda} \wr \boldsymbol{\Gamma}$-circle diagrams. That this is well-defined and homogeneous of degree 0 can be verified in the same manner as in [ES16a, Section 5].
In the special case $k = l = 0$ this simplifies to the ordinary multiplication in the Khovanov algebra K of type B from Section 3.1. Additionally, given a third sequence of

blocks $\boldsymbol{\Upsilon}$ with $\Upsilon_0 = \Lambda_k$, this multiplication is associative in the sense that the following diagram commutes, which again can be verified analogously to [ES16a, Section 5]:

$$\begin{tikzcd} K^{\boldsymbol{s}}_{\boldsymbol{\Upsilon}} \otimes K^{\boldsymbol{t}}_{\boldsymbol{\Lambda}} \otimes K^{\boldsymbol{u}}_{\boldsymbol{\Gamma}} \arrow[r, "1\otimes m"] \arrow[d, "m\otimes 1"] & K^{\boldsymbol{s}}_{\boldsymbol{\Upsilon}} \otimes K^{\boldsymbol{tu}}_{\boldsymbol{\Lambda}\wr\boldsymbol{\Gamma}} \arrow[d, "m"] \\ K^{\boldsymbol{st}}_{\boldsymbol{\Upsilon}\wr\boldsymbol{\Lambda}} \otimes K^{\boldsymbol{u}}_{\boldsymbol{\Gamma}} \arrow[r, "m"] & K^{\boldsymbol{stu}}_{\boldsymbol{\Upsilon}\wr\boldsymbol{\Lambda}\wr\boldsymbol{\Gamma}} \end{tikzcd} \tag{3.6}$$

Finally the linear map $*$ is antimultiplicative in the sense that the following diagram commutes (P denotes the flip $x \otimes y \mapsto y \otimes x$):

$$\begin{tikzcd} K^{\boldsymbol{t}}_{\boldsymbol{\Lambda}} \otimes K^{\boldsymbol{u}}_{\boldsymbol{\Gamma}} \arrow[r, "P\circ(*\otimes*)"] \arrow[d, "m"] & K^{\boldsymbol{u}^*}_{\boldsymbol{\Gamma}^*} \otimes K^{\boldsymbol{t}^*}_{\boldsymbol{\Lambda}^*} \arrow[d, "m"] \\ K^{\boldsymbol{tu}}_{\boldsymbol{\Lambda}\wr\boldsymbol{\Gamma}} \arrow[r, "*"] & K^{\boldsymbol{u}^*\boldsymbol{t}^*}_{\boldsymbol{\Gamma}^*\wr\boldsymbol{\Lambda}^*} \end{tikzcd} \tag{3.7}$$

Remark 3.18. Letting $\boldsymbol{\Upsilon} = \Lambda_k$ and $\boldsymbol{\Gamma} = \Lambda_0$ we see by (3.6) that the multiplication m turns $K^{\boldsymbol{t}}_{\boldsymbol{\Lambda}}$ into a $(K_{\Lambda_k}, K_{\Lambda_0})$-bimodule.
Recalling the primitive idempotents $e_\alpha \in K_{\Lambda_k}$ and $e_\beta \in K_{\Lambda_0}$, we have that

$$e_\alpha(a\boldsymbol{t}[\boldsymbol{\lambda}]b) = \begin{cases} (a\boldsymbol{t}[\boldsymbol{\lambda}]b) & \text{if } \underline{\alpha} = a, \\ 0 & \text{otherwise,} \end{cases} \tag{3.8}$$

$$(a\boldsymbol{t}[\boldsymbol{\lambda}]b)e_\beta = \begin{cases} (a\boldsymbol{t}[\boldsymbol{\lambda}]b) & \text{if } \overline{\beta} = b, \\ 0 & \text{otherwise.} \end{cases} \tag{3.9}$$

The following theorem generalizes the cellular structure of Khovanov's algebra to our setting and is very important for the following computations.

Theorem 3.19. *We follow the notation of* (3.5) *and suppose that we are given basis vectors* $(a\boldsymbol{t}[\boldsymbol{\lambda}]b) \in K^{\boldsymbol{t}}_{\boldsymbol{\Lambda}}$ *and* $(c\boldsymbol{u}[\boldsymbol{\mu}]d) \in K^{\boldsymbol{u}}_{\boldsymbol{\Gamma}}$. *Denote* $(a\boldsymbol{t}[\boldsymbol{\lambda}]b)$ *as* $\vec{a}\lambda b$ *where* $\vec{a} \coloneqq a\lambda_k t_k \lambda_{k-1} \dots \lambda_1 t_1$ *and* $\lambda \coloneqq \lambda_0$. *Similarly denote* $(c\boldsymbol{u}[\boldsymbol{\mu}]d)$ *as* $c\mu\vec{d}$ *with* $\mu \coloneqq \mu_l$ *and* $\vec{d} \coloneqq u_l \mu_{l-1} \dots \mu_1 u_1 \mu_0 d$. *The multiplication then satisfies*

$$(\vec{a}\lambda b)(c\mu\vec{d}) = \begin{cases} 0 & \textit{if } b \neq c^*, \\ s_{\vec{a}\lambda b}(\mu)(\vec{a}\mu\vec{d}) + (\dagger) & \textit{if } b = c^* \textit{ and } \vec{a}\mu \textit{ is oriented,} \\ (\dagger) & \textit{otherwise,} \end{cases} \tag{3.10}$$

where

(i) $(\dagger)$ *denotes a linear combination of basis vectors of* $K^{\boldsymbol{tu}}_{\boldsymbol{\Lambda}\wr\boldsymbol{\Gamma}}$ *of the form* $(a(\boldsymbol{tu})[\boldsymbol{\nu}]d)$ *for* $\boldsymbol{\nu} = \nu_{k+l} \dots \nu_0$ *with* $\nu_l > \mu_l, \nu_{l-1} \geq \mu_{l-1}, \dots, \nu_0 \geq \mu_0$,

(ii) *the scalar* $s_{\vec{a}\lambda b}(\mu) \in \{0, 1\}$ *depends only on* $\vec{a}\lambda b$ *and* μ, *but not on* $\vec{d}$,

(iii) *if* $b = \overline{\lambda} = c^*$ *and* $\vec{a}\mu$ *is oriented then* $s_{\vec{a}\lambda b}(\mu) = 1$,

(iv) *if* $k = 0$ *then* $s_{\vec{a}\lambda b}(\mu)$ *is equal to the scalar* $s_{a\lambda b}(\mu)$ *from* [ES16a, Theorem 7.1]

Proof. This follows by using the same arguments as [ES16a, Theorem 7.1], replacing a by $\vec{a}$. ☐

Corollary 3.20. *The product of any pair of basis vectors* $(a\boldsymbol{t}[\boldsymbol{\lambda}]b) \in K^{\boldsymbol{t}}_{\boldsymbol{\Lambda}}$ *and* $(c\boldsymbol{u}[\boldsymbol{\mu}]d) \in K^{\boldsymbol{u}}_{\boldsymbol{\Gamma}}$ *is a linear combination of basis vectors of* $K^{\boldsymbol{tu}}_{\boldsymbol{\Lambda}\wr\boldsymbol{\Gamma}}$ *of the form* $(a(\boldsymbol{tu})[\boldsymbol{\nu}]d)$ *for* $\boldsymbol{\nu} = \nu_{k+l}\dots\nu_0$ *with* $\lambda_k \leq \nu_{k+l}, \dots, \lambda_0 \leq \nu_l$ *and* $\nu_l \geq \mu_l, \dots, \nu_0 \geq \mu_0$.

Proof. By Theorem 3.19, $(a\boldsymbol{t}[\boldsymbol{\lambda}]b)(c\boldsymbol{u}[\boldsymbol{\mu}]d)$ is a linear combination of terms of the form $(a(\boldsymbol{tu})[\boldsymbol{\nu}]d)$ with $\nu_l \geq \mu_l, \dots, \nu_0 \geq \mu_0$. By using the map $*$ one can easily deduce from Theorem 3.19 that $(a\boldsymbol{t}[\boldsymbol{\lambda}]b)(c\mu d)$ is also a linear combination of terms of the form $(a(\boldsymbol{tu})[\nu]d)$ with $\lambda_k \leq \nu_{k+l}, \dots, \lambda_0 \leq \nu_l$. As the terms $(a(\boldsymbol{tu})[\boldsymbol{\nu}]d)$ are linearly independent the claim follows. ☐

Theorem 3.21. *Following the notation of* (3.5)*, the tensor product* $K^{\boldsymbol{t}}_{\boldsymbol{\Lambda}} \otimes_{K_{\Lambda_0}} K^{\boldsymbol{u}}_{\boldsymbol{\Gamma}}$ *is a well-defined* $(K_{\Lambda_k}, K_{\Gamma_0})$*-bimodule and we have that*

(i) *any vector* $(a\boldsymbol{t}[\boldsymbol{\lambda}]b) \otimes (c\boldsymbol{u}[\boldsymbol{\mu}]d) \in K^{\boldsymbol{t}}_{\boldsymbol{\Lambda}} \otimes_{K_{\Lambda_0}} K^{\boldsymbol{u}}_{\boldsymbol{\Gamma}}$ *is a linear combination of vectors of the form*
$$(a\boldsymbol{t}[\nu_{k+l}\dots\nu_l]\overline{\nu_l}) \otimes (\underline{\nu_l}\boldsymbol{u}[\nu_l\dots\nu_0]d) \tag{3.11}$$
where $(a(\boldsymbol{tu})[\boldsymbol{\nu}]d)$ *for* $\boldsymbol{\nu} = \nu_{k+l}\dots\nu_0$ *is an oriented* $\boldsymbol{\Lambda}\wr\boldsymbol{\Gamma}$*-circle diagram and we have furthermore* $\lambda_k \leq \nu_{k+l}, \dots, \lambda_0 \leq \nu_l$ *and* $\nu_l \geq \mu_l, \dots, \nu_0 \geq \mu_0$,

(ii) *a basis of* $K^{\boldsymbol{t}}_{\boldsymbol{\Lambda}} \otimes_{K_{\Lambda_0}} K^{\boldsymbol{u}}_{\boldsymbol{\Gamma}}$ *is given by all vectors* (3.11) *for oriented* $\boldsymbol{\Lambda}\wr\boldsymbol{\Gamma}$*-circle diagrams* $(a(\boldsymbol{tu}[\boldsymbol{\nu}])d)$ *and*

(iii) *the multiplication map* (3.5) *induces an isomorphism*
$$\overline{m}\colon K^{\boldsymbol{t}}_{\boldsymbol{\Lambda}} \otimes_{K_{\Lambda_0}} K^{\boldsymbol{u}}_{\boldsymbol{\Gamma}} \to K^{\boldsymbol{tu}}_{\boldsymbol{\Lambda}\wr\boldsymbol{\Gamma}}$$
of graded $(K_{\Lambda_k}, K_{\Gamma_0})$*-bimodules.*

Proof. We will prove (i) by contradiction, so let a and d be a cup and a cap diagram, such that the statement of (i) is wrong for some $\boldsymbol{\lambda}$, $\boldsymbol{\mu}$, b and c. Define the set S as
$$S := \left\{ (\boldsymbol{\lambda}, \boldsymbol{\mu}) \;\middle|\; \begin{array}{l} \boldsymbol{\lambda} = \lambda_k \dots \lambda_0 \text{ with } \lambda_i \in \Lambda_i, \\ \boldsymbol{\mu} = \mu_l \dots \mu_0 \text{ with } \mu_j \in \Gamma_j, \\ a\boldsymbol{t}[\boldsymbol{\lambda}] \text{ and } \boldsymbol{u}[\boldsymbol{\mu}]d \text{ are oriented} \end{array} \right\}.$$

We put a partial ordering on S by declaring that $(\boldsymbol{\lambda}, \boldsymbol{\mu}) \leq (\boldsymbol{\lambda}', \boldsymbol{\mu}')$ if $\lambda_i \leq \lambda'_i$ and $\mu_j \leq \mu'_j$ for all $0 \leq i \leq k$ and $0 \leq j \leq l$. For such a pair we define $K(\boldsymbol{\lambda}, \boldsymbol{\mu})$ to be the span of all vectors (3.11) with $\boldsymbol{\nu} = \nu_{k+l}\dots\nu_0$ satisfying $\lambda_k \leq \nu_{k+l}, \dots, \lambda_0 \leq \nu_l \geq \mu_l, \dots, \nu_0 \geq \mu_0$. As the set S is necessarily finite, we can chose $(\boldsymbol{\lambda}, \boldsymbol{\mu})$ maximal such that
$$(a\boldsymbol{t}[\boldsymbol{\lambda}]b) \otimes (c\boldsymbol{u}[\boldsymbol{\mu}]d) \notin K(\boldsymbol{\lambda}, \boldsymbol{\mu})$$

for some b and c. Note that we need to have $\lambda_0 \not< \mu_l$ or $\lambda_0 \not> \mu_l$. By using the antimultiplicative map $*$ we can assume without loss of generality that $\lambda_0 \not< \mu_l$. Using Theorem 3.19 and Corollary 3.20 we get

$$(a\boldsymbol{t}[\boldsymbol{\lambda}]\overline{\lambda_0})(\underline{\lambda_0}\lambda_0 b) = (a\boldsymbol{t}[\boldsymbol{\lambda}]b) + (\dagger),$$

where $(\dagger)$ denotes a linear combination of terms of the form $(a\boldsymbol{t}[\boldsymbol{\lambda}']b)$ with $(\boldsymbol{\lambda},\boldsymbol{\mu}) < (\boldsymbol{\lambda}',\boldsymbol{\mu})$ in our ordering. As $(\boldsymbol{\lambda},\boldsymbol{\mu})$ was chosen maximal, we observe that $(\dagger)\otimes(c\boldsymbol{u}[\boldsymbol{\mu}]d) \in K(\boldsymbol{\lambda}',\boldsymbol{\mu}) \subseteq K(\boldsymbol{\lambda},\boldsymbol{\mu})$. Thus

$$(a\boldsymbol{t}[\boldsymbol{\lambda}]\overline{\lambda_0})(\underline{\lambda_0}\lambda_0 b) \otimes (c\boldsymbol{u}[\boldsymbol{\mu}]d) = (a\boldsymbol{t}[\boldsymbol{\lambda}]\overline{\lambda_0}) \otimes (\underline{\lambda_0}\lambda_0 b)(c\boldsymbol{u}[\boldsymbol{\mu}]d)$$

cannot be contained in $K(\boldsymbol{\lambda},\boldsymbol{\mu})$. Hence we necessarily have that the product on the right hand side is nonzero, forcing $b = c^*$. Using Theorem 3.19 again we can rewrite the right hand side and get

$$(\underline{\lambda_0}\lambda_0 b)(b^*\boldsymbol{u}[\boldsymbol{\mu}]d) = s(\underline{\lambda_0}\boldsymbol{u}[\boldsymbol{\mu}]d) + (\dagger\dagger),$$

where $s = s_{\underline{\lambda_0}\lambda_0 b}(\mu_l)$ in case $\underline{\lambda_0}\mu_l$ is oriented and 0 otherwise. Furthermore $(\dagger\dagger)$ is a linear combination of $(\underline{\lambda_0}\boldsymbol{u}[\boldsymbol{\mu}']d)$'s with $(\boldsymbol{\lambda},\boldsymbol{\mu}) < (\boldsymbol{\lambda},\boldsymbol{\mu}')$. Again, by maximality, we deduce similarly to the above that $(a\boldsymbol{t}[\boldsymbol{\lambda}]\overline{\lambda_0}) \otimes (\dagger\dagger) \in K(\boldsymbol{\lambda},\boldsymbol{\mu})$. Thus we have

$$s(a\boldsymbol{t}[\boldsymbol{\lambda}]\overline{\lambda_0}) \otimes (\underline{\lambda_0}\boldsymbol{u}[\boldsymbol{\mu}]d) \notin K(\boldsymbol{\lambda},\boldsymbol{\mu})$$

and in particular $s \neq 0$. But s can only be nonzero if $\underline{\lambda_0}\mu_l$ is oriented. So we have $\lambda_0 \leq \mu_l$ and by our assumption additionally $\lambda_0 \not< \mu_l$, hence $\lambda_0 = \mu_l$. But then $(a\boldsymbol{t}[\boldsymbol{\lambda}]\overline{\lambda_0}) \otimes (\underline{\lambda_0}\boldsymbol{u}[\boldsymbol{\mu}]d)$ is of the form (3.11) for $\boldsymbol{\nu} = \lambda_k \ldots \lambda_1 \mu_l \ldots \mu_0$, yielding a contradiction.

In order to prove (ii) and (iii), first note that the multiplication is K_{Λ_0}-balanced by associativity, hence it induces a well-defined graded bimodule homomorphism $\overline{m}$. For this to be an isomorphism, it suffices to show that the restriction

$$\overline{m}\colon e_\alpha K^{\boldsymbol{t}}_{\boldsymbol{\Lambda}} \otimes_{K_{\Lambda_0}} K^{\boldsymbol{u}}_{\boldsymbol{\Gamma}} e_\beta \to e_\alpha K^{\boldsymbol{tu}}_{\boldsymbol{\Lambda}\wr\boldsymbol{\Gamma}} e_\beta$$

is an isomorphism for every fixed $\alpha \in \Lambda_k$ and $\beta \in \Gamma_0$. Note that after this restriction both sides are finite dimensional. The right hand side has a basis consisting of oriented $\boldsymbol{\Lambda}\wr\boldsymbol{\Gamma}$-circle diagrams $y(\nu) \coloneqq (\underline{\alpha}(\boldsymbol{tu})[\boldsymbol{\nu}]\overline{\beta})$. Note that by part (i) the left hand side has a generating set $X(\nu) \coloneqq (a\boldsymbol{t}[\nu_{k+l}\ldots\nu_l]\overline{\nu_l}) \otimes (\underline{\nu_l}\boldsymbol{u}[\nu_l\ldots\nu_0]d)$ indexed by exactly the same $\boldsymbol{\nu}$'s as the $y(\nu)$.

Using again the generalized cellular structure of Theorem 3.19 and Corollary 3.20 we see that $x(\nu)$ gets mapped to $y(\nu)$ plus some higher terms. But this means that $\overline{m}$ is surjective and that the $x(\nu)$'s are linearly independent. Therefore the $x(\nu)$'s form a basis and $\overline{m}$ is an isomorphism. $\square$

In the following theorem we will reduce the study of bimodules $K^{\boldsymbol{t}}_{\boldsymbol{\Lambda}}$ for arbitrary sequences $\boldsymbol{\Lambda} = \Lambda_k \ldots \Lambda_0$ and $\boldsymbol{t} = t_k \ldots t_1$ to the bimodules $K^{t}_{\Lambda\Gamma}$ for a single $\Lambda\Gamma$-

matching t, markedly simplifying our notation. For this denote the Frobenius algebra $\mathbb{C}[X]/(X^2)$ by R with the counit

$$\tau\colon R \to \mathbb{C}, \quad 1 \mapsto 0, X \mapsto 1.$$

By putting X in degree 1 and 1 in degree -1 we view this as a graded vector space, making the (co-)multiplication homogeneous of degree one.

Theorem 3.22. *Suppose we have a block sequence $\mathbf{\Lambda} = \Lambda_k \dots \Lambda_0$ and a proper $\mathbf{\Lambda}$-matching $\boldsymbol{t} = t_k \dots t_1$. Denote the reduction of $\boldsymbol{t}$ by u and let n be the number of internal circles removed in the reduction process. Then we have*

$$\begin{aligned} K_{\mathbf{\Lambda}}^{\boldsymbol{t}} &\cong K_{\Lambda_k \Lambda_0}^{u} \otimes R^{\otimes n} \langle \mathrm{caps}(t_1) + \dots + \mathrm{caps}(t_k) - \mathrm{caps}(u) \rangle \\ &\cong K_{\Lambda_k \Lambda_0}^{u} \otimes R^{\otimes n} \langle \mathrm{cups}(t_1) + \dots + \mathrm{cups}(t_k) - \mathrm{cups}(u) \rangle \end{aligned}$$

as graded $(K_{\Lambda_k}, K_{\Lambda_0})$-bimodules, viewing $K_{\Lambda_k \Lambda_0}^{u} \otimes R^{\otimes n}$ as a bimodule via the action on the first tensor factor.

Proof. Enumerate the n internal circles in some order and define a linear map

$$f\colon K_{\mathbf{\Lambda}}^{\boldsymbol{t}} \to K_{\Lambda_k \Lambda_0}^{u}, \quad (a\boldsymbol{t}[\boldsymbol{\lambda}]b) \mapsto (a\lambda_k u \lambda_0 b) \otimes x_1 \otimes \dots \otimes x_n,$$

where x_i is 1 if the i-th internal circle is oriented anticlockwise and X if it is oriented clockwise. This is clearly bijective as the orientation of the internal circles is determined by the x_i. It is a $(K_{\lambda_k}, K_{\lambda_0})$-bimodule homomorphism as the internal circles play no role in the bimodule structure and as by admissibility the tags get altered by an even number of undotted arcs (see also Lemma 3.7). Finally the map f is homogeneous of degree $\mathrm{caps}(t_1) + \dots + \mathrm{caps}(t_k) - \mathrm{caps}(u)$ by Lemma 3.12, giving the degree shift in the theorem. $\square$

This shows that in order to understand $K_{\mathbf{\Lambda}}^{\boldsymbol{t}}$ as a bimodule, it actually suffices to understand the bimodule $K_{\Lambda_k \Lambda_0}^{u}$ instead. This justifies why we are restricting ourselves to the latter case in the following section.

3.3 Projective functors

Definition 3.23. Let t be a proper $\Lambda\Gamma$-matching. Define the functor

$$G_{\Lambda\Gamma}^{t} := K_{\Lambda\Gamma}^{t} \langle -\mathrm{caps}(t) \rangle \otimes \_\colon \ \mathrm{mod}_{lf}(K_\Gamma) \to \mathrm{mod}_{lf}(K_\Lambda).$$

We call any functor which is isomorphic to a finite direct sum of the above functors (possibly shifted) a *projective functor.*

Remark 3.24. The degree shift in the definition ensures that $G_{\Lambda\Gamma}^{t}$ commutes with duality, see Theorem 3.34 below. Furthermore Theorem 3.21(iii) and Theorem 3.22 imply that the composition of projective functors is again projective.

Lemma 3.25. *Suppose that t is a proper $\Lambda\Gamma$-matching which does not contain any cups or caps. Then the functor $G^t_{\Lambda\Gamma}$ is an equivalence of categories.*

Proof. As t contains no cups or caps, it determines a bijection $f\colon \Lambda \to \Gamma$ which maps λ to the weight γ such that $\underline{\gamma}$ is the lower reduction of $\underline{\lambda}t$. Furthermore the induced map

$$f\colon K_\Lambda \to K_\Gamma, \quad \underline{\alpha}\lambda\overline{\beta} \mapsto \underline{f(\alpha)}f(\lambda)\overline{f(\beta)}$$

is an isomorphism of graded algebras (this follows as the position of $\circ$ and $\times$ play no role in the algebra structure). Under this identification the (K_Λ, K_Γ)-bimodule $K^t_{\Lambda\Gamma}$ is isomorphic to (K_Γ, K_Γ)-bimodule K_Γ. Since $G^t_{\Lambda\Gamma}$ is given by tensoring with $K^t_{\Lambda\Gamma}$, it is an equivalence. ☐

Theorem 3.26. *Let t be a proper $\Lambda\Gamma$-matching and let $\gamma \in \Gamma$. Then*

(i) $G^t_{\Lambda\Gamma}P(\gamma) \cong K^t_{\Lambda\Gamma}e_\gamma\langle -\operatorname{caps}(t)\rangle$ *as left K_Λ-modules,*

(ii) *the module $G^t_{\Lambda\Gamma}P(\gamma)$ is nonzero if and only if each upper line in $t\gamma\overline{\gamma}$ is oriented and*

(iii) *in this case moreover,*

$$G^t_{\Lambda\Gamma}P(\gamma) \cong P(\lambda) \otimes R^{\otimes n}\langle \operatorname{cups}(t) - \operatorname{caps}(t)\rangle$$

as graded left K_Λ-modules (K_Λ acts again on the right hand side only on the first factor), where $\lambda \in \Lambda$ is such that $\overline{\lambda}$ is the upper reduction of $t\overline{\gamma}$ and n denotes the number of upper circles removed in the reduction process.

Proof. For (i) note that

$$\begin{aligned} G^t_{\Lambda\Gamma}P(\gamma) &= K^t_{\Lambda\Gamma}\langle -\operatorname{caps}(t)\rangle \otimes_{K_\Gamma} P(\gamma) = K^t_{\Lambda\Gamma} \otimes_{K_\Gamma} K_\Gamma e_\gamma\langle -\operatorname{caps}(t)\rangle \\ &\cong K^t_{\Lambda\Gamma}e_\gamma\langle -\operatorname{caps}(t)\rangle. \end{aligned}$$

For the forward implication of (ii), note that for any weight ν such that $\nu\overline{\gamma}$ is oriented, the rays are oriented in the same ways as in $\gamma\overline{\gamma}$. Thus, if there exists an upper line in $t\gamma\overline{\gamma}$ which is not oriented, then there cannot exist an oriented $\Lambda\Gamma$-circle diagram of the form $a\mu t\nu\overline{\gamma}$. But these form a basis of $K^t_{\Lambda\Gamma}e_\gamma$ and hence $G^t_{\Lambda\Gamma}P(\gamma) = 0$ by (i). In order to finish the proof, suppose that each upper line of $t\gamma\overline{\gamma}$ is oriented properly. Enumerate the n upper circles in some order and define the map

$$f\colon K^t_{\Lambda\Gamma}e_\gamma \to K_\Lambda e_\lambda \otimes R^{\otimes n}, \quad (a\mu t\nu\overline{\gamma}) \mapsto (a\mu\overline{\lambda}) \otimes x_i \otimes \cdots \otimes x_n$$

where x_i is 1 (resp. X) if the i-th circle is oriented anticlockwise (resp. clockwise). This map is then an isomorphism of vector spaces. It is K_Λ-linear as every tag gets altered by an even number of undotted arcs (see Lemma 3.7), and moreover it is homogeneous of degree $\operatorname{cups}(t)$ by Lemma 3.15. By observing that $0 \neq P(\lambda) = K_\Lambda e_\lambda$ and $G^t_{\Lambda\Gamma}P(\gamma) \cong K^t_{\Lambda\Gamma}e_\gamma\langle -\operatorname{caps}(t)\rangle$, this finishes the proof of (ii) and (iii). ☐

Corollary 3.27. *The module $K^t_{\Lambda\Gamma}$ is projective as a left K_Λ-module as well as projective as a right K_Γ-module.*

Proof. By Theorem 3.26(i) and (iii) we have that $K^t_{\Lambda\Gamma} = \bigoplus_{\gamma\in\Gamma} K^t_{\Lambda\Gamma}e_\gamma$ is projective as a left K_Λ-module. Using the antimultiplicative map $*$, $K^t_{\Lambda\Gamma}$ being projective as a right K_Γ-modules is the same as $K^{t^*}_{\Gamma\Lambda}$ being a projective left K_Γ-module, but this was done above. $\square$

Corollary 3.28. *Projective functors are exact and preserve the property of being finitely generated.*

Proof. Use Corollary 3.27 and Theorem 3.26(iii). $\square$

The following theorem deals with the effect of a projective functor on standard modules $V(\mu)$ from (3.3).

Theorem 3.29. *Let t be a proper $\Lambda\Gamma$-matching and $\gamma \in \Gamma$.*

(i) *The K_Λ-module $G^t_{\Lambda\Gamma}V(\gamma)$ has a filtration*

$$\{0\} = M(0) \subset M(1) \subset \cdots \subset M(n) = G^t_{\Gamma\Lambda}V(\gamma)$$

such that $M(i)/M(i-1) \cong V(\mu_i)\langle \deg(\mu_i t\gamma) - \mathrm{caps}(t)\rangle$ for each i. In this case $\mu_1, \ldots, \mu_n$ denote the elements of the set $\{\mu \in \Lambda \mid \mu t\gamma$ oriented$\}$ ordered such that $\mu_i > \mu_j$ implies $i < j$.

(ii) *The module $G^t_{\Gamma\Lambda}V(\gamma)$ is nonzero if and only if each cup in $t\gamma$ is oriented.*

(iii) *Assuming* (ii), *the module $G^t_{\Gamma\Lambda}V(\gamma)$ is indecomposable with irreducible head isomorphic to $L(\lambda)\langle \deg(\lambda t\gamma) - \mathrm{caps}(t)\rangle$, where $\lambda \in \Lambda$ is the unique weight such that $\overline{\lambda}$ is the upper reduction of $t\overline{\gamma}$. In other words $\lambda t\gamma$ is oriented and all its caps are anticlockwise.*

Proof. We start this proof by claiming that the set

$$\{(a\mu_i t\gamma\overline{\gamma}) \otimes (\underline{\gamma}\gamma| \mid \text{for } i = 1, \ldots, n \text{ and all oriented cup diagrams } a\mu_i\}$$

is a basis for $G^t_{\Lambda\Gamma}V(\gamma)$. Using Theorem 3.21(ii), we see that $G^t_{\Lambda\Gamma}P(\gamma)$ has a basis given by vectors of the form $(a\mu t\nu\overline{\nu}) \otimes (\underline{\nu}\nu\overline{\gamma})$. But $V(\gamma)$ is the quotient of $P(\gamma)$ by the subspace spanned by the vectors $(c\nu\overline{\gamma})$ for $\nu > \gamma$. By Corollary 3.28 we see that $G^t_{\Lambda\Gamma}V(\gamma)$ is the quotient of $G^t_{\Lambda\Gamma}P(\gamma)$ by the subspace spanned by $(a\mu t\lambda b) \otimes (c\nu\overline{\gamma})$ for $\nu > \gamma$. But by Theorem 3.21(i) this subspace is already spanned by $(a\mu t\nu\overline{\nu}) \otimes (\underline{\nu}\nu\overline{\gamma})$ for $\nu > \gamma$. Thus $G^t_{\Lambda\Gamma}V(\gamma)$ has a basis given by the images of $(a\mu t\gamma\overline{\gamma}) \otimes (\underline{\gamma}\gamma\overline{\gamma})$ which coincides with our claim by definition of the cell module.
Now define $M(0) = \{0\}$ and inductively $M(i)$ to be the subspace generated by $M(i-1)$ and $\{(a\mu_i t\gamma\overline{\gamma}) \otimes (\underline{\gamma}\gamma| \mid$ for all oriented cup diagrams $a\mu_i\}$. This defines a filtration of $G^t_{\Lambda\Gamma}V(\gamma)$ by vector spaces with $M(n) = G^t_{\Lambda\Gamma}V(\gamma)$ by the above argumentation. That

the $M(i)$ are in fact K_Λ-submodules follows from Corollary 3.20, our assumption on the ordering of the μ_i, and the above paragraph.
The quotient $M(i)/M(i-1)$ has a basis given by

$$\{(c\mu_i t\gamma\overline{\gamma}) \otimes (\underline{\gamma}\gamma| \mid \text{ for all oriented cup diagrams } c\mu_i\}.$$

Theorem 3.19 says that

$$(a\lambda b)(c\mu_i t\gamma\overline{\gamma}) \otimes (\underline{\gamma}\gamma| \equiv \begin{cases} s_{a\lambda b}(\mu_i)(a\mu_i t\gamma\overline{\gamma}) \otimes (\underline{\gamma}\gamma| & \text{if } b = c^* \text{ and } a\mu_i \text{ oriented,} \\ 0 & \text{otherwise,} \end{cases}$$

working modulo $M(i-1)$. Looking at (3.3) we see that the map

$$M(i)/M(i-1) \to V(\mu_i), \quad (c\mu_i t\gamma\overline{\gamma}) \otimes (\underline{\gamma}\gamma| \mapsto (c\mu_i|$$

is an isomorphism of K_Λ-modules. Moreover it is homogeneous of degree $\deg(\mu_i t\gamma) - \operatorname{caps}(t)$ by definition. This proves the first statement.
For (ii) suppose that some cup in the diagram $t\gamma$ is not oriented. Then there exist no $\mu \in \Lambda$ such that $\mu t\gamma$ is oriented and thus $G^t_{\Lambda\Gamma}V(\gamma) = 0$ by part (i). For the converse note that if every cup in $t\gamma$ is oriented, $\lambda t\gamma$ is oriented for the weight λ defined in (iii). Hence $G^t_{\Lambda\Gamma}V(\gamma) \neq 0$.
In order to prove the last statement (ignoring the grading), note that the filtration defined in (i) is multiplicity free. Thus, as every cell module has multiplicity free head, we see easily that $G^t_{\Lambda\Gamma}V(\gamma)$ must have multiplicity free head. On the other hand $V(\gamma)$ is a quotient of $P(\gamma)$ and projective functors are exact (Corollary 3.28), so $G^t_{\Lambda\Gamma}V(\gamma)$ is a quotient of $G^t_{\Lambda\Gamma}P(\gamma)$. By Theorem 3.26(iii) we have that $G^t_{\Lambda\Gamma}P(\gamma)$ is a direct sum of $P(\lambda)$'s, where λ is exactly defined as in (iii). Hence, the head of $G^t_{\Lambda\Gamma}V(\gamma)$ is a direct sum of $L(\lambda)$'s. Putting these two facts together, we see that $G^t_{\Lambda\Gamma}V(\gamma)$ has irreducible head $L(\lambda)$ and is thus indecomposable. Looking again at the filtration in (i), one sees that this factor occurs with the claimed grading shift. $\square$

In Theorem 3.35 we will analyze the effect of projective functors on irreducible modules. For this we are interested in proving that the projective functors $G^t_{\Lambda\Gamma}$ and $G^{t^*}_{\Gamma\Lambda}$ form up to degree shift an adjoint pair (as in [BS10, Section 4] for type A), so that we can understand the composition factors of $G^t_{\Lambda\Gamma}L(\gamma)$ in terms of $G^{t^*}_{\Gamma\Lambda}P(\mu)$. For this we define a linear map

$$\phi\colon K^{t^*}_{\Gamma\Lambda} \otimes K^t_{\Lambda\Gamma} \to K_\Gamma \tag{3.12}$$

by declaring that $\phi := 0$ if t is not a proper $\Lambda\Gamma$-matching. If t is proper and given basis vectors $(a\lambda t^*\nu d) \in K^{t^*}_{\Gamma\Lambda}$ and $(d'\kappa t\mu b) \in K^t_{\Lambda\Gamma}$, we denote by c the upper reduction of t^*d. Then if $d' = d^*$ and all mirror image pairs of upper respectively lower circles in t^*d respectively d^*t are oriented in *opposite* ways in the corresponding basis vectors, we set

$$\phi((a\lambda t^*\nu d) \otimes (d'\kappa t\mu b)) := \pm(a\lambda c)(c^*\mu b) \tag{3.13}$$

and otherwise we set $\phi((a\lambda t^*\nu d)\otimes(d'\kappa t\mu b)) := 0$. The sign in (3.13) depends only on t and d and is defined inductively by the argument given in the next proof, i.e. by the induction argument in the next proof one can reconstruct the sign for each t and d.

Lemma 3.30. *The map $\phi\colon K^{t^*}_{\Gamma\Lambda}\otimes K^{t}_{\Lambda\Gamma}\to K_\Gamma$ is a homogeneous (K_Γ, K_Γ)-bimodule homomorphism of degree $-2\,\mathrm{caps}(t)$. Moreover it is K_Λ-balanced and thus induces a map $\overline{\phi}\colon K^{t^*}_{\Gamma\Lambda}\otimes_{K_\Lambda} K^{t}_{\Lambda\Gamma}\to K_\Gamma$.*

Proof. If t is not a proper $\Lambda\Gamma$-matching the claim is trivial, thus we assume in the following that t is proper.
First of all, we are going to show that ϕ is homogeneous of degree $-2\,\mathrm{caps}(t)$. For this take again basis vectors as in (3.13) (in every other case ϕ is 0 by definition). Suppose that p (resp. q) of the upper circles in t^*d are oriented clockwise (resp. anticlockwise) in $a\lambda t^*\nu d$. Then by our assumptions on the basis vectors q (resp. p) of the lower circles in d^*t are oriented clockwise (resp. anticlockwise) in $d^*\kappa t\mu b$. By Lemma 3.15 we have

$$\begin{aligned}\deg(a\lambda t^*\nu d) &= \deg(a\lambda c) + \mathrm{cups}(t^*) + p - q \quad\text{and}\\ \deg(d^*\kappa t\mu b) &= \deg(c^*\mu b) + \mathrm{caps}(t) + q - p.\end{aligned}$$

By definition of t^*, we have $\mathrm{cups}(t^*) = \mathrm{caps}(t)$, thus

$$\deg((a\lambda t^*\nu d)\otimes(d^*\kappa t\mu b)) = \deg((a\lambda c)(c^*\mu b)) + 2\,\mathrm{caps}(t). \tag{3.14}$$

Secondly, the map ϕ is a left K_Γ-homomorphism as in the proof of Theorem 3.26, which showed that mapping $(a\lambda t^*\nu d)$ to $(a\lambda c)$ is a left K_Γ-homomorphism, and one argues similarly for the right action.
Lastly we are going to prove that ϕ is K_Λ-balanced. For this we introduce the map

$$\omega\colon K^{t^*t}_{\Gamma\Lambda\Gamma}\to K_\Gamma$$

as follows. Take a basis vector $(a\lambda t^*\mu t\nu b)\in K^{t^*t}_{\Gamma\Lambda\Gamma}$. If any of its internal circles in the diagram t^*t are oriented anticlockwise, we declare that its image is 0. Otherwise we define u to be the reduction of t^*t and consider the diagram $a\lambda u\nu b$. This contains a symmetric middle section as u was the reduction of the symmetric diagram t^*t, so it makes sense to apply the surgery procedure to smooth this section out and obtain a linear combination of basis vectors of K_Γ. We define the image of $(a\lambda t^*\mu t\nu b)$ to be this linear combination. We claim that

$$\phi = \omega\circ m, \tag{3.15}$$

where m is the multiplication map $m\colon K^{t^*}_{\Gamma\Lambda}\otimes K^{t}_{\Lambda\Gamma}\to K^{t^*t}_{\Gamma\Lambda\Gamma}$ from (3.5). As we know that m is K_Λ-balanced (by associativity), this shows that ϕ is K_Λ-balanced.
In some sense, we are trying to prove that first reducing and then multiplying (ϕ) is "the same as" first multiplying and then reducing ($\omega\circ m$). The general idea of the proof is to replace t^*d by some easier $t_1^*d_1$ (for which we know the

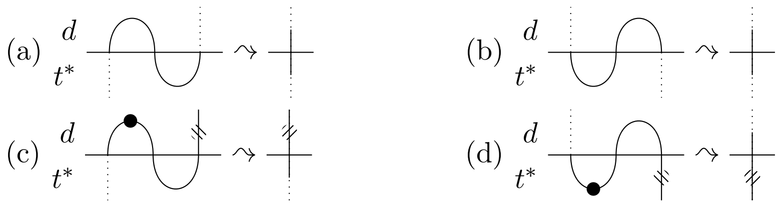
claim) such that both have the same upper reduction, and then trying to show that $\omega((a\lambda t^*\nu d)(d^*\kappa t\mu b)) = \omega((a\lambda t_1^*\nu_1 d_1)(d_1^*\kappa_1 t_1\mu b))$. But in general, the above equality holds only up to sign and that is why we incorporated a sign in the definition of ϕ.
To prove the claim, we proceed by induction on caps(t). If caps$(t) = 0$, then there are neither upper circles in t^*d nor internal circles in t^*t. Thus in this case applying the upper reduction to t^*d gives a bijection between the caps in t^*d and the caps in c, which is just given by reducing straight lines in t^*d. Hence the signs involved in surgery procedures will exactly be the same. Computing $\phi(a\lambda t^*\nu d) \otimes (d^*\kappa t\mu b) = (a\lambda c)(c^*\mu b)$ means that every cap in c gets eliminated by surgeries. On the other hand applying m eliminates each cap in d and ω eliminates then the remaining caps in t^*. Thus by the above comment, the results are the same.
For the induction step assume caps$(t) > 0$ and that (3.15) is proven for all smaller cases. We will consider five different cases depending on certain subpictures of t^*d, the last one being the general case.
Case 1: Suppose that t^*d contains a small circle, i.e. a circle consisting of only one cap and cup. If this circle in $t^*\nu d$ and its mirror image in $d^*\kappa t$ are oriented in the same way, then ϕ gives 0 by definition. On the other hand $(a\lambda t^*\nu d)(d^*\kappa t\mu b)$ produces 0 if both are oriented clockwise and the product produces an anticlockwise circle if both were oriented anticlockwise, but in this case ω produces 0. Thus we may assume that these two circles are oriented in opposite ways in $t^*\nu d$ and $d^*\kappa t$. Now we can remove these two circles (and the vertices involved) to obtain diagrams $a\lambda t_1^*\nu_1 d_1$ and $d_1^*\kappa_1 t_1\mu b$ with caps$(t_1) <$ caps(t). Using the definitions, one can easily verify that

$$\begin{aligned}\phi((a\lambda t^*\nu d) \otimes (d^*\kappa t\mu b)) &= \phi((a\lambda t_1^*\nu_1 d_1) \otimes (d_1^*\kappa_1 t_1\mu b)),\\ \omega((a\lambda t^*\nu d)(d^*\kappa t\mu b)) &= \omega((a\lambda t_1^*\nu_1 d_1)(d_1^*\kappa_1 t_1\mu b)).\end{aligned}$$

The first equality holds because the small circle is removed in the process of upper reduction, and thus it does not matter whether we remove it in the process of upper reduction or whether we remove it first and do the upper reduction after that. The second equality holds as merging the two small circles in a surgery for the left hand side produces exactly one small clockwise circle with no further signs, which then gets removed by ω. But these circles play no role for the other surgeries, hence it agrees with the right hand side. Using the induction hypothesis the right hand sides coincide, thus the left hand sides agree as well.
Case 2: Suppose that t^*d contains an upper line containing only one cup. Denote by $a\lambda t_1^*\nu_1 d_1$ and $d_1^*\kappa_1 t_1\mu b$ the diagrams where this upper line and its mirror image in d^*t get removed. When computing the product $(a\lambda t^*\nu d)(d^*\kappa t\mu b)$ one can apply the same surgery procedures as for $(a\lambda t_1^*\nu_1 d)(d^*\kappa_1 t_1\mu b)$. There is no further surgery needed as the upper line contains only one cup. Now notice that, when drawing $(a\lambda t^*\nu d)$ underneath $(d^*\kappa t\mu b)$ the upper line and its mirror image form a clockwise circle. This is not changed throughout the whole surgery procedure and $(a\lambda t^*\nu d)(d^*\kappa t\mu b)$ and $(a\lambda t_1^*\nu_1 d)(d_1^*\kappa_1 t_1\mu b)$ differ only by this clockwise circle, which is removed when applying ω. And as both of them clearly have the same upper reduction (upper lines get removed in this process) we get

$$\phi((a\lambda t^*\nu d)\otimes(d^*\kappa t\mu b)) = \phi((a\lambda t_1^*\nu_1 d_1)\otimes(d_1^*\kappa_1 t_1\mu b)) = \omega((a\lambda t_1^*\nu_1 d_1)(d_1^*\kappa_1 t_1\mu b))$$
$$= \omega((a\lambda t^*\nu d)(d^*\kappa t\mu b)).$$

Case 3: Suppose then that t^*d contains one of the following local pictures on the top number line of t^*d with a mirror image in d^*t. A dashed dot means that there may be a dot present and different dashing patterns correspond to different choices whether a dot is present or not. In any case, the parity of the number of dots stays the same.



Denote by $a\lambda t_1^*\nu_1 d_1$ and $d_1^*\kappa_1 t_1\mu b$ the diagrams obtained by straightening these curved lines as in the picture above. We are then again in the situation that $\mathrm{caps}(t_1) < \mathrm{caps}(t)$. In order to compute $(a\lambda t^*\nu d)(d^*\kappa t\mu b)$ we can apply exactly the same surgery procedures in the same order as for $(a\lambda t_1^*\nu_1 d_1)(d_1^*\kappa_1 t_1\mu b)$ and apply an additional one somewhere in the middle, which involves these curved lines we straightened. Figure 3.3 shows this additional surgery. Note that the dashed dots in the reduction process appear directly beneath each other when multiplying, thus they appear in pairs and get removed at the beginning of the multiplication process. Up to the point of the

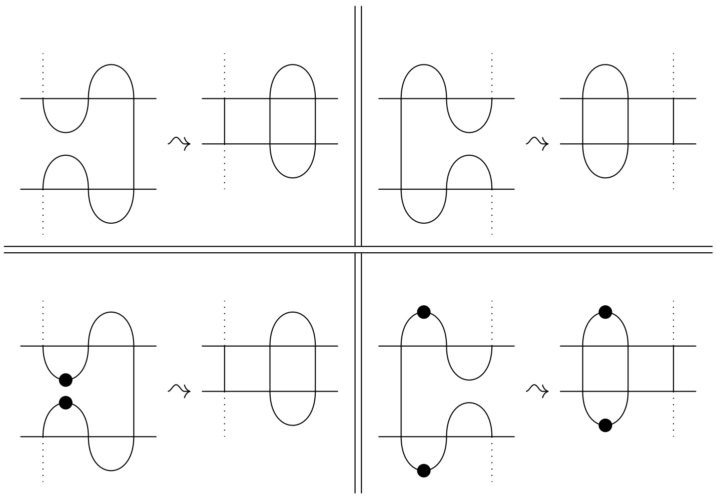
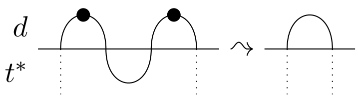
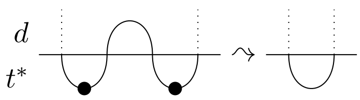


Figure 3.3: Additional surgeries for t^*d: *Case 3*

additional surgery procedure, the results of the surgeries applied so far is the same (except in the local spot that we changed). The additional surgery procedure is a split and produces one extra internal circle. We can concentrate on the case where this circle is oriented clockwise, as otherwise ω produces 0. But in this case the other component is oriented in the same way as before, thus leaving ourselves only with a few possible signs. Looking at the definition of the surgery procedure *Split*, one gets that the involved signs are $(-1)^{\mathrm{p}(i)+1}$ in all four cases. After this all the remaining surgeries produce exactly the same result (except for the additional circle produced by the split). This circle does not get altered by any other surgery and is later removed by ω. Thus we have $\omega((a\lambda t^*\nu d)(d^*\kappa t\mu b)) = (-1)^{\mathrm{p}(i)+1}\omega((a\lambda t_1^*\nu_1 d_1)(d_1^*\kappa_1 t_1\mu b))$. On the other hand t^*d and $t_1^*d_1$ clearly have the same upper reduction. Defining the involved sign in the definition of ϕ to be exactly $(-1)^{\mathrm{p}(i)+1}$ times the sign associated to $t_1^*d_1$, we conclude $\phi((a\lambda t^*\nu d) \otimes (d^*\kappa t\mu b)) = \phi((a\lambda t_1^*\nu_1 d_1) \otimes (d_1^*\kappa_1 t_1\mu b))$, thus finishing this case.

Case 4: Suppose that t^*d contains one of the following subpictures.

(a) 

(b) 

We can apply the indicated reduction and we denote the reduced diagrams by $a\lambda t_1^*\nu_1 d_1$ and $d_1^*\kappa_1 t_1\mu b$ respectively. Let us first look at the second case. The computation of $(a\lambda t^*\nu d)(d^*\kappa t\mu b)$ involves applying the same surgery procedures as for $(a\lambda t_1^*\nu_1 d_1)(d_1^*\kappa_1 t_1\mu b)$ and one additional surgery at the end. These first surgeries are actually the same because the orientations of every component agree and the tags are the same, as in this case we change two undotted cups and a cap into an undotted cup. Then in the end we apply the following surgery procedure: the circle which gets split is either oriented anticlockwise or clockwise, but it has the same orientation as the one in the reduced picture (on the right). This last additional surgery is a

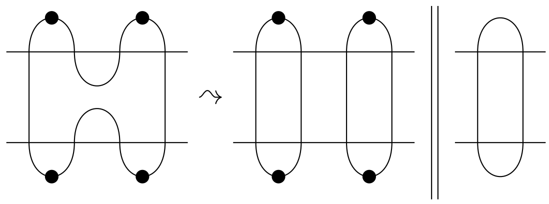


Figure 3.4: Additional surgery for t^*d in contrast to $t_1^*d_1$: *Case 4a*

split and it either splits an anticlockwise or a clockwise circle. If the circle is oriented anticlockwise, it produces the sum of two basis vectors and in each of these, one circle is oriented anticlockwise. Thus ω produces 0 and in the reduced picture (see right hand side of Figure 3.4) we also have an anticlockwise circle and hence ω produces 0 there as well. All together we have (as both pictures have the same upper reduction)

$$\phi((a\lambda t^*\nu d) \otimes (d^*\kappa t\mu b)) = \phi((a\lambda t_1^*\nu_1 d_1) \otimes (d_1^*\kappa_1 t_1\mu b)) = \omega((a\lambda t_1^*\nu_1 d_1)(d_1^*\kappa_1 t_1\mu b))$$

$$= 0 = \omega((a\lambda t^*\nu d)(d^*\kappa t\mu b)).$$

If on the other hand the circles is oriented clockwise, the split produces exactly two clockwise circles and the involved sign is $(-1)^{\mathrm{p}(i)+1}$. In the end comparing $(a\lambda t^*\nu d)(d^*\kappa t\mu b)$ with $(a\lambda t_1^*\nu_1 d_1)(d_1^*\kappa_1 t_1\mu b)$, we see that both agree up to the sign $(-1)^{\mathrm{p}(i)+1}$ and clockwise internal circles. But these internal clockwise circles get removed by ω, hence we get up to the sign the same result

$$\omega((a\lambda t^*\nu d)(d^*\kappa t\mu b)) = (-1)^{\mathrm{p}(i)+1}\omega((a\lambda t_1^*\nu_1 d_1)(d_1^*\kappa_1 t_1\mu b)).$$

On the other hand both pictures have the same upper reduction and thus defining the involved sign for t^*d to be $(-1)^{\mathrm{p}(i)+1}$ times the one associated to $t_1^*d_1$ we get

$$\phi((a\lambda t^*\nu d)\otimes(d^*\kappa t\mu b)) = (-1)^{\mathrm{p}(i)+1}\phi((a\lambda t_1^*\nu_1 d_1)(d_1^*\kappa_1 t_1\mu b)).$$

As $\mathrm{caps}(t_1) < \mathrm{caps}(t)$ we are done by induction.
Now for the last case we again have the same surgeries for $(a\lambda t^*\nu d)(d^*\kappa t\mu b)$ and $(a\lambda t_1^*\nu_1 d_1)(d_1^*\kappa_1 t_1\mu b)$ and an additional one for $(a\lambda t_1^*\nu_1 d_1)(d_1^*\kappa_1 t_1\mu b)$ and two additional ones for $(a\lambda t^*\nu d)(d^*\kappa t\mu b)$ somewhere in the middle (but for both at the same point) (see Figure 3.5). Up to this point the applied surgery procedures again give

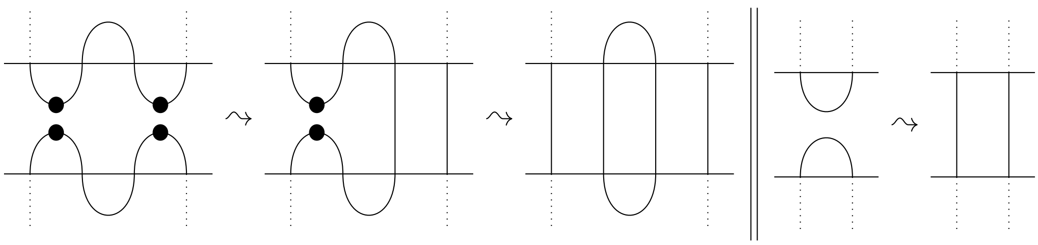


Figure 3.5: The additional surgeries for t^*d in contrast to $t_1^*d_1$: *Case 4b*

the same result, as the component which is reduced is oriented in the same way as before and no sign involved in the multiplication process gets changed. Both first additional surgeries are of the same type and produce the same diagram just differing in this local spot (see Figure 3.5) and maybe some additional signs. By looking at the definition of the multiplication one sees that the upcoming signs in a merge or a split turn out to be the same and for a reconnect at least one line would not be nonpropagating by admissibility, thus the results would be 0 anyway. The second additional surgery is then splitting off the circle in the middle in Figure 3.5. If the component is oriented clockwise, the split produces two clockwise oriented components and the involved sign is $(-1)^{\mathrm{p}(i)}$. So up to the sign and this clockwise oriented circle in between, the linear combinations agree. But ω removes the clockwise circle. If the component is oriented anticlockwise, the split produces a sum of two diagrams. In one the extra circle is oriented anticlockwise but then ω would produce 0. So we can concentrate on the summand, where the extra circle is oriented clockwise. Then the

other component is necessarily oriented anticlockwise (and thus as before) and the involved sign is $(-1)^{\mathrm{p}(i)}$.
All surgeries applied after these ones yield the same results, thus one finishes with the same linear combination of diagrams all just multiplied by $(-1)^{\mathrm{p}(i)}$. Again just as before the upper reduction of the reduced and the original picture is the same and furthermore $\mathrm{caps}(t_1) < \mathrm{caps}(t)$, and defining the sign for ϕ accordingly, we are done by induction and finished with this case.
Case 5: In the general setting, we may assume (using the base case of the induction, *Case 1* and *Case 2*) that we have cups in t^* but neither a small circle in t^*d nor an upper line containing only one cup. Then we have to have one of the subpictures

where a dashed dot means that a dot can be present or not. We may assume that we choose a picture such that the horizontal distance between the endpoints is minimal. This means that no attached cup or cap can end "inside" the cap or cup of the subpicture, i.e. one endpoint is at one of the dashed lines and the other one is either to the left or to the right of the picture. First observe that there cannot be two dots because then the picture would not be admissible, as the left dashed line would necessarily cut off one of the dots from the left boundary. If no dot is present we are either in *Case 3a* or *Case 3b*. If one dot is present, Figure 3.6 makes a case distinction between which of the arcs is dotted and what happens on the dotted line attached to the undotted arc. This concludes the proof as in each case we can apply one of *Case 3* and *Case 4* and for those we have seen the claim before. □

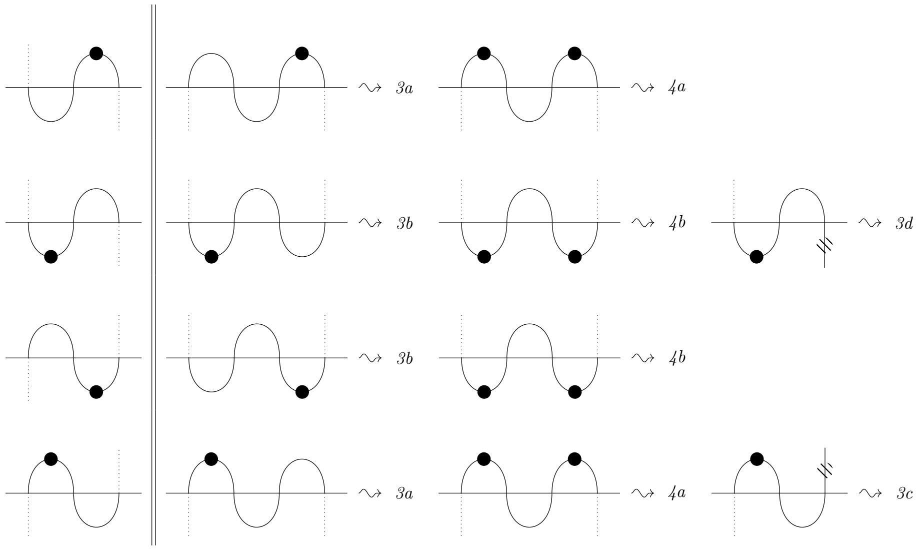


Figure 3.6: Case distinction for the general case

Theorem 3.31. *There is a graded (K_Λ, K_Γ)-bimodule isomorphism*

$$\hat{\phi}\colon K^t_{\Lambda\Gamma}\langle -2\operatorname{caps}(t)\rangle \xrightarrow{\sim} \operatorname{Hom}_{K_\Gamma}(K^{t^*}_{\Gamma\Lambda}, K_\Gamma)$$

given by sending $y \in K^t_{\Lambda\Gamma}$ to $\hat{\phi}(y)\colon K^{t^*}_{\Gamma\Lambda} \to K_\Gamma, x \mapsto \phi(x \otimes y)$.

Proof. First of all $\hat{\phi}$ is well-defined, since $\phi(\_\otimes y)$ is a left K_Γ-module homomorphism by Lemma 3.30. As ϕ is homogeneous of degree $-2\operatorname{caps}(t)$ by Lemma 3.30, $\hat{\phi}$ is homogeneous of degree 0. To check that it is a (K_Λ, K_Γ)-bimodule homomorphism, let $u \in K_\Lambda$, $v \in K_\Gamma$, $y \in K^t_{\Lambda\Gamma}$ and $x \in K^{t^*}_{\Gamma\Lambda}$. We then have

$$\begin{aligned}(u\hat{\phi}(y))(x) &\overset{\text{Def.}}{=} \hat{\phi}(y)(xu) \overset{\text{Def.}}{=} \phi(xu \otimes y) \overset{3.30}{=} \phi(x \otimes uy) \overset{\text{Def.}}{=} \hat{\phi}(uy)(x),\\ (\hat{\phi}(y)v)(x) &\overset{\text{Def.}}{=} (\hat{\phi}(y)(x))v \overset{\text{Def.}}{=} \phi(x \otimes y)v \overset{3.30}{=} \phi(x \otimes yv) \overset{\text{Def.}}{=} \hat{\phi}(yv)(x),\end{aligned}$$

thus $u\hat{\phi}(y) = \hat{\phi}(uy)$ and $\hat{\phi}(y)v = \hat{\phi}(yv)$.
It remains to show that $\hat{\phi}$ is a vector space isomorphism. For this it suffices to show that the restriction

$$\hat{\phi}\colon e_\lambda K^t_{\Lambda\Gamma} \to e_\lambda \operatorname{Hom}_{K_\Gamma}(K^{t^*}_{\Gamma\Lambda}, K_\Gamma) = \operatorname{Hom}_{K_\Gamma}(K^{t^*}_{\Gamma\Lambda} e_\lambda, K_\Gamma)$$

is an isomorphism. Additionally we can assume $e_\lambda K^t_{\Lambda\Gamma} \neq 0$ as it is equivalent to $K^{t^*}_{\Gamma\Lambda} e_\lambda \neq 0$. Let $\gamma \in \Gamma$ such that $\overline{\gamma}$ is the upper reduction of $t^*\overline{\lambda}$ and denote by n the number of upper circles removed in this reduction. Then using Theorem 3.26(i) and (iii), we have $K^{t^*}_{\Gamma\Lambda} e_\lambda \cong K_\Gamma e_\gamma \otimes R^{\otimes n}$ as left K_Γ-modules and similarly $e_\lambda K^t_{\Lambda\Gamma} \cong e_\gamma K_\Gamma \otimes R^{\otimes n}$ as right K_Γ-modules. Then $\hat{\phi}$ being an isomorphism is equivalent to the statement that the map

$$\begin{aligned}e_\gamma K_\Gamma \otimes R^{\otimes n} &\to \operatorname{Hom}_{K_\Gamma}(K_\Gamma e_\gamma \otimes R^{\otimes n}, K_\Gamma),\\ v \otimes x_1 \otimes \cdots \otimes x_n &\mapsto (u \otimes y_1 \otimes \cdots \otimes y_n \mapsto \varepsilon\tau(x_1 y_1)\cdots\tau(x_n y_n)uv),\end{aligned}$$

is an isomorphism where $\varepsilon \in \{\pm 1\}$ is the sign in the definition of ϕ associated to $t\overline{\lambda}$. To see that the isomorphisms actually transport $\hat{\phi}$ to the map above, observe that $\tau(x_i y_i)$ is 0 if x_i and y_i are either both 1 or X and 1 if they are different. But these correspond to orientations of the upper circles, thus ϕ produces 0 as well if some are oriented in the same way. If all mirror image pairs are oriented differently we multiply the upper reductions with an overall sign of ϵ which corresponds exactly to ϵuv. This reduces to showing that $e_\gamma K_\Gamma \to \operatorname{Hom}_{K_\Gamma}(K_\Gamma e_\gamma, K_\Gamma), v \mapsto (u \mapsto uv)$ is an isomorphism, which is obvious. □

Corollary 3.32. *There is a canonical isomorphism*

$$\operatorname{Hom}_{K_\Gamma}(K^{t^*}_{\Gamma\Lambda}, \_) \cong K^t_{\Lambda\Gamma}\langle -2\operatorname{caps}(t)\rangle \otimes_{K_\Gamma} \_$$

of functors from $\operatorname{mod}_{lf}(K_\Gamma)$ *to* $\operatorname{mod}_{lf}(K_\Lambda)$.

Proof. For any K_Γ-module M we have a natural homomorphism

$$K^t_{\Lambda\Gamma}\langle -2\,\mathrm{caps}(t)\rangle \otimes_{K_\Gamma} M \overset{3.31}{\cong} \mathrm{Hom}_{K_\Gamma}(K^{t^*}_{\Gamma\Lambda}, K_\Gamma) \otimes_{K_\Gamma} M \to \mathrm{Hom}_{K_\Gamma}(K^{t^*}_{\Gamma\Lambda}, M)$$

where the second map is an isomorphism because $K^{t^*}_{\Gamma\Lambda}$ is a projective left K_Γ-module by Corollary 3.27. □

Corollary 3.33. *We have an adjoint pair of functors*

$$(G^{t^*}_{\Gamma\Lambda}\langle \mathrm{cups}(t) - \mathrm{caps}(t)\rangle, G^t_{\Lambda\Gamma})$$

giving rise to a degree 0 *adjunction between* $\mathrm{mod}_{lf}(K_\Gamma)$ *and* $\mathrm{mod}_{lf}(K_\Lambda)$.

Proof. Use the standard adjunction $(K^{t^*}_{\Gamma\Lambda} \otimes_{K_\Lambda} \_, \mathrm{Hom}_{K_\Gamma}(K^{t^*}_{\Gamma\Lambda}, \_))$ in combination with Corollary 3.32 and recall the degree shift in Definition 3.23, where we defined projective functors. □

After finishing the part about the adjunction, we will show in the following theorem that our projective functors $G^t_{\Lambda\Gamma}$ commute with the duality in K. After that we analyze the effect of projective functors on simple modules.

Theorem 3.34. *Given any proper* $\Lambda\Gamma$*-matching* t *and any graded* K_Γ*-module* M*, there exists a natural isomorphism* $G^t_{\Lambda\Gamma}(M^\circledast) \cong (G^t_{\Lambda\Gamma}M)^\circledast$ *of graded* K_Λ*-modules.*

Proof. Recalling the grading shift in Definition 3.23 of the projective functors and recalling the grading on the dual, it suffices to construct a natural degree $-2\,\mathrm{caps}(t)$ isomorphism

$$K^t_{\Lambda\Gamma} \otimes_{K_\Gamma} (M^\circledast) \xrightarrow{\sim} (K^t_{\Lambda\Gamma} \otimes_{K_\Gamma} M)^\circledast.$$

In order to achieve this, let us first define the auxiliary map

$$\theta\colon K^t_{\Lambda\Gamma} \otimes (M^\circledast) \otimes K^t_{\Lambda\Gamma} \otimes M \to \mathbb{F}$$

by sending $x \otimes f \otimes y \otimes m$ to $f(\phi(x^* \otimes y)m)$, where ϕ is defined as in (3.13) and $*$ is the antimultiplicative linear map from (3.4). As ϕ is homogeneous of degree $-2\,\mathrm{caps}(t)$ by Lemma 3.30 and $*$ is of degree 0, we see immediately that θ is homogeneous of degree $-2\,\mathrm{caps}(t)$. For $u \in K_\Gamma$ we have

$$\begin{aligned}\theta(xu \otimes f \otimes y \otimes m) &= f(\phi((xu)^* \otimes y)m) = f(\phi(u^*x^* \otimes y)m) = f(u^*\phi(x^* \otimes y)m)\\ &= (uf)(\phi(x^* \otimes y)m) = \theta(x \otimes uf \otimes y \otimes m)\end{aligned}$$

and similarly

$$\theta(x \otimes f \otimes yu \otimes m) = f(\phi(x^* \otimes yu)m) = f(\phi(x^* \otimes y)um) = \theta(x \otimes f \otimes y \otimes um).$$

Thus θ factors over $(K^t_{\Lambda\Gamma} \otimes_{K_\Gamma} (M^\circledast)) \otimes (K^t_{\Lambda\Gamma} \otimes_{K_\Gamma} M) \to \mathbb{C}$, and hence we get an induced map of degree $-2\operatorname{caps}(t)$

$$\tilde{\theta}\colon K^t_{\Lambda\Gamma} \otimes_{K_\Gamma} (M^\circledast) \to (K^t_{\Lambda\Gamma} \otimes_{K_\Gamma} M)^\circledast$$

sending $x \otimes f$ to the function $\tilde{\theta}(x \otimes f)\colon y \otimes m \mapsto \theta(x \otimes f \otimes y \otimes m)$.
It remains to show that $\tilde{\theta}$ is K_Λ-linear and that it is an isomorphism. For linearity let $v \in K_\Lambda$. Then the following holds

$$\begin{aligned}(v\tilde{\theta}(x \otimes f))(y \otimes m) &= \tilde{\theta}(x \otimes f)(v^*y \otimes m) = f(\phi(x^* \otimes v^*y)m) = f(\phi(x^*v^* \otimes y)m)\\ &= f(\phi((vx)^* \otimes y)m) = \tilde{\theta}(vx \otimes f)(y \otimes m).\end{aligned}$$

In order to see that $\tilde{\theta}$ is a vector space isomorphism, it is enough to look for each $\lambda \in \Lambda$ at the restriction

$$\tilde{\theta}\colon e_\lambda K^t_{\Lambda\Gamma} \otimes_{K_\Gamma} (M^\circledast) \to e_\lambda(K^t_{\Lambda\Gamma} \otimes_{K_\Gamma} M)^\circledast.$$

We can identify $e_\lambda(K^t_{\Lambda\Gamma} \otimes_{K_\Gamma} M)^\circledast$ with $(e_\lambda K^t_{\Lambda\Gamma} \otimes_{K_\Gamma} M)^\circledast$, and may assume that $e_\lambda K^t_{\Lambda\Gamma} \neq 0$. Let $\gamma \in \Gamma$ be such that $\underline{\gamma}$ is the lower reduction of $\underline{\lambda}t$ and denote by n the number of lower circles removed in this process. By the dual version of Theorem 3.26(i) and (iii) we have $e_\lambda K^t_{\Lambda\Gamma} \cong e_\gamma K_\Gamma \otimes R^{\otimes n}$ as right K_Γ-modules. One traces $\tilde{\theta}$ through these isomorphisms and we are left to show that

$$(e_\gamma K_\Gamma \otimes_{K_\Gamma} (M^\circledast)) \otimes R^{\otimes n} \to ((e_\gamma K_\Gamma \otimes_{K_\Gamma} M) \otimes R^{\otimes n})^\circledast$$

sending $(u \otimes f) \otimes x_1 \otimes \cdots \otimes x_n$ to $(v \otimes m) \otimes y_1 \otimes \cdots \otimes y_n \mapsto \tau(x_1y_1)\cdots\tau(x_ny_n)f(u^*vm)$ is an isomorphism. Tracing $\tilde{\theta}$ through the isomorphisms is done in the same way as in the proof of Theorem 3.31. This reduces then to checking that $e_\gamma(M^\circledast) \xrightarrow{\cong} (e_\gamma M)^\circledast, e_\gamma f = f(e_\gamma \cdot \_) \mapsto (e_\gamma m \mapsto f(e_\gamma m))$, which is clearly an isomorphism. □

Now we have all the ingredients to explicitly state the effect of a projective functor on a simple module.

Theorem 3.35. *Given a proper* $\Lambda\Gamma$*-matching* t *and* $\gamma \in \Gamma$*, we have*

(i) *in the graded Grothendieck group of* $\operatorname{mod}_{lf}(K_\Lambda)$

$$[G^t_{\Lambda\Gamma}L(\gamma)] = \sum_\mu (q + q^{-1})^{n_\mu}[L(\mu)],$$

where n_μ *denotes the number of lower circles in* $\underline{\mu}t$ *and we sum over all* $\mu \in \Lambda$ *such that*

  (a) $\underline{\gamma}$ *is the lower reduction of* $\underline{\mu}t$*,*

  (b) *the rays of each lower line in* $\underline{\mu}\mu t$ *are properly oriented,*

(ii) *the module* $G^t_{\Lambda\Gamma}L(\gamma)$ *is nonzero if and only if all cups of* $t\gamma$ *are anticlockwise oriented, and*

(iii) *under the assumptions of* (ii) *define* $\lambda \in \Lambda$ *such that* $\overline{\lambda}$ *is the upper reduction of* $t\overline{\gamma}$ *or alternatively* $\lambda t\gamma$ *is oriented and every cup and cap is oriented anticlockwise. In this case* $G^t_{\Lambda\Gamma}L(\gamma)$ *is a self-dual indecomposable module with irreducible head* $L(\lambda)\langle -\operatorname{caps}(t)\rangle$.

Proof. In order to prove (i) we need to show that

$$(\dagger) := \sum_{j\in\mathbb{Z}} q^j \operatorname{Hom}_{K_\Lambda}(P(\mu), G^t_{\Lambda\Gamma}L(\gamma))_j$$

is nonzero if and only (a) and (b) are satisfied, and that in this case we have $(\dagger) = (q+q^{-1})^{n_\mu}$. Using Corollary 3.33 we have

$$(\dagger) = \sum_{j\in\mathbb{Z}} q^j \operatorname{Hom}_{K_\Gamma}(G^{t^*}_{\Gamma\Lambda}P(\mu)\langle \operatorname{cups}(t) - \operatorname{caps}(t)\rangle, L(\gamma))_j.$$

Theorem 3.26 tells us that $G^{t^*}_{\Gamma\Lambda}P(\mu)$ is nonzero if and only if (b) is satisfied and in this case it is isomorphic to $P(\beta)\otimes R^{\otimes n_\mu}\langle \operatorname{caps}(t) - \operatorname{cups}(t)\rangle$, where $\beta \in \Gamma$ is such that $\underline{\beta}$ is the lower reduction of $\underline{\mu}t$. Hence $(\dagger) \neq 0$ if and only if (a) and (b) are satisfied. In this case, noting that the two degree shifts cancel, we have

$$(\dagger) = \sum_{j\in\mathbb{Z}} q^j \operatorname{Hom}_{K_\Lambda}(P(\gamma)\otimes R^{\otimes n_\mu}, L(\gamma))_j = (q+q^{-1})^{n_\mu}$$

as claimed.

For (ii) and (iii) observe that as $G^t_{\Lambda\Gamma}$ is exact, $G^t_{\Lambda\Gamma}L(\gamma)$ is a quotient of $G^t_{\Lambda\Gamma}V(\gamma)$, and thus can only be nonzero if each cup of $t\gamma$ is oriented by Theorem 3.29(ii). In this case $G^t_{\Lambda\Gamma}V(\gamma)$ has irreducible head $L(\lambda)\langle \deg(\lambda t\gamma) - \operatorname{caps}(t)\rangle$ by Theorem 3.29(iii), where λ is the unique weight such that $\overline{\lambda}$ is the upper reduction of $t\overline{\gamma}$ or equivalently such that $\lambda t\gamma$ is oriented and all its caps are anticlockwise. Thus $G^t_{\Lambda\Gamma}L(\gamma)$ is zero or it has the same irreducible head. But this composition factor can only occur by (i) if λ satisfies (a) and (b), which then means exactly that each cup of $t\gamma$ is anticlockwise and hence $\deg(\lambda t\gamma) = 0$.

That $G^t_{\Lambda\Gamma}L(\gamma)$ is self-dual follows from Theorem 3.34 and the fact that $L(\gamma)^{\circledast} \cong L(\gamma)$. □

Corollary 3.36. *The functor* $G^t_{\Lambda\Gamma}$ *preserves finite dimensional modules.*

Proof. By Corollary 3.28 and by Theorem 3.35(i) it suffices to prove that there exist only finitely many μ such that $\underline{\gamma}$ is the lower reduction of $\underline{\mu}t$ and each lower line in $\underline{\mu}\mu t$ is properly oriented. In order to see this note that in this case we necessarily have that $\mu t\lambda$ is oriented and thus μ is determined on all positions on the number line except for the endpoint of caps in t. There exist only finitely many of those, thus there are only finitely many μ satisfying the desired properties and hence we conclude the proof. □

3.4 Nuclear diagrams and projective functors

In this section we are going to introduce nuclear circle diagrams, define an analogue of the projective functors incorporating nuclear diagrams and study these.

Definition 3.37. A nuclear circle diagram $a\lambda b \in K_\Lambda$ is an oriented circle diagram with at least one nonpropagating line. We denote by $\mathbb{I}_\Lambda \subseteq K_\Lambda$ the span of all nuclear circle diagrams.

Lemma 3.38. *The vector space $\mathbb{I}_\Lambda$ is a two-sided ideal in K_Λ.*

Proof. This is [ES16a, Proposition 5.3]. □

Using Lemma 3.38 above, we get an induced multiplication on $\tilde{K}_\Lambda \coloneqq K_\Lambda/\mathbb{I}_\Lambda$ turning this into a graded algebra. As for K, the $e_\lambda, \lambda \in \Lambda$ (or rather their equivalence classes) provide a set of local units. Thus in this algebra the simple modules are again characterized by $\lambda \in \Lambda$. They are one-dimensional and e_λ acts by 1 and every other circle diagram by 0. Furthermore the projective indecomposable modules are given by $(K_\Lambda/\mathbb{I}_\Lambda)e_\lambda$ and these are in fact self-dual and hence prinjective (see [ES17, Section II.4]). We will denote the simple and the projective indecomposable modules by $\overline{L}(\lambda)$ respectively $\overline{P}(\lambda)$. The statement from [ES16b, Theorem 6.10] that K is generated in degrees 0 and 1 directly gives us the following result.

Lemma 3.39. *The algebra $\tilde{K}$ is generated in degrees 0 and 1.*

Proof. By [ES16b, Theorem 6.10] K is generated in degrees 0 and 1 and as $\tilde{K}$ is quotient of K, it is generated in degrees 0 and 1 as well. □

On the next pages, we are going to extend the notion of nuclear morphisms to $\boldsymbol{\Lambda}$-circle diagrams and for this we fix notation as follows. Let $\boldsymbol{\Lambda} = \Lambda_k \dots \Lambda_0$ and $\boldsymbol{\Gamma} = \Gamma_l \dots \Gamma_0$ be sequences of blocks such that $\Lambda_0 = \Gamma_l$. Let $\boldsymbol{t} = t_k \dots t_1$ (resp. $\boldsymbol{u} = u_l \dots u_1$) be an oriented $\boldsymbol{\Lambda}$-matching (resp. $\boldsymbol{\Gamma}$-matching). As before denote the block sequence $\Lambda_k \dots \Lambda_1 \Gamma_l \dots \Gamma_0$ by $\boldsymbol{\Lambda} \wr \boldsymbol{\Gamma}$ and let $\boldsymbol{tu} = t_k \dots t_1 u_l \dots u_1$.

Definition 3.40. An oriented $\boldsymbol{\Lambda}$-circle diagram is called *nuclear* if it contains at least one nonpropagating strand. Denote the span of these circle diagrams by $\mathbb{I}^{\boldsymbol{t}}_{\boldsymbol{\Lambda}}$. Furthermore we will abbreviate $\tilde{K}^{\boldsymbol{t}}_{\boldsymbol{\Lambda}} \coloneqq K^{\boldsymbol{t}}_{\boldsymbol{\Lambda}}/\mathbb{I}^{\boldsymbol{t}}_{\boldsymbol{\Lambda}}$.

Lemma 3.41. *The multiplication m from* (3.5) *induces a degree preserving map*

$$\tilde{m} \colon \tilde{K}^{\boldsymbol{t}}_{\boldsymbol{\Lambda}} \otimes \tilde{K}^{\boldsymbol{u}}_{\boldsymbol{\Gamma}} \to \tilde{K}^{\boldsymbol{tu}}_{\boldsymbol{\Lambda}\wr\boldsymbol{\Gamma}}$$

which is associative and antimultiplicative in the same sense as in (3.6) *and* (3.7).

Proof. By the definition of the multiplication it is easy to see that under m, $\mathbb{I}^{\boldsymbol{t}}_{\boldsymbol{\Lambda}} \otimes K^{\boldsymbol{u}}_{\boldsymbol{\Gamma}}$ and $K^{\boldsymbol{t}}_{\boldsymbol{\Lambda}} \otimes \mathbb{I}^{\boldsymbol{u}}_{\boldsymbol{\Gamma}}$ are sent to $\mathbb{I}^{\boldsymbol{tu}}_{\boldsymbol{\Lambda}\wr\boldsymbol{\Gamma}}$, and thus the multiplication factors as claimed. It is degree preserving because m is and the subspaces $\mathbb{I}^{\boldsymbol{t}}_{\boldsymbol{\Lambda}}$ are homogeneous by definition. Antimultiplicativity and associativity follow directly from the analogous statements for m. □

Remark 3.42. In the special case that $\boldsymbol{u}$ is empty (and using the mirrored argument), we see that $\mathbb{I}^{\boldsymbol{t}}_{\boldsymbol{\Lambda}}$ is a $(K_{\Lambda_k}, K_{\Lambda_0})$-bisubmodule of $K^{\boldsymbol{t}}_{\boldsymbol{\Lambda}}$. In the subcase that $\boldsymbol{t}$ and $\boldsymbol{u}$ are empty we recover Lemma 3.38.

Lemma 3.43. *The map $\tilde{m}$ is $\tilde{K}_{\Lambda_0}$-balanced and thus induces a map*

$$\tilde{m}\colon \tilde{K}^{\boldsymbol{t}}_{\boldsymbol{\Lambda}} \otimes_{\tilde{K}_{\Lambda_0}} \tilde{K}^{\boldsymbol{u}}_{\boldsymbol{\Gamma}} \to \tilde{K}^{\boldsymbol{tu}}_{\boldsymbol{\Lambda}\wr\boldsymbol{\Gamma}}$$

which is in fact an isomorphism.

Proof. That it is $\tilde{K}_{\Lambda_0}$-balanced follows from the associativity of $\tilde{m}$ and hence it factors as desired.
In order to see that $\tilde{m}$ is an isomorphism, note that it is surjective because m is. For injectivity we first prove that the restriction of the multiplication map $\mathbb{I}^{\boldsymbol{t}}_{\boldsymbol{\Lambda}} \otimes_{\mathbb{C}} K^{\boldsymbol{u}}_{\boldsymbol{\Gamma}} + K^{\boldsymbol{t}}_{\boldsymbol{\Lambda}} \otimes_{\mathbb{C}} \mathbb{I}^{\boldsymbol{u}}_{\boldsymbol{\Gamma}} \to \mathbb{I}^{\boldsymbol{tu}}_{\boldsymbol{\Lambda}\wr\Gamma}$ is surjective. For this let $a(\boldsymbol{tu})[\boldsymbol{\nu}]b \in \mathbb{I}^{\boldsymbol{tu}}_{\boldsymbol{\Lambda}\wr\boldsymbol{\Gamma}}$. Define $\boldsymbol{\lambda} \coloneqq \nu_{k+l}\dots\nu_l$ and $\boldsymbol{\mu}' \coloneqq \nu_l\dots\nu_0$. Without loss of generality we may assume that one nonpropagating line ends at the bottom. Define c to be the upper reduction of $\boldsymbol{u}[\boldsymbol{\mu}']b$. Hence by definition of the upper reduction, $a\boldsymbol{t}[\boldsymbol{\lambda}]c$ contains a nonpropagating line, hence we have $a\boldsymbol{t}[\boldsymbol{\lambda}]c \in \mathbb{I}^{\boldsymbol{t}}_{\boldsymbol{\Lambda}}$. By definition of the upper reduction $c^*\boldsymbol{u}[\boldsymbol{\mu}']b$ is oriented. We define $\boldsymbol{\mu}$ to be the same as $\boldsymbol{\mu}'$ except that all components in $c^*\boldsymbol{u}b$ which lie partly in c^* are oriented anticlockwise. We claim then that $(a\boldsymbol{t}[\boldsymbol{\lambda}]c)(c^*\boldsymbol{u}[\boldsymbol{\mu}]b) = \pm a(\boldsymbol{tu})[\nu]b$. Observe that every surgery that needs to be applied is a merge and it always merges a component in $a\boldsymbol{t}[\boldsymbol{\lambda}]c$ with an anticlockwise circle in $c^*\boldsymbol{u}[\boldsymbol{\mu}]b$. But this means that the vertices belonging to the anticlockwise circle in $c^*\boldsymbol{u}[\boldsymbol{\mu}]b$ are exactly reoriented to agree with the parts in $\boldsymbol{\nu}$. Thus the surgery procedure produces up to possibly a sign the circle diagram $a(\boldsymbol{tu})[\boldsymbol{\nu}]b$, which finishes the proof of the claim.
Now consider the following commutative diagram (the horizontal maps are all induced by the multiplication)

$$\begin{tikzcd}
\mathbb{I}^{\boldsymbol{t}}_{\boldsymbol{\Lambda}} \otimes_{\mathbb{C}} K^{\boldsymbol{u}}_{\boldsymbol{\Gamma}} + K^{\boldsymbol{t}}_{\boldsymbol{\Lambda}} \otimes_{\mathbb{C}} \mathbb{I}^{\boldsymbol{u}}_{\boldsymbol{\Gamma}} \arrow[rr, two heads] \arrow[d, hook] & & \mathbb{I}^{\boldsymbol{tu}}_{\boldsymbol{\Gamma}\wr\boldsymbol{\Lambda}} \arrow[d, hook] \\
K^{\boldsymbol{t}}_{\boldsymbol{\Lambda}} \otimes_{\mathbb{C}} K^{\boldsymbol{u}}_{\boldsymbol{\Gamma}} \arrow[r, two heads] \arrow[d, two heads] & K^{\boldsymbol{t}}_{\boldsymbol{\Lambda}} \otimes_{K_{\Lambda_0}} K^{\boldsymbol{u}}_{\boldsymbol{\Gamma}} \arrow[r, "\bar{m}"] \arrow[d, dotted] & K^{\boldsymbol{tu}}_{\boldsymbol{\Lambda}\wr\boldsymbol{\Gamma}} \arrow[d, two heads] \\
\tilde{K}^{\boldsymbol{t}}_{\boldsymbol{\Lambda}} \otimes_{\mathbb{C}} \tilde{K}^{\boldsymbol{u}}_{\boldsymbol{\Gamma}} \arrow[r, two heads] & \tilde{K}^{\boldsymbol{t}}_{\boldsymbol{\Lambda}} \otimes_{\tilde{K}_{\Lambda_0}} \tilde{K}^{\boldsymbol{u}}_{\boldsymbol{\Gamma}} \arrow[r, "\tilde{m}"] & \tilde{K}^{\boldsymbol{tu}}_{\boldsymbol{\Lambda}\wr\boldsymbol{\Gamma}}
\end{tikzcd} .$$

The right and the left column are both short exact by definition and the map $\bar{m}$ is an isomorphism by Theorem 3.21.
Now suppose $x \in \tilde{K}^{\boldsymbol{t}}_{\boldsymbol{\Lambda}} \otimes_{\tilde{K}_{\Lambda_0}} \tilde{K}^{\boldsymbol{u}}_{\boldsymbol{\Gamma}}$ is mapped to 0 by $\tilde{m}$. Lift this to an element $x' \in K^{\boldsymbol{t}}_{\boldsymbol{\Lambda}} \otimes_{\mathbb{C}} K^{\boldsymbol{u}}_{\boldsymbol{\Gamma}}$. As $\tilde{m}(x) = 0$ we must have $m(x') \in \mathbb{I}^{\boldsymbol{tu}}_{\boldsymbol{\Lambda}\wr\boldsymbol{\Gamma}}$. By the above claim we find some $x'' \in \mathbb{I}^{\boldsymbol{t}}_{\boldsymbol{\Lambda}} \otimes_{\mathbb{C}} K^{\boldsymbol{u}}_{\boldsymbol{\Gamma}} + K^{\boldsymbol{t}}_{\boldsymbol{\Lambda}} \otimes_{\mathbb{C}} \mathbb{I}^{\boldsymbol{u}}_{\boldsymbol{\Gamma}}$ such that $m(x'') = m(x')$. Hence they agree in $K^{\boldsymbol{t}}_{\boldsymbol{\Lambda}} \otimes_{K_{\Lambda_0}} K^{\boldsymbol{u}}_{\boldsymbol{\Gamma}}$ as $\bar{m}$ is an isomorphism by Theorem 3.21(iii), and thus they also agree in $\tilde{K}^{\boldsymbol{t}}_{\boldsymbol{\Lambda}} \otimes_{\tilde{K}_{\Lambda_0}} \tilde{K}^{\boldsymbol{u}}_{\boldsymbol{\Gamma}}$. But as $x'' \in \mathbb{I}^{\boldsymbol{t}}_{\boldsymbol{\Lambda}} \otimes_{\mathbb{C}} K^{\boldsymbol{u}}_{\boldsymbol{\Gamma}} + K^{\boldsymbol{t}}_{\boldsymbol{\Lambda}} \otimes_{\mathbb{C}} \mathbb{I}^{\boldsymbol{u}}_{\boldsymbol{\Gamma}}$ it becomes 0 in $\tilde{K}^{\boldsymbol{t}}_{\boldsymbol{\Lambda}} \otimes_{\tilde{K}_{\Lambda_0}} \tilde{K}^{\boldsymbol{u}}_{\boldsymbol{\Gamma}}$

and as x' was a lift of x we have $0 = x \in \tilde{K}^{\boldsymbol{t}}_{\boldsymbol{\Lambda}} \otimes_{\tilde{K}_{\Lambda_0}} \tilde{K}^{\boldsymbol{u}}_{\boldsymbol{\Gamma}}$. Thus $\tilde{m}$ is injective, finishing the proof. $\square$

Theorem 3.44. *Let $\boldsymbol{t} = t_k \dots t_1$ be a proper $\boldsymbol{\Lambda}$-matching. Denote the reduction of $\boldsymbol{t}$ by u and let n be the number of internal circles getting removed in the reduction process. Then we have*

$$\begin{aligned}\tilde{K}^{\boldsymbol{t}}_{\boldsymbol{\Lambda}} &\cong \tilde{K}^{u}_{\Lambda_k\Lambda_0} \otimes R^{\otimes n}\langle \mathrm{caps}(t_1) + \dots + \mathrm{caps}(t_k) - \mathrm{caps}(u)\rangle \\ &\cong \tilde{K}^{u}_{\Lambda_k\Lambda_0} \otimes R^{\otimes n}\langle \mathrm{cups}(t_1) + \dots + \mathrm{cups}(t_k) - \mathrm{cups}(u)\rangle\end{aligned}$$

as graded $(\tilde{K}_{\Lambda_k}, \tilde{K}_{\Lambda_0})$-bimodules, viewing $\tilde{K}^{u}_{\Lambda_k\Lambda_0} \otimes R^{\otimes n}$ as a bimodule via acting on the first tensor factor.

Proof. Follow the proof of Theorem 3.22 by observing that $a\boldsymbol{t}b$ contains a nonpropagating line if and only if aub does and using that $\mathbb{I}^{\boldsymbol{t}}_{\boldsymbol{\Lambda}}$ is a $(K_{\Lambda_k}, K_{\Lambda_0})$-bisubmodule of $K^{\boldsymbol{t}}_{\boldsymbol{\Lambda}}$ by the proof of Lemma 3.41. $\square$

Definition 3.45. A $\Lambda\Gamma$-matching is called a *translation diagram* if the difference of the numbers of $\circ$'s and $\times$'s in Λ (resp. Γ) agrees. A $\boldsymbol{\Lambda}$-matching $\boldsymbol{t} = t_k \dots t_1$ is called a *translation diagram* if every t_i is.

Remark 3.46. If the $\boldsymbol{\Lambda}$-matching $\boldsymbol{t}$ is a translation diagram, so is its reduction.

From now on we will assume implicitly that every $\boldsymbol{\Lambda}$-matching is in fact a translation diagram. Furthermore we will only consider the idempotent truncation by super weight diagrams. For this we make the following definition:

Definition 3.47. Let $eK^{\boldsymbol{t}}_{\boldsymbol{\Lambda}}e$ be the subalgebra of $K^{\boldsymbol{t}}_{\boldsymbol{\Lambda}}$ spanned by all oriented stretched circle diagrams $a\lambda\boldsymbol{t}\mu b^*$, where a and b are super weight diagrams. Additionally we let $e\mathbb{I}^{\boldsymbol{t}}_{\boldsymbol{\Lambda}}e$ be the intersection of $eK^{\boldsymbol{t}}_{\boldsymbol{\Lambda}}e$ and $\mathbb{I}^{\boldsymbol{t}}_{\boldsymbol{\Lambda}}$ and $e\tilde{K}^{\boldsymbol{t}}_{\boldsymbol{\Lambda}}e := eK^{\boldsymbol{t}}_{\boldsymbol{\Lambda}}e/e\mathbb{I}^{\boldsymbol{t}}_{\boldsymbol{\Lambda}}e$. We observe that the multiplication on K_Λ induces a multiplication on $eK_\Lambda e$ as well and we get an induced $(eK_{\Lambda_k}e, eK_{\Lambda_0}e)$-bimodule structure on $eK^{\boldsymbol{t}}_{\boldsymbol{\Lambda}}e$ (and analogously for $e\tilde{K}^{\boldsymbol{t}}_{\boldsymbol{\Lambda}}e$). We will abuse notation and denote the simple and indecomposable projective modules for the algebra $eK_\Lambda e$ (resp. $e\tilde{K}_\Lambda e$) by $L(\lambda)$ and $P(\lambda)$ (resp. $\overline{L}(\lambda)$ and $\overline{P}(\lambda)$).
We define $eKe := \bigoplus_\Lambda eK_\Lambda e$, where the sum runs over all blocks and similarly $e\tilde{K}e := \bigoplus_\Lambda e\tilde{K}_\Lambda e$.
Using the projections $e\tilde{K}e \to e\tilde{K}_\Lambda e$ we might think of $e\tilde{K}^{\boldsymbol{t}}_{\boldsymbol{\Lambda}}e$ as an $e\tilde{K}e$-bimodule.

Remark 3.48. The algebra $e\tilde{K}e$ has a set of local units given by super weight diagrams. For a super weight diagram λ we obtain $\overline{P}(\lambda) = e\tilde{K}e_\lambda$ and thus in the Grothendieck group we have

$$[\overline{P(\lambda)}] = \sum_{\underline{\mu}\nu\overline{\lambda}} q^{\deg(\underline{\mu}\nu\overline{\lambda})}[L(\mu)] = \sum_{\underline{\mu}\overline{\lambda}} (q+q^{-1})^{n_\mu} q^{\mathrm{def}(\lambda)}[L(\mu)],$$

where n_μ denotes the number of circles in the circle diagram $\underline{\mu}\overline{\lambda}$ and we sum over all (oriented) not nuclear circle diagrams $\underline{\mu}\overline{\lambda}$ (resp. $\underline{\mu}\nu\overline{\lambda}$) for a super weight diagram μ.

Compare this also with Theorem 3.57 and Remark 3.58. Note that there are only finitely many μ such that $\underline{\mu}\overline{\nu}$ is oriented and not nuclear, and hence $\overline{P(\lambda)}$ is finite dimensional.

Using this notation we obtain the following result:

Lemma 3.49. *The map $\tilde{m}$ from Lemma 3.43 restricts to an isomorphism*

$$\tilde{m}_e \colon e\tilde{K}^{\boldsymbol{t}}_{\boldsymbol{\Lambda}}e \otimes_{e\tilde{K}_{\Lambda_0}e} e\tilde{K}^{\boldsymbol{u}}_{\boldsymbol{\Gamma}}e \to e\tilde{K}^{\boldsymbol{tu}}_{\boldsymbol{\Lambda}\wr\boldsymbol{\Gamma}}e.$$

Proof. This follows easily by noting that if we have an oriented stretched circle diagram $a\lambda t\mu b^*$ such that a is a super weight diagram, but if b is not, we necessarily have $a\lambda t\mu b^* \in \mathbb{I}^{\boldsymbol{t}}_{\boldsymbol{\Lambda}}$. Assume the layer numbers of a and b agree, i.e. $\kappa(a) = \kappa(b)$, then b would be a super weight diagram as well, because t is a translation diagram. So we have $\kappa(a) \neq \kappa(b)$ and this means that $a\lambda t\mu b^*$ has to contain a nonpropagating line. □

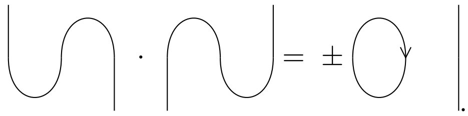
In Lemma 3.39 we have seen that $\tilde{K}$ is generated in degrees 0 and 1. In general idempotent truncations do not preserve this property. The following theorem shows that this property actually passes to the idempotent truncation $e\tilde{K}e$, and this will later be crucial for the analysis of radical filtrations of indecomposable modules.

Theorem 3.50. *The algebra $e\tilde{K}e$ is generated by its degree* 0 *and* 1 *part.*

Remark 3.51. The basic idea of this proof is the same as [ES16b, Theorem 6.10]. However Ehrig and Stroppel use some reduction process in the finite case to only consider circle diagrams without lines. We will not do this for two reasons. First our weight diagrams are infinite opposed to finite, so one would first need to do some reduction to finite weight diagrams to adapt this idea to our setting. Secondly we want to make sure that in every step we actually use only circle diagrams which actually live in $e\tilde{K}e$.

Proof. We will prove the statement via induction. For this we are going to change weight diagrams locally. Every local change will either change the positions between a ray and a cup or the positions of two cups relative to each other. In particular, every local change in this proof preserves the property of being a super weight diagram, and we will not mention this later. We first define elements $X_{i,\lambda}$ which are based on the circle diagram $\underline{\lambda}\lambda\overline{\lambda}$. In case that the vertex at position i is not part of a circle we set $X_{i,\lambda} := 0$, and otherwise we reverse the orientation of this circle in contrast to $\underline{\lambda}\lambda\overline{\lambda}$. The proof of this theorem is split into two parts. First we are going to show that the elements $X_{i,\lambda}$ for every i and λ are generated by degree 1 and 0 elements.
The second part is going to be an induction over the degree, where the first part allows us to consider only anticlockwise oriented circles.
Suppose that we are given $X_{i,\lambda}$. We may assume $X_{i,\lambda} \neq 0$, as otherwise the claim is trivial. Furthermore denote the circle containing the vertex i by C. We are going to consider three different cases, depending on what happens directly to the right of C. Either there is a line, the starting point of a cup, or the endpoint of a cup.

If there is a line to the right of C, we can look schematically (meaning that there might be dots involved, which we omit here) at the picture



Let μ be the weight such that $\underline{\mu}$ agrees with $\underline{\lambda}$ except that the cup belonging to C and the line to the right of it are swapped. If the cup and the line in $\underline{\lambda}$ contain a dot, we choose $\underline{\mu}$ such that it has no dot on either of them. If exactly one of the cup and the line are dotted in $\underline{\lambda}$ we require the corresponding line in $\underline{\mu}$ to be dotted. In other words, we want to have an even number of dots on this curved line in the above picture and the picture should be admissible.
Then the circle diagram $\underline{\lambda}\overline{\mu}$ admits a unique degree 1 orientation ν (i.e. every circle is oriented anticlockwise). Then by the definition of the surgery procedure we have

$$\underline{\lambda}\nu\overline{\mu} \cdot \underline{\mu}\nu\overline{\lambda} = \pm X_{i,\lambda}.$$

If there is a circle directly to the right of C we look schematically at Figure 3.7. We

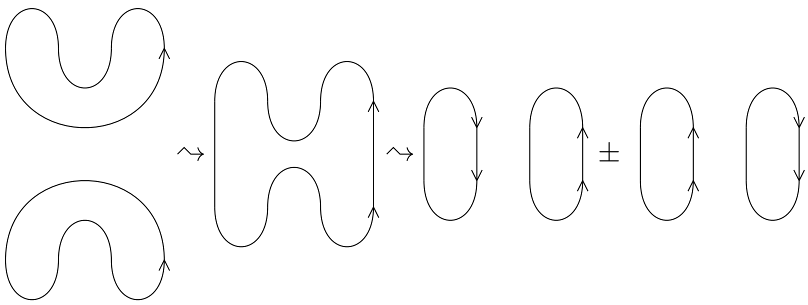


Figure 3.7: The case that a circle is directly to the right of C

choose μ such that there are two nested cups in $\underline{\mu}$ instead of the two next to each other. We may equip the outer cup in $\underline{\mu}$ with a dot if exactly one of the cups in $\underline{\lambda}$ is dotted. If we denote the unique orientation of degree 1 by ν, we have by definition of the surgery procedures (see picture above)

$$\underline{\lambda}\nu\overline{\mu} \cdot \underline{\mu}\nu\overline{\lambda} = \pm X_{i,\lambda} \pm X_{i+2,\lambda}.$$

Now we can repeat the argument for $X_{i+2,\lambda}$ and see that $X_{i+2,\lambda}$ is generated by degree 1 elements. Hence this is true for $X_{i,\lambda}$. Note that this recursion has to stop at some point as $\underline{\lambda}$ has only finitely many cups.
Lastly the cup corresponding to C may be nested in some other cup. Then we can proceed as indicated in Figure 3.8. We choose μ such that there are two cups in $\underline{\mu}$ next to each other instead of two nested ones. We may equip the left cup in $\underline{\mu}$ with a

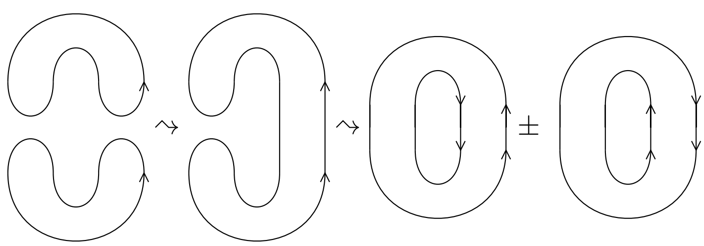
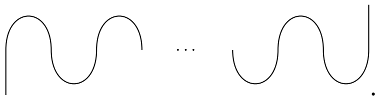


Figure 3.8: The case that the cup of C is nested in some other cup

dot if exactly one of the cups in $\underline{\lambda}$ is dotted. If we denote the unique orientation of degree 1 by ν, we have by definition of the surgery procedures (see picture above)

$$\underline{\lambda}\nu\overline{\mu} \cdot \underline{\mu}\nu\overline{\lambda} = \pm X_{i,\lambda} \pm X_{j,\lambda},$$

where j denotes a vertex belonging to the outer cup. Now similar as before we can repeat the argument for $X_{j,\lambda}$ and see that $X_{j,\lambda}$ is generated by degree 1 elements and hence $X_{i,\lambda}$ as well. Note that this recursion has to stop at some point as $\underline{\lambda}$ has only finitely many cups. This finishes the first part of the proof.
The second step is to show the general statement. We prove this via induction over the degree of the circle diagram. If the degree is 0 or 1 the statement is trivial, so let $\underline{\lambda}\nu\overline{\mu}$ be any circle diagram of degree > 1. By the first step, we may assume that ν is the orientation $\nu_{\min}$ of $\underline{\lambda}\overline{\mu}$ of minimal degree, as any other orientation arises from $\underline{\lambda}\nu_{\min}\overline{\mu}$ by multiplying with some $X_{i,\lambda}$. Take any component C of $\underline{\lambda}\nu\overline{\mu}$ of degree ≥ 1. This is either a circle or a line.
If it is a line, it (or its horizontal mirror image) looks schematically like



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We let μ' be the weight such that $\underline{\mu'}$ differs from $\underline{\lambda}$ in the way that the ray (corresponding to the line) is swapped with the cup to the right. The cup and the ray are decorated with dots, in the unique way such that $\underline{\mu'}\overline{\mu}$ is orientable. We furthermore let ν' be the unique orientation of $\underline{\lambda}\overline{\mu'}$ of degree 1 and ν'' be the unique orientation of minimal degree of $\underline{\mu'}\overline{\mu}$. This then looks locally as in Figure 3.9. Looking at the above pictures and the definition of the surgery procedures one easily checks that

$$\underline{\lambda}\nu'\overline{\mu'} \cdot \underline{\mu'}\nu''\overline{\mu} = \pm\underline{\lambda}\nu\overline{\mu}.$$

By Lemma 3.9 we have that the degree of $\underline{\mu'}\nu''\overline{\mu}$ is one less than the degree of $\underline{\lambda}\nu\overline{\mu}$, hence it is generated by degree 0 and 1 elements by induction. Therefore we see that $\underline{\lambda}\nu\overline{\mu}$ is generated by degree 0 and 1 elements.
However, if the component C is a circle, it necessarily consists of at least two cups and caps by Lemma 3.9, and we need to have a pair of cups or a pair of caps γ_1 and

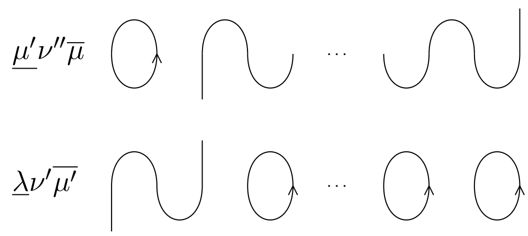


Figure 3.9: The reduction process for a line

γ_2 nested in each other. Without loss of generality, we may assume that this is a pair of cups. We choose γ_1 such that it is not contained in any other cup and γ_2 such that it is only contained in γ_1. Figure 3.10 gives an overview about our choices of γ_1 and γ_2. Then we choose μ' such that $\underline{\mu}'$ is the same as $\underline{\lambda}$ except that these nested

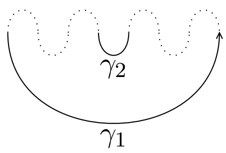


Figure 3.10: The choice of γ_1 and γ_2 in C

cups are replaced by two neighbored ones. Figure 3.11 describes the definition of μ'. Then $\underline{\lambda}\overline{\mu'}$ admits a unique orientation of degree 1, which we call ν'. Additionally, we define ν'' to be the orientation of minimal degree of $\underline{\mu}'\overline{\mu}$. Then by construction

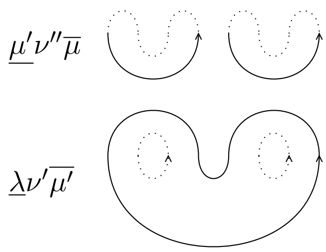


Figure 3.11: Definition of μ'.

and Lemma 3.9 the degree of $\underline{\mu}'\nu''\overline{\mu}$ is one less than the degree of $\underline{\lambda}\nu\overline{\mu}$. Hence it is generated by degree 0 and 1 elements by induction. Furthermore by the definition of the surgery procedure we have

$$\underline{\lambda}\nu'\overline{\mu'} \cdot \underline{\mu}'\nu''\overline{\mu} = \pm\underline{\lambda}\nu\overline{\mu}. \tag{3.16}$$

Thus we see that $\underline{\lambda}\nu\overline{\mu}$ is generated by degree 1 and 0 elements, finishing the proof. $\square$

Definition 3.52. Let t be a proper $\Lambda\Gamma$-matching and define $\tilde{G}^t_{\Lambda\Gamma}$ to be the functor $e\tilde{K}^t_{\Lambda\Gamma}e\langle -\operatorname{caps}(t)\rangle \otimes_{e\tilde{K}_\Gamma e} \_$. We call (possibly shifted) direct sums of these functors *projective functors* as well.

Theorem 3.53. *Let t be a proper $\Lambda\Gamma$-matching and let $\gamma \in \Gamma$. Then*

(i) $\tilde{G}^t_{\Lambda\Gamma}\overline{P}(\gamma) \cong (e\tilde{K}^t_{\Lambda\Gamma}e)e_\gamma\langle -\operatorname{caps}(t)\rangle$ *as left $e\tilde{K}_\Lambda e$-modules,*

(ii) *the module $\tilde{G}^t_{\Lambda\Gamma}\overline{P}(\gamma)$ is nonzero if and only if there exists no upper line in $t\overline{\gamma}$ and*

(iii) *supposing that* (ii) *holds, there is an isomorphism*

$$\tilde{G}^t_{\Lambda\Gamma}\overline{P}(\gamma) \cong \overline{P}(\lambda) \otimes R^{\otimes n}\langle \operatorname{cups}(t) - \operatorname{caps}(t)\rangle$$

of graded left $e\tilde{K}_\Lambda e$-modules ($e\tilde{K}_\Lambda e$ acts on the right hand side only on the first factor), where $\lambda \in \Lambda$ is such that $\overline{\lambda}$ is the upper reduction of $t\overline{\gamma}$ and n denotes the number of upper circles removed in the reduction process.

Proof. For (i) note that

$$\begin{aligned}\tilde{G}^t_{\Lambda\Gamma}\overline{P}(\gamma) &= e\tilde{K}^t_{\Lambda\Gamma}e\langle -\operatorname{caps}(t)\rangle \otimes_{e\tilde{K}_\Gamma e} \overline{P}(\gamma) = e\tilde{K}^t_{\Lambda\Gamma}e \otimes_{e\tilde{K}_\Gamma e} (e\tilde{K}_\Gamma e)e_\gamma\langle -\operatorname{caps}(t)\rangle \\ &\cong (e\tilde{K}^t_{\Lambda\Gamma}e)e_\gamma\langle -\operatorname{caps}(t)\rangle.\end{aligned}$$

For the forward implication of (ii) note that if there exists an upper line in $t\overline{\gamma}$, any oriented $\Lambda\Gamma$-circle diagram $a\lambda t\nu\overline{\gamma}$ contains an upper line and thus is 0 in $e\tilde{K}^t_{\Lambda\Gamma}e$ by definition. But by (i) these form a basis of $\tilde{G}^t_{\Lambda\Gamma}\overline{P}(\gamma)$ and hence $\tilde{G}^t_{\Lambda\Gamma}\overline{P}(\gamma) = 0$.
In order to finish the proof, suppose that there exists no upper line in $t\overline{\gamma}$. Enumerate the n upper circles in some order and define the map

$$f\colon (e\tilde{K}^t_{\Lambda\Gamma}e)e_\gamma \to (e\tilde{K}_\Lambda e)e_\lambda \otimes R^{\otimes n}, \quad (a\mu t\nu\overline{\gamma}) \mapsto (a\mu\overline{\lambda}) \otimes x_i \otimes \cdots \otimes x_n$$

where x_i is 1 respectively X if the i-th circle is oriented anticlockwise respectively clockwise. This map is well-defined because taking the upper reduction only removes upper lines, which are not present by assumption. Additionally the map f is then an isomorphism of left $e\tilde{K}_\Lambda e$-modules and moreover it is homogeneous of degree $\operatorname{cups}(t)$ by Lemma 3.15 (use the same argument as in Theorem 3.26(iii)). Since $0 \neq \overline{P}(\lambda) = (e\tilde{K}_\Lambda e)e_\lambda$ and $\tilde{G}^t_{\Lambda\Gamma}\overline{P}(\gamma) \cong (e\tilde{K}^t_{\Lambda\Gamma}e)e_\gamma\langle -\operatorname{caps}(t)\rangle$ this finishes the proof of (ii) and (iii). $\square$

Lemma 3.54. *The map ϕ from* (3.13) *induces a homogeneous $(e\tilde{K}_\Gamma e, e\tilde{K}_\Gamma e)$-bimodule map $\tilde{\phi}\colon e\tilde{K}^{t^*}_{\Gamma\Lambda}e \otimes e\tilde{K}^t_{\Lambda\Gamma}e \to e\tilde{K}_\Gamma e$ of degree $-2\operatorname{caps}(t)$, which is also $e\tilde{K}_\Lambda e$-balanced.*

Proof. Let $x := a\gamma t^*\mu d \otimes d^*\mu' t\gamma' b \in e\mathbb{I}^{t^*}_{\Gamma\Lambda}e \otimes eK^t_{\Lambda\Gamma}e$. We want to show that $\phi(x) \in e\mathbb{I}_\Gamma e$. First of all observe that each basis vector of $e\mathbb{I}^t_{\Gamma\Lambda}e$ contains at least one upper and one lower line by definition and assumptions on the matching t. Now note that the process of upper reduction preserves lower lines, and thus if $a\gamma t^*\mu d \in e\mathbb{I}^t_{\Gamma\Lambda}e$ and if c

denotes the upper reduction of t^*d, then $a\gamma c$ contains a nonpropagating line ending at the bottom. But this line is preserved under surgeries for $(a\gamma c)(c^*\gamma' b)$ and hence $\phi(x) = \pm(a\gamma c)(c^*\gamma' b) \in e\mathbb{I}_\Gamma e$. Using this and the dual argument for $eK^{t^*}_{\Gamma\Lambda}e \otimes e\mathbb{I}^t_{\Lambda\Gamma}e$, we see that ϕ indeed factors as claimed in the statement of the lemma. The remaining properties follow from Lemma 3.30. □

From this we can deduce analogous results to 3.31–3.33 with the same proofs by replacing Theorem 3.26 with Theorem 3.53, resulting in the following corollary.

Corollary 3.55. *We have an adjoint pair of functors*

$$(\tilde{G}^{t^*}_{\Gamma\Lambda}\langle \mathrm{cups}(t) - \mathrm{caps}(t)\rangle, \tilde{G}^t_{\Lambda\Gamma})$$

giving rise to a degree 0 *adjunction between* $\mathrm{mod}_{lf}(e\tilde{K}_\Gamma e)$ *and* $\mathrm{mod}_{lf}(e\tilde{K}_\Lambda e)$.

From Lemma 3.54 we get with the same proof as in Theorem 3.34 (using that $\mathbb{I}^t_{\Lambda\Gamma}$ and $\mathbb{I}_\Lambda$ are preserved under $\circledast$) the following theorem.

Theorem 3.56. *Given any proper* $\Lambda\Gamma$*-matching* t *and any graded* $\tilde{K}_\Gamma$*-module* M*, there exists a natural isomorphism* $\tilde{G}^t_{\Lambda\Gamma}(M^\circledast) \cong (\tilde{G}^t_{\Lambda\Gamma}M)^\circledast$ *of graded* $\tilde{K}_\Lambda$*-modules.*

Now we have all the ingredients to state the equivalent of Theorem 3.35 in the setting of nuclear diagrams.

Theorem 3.57. *Suppose we are given a proper* $\Lambda\Gamma$*-matching* t *and* $\gamma \in \Gamma$*. Then*

(i) *in the graded Grothendieck group of* $\mathrm{mod}_{lf}(e\tilde{K}_\Lambda e)$

$$[\tilde{G}^t_{\Lambda\Gamma}\overline{L}(\gamma)] = \sum_\mu (q+q^{-1})^{n_\mu}[\overline{L}(\mu)],$$

where n_μ *denotes the number of lower circles in* $\underline{\mu}t$ *and we sum over all* $\mu \in \Lambda$ *such that*

  (a) $\underline{\gamma}$ *is the lower reduction of* $\underline{\mu}t$,

  (b) *there exists no lower line in* $\underline{\mu}t$,

(ii) *the module* $\tilde{G}^t_{\Lambda\Gamma}\overline{L}(\gamma)$ *is nonzero if and only if all cups of* $t\gamma$ *are anticlockwise oriented and*

(iii) *under the assumptions of* (ii) *define* $\lambda \in \Lambda$ *such that* $\overline{\lambda}$ *is the upper reduction of* $t\overline{\gamma}$ *or alternatively* $\lambda t\gamma$ *is oriented and every cup and cap is oriented anticlockwise. In this case* $\tilde{G}^t_{\Lambda\Gamma}\overline{L}(\gamma)$ *is a self-dual indecomposable module with irreducible head* $\overline{L}(\lambda)\langle -\mathrm{caps}(t)\rangle$.

Proof. In order to prove (i) we need to show that

$$(\dagger) := \sum_{j\in\mathbb{Z}} q^j \operatorname{Hom}_{e\tilde{K}_\Lambda e}(\overline{P}(\mu), \tilde{G}^t_{\Lambda\Gamma}\overline{L}(\gamma))_j$$

is nonzero if and only (a) and (b) are satisfied, and that in this case we have $(\dagger) = (q+q^{-1})^{n_\mu}$. Using Corollary 3.55 we have

$$(\dagger) = \sum_{j\in\mathbb{Z}} q^j \operatorname{Hom}_{e\tilde{K}_\Gamma e}(\tilde{G}^{t^*}_{\Gamma\Lambda}\overline{P}(\mu)\langle \mathrm{cups}(t) - \mathrm{caps}(t)\rangle, \overline{L}(\gamma))_j.$$

Theorem 3.53 tells us that $\tilde{G}^{t^*}_{\Gamma\Lambda}\overline{P}(\mu)$ is nonzero if and only if (b) is satisfied and that in this case it is isomorphic to $\overline{P}(\beta)\otimes R^{\otimes n_\mu}\langle \mathrm{caps}(t) - \mathrm{cups}(t)\rangle$, where $\beta\in\Gamma$ is such that $\underline{\beta}$ is the lower reduction of $\underline{\mu}\overline{t}$. Hence $(\dagger)\neq 0$ if and only if (a) and (b) are satisfied. In this case, noting that the two degree shifts cancel, we have

$$(\dagger) = \sum_{j\in\mathbb{Z}} q^j \operatorname{Hom}_{e\tilde{K}_\Gamma e}(\overline{P}(\gamma)\otimes R^{\otimes n_\mu}, \overline{L}(\gamma))_j = (q+q^{-1})^{n_\mu}$$

as claimed.

For (ii) and (iii) observe that a Jordan–Hölder series for $\tilde{G}^t_{\Lambda\Gamma}\overline{L}(\gamma)$ as $e\tilde{K}_\Lambda e$-module is the same as one Jordan–Hölder series as $eK_\Lambda e$-module. Thus we can look at a Jordan–Hölder series as $eK_\Lambda e$-modules. Now note that $(e\mathbb{I}_\Gamma e)L(\gamma) = 0$ and thus $\tilde{G}^t_{\Lambda\Gamma}\overline{L}(\gamma) = (G^t_{\Lambda\Gamma}L(\gamma))/((e\mathbb{I}_\Lambda e)G^t_{\Lambda\Gamma}L(\gamma))$. Hence $\tilde{G}^t_{\Lambda\Gamma}\overline{L}(\gamma)$ is a quotient of $G^t_{\Lambda\Gamma}L(\gamma)$ as $eK_\Lambda e$-modules and thus it can be only nonzero if each cup of $t\gamma$ is oriented anticlockwise by Theorem 3.35(ii). But in this case $G^t_{\Lambda\Gamma}L(\gamma)$ has irreducible head $L(\lambda)\langle -\mathrm{caps}(t)\rangle$, where λ is such that $\lambda t\gamma$ oriented and all every cup and cap is oriented anticlockwise. So $\tilde{G}^t_{\Lambda\Gamma}\overline{L}(\gamma)$ is zero or it has the same irreducible head. But this composition factor can only occur by (i) if λ satisfies (a) and (b). These two conditions are automatically satisfied as any nonpropagating line needs to have a clockwise oriented cup or cap in t.

That $\tilde{G}^t_{\Lambda\Gamma}\overline{L}(\gamma)$ is self-dual follows from Theorem 3.56 and the fact that $\overline{L}(\gamma)^{\circledast} \cong \overline{L}(\gamma)$. □

Remark 3.58. Note that the formulas in Remark 3.48 and Theorem 3.57 share many similarities. Let $\gamma\in\Gamma$ and $\mu\in\Lambda$ denote any super weight diagrams. Define t to be $\overline{\mu}\underline{\gamma}$, i.e. we draw the cap diagram of μ under the cup diagram of γ and connect the rays from left to right. A basis for $e\tilde{K}^t_{\Lambda\Gamma}e$ is given by $a\nu t\eta b$. As we apply $\tilde{G}^t_{\Lambda\Gamma}$ to the irreducible module $\overline{L}(\gamma)$, it follows from the definition of $\overline{L}(\gamma)$ that a basis for $\tilde{G}^t_{\Lambda\Gamma}\overline{L}(\gamma)$ is given by $a\nu t\gamma\overline{\gamma}$. By definition of t this can then be easily identified with the basis $a\nu\overline{\mu}$ for $P(\mu)$. Thus we have $\tilde{G}^t_{\Lambda\Gamma}\overline{L}(\gamma)\cong\overline{P}(\mu)\langle -\mathrm{def}(\mu)\rangle$. Note the degree shift in Definition 3.52 and that $\mathrm{caps}(t) = \mathrm{def}(\mu)$.

In this case, we also see that the assumption (a) in Theorem 3.57 is automatically satisfied.

Remark 3.59. Corollary 3.55 tells us, in particular, that $\tilde{G}^t_{\Lambda\Gamma}$ is exact and then we can use the same argument as in Corollary 3.36, replacing Theorem 3.35(i) by Theorem 3.57(i) to show that $\tilde{G}^t_{\Lambda\Gamma}$ preserves finite dimensional modules.

4 Equivalence of categories between $\mathcal{F}$ and Khovanov's algebra

In this chapter we are going to prove the main theorem stated in the introduction. We will relate the category $\mathcal{F}$ and i-translation with a Khovanov algebra of type B and the corresponding projective functors. We will not prove the whole theorem but rely on a black box to simplify this proof. This black box involves only complicated calculations and would not be enlightening to see. We will however indicate how one could prove it. Throughout this chapter eKe will denote the idempotent truncation of K by the super weight diagrams and $e\tilde{K}e$ the quotient of eKe by the nuclear ideal, similar to Section 3.4. We fix r, $n \in \mathbb{Z}_{\geq 0}$, set $m := \lfloor \frac{r}{2} \rfloor$ and furthermore $\delta = r - 2n$.

Definition 4.1. Let $i \in \mathbb{Z} + \frac{\delta+1}{2}$. Given a block Γ of Deligne weight diagrams, suppose that the number line of Γ agrees at the vertices $|i| \pm \frac{1}{2}$ with the bottom line of one of the pictures C in Figure 4.1. By adding vertical strands C can be extended to a unique $\Gamma_{t_i^C}\Gamma$-matching t_i^C, where $\Gamma_{t_i^C}$ is the block which is obtained from Γ, when replacing the symbols at positions $|i| \pm \frac{1}{2}$ with the top of the picture C.
We then define the functor $\Theta_i^\Gamma := \bigoplus_C G^{t_i^C}_{\Gamma_{t_i^C}\Gamma}\colon \mathrm{mod}_{lf}(K_\Gamma) \to \mathrm{mod}_{lf}(K)$, where the direct sum runs through all possible pictures, which can be put at positions $|i| \pm \frac{1}{2}$ onto Γ. We remark here, that whenever $i \neq -\frac{1}{2}$ there is always at most one choice and if $i = -\frac{1}{2}$ and the block sequence of Γ starts with $\diamond\circ$ we have two choices.
Given this we define $\Theta_i\colon \mathrm{mod}_{lf}(K) \to \mathrm{mod}_{lf}(K)$ as $\bigoplus_\Gamma \Theta_i^\Gamma$.
In the same way we define $\tilde{\Theta}_i^\Gamma := \bigoplus_C \tilde{G}^{t_i^C}_{\Gamma_{t_i^C}\Gamma}\colon e\tilde{K}_\Gamma e\text{-mod} \to e\tilde{K}e\text{-mod}$ and $\tilde{\Theta}_i := \bigoplus_\Gamma \tilde{\Theta}_i^\Gamma\colon e\tilde{K}e\text{-mod} \to e\tilde{K}e\text{-mod}$.

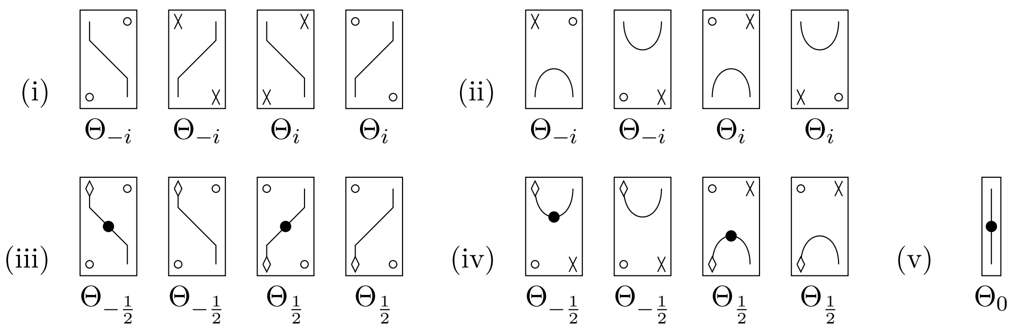


Figure 4.1: Local moves

Definition 4.2. Define $T_d \coloneqq \bigoplus_{\mathbf{i}\in(\mathbb{Z}+\frac{\delta+1}{2})^d} \Theta_{\mathbf{i}} L(\emptyset_\delta)$, where $\Theta_{\mathbf{i}} \coloneqq \Theta_{i_d} \dots \Theta_{i_1}$ if $\mathbf{i} = (i_1, \dots i_d)$.

The following theorem is the previously mentioned black box:

Theorem 4.3. *There exist isomorphisms of algebras $\xi_d \colon \mathrm{Br}_d(\delta) \xrightarrow{\cong} \mathrm{End}_K(T_d)$ such that the following diagram commutes*

$$\begin{tikzcd} \mathrm{Br}_d(\delta) \arrow[r, "\xi_d"] \arrow[d, "i\text{-ind}"] & \mathrm{End}_K(T_d) \arrow[d, "\Theta_i"] \\ \mathrm{Br}_{d+1}(\delta) \arrow[r, "\xi_{d+1}"] & \mathrm{End}_K(T_{d+1}) \end{tikzcd}.$$

Idea of Proof. The problem is that the idempotents picking out the eigenspaces for the i-induction are not part of the definition of the Brauer algebra and very hard to handle. So one would like to find a variant of the Brauer algebra that has these idempotents build in the definition. This is the algebra $G_d(\delta)$ provided by [Li14]. This algebra $G_d(\delta)$ is the Brauer analogue of the cyclotomic Khovanov–Lauda–Rouquier algebra R_d in [BK09] and plays the same role as R_d for the degenerate affine Hecke algebra. The definition of $G_d(\delta)$ is however much more complicated and involved.
Via the same definition of i-induction for $G_d(\delta)$ one can easily verify that the isomorphism between $\mathrm{Br}_d(\delta)$ and $G_d(\delta)$ in [Li14] is compatible with i-induction.
On the other hand we have the so called cup-cap algebra $C_d(\delta)$. It consists out of so called oriented stretched circle diagrams of height d and a multiplication, which is also given by some surgery procedure, for details see [ES21, Section 11] and [Mkr20, Section 4]. This can then easily be identified with $\mathrm{End}_K(T_d)$ using the diagrammatic description of Θ_i. For the cup-cap algebra one can also define a version of Θ_i which would be given by inserting the local moves in the middle of an oriented stretched circle diagram.
The most difficult and important part will be the identification of $G_d(\delta)$ with $C_d(\delta)$ and the compatibility of this with i-induction and Θ_i. An explicit isomorphism can be found in [Mkr20]. One would then only have to check that this is swaps i-induction and Θ_i. This would be a very tedious and long calculation, which will be omitted here. $\square$

Now we have all the ingredients to prove the main theorem from the introduction.

Theorem 4.4. *We have an equivalence of categories $\Psi \colon (e\tilde{K}e)$-mod $\to \mathcal{F}$ such that $\theta_i \circ \Psi \cong \Psi \circ \tilde{\Theta}_i$, which maps $\overline{L}(\lambda_\varepsilon^{\oslash})$ to $L(\lambda, \varepsilon)$ for every $(\lambda, \varepsilon) \in s\Gamma_\delta(m,n) \cong X^+(\mathrm{OSp}(r|2n))$.*

Proof. By Theorem 4.3, we have algebra isomorphisms $\psi_d \colon \mathrm{End}_K(T_d) \to \mathrm{Br}_d(\delta)$ for every $d \geq 0$ such that $\psi_{d+1} \circ \Theta_i = i\text{-ind} \circ \psi_d$.
We can take the direct limit of $\mathrm{End}_k(T_d)$ with respect to the inclusion

$$\bigoplus_{i\in\mathbb{Z}+\frac{\delta+1}{2}} \Theta_i \colon \mathrm{End}_k(T_d) \to \mathrm{End}_k(T_{d+1})$$

and we can take the direct limit of $\mathrm{Br}_d(\delta)$ with respect to the natural inclusion $\mathrm{Br}_d(\delta) \to \mathrm{Br}_{d+1}(\delta), f \mapsto f \otimes 1$. Note that this natural inclusion is the same as $\bigoplus_{i \in \mathbb{Z}+\frac{\delta+1}{2}} i\text{-ind}$ and thus we obtain an isomorphism

$$\psi\colon \varinjlim \mathrm{End}_k(T_d) \to \varinjlim \mathrm{Br}_d(\delta) \tag{4.1}$$

with $\psi \circ \Theta_i = i\text{-ind} \circ \psi$.
By (1.19) we have a surjective algebra homomorphism $\mathrm{Br}_d(\delta) \to \mathrm{End}_{\mathcal{F}}(V^{\otimes d})$. Taking the direct limit of $\mathrm{End}_{\mathcal{F}}(V^{\otimes d})$ with respect to the embedding $f \mapsto f \otimes 1$, we obtain a surjective algebra homomorphism

$$\Phi\colon \varinjlim \mathrm{Br}_d(\delta) \to \varinjlim \mathrm{End}_{\mathcal{F}}(V^{\otimes d}) \tag{4.2}$$

such that $\Phi \circ i\text{-ind} = \theta_i \circ \Phi$ (the compatibility follows from Theorem 1.39).
Putting this together we obtain a surjective algebra homomorphism

$$\Psi' \coloneqq \Phi \circ \psi\colon \varinjlim \mathrm{End}_k(T_d) \to \varinjlim \mathrm{End}_{\mathcal{F}}(V^{\otimes d}) \tag{4.3}$$

such that $\theta_i \circ \Psi' = \Psi' \circ \Theta_i$.
Now we take a look at the algebra $A_{(r|2n)}$ which we define as

$$A_{(r|2n)} \coloneqq \bigoplus_{(\lambda,\varepsilon),(\mu,\varepsilon') \in X^+(\mathrm{OSp}(r|2n))} \mathrm{Hom}_{\mathcal{F}}(P(\lambda,\varepsilon), P(\mu,\varepsilon')).$$

Let $f \in A_{(r|2n)}$. By definition this can be identified with some $f \in \mathrm{End}_{\mathcal{F}}(P,P)$ for some projective module P which is the direct sum of finitely many nonisomorphic indecomposable projective objects in $\mathcal{F}$. We may assume that they lie in the same block. Then by [ES21, Proposition 5.10] there exists $V^{\otimes d}$ containing P as a summand, and thus we can consider f as an endomorphism of $V^{\otimes d}$. In this way we can realize $A_{(r|2n)}$ as a subalgebra of $\varinjlim \mathrm{End}_{\mathcal{F}}(V^{\otimes d})$.
By Theorem 2.24 and Theorem 2.31 Ψ' restricts to a surjective algebra homomorphism $\bar{\Psi}\colon eKe \to A_{(r|2n)}$ which identifies the idempotent corresponding to $P(\lambda,\varepsilon) \in \mathcal{F}$ with the idempotent corresponding to the super weight diagram associated to (λ,ε)[1]. We have a commutative diagram

$$\begin{tikzcd}
\varinjlim \mathrm{End}_K(T_d) \arrow[r, "\Psi'"] & \varinjlim \mathrm{End}_{\mathcal{F}}(V^{\otimes d}) \\
eKe \arrow[u, hook, "\iota_1"] \arrow[r, "\bar{\Psi}"] & A_{(r|2n)}, \arrow[u, hook, "\iota_2"]
\end{tikzcd}$$

when we identify $eKe = \bigoplus_{\lambda,\mu} \mathrm{Hom}_K(P(\lambda), P(\mu))$, where we sum over pairs of super weight diagrams with a subalgebra of $\varinjlim \mathrm{End}_K(T_d)$. We clearly have $\theta_i \circ \iota_2 = \iota_2 \circ \theta_i$,

[1]Actually it identifies the idempotent corresponding to the super weight diagram with the reversed sign rule (see Remark 2.35) associated to (λ,ε). But changing the parity of the dot on the leftmost ray in $e\tilde{K}e$ is an automorphism, so we just twist in the end by this automorphism and obtain the desired result. On the super side, this would correspond to tensoring with $L(\emptyset,-)$.

but unfortunately we do not have $\Theta_i \circ \iota_1 = \iota_1 \circ e\Theta_i e$. The problem is that in K (and thus in $\varinjlim \mathrm{End}_K(T_d)$) we are allowed to have circle diagrams $\underline{\lambda}\nu\overline{\mu}$ such that $\kappa(\lambda) \neq \kappa(\mu)$, but in eKe this is not possible by Definition 3.47 as it is the idempotent truncation by super weight diagrams and by Definition 2.33 every super weight diagram μ satisfies $\kappa(\mu) = \min(m,n)$. By [ES21, Proposition 8.8], we know that Θ_i never decreases the layer number, but it might increase it. However, in this case Ψ' produces 0 by Theorem 2.24. Thus we have $\theta_i \circ \bar{\Psi} = \bar{\Psi} \circ e\Theta_i e$.
By [ES21, Lemma 10.4] and Theorem 2.24 $\bar{\Psi}$ factors through the nuclear ideal $e\mathbb{I}e$ giving rise to $\Psi\colon e\tilde{K}e \to A_{(r|2n)}$. By additionally looking at [ES21, Proposition 8.8] and the definition of $\tilde{\Theta}_i$ we also see that $\theta_i \circ \Psi = \Psi \circ \tilde{\Theta}_i$. By [ES17, Theorem 5.1] Ψ is an isomorphism, so we get an equivalence of categories $\Psi\colon e\tilde{K}e\text{-mod} \to \mathcal{F}$ such that $\theta_i \circ \Psi \cong \Psi \circ \tilde{\Theta}_i$. This equivalence maps $\overline{L}(\lambda_\varepsilon^{\oslash})$ to $L(\lambda,\varepsilon)$ as the isomorphism $\phi\colon e\tilde{K}e \to A_{(r|2n)}$ identifies the idempotent corresponding to $P(\lambda,\varepsilon)$ in $A_{(r|2n)}$ with $e_{\lambda_\varepsilon^{\oslash}}$ in $e\tilde{K}e$. $\square$

Remark 4.5. If we summarize our results so far in terms of understanding direct summands of $V^{\otimes d}$, we know by Definition 1.38 that it suffices to understand $\theta_{i_1} \circ \cdots \circ \theta_{i_d} L(\emptyset,+)$. By Theorem 4.4 this is the same as $\Psi \circ \tilde{\Theta}_{i_1} \circ \cdots \circ \tilde{\Theta}_{i_d} L((\emptyset_\delta)_+^{\oslash})$. Forgetting the grading on $e\tilde{K}e$ we know that (by Definition 4.1) $\tilde{\Theta}_i$ is given by tensoring with some $\bigoplus_j e\tilde{K}^{t_j}_{\Lambda_j\Gamma_j}e$ for certain blocks Λ_j and Γ_j and $\Lambda_j\Gamma_j$-matchings t_j. Note that by definition each of these t_j is a translation diagram. Lemma 3.43 then tells us that $\tilde{\Theta}_{i_1} \circ \cdots \circ \tilde{\Theta}_{i_d}$ is actually given by $\bigoplus_j e\tilde{K}^{\boldsymbol{t}_j}_{\boldsymbol{\Lambda}_j}e$ for some sequences of blocks $\boldsymbol{\Lambda}_j$ and $\boldsymbol{\Lambda}_j$-matchings $\boldsymbol{t}_j$. Finally using Theorem 3.44 we see that the sum $\bigoplus_j e\tilde{K}^{\boldsymbol{t}_j}_{\boldsymbol{\Lambda}_j}e$ can be reduced to $\bigoplus_{j'} e\tilde{K}^{t_{j'}}_{\Lambda_{j'}\Gamma_{j'}}e$. Furthermore by Theorem 3.57(iii) we know that $e\tilde{K}^{t_{j'}}_{\Lambda_{j'}\Gamma_{j'}}e \otimes_{e\tilde{K}e} \overline{L}((\emptyset_\delta)_+^{\oslash})$ is indecomposable.
As the equivalence Ψ from Theorem 4.4 is necessarily additive, every indecomposable summand of $V^{\otimes d}$ is then of the form $\Psi(e\tilde{K}^t_{\Lambda\Gamma}e \otimes_{e\tilde{K}e} \overline{L}((\emptyset_\delta)_+^{\oslash}))$ for some blocks Λ, Γ and a $\Lambda\Gamma$-matching t. This is the same (forgetting the grading) as writing that every indecomposable summand is of the form $\Psi(\tilde{G}^t_{\Lambda\Gamma}\overline{L}((\emptyset_\delta)_+^{\oslash}))$.
Conversely every such choice of Λ, Γ and t gives in this way an indecomposable summand in some $V^{\otimes d}$.

5 Applications

In this chapter we are going are going to prove some results about the indecomposable modules appearing in $V^{\otimes d}$. For this recall that the indecomposable modules in $V^{\otimes d}$ are parametrized by partitions which give rise to Deligne weight diagrams. The indecomposable summands are then given by $\{\mathbb{F}\,\mathrm{R}_\delta(\lambda) \mid \kappa(\lambda_\delta) \leq \min(m,n)\}$ by Theorem 2.24.
By Remark 4.5 we know that each $\mathbb{F}\,\mathrm{R}_\delta(\lambda)$ arises as $\Psi(\tilde{G}^t_{\Lambda\Gamma}\overline{L}((\emptyset_\delta)^{\oslash}_{+}))$ for some blocks Λ and Γ in $e\tilde{K}e$ and some $\Lambda\Gamma$-matching t.

Theorem 5.1. *Given a Deligne weight diagram λ_δ, the module $\mathbb{F}\,\mathrm{R}_\delta(\lambda)$ is self-dual, in particular it has simple head and socle and these two agree.*

Proof. We know by Theorem 3.57 that $\tilde{G}^t_{\Lambda\Gamma}\overline{L}((\emptyset_\delta)^{\oslash}_{+})$ has simple head and is indecomposable and thus also $\mathbb{F}\,\mathrm{R}_\delta(\lambda) \cong \Psi(\tilde{G}^t_{\Lambda\Gamma}\overline{L}((\emptyset_\delta)^{\oslash}_{+}))$.
An argument why $\mathbb{F}\,\mathrm{R}_\delta(\lambda)$ is self-dual can be found in [ES21, Theorem 12.1] and we are going to repeat it here. The claim clearly holds for $\emptyset$ and if $\mathbb{F}\,\mathrm{R}_\delta(\lambda)$ is self-dual, so is $\mathbb{F}(\mathrm{R}_\delta(\lambda) \boxtimes \mathrm{R}_\delta(\square)) \cong \mathbb{F}\,\mathrm{R}_\delta(\lambda) \otimes V$ using [Mus12, Proposition 13.7.2]. The claim then follows for any $\mathbb{F}\,\mathrm{R}_\delta(\mu)$ for which $\mathrm{R}_\delta(\mu)$ can be obtained via i-induction from $\mathrm{R}_\delta(\lambda)$ by Theorem 1.39 and [Mus12, Proposition 13.7.1] using [ES21, Remark 8.12] which states that i-translation is given by $\_ \otimes V$ followed by a projection onto some block (for $i = -\frac{1}{2}$ the functor decomposes further into a sum of two of these types, see also Definition 4.1). $\square$

Proposition 5.2. *The radical and socle filtration of $\tilde{G}^t_{\Lambda\Gamma}\overline{L}((\emptyset_\delta)^{\oslash}_{+})$ agrees with the grading filtration. In particular the radical and socle filtration of $\mathbb{F}\,\mathrm{R}_\delta(\lambda)$ is induced by the grading filtration on $\tilde{G}^t_{\Lambda\Gamma}\overline{L}((\emptyset_\delta)^{\oslash}_{+})$, it is rigid and has Loewy length $ll(\mathbb{F}\,\mathrm{R}_\delta(\lambda)) = 2d(\lambda_\delta)+1$, where $d(\lambda_\delta)$ denotes the number of caps in the cap diagram of λ_δ.*

Proof. Let us abbreviate $X \coloneqq \tilde{G}^t_{\Lambda\Gamma}\overline{L}((\emptyset_\delta)^{\oslash}_{+})$. This admits a filtration by submodules $X(j)$ which is spanned by all graded pieces of degree $\geq j$ as $e\tilde{K}e$ is a nonnegatively graded algebra. So for some $m < n$ we have

$$X = X(m) \supset X(m+1) \supset \cdots \supset X(n)$$

and $X(n+1) = 0$. Using Theorem 3.50 and the fact that the degree 0 part of the Khovanov algebra is semisimple (it is a direct sum of irreducible modules) we can apply Proposition 5.3 below. In conjunction with Theorem 3.57(iii) we obtain that the radical and socle filtration agree and that m respectively n has to be $-\operatorname{caps}(t)$ respectively $\operatorname{caps}(t)$. The Loewy length $ll(X)$ is thus given by $2\operatorname{caps}(t)+1$. This then translates to $\mathbb{F}\,\mathrm{R}_\delta(\lambda)$ by using Ψ from Theorem 4.4. It remains to see that

$\operatorname{caps}(\underline{t}) = d(\lambda_\delta)$. But [ES21, Theorem 8.7] tells us that $\overline{\lambda_\delta}$ is the upper reduction of $t\overline{\emptyset_\delta}$. Thus as $\overline{\emptyset_\delta}$ is cap-free (by Example 2.19 and Definition 2.10) we have that $\operatorname{caps}(t) = d(\lambda_\delta)$ as no cap in t gets removed during the process of upper reduction (see Definition 3.13). □

The previous proof relies on the following general fact:

Proposition 5.3 ([BGS96, Prop. 2.4.1])**.** *Let A be a graded ring such that*

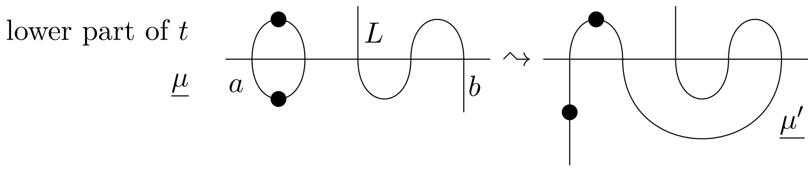
(i) *A_0 is semisimple and*

(ii) *A is generated by A_1 over A_0.*

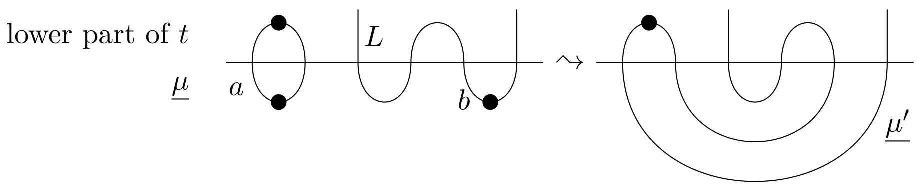
Let $M \in A$-Mod be a graded A-module of finite length. If $\operatorname{soc} M$ (resp. $M/\operatorname{rad} M$) is simple, then the socle (resp. radical) filtration on M coincides with the grading filtration, up to shift.

Proposition 5.4. *The highest weight constituent of $\mathbb{F}\,\mathrm{R}_\delta(\lambda)$ sits inside the middle Loewy layer with multiplicity* 1*.*

Proof. Abbreviate $\gamma = (\emptyset_\delta)^{\oslash}_{+}$ and $X := \tilde{G}^t_{\Lambda\Gamma}\overline{L}(\gamma)$. We are going to prove the statement using $\mathbb{F}\,\mathrm{R}_\delta(\lambda) \cong \Psi(X)$. In Proposition 5.2 we proved that the socle filtration coincides with the grading filtration on X. Theorem 3.57(i) gives an explicit description of the grading filtration, where $\overline{L}(\mu)$ appears if $\underline{\gamma}$ is the lower reduction of $\underline{\mu}t$ and there exists no lower line in $\underline{\mu}t$. If there are additionally no lower circles in $\underline{\mu}t$, $\overline{L}(\mu)$ appears in the middle Loewy layer. Therefore it suffices to show that whenever we have a super weight diagram μ satisfying that $\underline{\mu}t$ has lower reduction $\underline{\gamma}$ and contains no lower line but at least one lower circle, we find a super weight diagram μ' such that the associated highest weight of μ' is bigger than μ and that $\mu' t$ has lower reduction $\underline{\gamma}$ and contains no lower line and less lower circles.
The general idea is that for such a lower circle we will construct a different cup diagram, which merges this lower circle with a line and thus removes it.
So let μ be a super weight diagram such that $\underline{\mu}t$ has lower reduction $\underline{\gamma}$ and contains no lower line but a lower circle. Choose C to be the rightmost lower circle in $\underline{\mu}t$ and let a be the cup of $\underline{\mu}$ which ends at the rightmost vertex of C. We denote the first ray or dotted cup to the right of C by b and denote the component containing b by L. We have to distinguish two cases, depending on b. *Case 1* will be that b is a ray and *Case 2* that b is a dotted cup.
In *Case 1* (i.e. b is a ray) there may be some cups in $\underline{\mu}$ which lie between the end of a and b but by assumption these are all undotted. Then we define μ' to be the super weight diagram such that $\underline{\mu'}$ agrees with $\underline{\mu}$ except that a and b are replaced by a ray ending at the left endpoint of a and a cup connecting the right endpoint of a with the endpoint of b. The cup will always be undotted and the ray has a dot according to the parity of the dots on a and b.



Then by construction $\underline{\mu'}t$ has lower reduction $\underline{\gamma}$, contains no lower lines and fewer lower circles. Note that all weight diagrams that we are looking at are super weight diagrams, i.e. they have layer number $\min(m, n)$. This means that when passing to hook weight diagrams, the cups stay the same and every symbol corresponding to a ray gets changed into a $\wedge$. Therefore, the symbol corresponding to the ray b in μ^∞ is always $\wedge$ and the endpoints of a are labelled $\vee\wedge$ if a is undotted and $\wedge\wedge$ if it is dotted. The same points in μ' are always labelled $\wedge \vee \wedge$. The $\wedge$ comes from the new ray in the left and $\vee\wedge$ are the endpoints of the new undotted cup. Thus we either change $\vee \wedge \wedge$ or $\wedge \wedge \wedge$ into $\wedge \vee \wedge$ depending on whether a was undotted or dotted. However both changes make the hook weight diagrams bigger by Remark 2.28.
In the second case b is a dotted cup. Then we let μ be the super weight diagram such that $\underline{\mu'}$ agrees with $\underline{\mu}$ except that the two cups a and b are replaced with two nested cups. The inner one will be undotted and the outer one will be dotted in a way such that the parity of dots agrees with the parity of dots on a and b.



By construction $\underline{\mu'}u$ has lower reduction $\underline{\gamma}$, contains no lower lines and has fewer lower circles. When looking at hook weight diagrams the endpoints of the dotted cup b are labelled $\wedge\wedge$. If a is undotted, its endpoints are labelled $\vee\wedge$. Then these four points in μ' are labelled (from left to right) $\wedge \vee \wedge\wedge$ by construction. If a is dotted, its endpoints are labelled $\wedge\wedge$ and thus by our construction we turn $\wedge \wedge \wedge\wedge$ into $\vee \vee \wedge\wedge$. However both changes make the hook weight diagrams bigger by Remark 2.28. $\square$

Proposition 5.5. *Each block has a unique irreducible* $\mathbb{F}\,\mathrm{R}_\delta(\lambda)$.

Proof. By Proposition 5.2, we have that $\mathbb{F}\,\mathrm{R}_\delta(\lambda)$ is irreducible if and only if $d(\lambda_\delta) = 0$, i.e. the cup diagram $\underline{\lambda_\delta}$ contains no cup. Now recall that by Definition 2.3 any block is uniquely determined by the positions of $\circ$ and $\times$ as well as the parity of the number of $\wedge$'s. Note that for each of these choices we can create exactly one Deligne weight diagram λ_δ such that its cup diagram does not contain a cup.
This follows because the positions of $\circ$ and $\times$ are fixed and by observing that whenever we have at least two $\wedge$'s, we necessarily create a cup. Therefore, we have at most one $\wedge$. In the case of one $\wedge$, note further that no $\vee$ can appear to the left of it without creating a cup, thus $\wedge$ has to appear on the leftmost free position. Hence we only

have one choice. It is additionally easy to see that this choice of a weight diagram gives rise to an irreducible $\mathbb{F}\,\mathrm{R}_\delta(\lambda)$. $\square$

5.1 Kazhdan–Lusztig polynomials of type B and Deligne–Kostant weights

In [BS10, Section 5] Brundan and Stroppel defined certain polynomials $p_{\lambda,\mu}(q)$ associated to weight diagrams λ, μ in type A via labellings of diagrams. These also agree (up to a scaling factor) with the polynomials $Q_w^v(q)$ defined in [LS81, Section 6], which are shown in [LS81, Théorème 7.8] to be equal to the (geometric) Kazhdan–Lusztig polynomials associated to Grassmannians in the sense of [KL79]. In [Sey17] the diagrammatic definition was extended for weight diagrams of type D, which is very similar to our situation. The only difference is that the weight diagrams of type D are finite, whereas ours are infinite.
In [BS10, Section 5] Brundan and Stroppel related the polynomials $p_{\lambda,\mu}(q)$ to the dimension of some extension group and gave a diagrammatical condition when these polynomials are monomials (for a fixed μ). The same results were obtained in [Sey17, Section 5] for type D and we will prove these statements also for type B. The main advantage of the diagrammatical description of the Kazhdan–Lusztig polynomials $p_{\lambda,\mu}(q)$ in [BS10] is that the definition also makes sense for unbounded weights and this gives us the possibility to directly adapt the proofs of [Sey17, Section 5].
Throughout this section we fix $\delta \in \mathbb{Z}$ and consider two Deligne weight diagrams $\lambda \leq \mu$.

Definition 5.6. A cap γ in $\overline{\mu}$ is called *D-nested* inside a cap γ' if either γ lies under γ', or γ' is dotted and γ lies to the left of γ'.
Suppose we are given two Deligne weight diagrams $\lambda_\delta \leq \mu_\delta$ such that $l_0(\lambda,\mu) = 2k$. A *λ-labelling* C of the oriented cap diagram $\mu\overline{\mu}$ assigns to every cap a natural number, such that the following properties are satisfied:

(i) If the left end of an undotted cap is at position i, its label is at most $l_i(\lambda,\mu)$.

(ii) The label of any dotted cap is even and at most $l_0(\lambda,\mu)$.

(iii) If a cap γ is D-nested inside another cap γ', the label of γ is greater or equal to the label of γ'.

(iv) A cap may only have an odd label if there is some other cap above it or to the left of it, which has a strictly smaller label, or if there is a ray to the left of it.

We denote the set of λ-labellings of $\mu\overline{\mu}$ by $D(\lambda,\mu)$. The *value* $|C|$ of a labelling $C \in D(\lambda,\mu)$ is defined to be the sum of the labels in C.

Definition 5.7. For two Deligne weight diagram $\lambda \leq \mu$ we define the *dual Kazhdan–Lusztig polynomial* $p_{\lambda,\mu}(q)$ to be

$$p_{\lambda,\mu}(q) = q^{l(\lambda,\mu)} \sum_{C \in D(\lambda,\mu)} q^{-2|C|}.$$

Theorem 5.8. *The dual Kazhdan–Lusztig polynomials $p_{\lambda,\mu}(q)$ satisfy the following recursive relations.*

(i) *If $\lambda = \mu$, then $p_{\lambda,\mu}(q) = 1$ and if $\lambda \nleq \mu$ then $p_{\lambda,\mu}(q) = 0$.*

(ii) *If $\lambda < \mu$, let B be a Bruhat move that can be applied to λ at positions i and j. Then*

$$p_{\lambda,\mu}(q) = \begin{cases} p_{\partial_{i,j}(\lambda),\partial_{i,j}(\mu)}(q) + q p_{B(\lambda),\mu}(q) & \text{if } B \text{ can be applied to } \mu, \\ q p_{B(\lambda),\mu}(q) & \text{if } B \text{ cannot be applied to } \mu, \end{cases}$$

where $\partial_{i,j}(\gamma)$ denotes the weight diagram, which is obtained from γ by deleting the positions i and j.

(iii) *If $\partial(\gamma)$ denotes the weight diagram obtained from γ by deleting all symbols $\circ$ and $\times$, we have for $\lambda \leq \mu$*

$$p_{\lambda,\mu}(q) = p_{\partial(\lambda),\partial(\mu)}(q).$$

Proof. Part (iii) is obvious from the definition. Parts (i) and (ii) are proven in [Sey17, Theorem 5.3.4]. This proof considers only the finite case (without any $\circ$'s and $\times$'s) but it also works in this case as Deligne weight diagrams are admissible. This means that at some point there exist only $\vee$'s and undotted rays in the weight diagram and the associated cup diagram of λ (resp. μ). But these are irrelevant in any λ-labelling $\mu\overline{\mu}$ and so we can reduce to the finite case. There one starts with $\lambda < \mu$ and assume that the claim holds for all $\lambda < \lambda' \leq \mu$. One then applies a Bruhat move B to λ such that $B(\lambda) \leq \mu$ and relates $B(\lambda)$-labellings with λ-labellings.
As our set Λ_δ is upper finite, this induction also works in our case and we can use the same proof. □

Theorem 5.9. *For every standard module $V(\lambda)$ there exists a linear projective resolution*

$$\cdots \to P^k \to P^{k-1} \to \cdots \to P^1 \to P^0 \to V(\lambda)$$

such that $P^k \cong \bigoplus_\mu p_{\lambda,\mu}^{(k)} P(\mu)\langle k\rangle$, where $p_{\lambda,\mu}^{(k)}$ denotes the coefficient of q^k in $p_{\lambda,\mu}(q)$.

Proof. The proof is the same as in the finite case (see [Sey17, Theorem 4.3.2] or [BS10, Theorem 5.3] for a proof in type A) but we believe that it is a nice application of the theory of geometric bimodules and projective functors, so we are going to recall it here.
The claim is shown by a nested induction. First we do an induction on the number of caps of $\overline{\lambda}$ and secondly one on the Bruhat order. If $\#(\text{caps}(\overline{\lambda})) = 0$, then λ is maximal. Hence it suffices to consider maximal weights for the induction beginning. But in this case we have $P(\lambda) = V(\lambda)$ and the claim holds.
Now suppose that $\#(\text{caps}(\overline{\lambda})) > 0$ and assume the claim for all μ with less caps and all $\lambda' > \lambda$. As we are only considering Deligne weight diagrams which are admissible, the number of caps is actually finite.

As λ is not maximal, we can apply a Bruhat move B at positions i and j to λ. This corresponds to a cap C in $\overline{\lambda}$. Now let Γ be the block given by the same positions of $\circ$ and $\times$ as in Λ but an additional $\circ$ at position i and a $\times$ at position j. The parity of the number of $\wedge$'s is the same as Λ if C is undotted and different if C is dotted. Then let t be the $\Lambda\Gamma$-matching given by a cap connecting positions i and j (with the same parity of dots as C) and vertical strands everywhere else. Let μ be the weight diagram obtained from λ by putting a $\circ$ at position i and a $\times$ at position j.
We observe that there are exactly two weights γ such that $\gamma t\mu$ is oriented, namely $\gamma = \lambda$ and $\gamma = \lambda'$ (the former corresponds to orienting C anticlockwise and the latter to a clockwise orientation). So by Theorem 3.29 (as $\lambda' > \lambda$) there is a short exact sequence

$$0 \longrightarrow V(\lambda') \xrightarrow{f} G^t_{\Lambda\Gamma} V(\mu) \longrightarrow V(\lambda)\langle -1\rangle \longrightarrow 0.$$

By the induction hypothesis we have constructed a linear projective resolution $P^\bullet(\lambda')$ of $V(\lambda')$ and as $\overline{\mu}$ contains less caps than $\overline{\lambda}$ we also have constructed a linear projective resolution $P^\bullet(\mu)$ of $V(\mu)$. We have $P^k(\lambda') = \bigoplus_{\gamma\in\Lambda} p^{(k)}_{\lambda',\gamma} P(\gamma)\langle k\rangle$ and $P^k(\mu) = \bigoplus_{\xi\in\Gamma} p^{(k)}_{\mu,\xi} P(\xi)\langle k\rangle$. By Corollary 3.28, applying $G^t_{\Lambda\Gamma}$ to $P^\bullet(\mu)$ gives a projective resolution $G^t_{\Lambda\Gamma}P^\bullet(\mu)$ of $G^t_{\Lambda\Gamma}V(\mu)$ with $G^t_{\Lambda\Gamma}P^k(\mu) \cong \bigoplus_{\xi\in\Gamma} p^{(k)}_{\mu,\xi} G^t_{\Lambda\Gamma} P(\xi)\langle k\rangle$. But by Theorem 3.26, we have $G^t_{\Lambda\Gamma}P(\xi) \cong P(\nu)\langle -1\rangle$, where ν is obtained from ξ by replacing $\circ\times$ with $\vee\wedge$ (if the chosen cap is undotted) or $\wedge\wedge$ (if the chosen cap is dotted) at position i and j.
Then by [Sey17, Proposition 4.3.1] the cone of f is a projective resolution of $V(\lambda)\langle -1\rangle$. Now note that

$$\mathrm{Cone}(f)^k = P(\lambda')^{k-1} \oplus G^t_{\Lambda\Gamma}P(\mu)^k = \bigoplus_{\gamma\in\Lambda} p^{(k-1)}_{\lambda',\gamma} P(\gamma)\langle k-1\rangle \oplus \bigoplus_{\nu\in\Lambda_B} p^{(k)}_{\mu,\nu_{\circ\times}} P(\nu)\langle k\rangle,$$

where Λ_B denotes all weights in Λ such that the Bruhat move B can be applied and $\nu_{\circ\times}$ is the weight ν with $\circ$ at position i and $\times$ at position j. By Theorem 5.8(iii) we see that $p^{(k)}_{\mu,\nu_{\circ\times}} = p^{(k)}_{\partial_{i,j}(\lambda),\partial_{i,j}(\nu)}$ and so the occurring recursive formulas agree with Theorem 5.8(ii), finishing the proof. □

Definition 5.10. A Deligne weight diagram μ is called a *Deligne–Kostant weight* if

$$\sum_{k\geq 0} \dim \mathrm{Ext}^k_K(V(\lambda), L(\mu)) \leq 1$$

for all λ in the same block as μ.

This definition agrees with the definition of Kostant weights for the Khovanov algebra of type A given by Brundan and Stroppel in [BS10, Section 7]. We speak of Deligne–Kostant weights to distinguish them from Kostant weights in the sense of [GH21], which we will use in Section 5.2.

Proposition 5.11. *A weight diagram μ is a Deligne–Kostant weight if and only if $p_{\lambda,\mu}(q) = q^{l(\lambda,\mu)}$ for all $\lambda \leq \mu$.*

Proof. This is a direct application of Theorem 5.9. Note that first of all

$$\sum_{k\geq 0} \dim \operatorname{Ext}^k_K(V(\lambda), L(\mu)) = p_{\lambda,\mu}(1) \in \mathbb{Z} \cup \{\infty\}$$

if $\lambda \leq \mu$ and 0 otherwise. So we only have to consider the case $\lambda \leq \mu$. There we observe that the term $q^{l(\lambda,\mu)}$ always occurs (just take the 0-labelling). Thus μ is a Deligne–Kostant weight if and only if $p_{\lambda,\mu}(q) = q^{l(\lambda,\mu)}$ for all $\lambda \leq \mu$. □

We can also directly characterize Deligne–Kostant weights in terms of the weight diagram.

Definition 5.12. Let χ be a finite sequence of $\wedge$'s and $\vee$'s. A weight diagram λ is called χ-avoiding if χ does not occur as a subsequence of λ.

Proposition 5.13. *For a Deligne weight diagram μ the following are equivalent:*

(i) *μ is a Deligne–Kostant weight.*

(ii) *μ is $\vee\wedge$-avoiding, $\wedge\wedge$-avoiding and $\diamond\wedge$-avoiding.*

(iii) *$\overline{\mu}$ contains no caps.*

(iv) *μ_δ is maximal in the Bruhat order from Definition 2.4.*

Proof. That (ii) is equivalent to (iii) follows directly from Definition 2.10. The equivalence of (ii) and (iv) is a direct consequence of Definition 2.4. If there are no caps in $\overline{\mu}$, we only have the 0-labelling, so (iii) directly implies (i).
Lastly it suffices to construct a nontrivial labelling for some $\lambda \leq \mu$ in case of a subsequence $\vee\wedge$ or $\wedge\wedge$. For this we may assume that μ does not contain any $\circ$ or $\times$. If μ contains $\vee\wedge$, we choose $i < i+1 < j < k$ labelled $\vee \wedge \vee\vee$ (this is always possible as μ is admissible) and define λ to be obtained from μ by replacing the $\vee$'s at positions j and k by $\wedge$'s. Then we have $l_i(\lambda, \mu) = 2$ and $\overline{\mu}$ has a small cap at positions i, $i+1$. Labelling this cap 2 and all other ones by 0 gives a nontrivial λ-labelling of $\mu\overline{\mu}$.
If μ however contains $\wedge\wedge$ and no $\vee\wedge$, the first two symbols are necessarily $\wedge\wedge$. Let λ be the weight obtained from μ by replacing two $\vee$'s from μ by $\wedge$'s. Then $\lambda \leq \mu$ and $l_0(\lambda, \mu) = 2$. Labelling the small dotted cup of μ coming from $\wedge\wedge$ by 2 and all other ones by 0 we obtain a nontrivial λ-labelling of $\mu\overline{\mu}$. □

5.2 Characterizations of direct summands $L(\lambda)$ in $V^{\otimes d}$

proposition B in the introduction gives quite different characterizations of those weights λ such that $L(\lambda)$ appears as a direct summand of some $V^{\otimes d} \otimes (V^*)^{\otimes d'}$ for $\mathfrak{gl}(m|n)$. The aim of this section is to prove a similar statement for $\mathrm{OSp}(r|2n)$. So we

are interested in the cases such that $\mathbb{F}\,\mathrm{R}_\delta(\mu) \cong L(\lambda, \varepsilon)$ for some partition μ and some $(\lambda, \varepsilon) \in s\Gamma_\delta(m, n)$.
This gives us two different viewpoints to characterize these summands. For example we could classify those Deligne weight diagrams μ such that $\mathbb{F}\,\mathrm{R}_\delta(\mu)$ is irreducible, or we could classify those pairs (λ, ε) such that $L(\lambda, \varepsilon)$ appears as a direct summand in some $V^{\otimes d}$.
For the first point of view we get the following characterization

Corollary 5.14. *For a Deligne weight diagram μ_δ the following are equivalent.*

- $\mathbb{F}\,\mathrm{R}_\delta(\mu)$ *is irreducible.*
- $\overline{\mu_\delta}$ *contains no caps.*
- μ_δ *is $\vee\wedge$, $\wedge\wedge$ and $\diamond\wedge$-avoiding.*
- μ_δ *is maximal in the Bruhat order from Definition 2.4.*
- μ_δ *is a Deligne–Kostant weight.*
- $p_{\lambda_\delta,\mu_\delta}(q)$ *is a monomial for all Deligne weight diagrams $\lambda_\delta \leq \mu_\delta$.*

Proof. The equivalence of the first two properties is Proposition 5.2. The middle four are equivalent by Proposition 5.13 and the last two by Proposition 5.11. ☐

The main idea for the classification of the (λ, ε) is to compute $\mu_\delta^\dagger$ of a Deligne weight diagram μ_δ with irreducible $\mathbb{F}\,\mathrm{R}_\delta(\mu_\delta)$. This will give the highest weights of the irreducible indecomposable summands.
For this we first introduce the sign of a weight diagram.

Definition 5.15. Suppose that δ is odd. For a weight diagram λ of hook partition type we define $\mathrm{sgn}(\lambda)$ as follows: For each $\circ$ and $\times$ appearing in λ we count the number of symbols $\vee$ and $\wedge$ to the left of it and denote their sum X. We set $\mathrm{sgn}(\lambda) := (-1)^{X + \#\vee(\lambda)}$. If δ is even, we set $\mathrm{sgn}(\lambda) := +$ for a weight diagram λ of hook partition type.

This definition allows us to characterize explicitly the irreducible direct summands in terms of weight diagrams.

Theorem 5.16. *For $(\lambda, \varepsilon) \in X^+(\mathrm{OSp}(r|2n))$, the corresponding irreducible module $L(\lambda, \varepsilon)$ is a direct summand in some $V^{\otimes d}$ if and only if λ is typical, or it is $?\vee$ avoiding for $? \in \{\diamond, \vee, \wedge\}$ and $\varepsilon = \mathrm{sgn}(\lambda^\infty)$.*

Proof. Note that every such $L(\lambda, \varepsilon)$ is necessarily isomorphic to some $\mathbb{F}\,\mathrm{R}(\mu_\delta)$ and thus $(\lambda, \varepsilon) = \mu_\delta^\dagger$ by Theorem 2.31. Now observe that $\mathbb{F}\,\mathrm{R}(\mu_\delta)$ is irreducible if and only if there are no caps in $\overline{\mu_\delta}$ by Proposition 5.2. It is easy to check using the definition of the associated cap diagram that $\overline{\mu_\delta}$ contains no caps if and only if μ_δ is $\wedge\wedge$-, $\vee\wedge$- and $\diamond\wedge$-avoiding (see also Corollary 5.14). When removing all $\circ$'s and $\times$'s μ_δ has to look like $\vee \vee \vee \dots$, $\wedge \vee \vee \dots$ or $\diamond \vee \vee \dots$. Then one only needs to determine $\mu_\delta^\dagger$. We remark here that the case distinction comes from the distinction between projective and nonprojective weight diagrams (i.e. the typical and atypical case). For the sign in the odd case, note that $\mathrm{sgn}(\lambda^\infty)$ is the same as $(-1)^{|\mu|}$. $\square$

Translating the definition of [GH21, Definition 3.5.3] to the combinatorics of Ehrig and Stroppel which we are using here, we obtain the following definition of *Kostant weight*.

Definition 5.17. We call $\lambda \in X^+(\mathfrak{osp}(r|2n))$ a *Kostant weight* if the associated weight diagram λ^∞ is $\vee$-avoiding.

Remark 5.18. Note that every pair (λ, ε) for a typical highest weight λ (which means $\min(\#\circ(\lambda^\infty), \#\times(\lambda^\infty)) = \min(n, m)$) is automatically $\vee$ avoiding by Remark 2.28. On the other hand if we have $\min(\#\circ(\lambda^\infty), \#\times(\lambda^\infty)) < \min(n, m)$ (the atypical case), $L(\lambda, \varepsilon)$ (for $\varepsilon = \mathrm{sgn}(\lambda^\infty)$) appears as a direct summand if and only if it is $\vee$-avoiding except for maybe the first position.

In the following paragraph we are going to define a twist, which turns the first symbol different from $\circ$ or $\times$ upside down. We will use this to relate Kostant weights in the sense of [GH21] with the weights λ such that $L(\lambda, \mathrm{sgn}(\lambda^\infty))$ appears as a direct summand in $V^{\otimes d}$.

Given a super weight diagram λ we can look at the leftmost position where a $\diamond$, $\vee$ or $\wedge$ occurs. We denote the weight diagram which is obtained by turning this symbol upside down by $\lambda^\square$. Comparing $\underline{\lambda}$ and $\underline{\lambda^\square}$, this means that they agree if $\diamond$ is present and that we change the parity of dots on the leftmost component otherwise. Thus if $\underline{\lambda}\gamma\overline{\mu}$ is an oriented circle diagram, so is $\underline{\lambda^\square}\gamma^\square\overline{\mu^\square}$, and this amounts to an isomorphism $\square \colon e\tilde{K}e \to e\tilde{K}e$. Hence by Theorem 4.4 we get a self-equivalence $\square \colon \mathcal{F} \to \mathcal{F}$. We define for $(\lambda, \varepsilon) \in X^+(\mathrm{OSp}(r|2n))$ the hook weight diagram $(\lambda, \varepsilon)^\square = (\lambda^\square, \varepsilon^\square)$ by first taking the associated super weight diagram $\lambda_\varepsilon^{\oslash}$, applying $\square$ and going back to hook weight diagrams.

The map $\square$ can also be defined on the supergroup side. Every block of $\mathrm{OSp}(r|2n)$ is equivalent to the principal block of $\mathrm{OSp}(2k+1|2k)$ or $\mathrm{OSp}(2k|2k)$. This can be achieved via transporting this block through Ψ from Theorem 4.4 to $e\tilde{K}e$-mod. There we can remove all $\circ$'s and $\times$'s as they play no role in the module structure and transport back (this is very similar to [GS10, Theorem 2] which relates blocks of $\mathfrak{osp}(r|2n)$ to principal blocks in $\mathfrak{osp}(2k+1|2k)$, $\mathfrak{osp}(2k|2k)$ or $\mathfrak{osp}(2k+2|2k)$).

For the principal block of $\mathrm{OSp}(2k+1|2k)$ the map $\square$ is then just given by $\theta_0 = \mathrm{pr}_{\chi_0}(\_ \otimes V)$ and for the one of $\mathrm{OSp}(2k|2k)$ this is just the identity (all blocks containing a $\diamond$ are equivalent to this one and turning $\diamond$ upside down changes nothing).

In general $\square$ is defined by identifying a block with the corresponding principal block under the identification above, applying the explicit description of $\square$ there and transferring back to the original block.

Putting everything together we obtain the following corollary:

Corollary 5.19. *For $\lambda \in X^+(\mathfrak{osp}(r|2n))$ the following are equivalent:*

- *$L(\lambda, \varepsilon)$ is a direct summand of some $V^{\otimes d}$, where $\varepsilon = \mathrm{sgn}(\lambda^\infty)$ if λ is atypical and $\varepsilon \in \{\pm\}$ otherwise.*
- *λ or $\lambda^\square$ is a Kostant weight.*

And if r is odd or $\mathrm{at}(\lambda) > 1$ this is equivalent to

- *$L(\lambda, \varepsilon)$ or $L(\lambda^\square, \varepsilon^\square)$ satisfies the Kac–Wakimoto condition.*

Remark 5.20. In [GH21, Remark 3.5.4], Gorelik and Heidersdorf stated that a weight λ for odd r or λ of atpyicality > 1 satisfies the Kac–Wakimoto conditions if and only if it is a Kostant weight in their sense. Here a weight λ satisfies the Kac–Wakimoto condition if it is the highest weight of some irreducible module L with respect to a base $\Sigma \supseteq S$ of simple roots, such that S consists out of exactly $\mathrm{at}(\lambda)$ mutually orthogonal isotropic roots and $(S, S) = (S, \lambda + \rho) = 0$.
In [CK17, Theorem 5.2], Cheng and Kwon proved that the Kac–Wakimoto conditions imply the Kac–Wakimoto character formula, i.e.

$$Re^\rho \mathrm{ch} L(\lambda) = j^{-1} \sum_{w \in W} \mathrm{sgn}(w) w \Big(\frac{e^{\lambda+\rho}}{\prod_{\beta \in S}(1 + e^{-\beta})} \Big) \tag{5.1}$$

where R denotes the Weyl denominator $\frac{\prod_{\alpha \in \Phi_0^+}(e^{\alpha/2} - e^{-\alpha/2})}{\prod_{\beta \in \Phi_1^+}(e^{\beta/2} - e^{-\beta/2})}$, W denotes the Weyl group of $\mathfrak{osp}(r|2n)$ (which is the Weyl group of $\mathfrak{so}(r) \oplus \mathfrak{sp}(2n)$), and j is some scalar. For details see [CK17] or [GH21].

6 Koszulity

Definition 6.1. A positively graded algebra $A = \bigoplus_{j\geq 0} A_j$ such that A_0 is semisimple is called a *Koszul algebra* if every irreducible direct summand L of A_0 considered as a graded left A-module admits a linear projective resolution that is a projective resolution $\ldots \longrightarrow P^2 \longrightarrow P^1 \longrightarrow P^0 \longrightarrow\!\!\!\!\rightarrow L$ such that P^i is generated by its degree i component (i.e. $P^i = AP^i_i$).

In this chapter we are going to prove that the algebra K is Koszul. The Khovanov algebras of type A respectively type D are known to be Koszul. For type A we refer for example to [BS10, Section 5] and for type D to [Sey17, Section 4] for direct proofs of this. The results could also be obtained via identifying it with (parabolic) blocks of category $\mathcal{O}$ (see [ES16a, Theorem 9.1]), where the result is known (see e.g. [BGS96]). However, in our setting there is not (yet) an equivalence to some category $\mathcal{O}$ for which we could translate the result. Thus we have to prove the statement directly by relating it to the type D case. We will prove a more general result and for this we introduce the following definition.

Definition 6.2. Let A be a graded algebra such that the category of locally finite dimensional graded modules is upper finite based highest weight with labelling set Λ. For λ and $\mu \in \Lambda$ we define $\operatorname{diff}(\lambda, \mu)$ to be the minimum number m such that there is a sequence $\mu = \nu_0, \nu_1, \ldots, \nu_m = \lambda$ together with nonzero degree 1 elements a_i in $e_{\nu_i} A e_{\nu_{i+1}}$.

Remark 6.3. This definition is a slight generalization of the idea that $e_\mu A e_\lambda \subseteq A_{\geq \operatorname{diff}(\lambda,\mu)}$.
The main idea is that a map $P(\nu_i)\langle 1\rangle \to P(\nu_{i+1})$ is given by right multiplication with a degree 1 element in $e_{\nu_i} A e_{\nu_{i+1}}$. Thus in a linear projective resolution of $L(\lambda)$ (if it exists) $P(\mu)$ cannot appear in homological degrees $\leq \operatorname{diff}(\lambda, \mu)$.

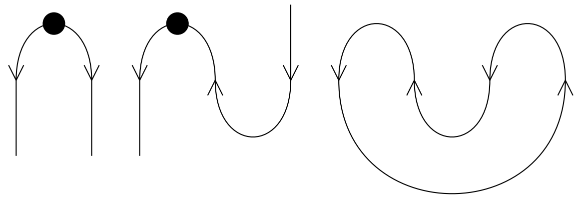
Definition 6.4. Let A be a positively graded algebra such that the category of locally finite dimensional graded modules is upper finite highest weight with labelling set Λ. A subset $\Gamma \subseteq \Lambda$ is called an upper set if $\lambda \geq \mu$ and $\mu \in \Gamma$ imply $\lambda \in \Gamma$.
Let now $\Lambda_0 \subset \Lambda_1 \subset \ldots$ be an increasing sequence of finite upper sets such that $\bigcup_{i\in\mathbb{N}} \Lambda_i = \Lambda$. We call A *quasi-finite (with respect to* $\Lambda_0 \subset \Lambda_1 \subset \ldots$*)* if for every $\lambda \in \Lambda$ and $k \geq 0$ there exists $L \in \mathbb{N}$ such that $\operatorname{diff}(\lambda, \mu) \geq k$ for every $\mu \notin \Lambda_L$.
If A is quasi finite and the algebras $A_n := e_n A e_n$, where $e_n = \sum_{\lambda\in\Lambda_n} e_\lambda$, are Koszul for all n, we call A *quasi-finite Koszul.*

We fix $\delta \in \mathbb{Z}$. Then the category $\operatorname{mod}_{lf}(K)$ of locally finite dimensional modules over the Khovanov algebra K of type B is upper finite highest weight with labelling set

Λ_δ in the sense of [BS21] by Theorem 3.6. We define the set $\Lambda_i \subset \Lambda_\delta$ for $i \in \mathbb{N}_0$ to be all those Deligne weight diagrams λ such that $\lambda(j) = \vee$ for all $i \leq j \in L$. These are clearly finite and an upper set by Definition 2.4. All associated cup diagrams have only undotted rays at position $\geq i$ and thus these actually play no role for the multiplicative structure. Thus the algebra e_iKe_i for $e_i = \sum_{\lambda\in\Lambda_i} e_\lambda$ can be identified with weight diagrams only having length i. These algebras were defined and studied in detail in [ES16a]. Ehrig and Stroppel proved in [ES16a, Theorem 9.1] that blocks of these algebras are equivalent to $\mathcal{O}_0^{\mathfrak{p}}(\mathfrak{so}(2k))$, the principal block of parabolic category $\mathcal{O}$ for the Lie algebra $\mathfrak{so}(2k)$. That $\mathcal{O}_0^{\mathfrak{p}}(\mathfrak{so}(2k))$ is Koszul is known, see e.g. [BGS96]. In [Sey17] it is proven that the algebras e_nKe_n are Koszul directly without identifying it with category $\mathcal{O}$. The following Lemma shows that K fits into the setting of Definition 6.4.

Lemma 6.5. *The algebra K is quasi finite Koszul with respect to $\Lambda_0 \subset \Lambda_1 \subset \cdots \subset \Lambda_\delta$.*

Proof. In the previous paragraph, we saw that the Λ_i are finite upper sets, their union is clearly Λ_δ and we saw that e_iKe_i for $i \in \mathbb{N}_0$ is Koszul. Thus it only remains to show that K is quasi finite, i.e. that for every $\lambda \in \Lambda_\delta$ and $k \geq 0$ there exists an L such that $\mathrm{diff}(\lambda, \mu) \geq k$ for all $\mu \notin \Lambda_L$.
If n_λ is the smallest integer such that $\lambda \in \Lambda_{n_\lambda}$, we can choose $L = n_\lambda + 2k$. Then $\mathrm{diff}(\lambda, \mu) \geq k$ follows ultimately because every oriented circle diagram $\underline{\nu}\nu''\overline{\nu'}$ of degree one consists of (apart from degree 0 components) exactly one of these components (or their mirrors or a different number of dots in the third picture):



In the first case this means that ν and ν' agree up to $\vee\vee$ and $\wedge\wedge$ at the positions coming from the degree one component. In the third case the rightmost $\wedge$ appears in the same positions in ν and ν' by Definition 3.3 as every circle is oriented anticlockwise In the second case we have $\vee\vee\wedge$ in contrast to $\wedge\wedge\vee$, which means that the rightmost $\wedge$ moves at most one position to the right. In the first case ν' is obtained from ν by changing $\vee\vee$ into $\wedge\wedge$ at the positions of the degree one component. Note that every component that lies inside or to the left of this line has to be a small anticlockwise circle. This means that the right vertex of each of these circles has to be a $\wedge$ in ν and ν'. Hence, the positions of the rightmost $\wedge$'s in ν and ν' differ by at most 2. Thus we see that with every degree one element, we can move the rightmost up at most two places to the right. But any $\mu \notin \Lambda_L$ for $L = n_\lambda + 2k$ has a $\wedge$ to the right of the L-th position.
Hence K is quasi-finite Koszul. $\square$

Theorem 6.6. *Let A be a quasi-finite Koszul algebra. Then A is Koszul.*

Before we give the proof below, we first outline the idea. If $\lambda \in \Lambda_i$, we have the simple object ${}^nL(\lambda)$ inside e_nAe_n-mod for all $n \geq i$. In each of those there exists a linear projective resolution of ${}^nL(\lambda)$. Using induction functors $e_mAe_n \otimes_{e_nAe_n} \_$ we can relate all these linear projective resolutions. Ultimately we construct a directed sequence of chain complexes in $\mathrm{mod}_{lf}(A)$ (the induction functors are not necessarily exact) and take its colimit. We then prove that this colimit is in fact a linear projective resolution for $L(\lambda)$ in $\mathrm{mod}_{lf}(A)$. That the colimit gives actually a resolution uses that A is quasi finite. We use this assumption to show that all the structure morphisms in each homological degree turn eventually into isomorphisms and thus for exactness it suffices to look at the restriction to e_NAe_N for big enough $N >> 0$. But there we know that it is exact.

Proof of Theorem 6.6. For any weight λ we write $\lambda \preceq n$ if $\lambda \in \Lambda_n$ and by n_λ we denote the smallest n such that $\lambda \preceq n$. For $\lambda \preceq n$ we write ${}^nP(\lambda)$ for the indecomposable projective associated to λ in A_n-mod and we use $P(\lambda)$ to denote the indecomposable projective associated to λ in $\mathrm{mod}_{lf}(A)$. We use similar notation for the standard and irreducible modules.
By $j^n\colon \mathrm{mod}_{lf}(A) \to A_n$-mod we denote the Serre quotient functor. This admits a left adjoint $j^n_!\colon A_n\text{-mod} \to \mathrm{mod}_{lf}(A)$ which is given by $Ae_n \otimes_{A_n} \_$ and a right adjoint $j^{n,*}\colon A_n\text{-mod} \to \mathrm{mod}_{lf}(A)$ given by $\mathrm{Hom}_{A_n}(e_nK, \_)$.
Furthermore for $n \leq m$ we have Serre quotient functors $j^n_m\colon A_m\text{-mod} \to A_n\text{-mod}$ and the corresponding left respectively right adjoints $j^n_{m,!} := e_mAe_n \otimes_{A_n} \_$ and $j^{n,*}_m := \mathrm{Hom}_{A_n}(e_nAe_m, \_)$. The left adjoint is called *standardization functor* and the right adjoint is called *costandardization functor* These satisfy the following commutation relations ($n < m < k$):

$$\begin{array}{rr} j^n_m \circ j^m_k = j^n_k & j^n_m \circ j^m = j^n \\ j^m_{k,!} \circ j^n_{m,!} = j^n_{k,!} & j^m_! \circ j^n_{m,!} = j^n_! \\ j^{m,*}_k \circ j^{n,*}_m = j^{n,*}_k & j^{m,*} \circ j^{n,*}_m = j^{n,*} \end{array} \tag{6.1}$$

Additionally for $\lambda \prec m$ and $n < m$ the functor j^n_m sends ${}^mP(\lambda)$, ${}^mV(\lambda)$ respectively ${}^mL(\lambda)$ to ${}^nP(\lambda)$, ${}^nV(\lambda)$ respectively ${}^nL(\lambda)$ if $\lambda \preceq n$ and ${}^mV(\lambda)$ and ${}^mL(\lambda)$ to 0 otherwise. The same holds if we leave out the m.
The standardization functor $j^n_{m,!}$ sends ${}^nP(\lambda)$ to ${}^mP(\lambda)$ for $\lambda \prec n$ and we have that $j^n_m \circ j^n_{m,!} = \mathrm{id}$. Again the same holds true if we leave out the m.
By assumption the categories A_m-mod are finite highest weight categories and Koszul, so for $\lambda \preceq m$ we have a linear projective resolution of ${}^mL(\lambda)$ in A_m-mod, which is unique up to isomorphism. We denote this by (where $P^k_m(\lambda)$ is generated in degree k)

$$P^\bullet_m(\lambda)\colon \; \cdots \to P^k_m(\lambda) \to P^{k-1}_m(\lambda) \to \cdots \to P^1_m(\lambda) \to P^0_m(\lambda) \to {}^mL(\lambda).$$

Now if $\lambda \preceq n$ for some $n < m$ by exactness of j^n_m we get an exact sequence

$$\cdots \to j^n_m P^k_m(\lambda) \to j^n_m P^{k-1}_m(\lambda) \to \cdots \to j^n_m P^1_m(\lambda) \to j^n_m P^0_m(\lambda) \to {}^nL(\lambda).$$

Therefore by [Wei94, Thm. 2.2.6] there is a map (unique up to homotopy)

$$\iota_m^n(\lambda)\colon P_n^\bullet(\lambda) \to j_m^n P_m^\bullet(\lambda)$$

and we can choose these maps $\iota_m^n(\lambda)$ such that for $n \leq m \leq k$ the following diagram commutes

$$\begin{tikzcd}
P_n^\bullet(\lambda) \arrow[r, "\iota_m^n(\lambda)"] \arrow[dr, "\iota_k^n(\lambda)"'] & j_m^n P_m^\bullet(\lambda) \arrow[d, "j_m^n(\iota_k^m(\lambda))"] \\
& j_k^n P_k^\bullet(\lambda) = j_m^n j_k^m P_k^\bullet(\lambda)
\end{tikzcd}.$$

Note that for every $r \geq 0$ there is an $L \in \mathbb{N}$ such that for $\mu \not\preceq L$ but $\mu \preceq m$ the module ${}^mP(\mu)$ can only appear in homological degrees r and higher by assumption on $\operatorname{diff}(\lambda, \mu)$. But this means that if $n > L$ and $r' < r$, any indecomposable summand of $P_m^{r'}(\lambda)$ has to be of the form ${}^mP(\mu)$ for some $\mu \preceq L$ and similarly any indecomposable summand of $P_n^{r'}(\lambda)$ has to be of the form ${}^nP(\mu)$. But if $\mu \preceq L$ (and hence $\mu \preceq n$) we have that $j_m^n {}^mP(\mu) = {}^nP(\mu)$. Therefore

$$j_m^n P_m^{r-1}(\lambda) \longrightarrow \ldots \longrightarrow j_m^n P_m^1(\lambda) \longrightarrow j_m^n P_m^0(\lambda) \longrightarrow {}^nL(\lambda)$$

is a beginning of a linear projective resolution of ${}^nL(\lambda)$ and by uniqueness of a linear projective resolution, $\iota_m^n(\lambda)$ has to be an isomorphism in degrees $< r$.
Using the unit η_m^n of the adjunction $j_{m,!}^n \vdash j_m^n$ we get a morphism $g_m^n(\lambda)$ of resolutions as the composition

$$j_{m,!}^n P_n^\bullet(\lambda) \xrightarrow{j_{m,!}^n(\iota_m^n(\lambda))} j_{m,!}^n j_m^n P_m^\bullet(\lambda) \xrightarrow{(\eta_m^n)_{P_m^\bullet(\lambda)}} P_m^\bullet(\lambda).$$

We observe that for $\mu \preceq n$ we have $j_{m,!}^n j_m^n {}^mP(\mu) = {}^mP(\mu)$ so $g_m^n(\lambda)$ is for $m \geq n > L$ an isomorphism in degrees $< r$. Additionally, we look at the diagram

$$\begin{tikzcd}
j_{k,!}^n P_n^\bullet(\lambda) = j_{k,!}^m j_{m,!}^n P_n^\bullet(\lambda) \arrow[r, "j_{k,!}^n(\iota_m^n(\lambda))"] \arrow[d, "j_{k,!}^n(\iota_k^n(\lambda))"] & j_{k,!}^m j_{m,!}^n j_m^n P_m^\bullet(\lambda) \arrow[r, "j_{k,!}^m(\eta_m^n)"] \arrow[d, "j_{k,!}^n j_m^n(\iota_k^m(\lambda))"] & j_{k,!}^m P_m^\bullet(\lambda) \arrow[d, "j_{k,!}^m(\iota_k^m(\lambda))"] \\
j_{k,!}^n j_k^n P_k^\bullet(\lambda) \arrow[r, "="] \arrow[dr, "\eta_k^n"'] & j_{k,!}^m j_{m,!}^n j_m^n j_k^m P_k^\bullet(\lambda) \arrow[r, "j_{k,!}^m(\eta_m^n)"] & j_{k,!}^m j_k^m P_k^\bullet(\lambda) \arrow[dl, "\eta_k^m"] \\
& P_k^\bullet(\lambda) &
\end{tikzcd}.$$

In order to make the diagram a bit more clear we suppressed for the unit η_m^n the index describing the object. By our choice of the maps $\iota_m^n(\lambda)$ the left square commutes. By naturality of η_m^n the right square commutes as well. The lower triangle commutes as the adjunction $j_{k,!}^n \vdash j_k^n$ is given as the composition of the adjunctions $j_{m,!}^n \vdash j_m^n$ and $j_{k,!}^m \vdash j_k^m$. Now we observe that we have two possibilities to go from $j_{k,!}^n P_n^\bullet(\lambda)$ to $P_k^\bullet(\lambda)$. One is given by $g_k^n(\lambda)$ and the other one by $g_k^m(\lambda) \circ j_{k,!}^m g_m^n(\lambda)$. So we obtain $g_k^n(\lambda) = g_k^m(\lambda) \circ j_{k,!}^m g_m^n(\lambda)$.

By defining $f^n_m(\lambda) := j^m_! g^n_m(\lambda)$ we get morphisms of resolutions of A-modules

$$j^n_! P^\bullet_n(\lambda) \xrightarrow{f^n_m(\lambda)} j^m_! P^\bullet_m(\lambda).$$

From $g^n_k(\lambda) = g^m_k(\lambda) \circ j^m_{k,!} g^n_m(\lambda)$ we get $f^n_m \circ f^m_k = f^n_k$ for $n \leq m \leq k$. We claim that the direct limit of this sequence is a linear projective resolution of $L(\lambda)$ as an A-module. First note that for $m, n > L$ the morphism $f^n_m(\lambda)$ is an isomorphism in homological degrees $< r$ (because $g^n_m(\lambda)$ is one in these degrees). This means that in homological degrees $r' < r$ the direct limit is actually a projective module which is generated in degree r'. So in each homological degree, we have by construction a finite direct sum of indecomposable projectives, and so this is a chain complex of locally finite dimensional modules.
Let us take a closer look at this resulting chain complex

$$\cdots \to \varinjlim j^n_! P^s_n(\lambda) \to \cdots \to \varinjlim j^n_! P^1_n(\lambda) \to \varinjlim j^n_! P^0_n(\lambda) \to \varinjlim j^n_! L(\lambda) \to 0.$$

An object M in $\mathrm{mod}_{lf}(A)$ is zero if and only if $j^m M = 0$ for all $m \geq 0$. As taking homology commutes with exact functors, we observe that the above chain complex is exact if and only if $j^m \varinjlim j^n_! P^\bullet_n$ is exact for all $m \geq 0$.
The restriction functor j^m admits a left adjoint $(j^{m,*})$ and so it commutes with direct limits. Hence we have

$$j^m \varinjlim j^n_! P^s_n(\lambda) = \varinjlim j^m j^n_! P^s_n(\lambda).$$

Using the commutations relations (6.1) we see that

$$j^m j^n_! = \begin{cases} j^n_{m,!} & \text{if } n < m, \\ \mathrm{id} & \text{if } n = m \text{ and} \\ j^m_n & \text{if } n > m, \end{cases}$$

and thus the right hand side of the above equation is the direct limit of the complex

$$j^0_{m,!} P^s_0(\lambda) \to \cdots \to j^{m-1}_{m,!} P^s_{m-1} \to P^s_m(\lambda) \to j^m_{m+1} P^s_{m+1}(\lambda) \to j^m_{m+2} P^s_{m+2}(\lambda) \to \ldots.$$

Now by our previous argument there exists for given r some integer L such that for all $l > L$ the homomorphisms $j^m_l P^s_l(\lambda) \to j^m_{l+1} P^s_{l+1}(\lambda)$ are isomorphisms for $s < r$. Thus the first r terms in $j^m \varinjlim j^n_! P^\bullet_n(\lambda)$ agree with the first r terms in $j^m_l P^\bullet_l(\lambda)$ and thus are exact. As r was chosen arbitrarily we see that $j^m \varinjlim j^n_! P^\bullet_n(\lambda)$ is exact for every m and hence also $\varinjlim j^n_! P^\bullet_n(\lambda)$ which is then a linear projective resolution of $L(\lambda)$. Hence A is Koszul. □

Corollary 6.7. *The algebra K is Koszul.*

Proof. The algebra K is quasi-finite Koszul by Lemma 6.5 and thus Koszul by Theorem 6.6. □

We have now proven that the Khovanov algebra of type B is Koszul. Similar statements are known to be true for the Khovanov algebras of type A and D.
Ultimately we would also like to prove that $e\tilde{K}e$ is a Koszul algebra as this then show that the category of finite dimensional $\mathrm{OSp}(r|2n)$-representations is Koszul as well by Theorem 4.4. However, the tools that exist so far do not seem sufficient to prove this statement. The approach that was taken in this thesis also does not seem to provide new information on the Koszulity of $\mathrm{OSp}(r|2n)$.
So for future research it would be very interesting to try and tackle this problem from a new point of view, establishing the Koszulity of $\mathrm{OSp}(r|2n)$ and develop on the fly new tools for future work.

7 Explicit examples

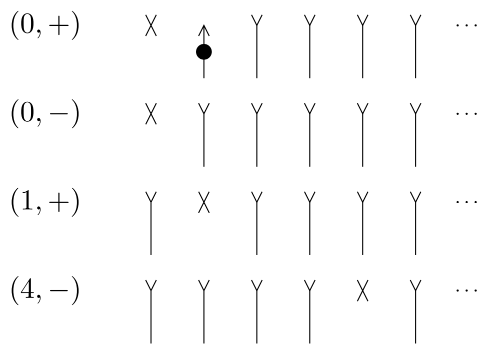
The results on projective modules could also be deduced using solely the diagrammatic description from [ES21, Theorem 8.10] and the computations for the irreducible module would have been feasible by hand as well. For the cases of OSp(1|2), OSp(3|2) and OSp(2|2) every indecomposable summand is either projective or irreducible, and so we actually do not need Theorem 4.4 and the Khovanov algebra of type B for this. However, for bigger examples (which are then too big to compute by hand) we would also gain explicit results for the nonirreducible nonprojective $\mathbb{F}\,\mathrm{R}_\delta(\mu)$. We included the small examples anyway to give the reader the idea how everything fits together.

7.1 The semisimple case: OSp(1|2)

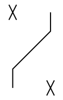
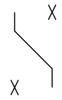
The category of finite dimensional representations of OSp(1|2) is the easiest case one can consider as it is semisimple, but even in this case it illustrates the power of the combinatorial nature of Khovanov's arc algebra of type B. We can actually obtain closed formulas for the multiplicities of a simple module in $V^{\otimes d}$.
Even though the results have been previously known, for example using the correspondence between $\mathfrak{osp}(1|2)$ and $\mathfrak{so}(3)$ from [RS82], we think it is nonetheless interesting in this rather easy example to see an application of the Khovanov algebra of type B.
By Lemma 1.7 and Proposition 1.18 we know that the finite dimensional representations of OSp(1|2) are labelled by (n, ε) with $n \in \mathbb{N}_0$ and $\varepsilon \in \{\pm\}$.

7.1.1 Translating to Khovanov's arc algebra of type D

Translating (n, ε) into a super weight diagram (using Definition 2.33), we obtain a weight diagram consisting of a $\times$ at position $n + \frac{1}{2}$ and a $\vee$ at every other position except for maybe the first free one. There we put in case n is even a $\wedge$ if $\varepsilon = +$ and a $\vee$ if $\varepsilon = -$. If n is odd, we just reverse the previous assignment, for example:



As OSp(1|2) is semisimple (or equivalently as $\min(m,n)=0$) there are no cups and caps involved. If we consider the associated cup respectively cap diagrams to these weights, we obtain diagrams consisting of one free vertex and apart from that only lines where the leftmost one might be dotted. From this it is fairly easy to see that the only circle diagrams we can build are the e_λ, where λ is one of the super weight diagrams from the previous paragraph, and furthermore we cannot have any nuclear diagrams as we have no cups or caps. Then by Theorem 4.4 we know that the category of finite dimensional OSp(1|2)-modules is equivalent to $e\tilde{K}e$-mod.
In order to later analyze the effect of $\_\otimes V$ we first take a look at the geometric bimodules $K^t_{\Lambda\Gamma}$. Now observe that $G^t_{\Lambda\Gamma}P(\gamma)$ will be 0 whenever t contains a cup or a cap. Thus the only relevant t's look locally like

θ_{-i}:  θ_i:  θ_0: 

where the i means that it involves the positions $i+\frac{1}{2}$ and $i-\frac{1}{2}$. The last picture can only be present on the vertex $\frac{1}{2}$. Apart from these involved vertices t consists only of straight lines. The geometric bimodules $K^t_{\Lambda\Gamma}$ are thus also one-dimensional and by Theorem 3.53 using $L(\mu)=P(\mu)$ we have that $G^t_{\Lambda\Gamma}L(\gamma)=L(\lambda)$ where $\overline{\lambda}$ is the upper reduction of $t\overline{\gamma}$. In this case the upper reduction for the first two picture is obtained by swapping the $\times$ one position to the left respectively right, and in the first picture we change a $\vee$ at position $\frac{1}{2}$ into a $\wedge$ and vice versa.
Observe that for each $L(\gamma)$ we have three projective functors producing something nonzero if the $\times$ is not at position $\frac{1}{2}$ and only θ_1 if it is at position $\frac{1}{2}$.

7.1.2 Decomposition of $V^{\otimes d}$ into irreducible summands

Translating the results of the previous paragraph, we obtain for our translation functors θ_i

$$\theta_i L(n,\varepsilon)=\begin{cases} L(n+1,-\varepsilon) & \text{if } i=n+1,\\ L(n,-\varepsilon) & \text{if } i=0 \text{ and } n>0,\\ L(n-1,-\varepsilon) & \text{if } i=-n \text{ and } n>0,\\ 0 & \text{otherwise.}\end{cases} \tag{7.1}$$

With this at hand and using $\_\otimes V=\bigoplus_{i\in\mathbb{Z}}\theta_i$, we can directly write down the first decompositions of $V^{\otimes d}$ into irreducibles.

$$\begin{aligned} V^{\otimes 0} &= L(0,+)\\ V^{\otimes 1} &= L(1,-)\\ V^{\otimes 2} &= L(2,+)\oplus L(1,+)\oplus L(0,+)\\ V^{\otimes 3} &= L(3,-)\oplus L(2,-)^{\oplus 2}\oplus L(1,-)^{\oplus 3}\oplus L(0,-)^{\oplus 1}\end{aligned}$$

Now note that in $V^{\otimes d}$ the signs of all irreducible summands are the same depending on the parity of d. Moreover, the multiplicity of $L(n,\varepsilon)$ in $V^{\otimes d}$ is either 0 (if d even and $\varepsilon = -$ or d odd and $\varepsilon = +$) or it agrees with the multiplicity of $L(n)$ in $V^{\otimes d}$ as SOSp(1|2)-modules. Define $m(n,d)$ to be multiplicity of $L(n)$ in $V^{\otimes d}$, or equivalently the multiplicity of $L(n,\varepsilon)$ for the correct choice (see above) of ε. Using (7.1) we quickly get the following recurrence relations for $m(n,d)$:

$$\begin{aligned}
&m(0,0) = 1,\\
&m(0,d) = m(1,d-1),\\
&m(n,d) = m(n+1,d-1) + m(n,d-1) + m(n-1,d-1) \quad \text{if } n > 0.
\end{aligned}$$

Using some tricks and combinatorics, one gets then for $m(n,d)$ the explicit formulas

$$\begin{aligned}
m(n,d) &= \sum_{i=n}^{d} (-1)^{i+n} \tbinom{d}{i} \binom{i}{\lfloor \frac{i-n}{2} \rfloor}\\
&= \sum_{j=0}^{\frac{d-n}{2}} \frac{d!}{j!\,(n+j)!\,(d-n-2j)!} - \sum_{j=0}^{\frac{d-n-1}{2}} \frac{d!}{j!\,(n+1+j)!\,(d-n-2j-1)!}\\
&= T(n,d) - T(n+1,d)
\end{aligned}$$

where $T(n,d)$ denotes the coefficient of x^{n+d} in the expansion of $(1+x+x^2)^d$. The number $T(n,d)$ denotes also the number of possible outcomes of elections with d votes of three parties A, B and C such that B obtains n votes more than A.

7.2 The smallest nonsemisimple odd case: OSp(3|2)

As the super world is in general not semisimple, we want to include the example OSp(3|2) as well. In this case we are not able to give closed formulas, but we will present recursive ones for the computation of multiplicities of simples and of indecomposables in $V^{\otimes d}$.

7.2.1 The irreducible representations of $\mathfrak{osp}(3|2)$ and OSp(3|2)

According to (1.4) we choose the simple roots

$$\varepsilon_1 - \delta_1, \delta_1$$

and $\rho = (-\frac{1}{2} \mid \frac{1}{2})$. Let $\lambda \in \mathfrak{h}^*$ be a weight and write $\lambda + \rho = a\varepsilon_1 + b\delta_1$. Then λ is integral dominant by Lemma 1.7 if and only if $a, b \in \frac{1}{2} + \mathbb{Z}$ and either $a, b \geq \frac{1}{2}$ or $-a = b = \frac{1}{2}$. Rephrasing this means that if $\lambda = (a|b)$ is integral dominant we either have $a = b = 0$ or $a \geq 1$ and $b \geq 0$ where a and b are integers. These can be identified with $(1,1)$-hook partitions via $(a|b) \mapsto (a, 1^b)^t$.

By Proposition 1.18 the irreducible modules for OSp(3|2) are labeled by $(\lambda, \pm)$ where λ is a $(1,1)$-hook partition.

7.2.2 Translating to Khovanov's arc algebra of Type D

From highest weights to super weight diagrams

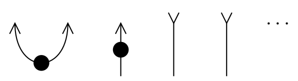
Let $\lambda = (k, 1^l)$ be a $(1,1)$-hook partition. We will distinguish three cases, the first being that $k \neq 0$ and $k - 1 \neq l$. The associated flipped weight diagram then looks like

$$\wedge \quad \cdots \quad \wedge \quad \overset{k-\frac{1}{2}}{\circ} \quad \wedge \quad \cdots \quad \wedge \quad \overset{l+\frac{1}{2}}{\times} \quad \wedge \quad \cdots .$$

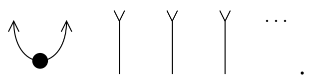
The positions of $\circ$ and $\times$ are swapped if $l < k - 1$. The corresponding super weight diagram to (λ, ε) is created by replacing all $\wedge$'s with $\vee$'s except for possibly the leftmost one. There we leave the $\vee$ if $k + l$ is even and $\varepsilon = -$, or if $k + l$ is odd and $\varepsilon = +$. In all other cases we change the leftmost vertex to $\wedge$.
In the case $\lambda = \emptyset$ we get the flipped weight diagram

$$\wedge \quad \wedge \quad \wedge \quad \wedge \quad \cdots .$$

The associated super weight diagram is



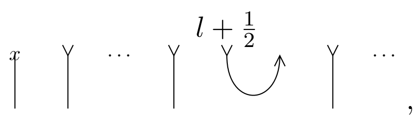
for $(\emptyset, +)$, and for $(\emptyset, -)$ we get



The last case is $k = l + 1$. In this case the flipped weight diagram is

$$\wedge \quad \cdots \quad \wedge \quad \overset{l+\frac{1}{2}}{\vee} \quad \wedge \quad \cdots .$$

The associated super weight diagrams are thus given by



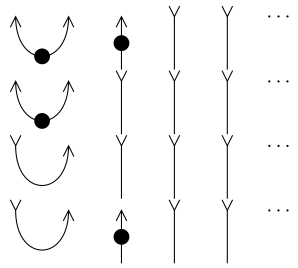
where $x = \vee$ if $\varepsilon = +$, and $x = \wedge$ if $\varepsilon = -$ (in this case there is also a dot on the leftmost ray). Note that if $l = 0$ the x appears at position $\frac{5}{2}$.
If we want to directly go from highest weights to super weight diagrams, we get the following connection:

$(0|0,+)$

$(0|0,-)$

$(1|0,+)$

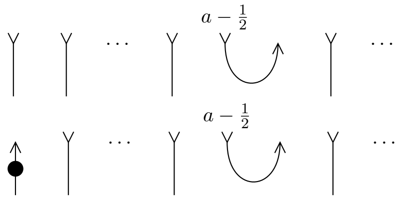
$(1|0,-)$



For $a > 1$:

$(a|a-1,+)$

$(a|a-1,-)$



For $b \neq a-1$ and $(a|b) \neq (0|0)$:
There is a $\circ$ at position $a - \frac{1}{2}$ and a $\times$ at position $b + \frac{1}{2}$. We have a dot on the leftmost ray if $a + b$ is even and $\varepsilon = +$ or if $a + b$ is odd and $\varepsilon = -$.
In all other cases we have no dot.

So our super weight diagrams either consist only of $\vee$'s and $\wedge$'s (then the cup diagram has one cup) or it has exactly one $\circ$ and one $\times$ (then the cup diagram has no cup). If λ belongs to the second group, we have $L(\lambda) = P(\lambda)$ and this forms a semisimple block. The super weight diagrams of the first form give rise to two blocks, the one containing $L(0|0,+)$ and $L(a|a-1,+)$ for $a > 0$ (where we have an even number of dots) and the one containing $L(0|0,-)$ and $L(a|a-1,-)$ for $a > 0$ (where we have an odd number of dots).
Looking at the diagrammatics, we can easily write down the socle (resp. radical) filtration of the nonirreducible projectives. To make things more clearly, we write the highest weight with the sign instead of the super weight diagram. The translation between these two can be found in the previous paragraph.

$P(0\|0,\pm)$	$P(1\|0,\pm)$	$P(2\|1,\pm)$	$P(k\|k-1,\pm)$ for $k > 2$
$L(0\|0,\pm)$	$L(1\|0,\pm)$	$L(2\|1,\pm)$	$L(k\|k-1,\pm)$
$L(2\|1,\pm)$	$L(2\|1,\pm)$	$L(0\|0,\pm)$ $L(1\|0,\pm)$ $L(3\|2,\pm)$	$L(k-1\|k-2,\pm)$ $L(k+1\|k,\pm)$
$L(0\|0,\pm)$	$L(1\|0,\pm)$	$L(2\|1,\pm)$	$L(k\|k-1,\pm)$

Translation functors

In this section we are going to give formulas for the decomposition of $V^{\otimes d}$ into indecomposable summands. As $V \otimes \_ = \bigoplus_{i\in\mathbb{Z}} \theta_i$ decomposes into translation functors, we are only going to describe the decomposition $\theta_i M$ into indecomposable summands for an indecomposable summand M of $V^{\otimes d}$. First of all note that every indecomposable summand in $V^{\otimes d}$ is actually projective or irreducible (as $m = n = 1$, see Proposition 5.2 and the comment just after Theorem 2.24). By Proposition 5.5 we know that every block contains a unique $L(\lambda, \varepsilon)$ that appears as a direct summand. In every typical block, this is also the corresponding indecomposable projective, and for the two atypical blocks we know that $V^{\otimes 0} = L(0|0,+)$ and $V^{\otimes 1} = L(1|0,-)$, it suffices to consider translation functors for $L(0|0,+)$, $L(1|0,-)$ and indecomposable projectives. From the weight diagram $\cdots$ we see that the only applicable local move (see Figure 4.1) is given by , and thus

$$\theta_i L(0|0,+) = \begin{cases} L(1|0,-) & \text{if } i = 0, \\ 0 & \text{otherwise,} \end{cases}$$

by Theorem 3.57. For $\cdots$ we find three applicable local moves namely , and , where the last two are applied at positions $\frac{1}{2}$ and $\frac{3}{2}$. Again by Theorem 3.57 we get

$$\theta_i L(1|0,-) = \begin{cases} L(0|0,+) & \text{if } i = 0, \\ L(1|1,+) = P(1|1,+) & \text{if } i = -1, \\ L(2|0,+) = P(2|0,+) & \text{if } i = 1, \\ 0 & \text{otherwise.} \end{cases}$$

So only the effects of translation functors on indecomposable projectives are left to establish, which can be deduced easily from the diagrammatic description above and Theorem 3.53.

For $(a|a-1)$ and $a > 1$ we have

$$\theta_i P(a|a-1,\varepsilon) = \begin{cases} P(a|a-1,-\varepsilon) & \text{if } i = 0, \\ P(a|a,-\varepsilon)^{\oplus 2} & \text{if } i = -a, \\ P(a+1|a-1,-\varepsilon)^{\oplus 2} & \text{if } i = a, \\ P(a+2|a,-\varepsilon) & \text{if } i = a+1, \\ P(a+1|a+1,-\varepsilon) & \text{if } i = -a-1, \\ P(a|a-2,-\varepsilon) & \text{if } i = a-1, \\ P(a-1|a-1,-\varepsilon) & \text{if } i = -a+1, \\ 0 & \text{otherwise.} \end{cases}$$

For $(0|0)$ we have

$$\theta_i P(0|0,\varepsilon) = \begin{cases} P(1|0,-\varepsilon) & \text{if } i = 0, \\ P(2|2,-\varepsilon) & \text{if } i = -2, \\ P(3|1,-\varepsilon) & \text{if } i = 2, \\ 0 & \text{otherwise.} \end{cases}$$

For $(1|0)$ we have

$$\theta_i P(1|0,\varepsilon) = \begin{cases} P(0|0,-\varepsilon) & \text{if } i = 0, \\ P(1|1,-\varepsilon)^{\oplus 2} & \text{if } i = -1, \\ P(2|0,-\varepsilon)^{\oplus 2} & \text{if } i = 1, \\ P(2|2,-\varepsilon) & \text{if } i = -2, \\ P(3|1,-\varepsilon) & \text{if } i = 2, \\ 0 & \text{otherwise.} \end{cases}$$

For $(a|b) \neq (0|0)$ and $b \neq a-1$ we have

$$\theta_i P(a|b,\varepsilon) = \begin{cases} P(a|b,-\varepsilon) & \text{if } i = 0 \text{ and } a > 1 \text{ and } b > 0, \\ P(a+1|b,-\varepsilon) & \text{if } i = a \text{ and } b \neq a, \\ P(a-1|b,-\varepsilon) & \text{if } i = -a+1 \text{ and } a > 1, \\ P(a|b-1,-\varepsilon) & \text{if } i = b, \\ P(a|b+1,-\varepsilon) & \text{if } i = -b-1 \text{ and } b+2 \neq a, \\ 0 & \text{otherwise.} \end{cases}$$

7.3 The smallest even case: $\mathrm{OSp}(2|2)$

This case is in some sense similar to analyzing $\mathrm{OSp}(3|2)$ like in the previous paragraph, but it also illustrates the usage of $\diamond$ in weight diagrams.

7.3.1 The irreducible representations of $\mathfrak{osp}(2|2)$ and $\mathrm{OSp}(2|2)$

According to (1.7) we choose the simple roots $\delta_1 - \varepsilon_1$, $\delta_1 + \varepsilon_1$ and have $\rho = (0|0)$. Let $\lambda = a\varepsilon_1 + b\delta_1 \in \mathfrak{h}^*$. By Lemma 1.7 λ is integral dominant if and only if $a, b \in \mathbb{Z}$ and either $b > 0$ or $a = b = 0$. When inducing the irreducible representation $L^{\mathfrak{g}}(a|b)$ of $\mathfrak{osp}(2|2)$ to a representation M of $\mathrm{OSp}(2|2)$ we must distinguish two cases. If $a = 0$, the representation M decomposes into $L(0|b,+)$ and $L(0|b,-)$. If $a \neq 0$, the representation M is irreducible and isomorphic to the induced one from $L^{\mathfrak{g}}(-a|b)$ and we denote it by $L((a|b)^G)$. By Proposition 1.22 these are all irreducible modules, which appear.

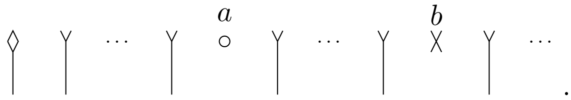
7.3.2 Translating to Khovanov's algebra of type D

From highest weights to super weight diagrams

For the study of super weight diagrams we distinguish some cases. First assume that our highest weight is denoted by $(a|b)^G$ with $a > 0$ and $a \neq b$. Then the associated flipped weight diagram looks like

$$\begin{array}{ccccccccccc} & & & & a & & & & b & & \\ \Diamond & \wedge & \cdots & \wedge & \circ & \wedge & \cdots & \wedge & \times & \wedge & \cdots \end{array}.$$

The corresponding super weight diagram is obtained from this by replacing all $\wedge$'s with $\vee$'s. These all give rise to a semisimple block in Khovanov's arc algebra.



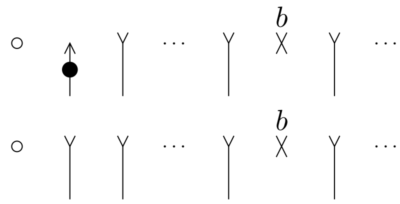
In the case of $(0|b,\varepsilon)$ with $b \neq 0$ the associated flipped diagram looks like

$$\circ \quad \wedge \quad \cdots \quad \wedge \quad \overset{b}{\times} \quad \wedge \quad \cdots .$$

When passing to super weight diagrams, we again change all $\wedge$'s to $\vee$'s except for maybe the leftmost one. This stays a $\wedge$ if $\varepsilon = +$ and gets changed to $\vee$ if $\varepsilon = -$. Similar to the previous case, these all give rise to a semisimple block.

$(0|b,+)$

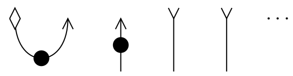
$(0|b,-)$



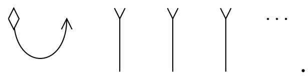
All remaining ones lie in the same block, but we distinguish whether we have $(0|0,\varepsilon)$ or $(a|a)^G$ for $a > 0$. The flipped weight diagram for $(0|0)$ is given by

$$\lozenge \quad \wedge \quad \wedge \quad \wedge \quad \cdots$$

In case of $(0|0,+)$ the super weight diagram is given by



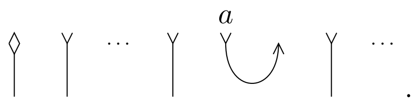
and for $(0|0,-)$ we get



For $(a|a)^G$ with $a > 0$ the flipped weight diagram is given by

$$\lozenge \quad \wedge \quad \cdots \quad \wedge \quad \overset{a}{\vee} \quad \wedge \quad \cdots$$

and the associated super weight diagram is given by



Only this last block is nonsemisimple, but looking at the diagrammatics, we can easily establish the socle (reps. radical) filtration of the indecomposable projectives. The following table presents these (we replaced the super weight diagrams by the highest weights):

$P(0\|0,\pm)$	$P((1\|1)^G)$	$P((k\|k)^G)$ for $k>1$
$L(0\|0,\pm)$	$L((1\|1)^G)$	$L((k\|k)^G)$
$L((1\|1)^G)$	$L(0\|0,+)\ L(0\|0,-)\ L((2\|2)^G)$	$L((k-1\|k-1)^G)\ L((k+1\|k+1)^G)$
$L(0\|0,\pm)$	$L((1\|1)^G)$	$L((k\|k)^G)$

Remark 7.1. By identifying $P(0|0,+)$ with $P(0|0,+)$, $P(0|0,-)$ with $P(1|0,+)$ and $P((a|a)^G)$with $P(a+1|a,+)$ we see that the principal block of OSp(2|2) and the principal block of OSp(3|2) are equivalent.

Translation functors

Similar to the argumentation for OSp(3|2) (as $m=n=1$) all indecomposable summands of $V^{\otimes d}$ are either irreducible or projective. As $V^{\otimes 0}=L(0|0,+)$ is not projective, we actually know by Proposition 5.5 and our knowledge of the blocks that this is the only summand which is not projective, so except for $\theta_i L(0|0,+)$ we only need to deal with indecomposable projectives.
For the irreducible $L(0|0,+)$, one easily sees using Theorem 3.57 that

$$\theta_i L(0|0,+) = \begin{cases} L(0|1,+) = P(0|1,+) & \text{if } i=-\frac{1}{2}, \\ 0 & \text{otherwise.} \end{cases}$$

The study of translation functors on projective objects is explicitly written down in Theorem 3.53 and we will just state the results here.

For $(1|1)^G$ we have

$$\theta_i P((1|1)^G) = \begin{cases} P(0|1,+)\oplus P(0|1,-) & \text{if } i=-\dfrac{1}{2}, \\ P((1|2)^G)^{\oplus 2} & \text{if } i=-\dfrac{3}{2}, \\ P((2|1)^G)^{\oplus 2} & \text{if } i=\dfrac{3}{2}, \\ P((2|3)^G)^{\oplus 2} & \text{if } i=-\dfrac{5}{2}, \\ P((3|2)^G)^{\oplus 2} & \text{if } i=\dfrac{5}{2}, \\ 0 & \text{otherwise,} \end{cases}$$

For $(0|0)$ we have

$$\theta_i P(0|0,\pm) = \begin{cases} P(0|1,\pm)^{\oplus 2} & \text{if } i = -\frac{1}{2}, \\ P((1|2)^G) & \text{if } i = -\frac{3}{2}, \\ P((2|1)^G) & \text{if } i = \frac{3}{2}, \\ 0 & \text{otherwise,} \end{cases}$$

For $(0|1)$ we have

$$\theta_i P(0|1,\pm) = \begin{cases} P(0|0,\pm)^{\oplus 2} & \text{if } i = \frac{1}{2}, \\ P(0|2,\pm) & \text{if } i = -\frac{3}{2}, \\ 0 & \text{otherwise.} \end{cases}$$

For $a > 1$ we have

$$\theta_i P((a|a)^G) = \begin{cases} P((a|a+1)^G)^{\oplus 2} & \text{if } i = -a - \frac{1}{2}, \\ P((a+1|a)^G)^{\oplus 2} & \text{if } i = a + \frac{1}{2}, \\ P((a-1|a)^G) & \text{if } i = -a + \frac{1}{2}, \\ P((a|a-1)^G) & \text{if } i = a - \frac{1}{2}, \\ P((a+1|a+2)^G) & \text{if } i = -a - \frac{3}{2}, \\ P((a+2|a+1)^G) & \text{if } i = a + \frac{3}{2}, \\ 0 & \text{otherwise,} \end{cases}$$

$$\theta_i P(0|a,\pm) = \begin{cases} P((1|0)^G) & \text{if } i = \frac{1}{2}, \\ P(0|a+1,\pm) & \text{if } i = -a - \frac{1}{2}, \\ P(0|a-1,\pm) & \text{if } i = a - \frac{1}{2}, \\ 0 & \text{otherwise,} \end{cases}$$

$$\theta_i P((1|a)^G) = \begin{cases} P(0|a,+) \oplus P(0|a,-) & \text{if } i = \frac{1}{2}, \\ P((1|a+1)^G) & \text{if } i = -a - \frac{1}{2}, \\ P((1|a-1)^G) & \text{if } i = a - \frac{1}{2}, \\ 0 & \text{otherwise.} \end{cases}$$

For $a > 1$ and $b \neq a$ we have

$$\theta_i P((a|b)^G) = \begin{cases} P((a-1|b)^G) & \text{if } i = -a + \frac{1}{2}, \\ P((a+1|b)^G) & \text{if } i = a + \frac{1}{2} \text{ and } a+1 \neq b, \\ P((a|b-1)^G) & \text{if } i = b - \frac{1}{2}, \\ P((a|b+1)^G) & \text{if } i = -b - \frac{1}{2} \text{ and } a-1 \neq b, \\ 0 & \text{otherwise.} \end{cases}$$

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