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$G(\ell, k, d)$-modules via groupoids

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Abstract In this note, we describe a seemingly new approach to the complex representation theory of the wreath product $G \wr S_d$, where G is a finite abelian group. The approach is motivated by an appropriate version of Schur–Weyl duality. We construct a combinatorially defined groupoid in which all endomorphism algebras are direct products of symmetric groups and prove that the groupoid algebra is isomorphic to the group algebra of $G \wr S_d$. This directly implies a classification of simple modules. As an application, we get a Gelfand model for $G \wr S_d$ from the classical involutive Gelfand model for the symmetric group. We describe the Schur–Weyl duality which motivates our approach and relate it to various Schur–Weyl dualities in the literature. Finally, we discuss an extension of these methods to all complex reflection groups of type $G(\ell, k, d)$.

Keywords Schur–Weyl duality · Wreath product · Simple module · Groupoid

1 Introduction

A very important class of finite groups form wreath products of the form $G \wr S_d$, where S_d is the symmetric group and G is abelian. The study of its representation theory is a classical topic. The first major results, e.g., the classification of simple modules, were

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already obtained by Specht in his thesis [38]. Since then the theory was revised on various occasions, in particular in case of G being a cyclic group, see [5,20,27,29] and references therein. This note contributes yet another approach which, from our point of view, simplifies the theory and makes several results, in particular, on the combinatorics of simple modules and on Gelfand models, especially transparent.

Our approach originates in an attempt to understand various Schur–Weyl dualities appearing in [4,9,32] in which on one side we have an action of a direct product of general linear groups, while on the other side we have a non-faithful action of the Coxeter group of type B, respectively, D. While looking for similar results in the literature, we discovered that analogous Schur–Weyl dualities already appeared in [26,28] and also in the context of Ariki–Koike algebras in [3,16,31,36]. These dualities have a common structure, which suggests the substitution of the Coxeter group of type B by a certain combinatorially defined groupoid, see Sect. 2.2 for a precise definition of the latter. The main observation of the present note is that this groupoid can be used to describe the representation theory of the Coxeter group of type B and, more generally, of the wreath products of the form $\mathbf{C}_\ell \wr S_d$, where $\mathbf{C}_\ell$ is a cyclic group of order ℓ or any complex reflection group of type $G(\ell, k, d)$. The transparent combinatorial structure of the groupoid proposes a straightforward reduction in all statement to type A, that is, to the case of direct products of symmetric groups. An explicit construction of all irreducible representations for $G(\ell, k, d)$ can be found in Proposition 3 and Theorem 17. The symmetric group S_d, the Weyl group of type B_d, and the Weyl group of type D_d are the special examples $G(1, 1, d)$, $G(2, 1, d)$, and $G(2, 2, d)$, respectively.

In Sect. 2.2, we define our main object of study, that is, a finite groupoid $\mathscr{G}_{(\ell,d)}$, and in Sect. 2.5 we show that its algebra (over $\mathbb{C}$) is isomorphic to the group algebra of $\mathbf{C}_\ell \wr S_d$. Consequently, we immediately get a classification and explicit construction of simple $\mathbf{C}_\ell \wr S_d$-modules (see Sect. 2.4), which does not even involve any counting of the number of conjugacy classes (the latter being one of the ingredients in all classical approaches). The indexing set of simple modules is the set of ℓ-multi-partitions of d. Moreover, our construction immediately gives a basis of all simple modules indexed by all standard ℓ-multi-tableaux of the corresponding type, see Sect. 2.4. We connect our construction of simple modules to the one from [29] which uses induction from generalized Young subgroups. Finally, we also provide in Sect. 2.7 a straightforward construction of an involutive Gelfand model for $\mathbf{C}_\ell \wr S_d$ (that is, a multiplicity-free direct sum of all simple modules), significantly simplifying the previous approaches from [2,6]. In Sect. 3.4, we give a short proof of the Schur–Weyl duality, which motivated our approach (as we mentioned before, several (quantum) versions of this duality exist in the literature). In Sect. 3.7, we use this duality to justify that our results naturally extent to the case $G \wr S_d$, where G is any finite abelian group. Finally, in Sect. 4, we extend most of the results to all complex reflection groups $G(\ell, k, d)$.

We note that our approach generalizes to the quantum group setting. However, to prevent that the main idea of the proof is buried in technical details, we stick to the non-quantized situation.

2 Modules over generalized symmetric groups

2.1 Generalized symmetric groups

We denote by $\mathbb{Z} \supseteq \mathbb{Z}_{\geq 0} \supseteq \mathbb{Z}_{>0}$ the sets of all integers, all nonnegative integers, and all positive integers, respectively. For $n \in \mathbb{Z}_{\geq 0}$, we denote by $\underline{n}$ the set $\{1, 2, \ldots, n\}$ (with $\underline{0} = \varnothing$). Throughout the paper, we fix as ground field the field $\mathbb{C}$ of complex numbers and abbreviate $\otimes_{\mathbb{C}}$ as $\otimes$.

For $\ell \in \mathbb{Z}_{>0}$, let $\mathbf{C}_\ell$ be the group of all complex ℓth roots of unity. The group $\mathbf{C}_\ell$ is cyclic, and we fix some generator $\xi_\ell \in \mathbf{C}_\ell$, that is, a primitive ℓth root of unity.

Given a set X, we denote by $S(X)$ the symmetric group on X and abbreviate $S_d := S(\underline{d})$ for any $d \in \mathbb{Z}_{>0}$. For $\mathbf{d} = (d_1, d_2, \ldots, d_k) \in \mathbb{Z}_{\geq 0}^k$ set

$$S_{\mathbf{d}} := S_{d_1} \times S_{d_2} \times \cdots \times S_{d_k}. \tag{1}$$

Given $n \in \mathbb{Z}_{\geq 0}$ and a partition $\mu \vdash n$, we denote by $\mathscr{S}_\mu$ the (irreducible) Specht S_n-module corresponding to μ. For $\mathbf{n} = (n_1, n_2, \ldots, n_k) \in \mathbb{Z}_{\geq 0}^k$ and a multi-partition $\boldsymbol{\mu} = (\mu_1, \mu_2, \ldots, \mu_k)$ such that $\mu_i \vdash n_i$ for all i, we denote by $\mathscr{S}_{\boldsymbol{\mu}}$ the $S_{\mathbf{n}}$-module $\mathscr{S}_{\mu_1} \otimes \mathscr{S}_{\mu_2} \otimes \cdots \otimes \mathscr{S}_{\mu_k}$.

From now on, we **fix** $d \in \mathbb{Z}_{\geq 0}$ and $\ell \in \mathbb{Z}_{>0}$ and consider the wreath product $S(\ell, d) := \mathbf{C}_\ell \wr S_d$, also known as a *generalized symmetric group*. The group $S(\ell, d)$ is naturally identified with the group of all complex $d \times d$-matrices X that satisfy the following two conditions:

- Each row and each column of X contains exactly one nonzero entry.
- Each nonzero entry of X is an element of $\mathbf{C}_\ell$.

We have $|S(\ell, d)| = \ell^d \cdot d!$. The group $S(\ell, d)$ is a complex reflection group usually denoted by $G(\ell, 1, n)$.

The group $S(\ell, d)$ has a presentation with generators $s_0, s_1, \ldots, s_{d-1}$ and relations

$$
\begin{array}{lll}
s_0^\ell & = e; & \\
s_i^2 & = e, & i = 1, 2, \ldots, d-1; \\
s_0 s_1 s_0 s_1 & = s_1 s_0 s_1 s_0; & \\
s_i s_{i+1} s_i & = s_{i+1} s_i s_{i+1}, & i = 1, 2, \ldots, d-2; \\
s_i s_j & = s_j s_i, & |i - j| > 1, \quad i, j = 0, 1, \ldots, d-1.
\end{array}
\tag{2}
$$

An isomorphism with the earlier description is given by sending s_0 to the diagonal $d \times d$-matrix in which the $(1, 1)$-entry is ξ_ℓ and all other diagonal entries are equal to 1, and sending s_i for $i = 1, 2, \ldots, d-1$ to the permutation matrix corresponding to the transposition $(i, i+1)$.

Some classical special cases are: The group $S(1, d) \cong S_d$ is the Weyl group of type A_{d-1} and $S(2, d)$ is the Weyl group of type B_d and C_d.

2.2 The groupoid $\mathscr{G}_{(\ell, d)}$

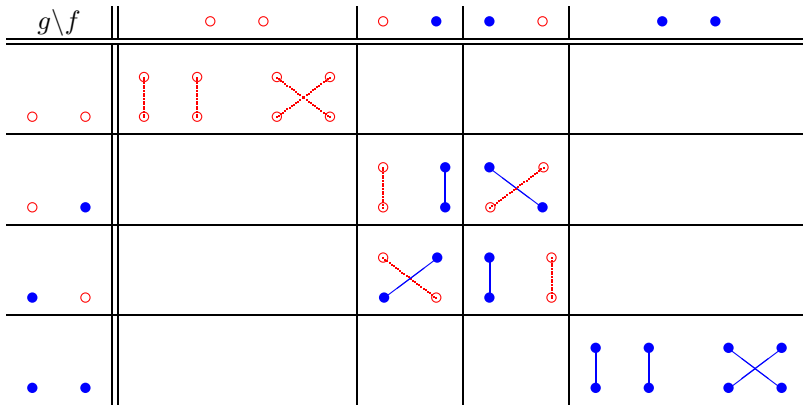Consider a category $\mathscr{G}_{(\ell, d)}$ defined as follows:

Fig. 1 The sets of morphisms $\mathscr{G}_{(\ell,d)}(f,g)$ from Example 1

- Objects of $\mathscr{G}_{(\ell,d)}$ are all maps $f : \underline{d} \to \underline{\ell}$.
- For two objects f and g, the set of morphisms $\mathscr{G}_{(\ell,d)}(f,g)$ consists of all bijections $\sigma : \underline{d} \to \underline{d}$ such that $g \circ \sigma = f$.
- The identity morphism $e_f \in \mathscr{G}_{(\ell,d)}(f,f)$ is the identity map $\mathrm{Id}_{\underline{d}} : \underline{d} \to \underline{d}$.
- Composition of morphisms is given by composition of maps.

It is convenient to think of objects in $\mathscr{G}_{(\ell,d)}$ as $\underline{\ell}$-colorings of elements in $\underline{d}$, that is, as ordered sequences of d dots colored in ℓ colors. Then, morphisms in $\mathscr{G}_{(\ell,d)}$ are color-preserving bijections. We usually represent them in terms of colored permutation diagrams (read from top to bottom), see Fig. 1.

Example 1 Let $d = 2$ and $\ell = 2$. We depict colors as follows: $1 = $ red and $2 = $ blue. Then, $\mathscr{G}_{(\ell,d)}$ has four objects, namely

$$\circ \quad \circ \qquad\qquad \circ \quad \bullet \qquad\qquad \bullet \quad \circ \qquad\qquad \bullet \quad \bullet \quad ,$$

and the elements of $\mathscr{G}_{(\ell,d)}(f,g)$ are given in Fig. 1 (for convenience, all red strands are dashed and all red points are circled).

For an object $f \in \mathscr{G}_{(\ell,d)}$, the *type* of f is defined as

$$\boldsymbol{\lambda}_f := (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$$

where $\lambda_i := |\{x \in \underline{d} \mid f(x) = i\}|$ for $i = 1, 2, \ldots, \ell$. As $\lambda_1 + \lambda_2 + \cdots + \lambda_\ell = d$ and all $\lambda_i \in \mathbb{Z}_{\geq 0}$, the type $\boldsymbol{\lambda}_f$ is a *composition of d with ℓ parts*. We denote by $\Lambda(\ell,d)$ the set of all compositions of d with ℓ parts, that is, all $(a_1, a_2, \ldots, a_\ell) \in \mathbb{Z}_{\geq 0}^\ell$ such that $a_1 + a_2 + \cdots + a_\ell = d$. Then, $\Lambda(\ell,d)$ is exactly the set of all possible types for objects in $\mathscr{G}_{(\ell,d)}$.

Proposition 2 *Let $f, g \in \mathscr{G}_{(\ell,d)}$.*

(i) We have $\mathscr{G}_{(\ell,d)}(f,g) \neq \varnothing$ if and only if $\boldsymbol{\lambda}_f = \boldsymbol{\lambda}_g$.

(ii) If $\boldsymbol{\lambda}_f = \boldsymbol{\lambda}_g = (\lambda_1, \lambda_2, \ldots, \lambda_\ell) =: \boldsymbol{\lambda}$, then $|\mathscr{G}_{(\ell,d)}(f,g)| = \prod_{i=1}^{\ell} \lambda_i! =: \boldsymbol{\lambda}!$.

(iii) *We have* $\left| \coprod_{f_1, f_2 \in \mathscr{G}_{(\ell,d)}} \mathscr{G}_{(\ell,d)}(f_1, f_2) \right| = \ell^d \cdot d!$, *where* $f_1, f_2 \in \mathscr{G}_{(\ell,d)}$.

Proof Claims (i) and (ii) follow directly from the definitions. Claim (iii) follows from the observation that there is a natural bijection between the set of all morphisms in $\mathscr{G}_{(\ell,d)}$ and the set of all colored permutations of d dots with ℓ colors. $\qquad\square$

Note that $\mathscr{G}_{(\ell,d)}$ is a groupoid. For $\lambda \in \Lambda(\ell, d)$ denote by $\mathscr{G}^\lambda_{(\ell,d)}$ the full subcategory of $\mathscr{G}_{(\ell,d)}$ generated by all objects of type λ. Then, $\mathscr{G}^\lambda_{(\ell,d)}$ is a subgroupoid and is a connected component in $\mathscr{G}_{(\ell,d)}$ in the sense that $\mathscr{G}^\lambda_{(\ell,d)}(f, g) \neq \varnothing$ for any $f, g \in \mathscr{G}^\lambda_{(\ell,d)}$ and

$$\mathscr{G}_{(\ell,d)} = \coprod_{\lambda \in \Lambda(\ell,d)} \mathscr{G}^\lambda_{(\ell,d)}.$$

If f has type λ, then there is an isomorphism of groups $\mathscr{G}_{(\ell,d)}(f, f) \cong S_\lambda$.

2.3 The linearization of $\mathscr{G}_{(\ell,d)}$

For a set X, we denote by $\mathbb{C}[X]$ the formal complex vector space with the elements of X as basis. If $X = \varnothing$, then we have $\mathbb{C}[X] = 0$.

Note that the groupoid $\mathscr{G}_{(\ell,d)}$ is a finite category. Denote by $\mathbb{C}\mathscr{G}_{(\ell,d)}$ the $\mathbb{C}$-linear category generated by $\mathscr{G}_{(\ell,d)}$. This means the following:

- Objects in $\mathbb{C}\mathscr{G}_{(\ell,d)}$ are the same as in $\mathscr{G}_{(\ell,d)}$.
- For two objects f and g, we have $\mathbb{C}\mathscr{G}_{(\ell,d)}(f, g) := \mathbb{C}[\mathscr{G}_{(\ell,d)}(f, g)]$.
- The identity elements in $\mathbb{C}\mathscr{G}_{(\ell,d)}$ are given by the identity elements in $\mathscr{G}_{(\ell,d)}$.
- Composition in $\mathbb{C}\mathscr{G}_{(\ell,d)}$ is induced from composition in $\mathscr{G}_{(\ell,d)}$ by bilinearity.

We remark that $\mathbb{C}\mathscr{G}_{(\ell,d)}$ is no longer a groupoid. If $f \in \mathbb{C}\mathscr{G}_{(\ell,d)}$ is of type λ, then $\mathbb{C}\mathscr{G}_{(\ell,d)}(f, f) \cong \mathbb{C}[S_\lambda]$, the group algebra of S_λ.

2.4 Simple finite-dimensional $\mathbb{C}\mathscr{G}_{(\ell,d)}$-modules

Consider the category $\mathbb{C}\mathscr{G}_{(\ell,d)}$-mod of $\mathbb{C}$-linear functors from $\mathbb{C}\mathscr{G}_{(\ell,d)}$ to the category of finite-dimensional complex vector spaces. Objects in $\mathbb{C}\mathscr{G}_{(\ell,d)}$-mod are called $\mathbb{C}\mathscr{G}_{(\ell,d)}$-*modules*. Morphism in $\mathbb{C}\mathscr{G}_{(\ell,d)}$-mod are natural transformations of functors.

For $\lambda \in \Lambda(\ell, d)$, consider the set $\mathbf{T}_\lambda$ of all ℓ-multi-partitions $\mathbf{p}$ of shape λ, that is, all ℓ-tuples $\mathbf{p} = (p_1, p_2, \ldots, p_\ell)$ of partitions such that $p_i \vdash \lambda_i$ for all i. For f, g of type λ let $\sigma_{(f,g)}$ be the unique *order-preserving* element in $\mathscr{G}_{(\ell,d)}(f, g)$ in the sense that it has the following property:

For all $i, j \in \underline{d}$ satisfying $i < j$ and $f(i) = f(j)$, we have $\sigma_{(f,g)}(i) < \sigma_{(f,g)}(j)$.

The element $\sigma_{(f,g)}$ should be thought of as a "canonical" isomorphism between the objects f and g. Clearly, for f, g, h of type λ we have $\sigma_{(g,h)}\sigma_{(f,g)} = \sigma_{(f,h)}$ and

$\sigma_{(f,f)} = e_f$. We also denote by f_λ the unique object of type λ in which all colors are assigned in the natural order from 1 to d, that is,

$$f_\lambda(1) = f_\lambda(2) = \cdots = f_\lambda(\lambda_1) = 1,$$
$$f_\lambda(\lambda_1 + 1) = f_\lambda(\lambda_1 + 2) = \cdots = f_\lambda(\lambda_1 + \lambda_2) = 2,$$
$$\cdots.$$

This element is our "canonical" object in $\mathscr{G}^\lambda_{(\ell,d)}$. We fix the evident identification of S_λ with $\mathscr{G}_{(\ell,d)}(f_\lambda, f_\lambda)$.

For $\mathbf{p} \in \mathbf{T}_\lambda$, define a $\mathbb{C}\mathscr{G}_{(\ell,d)}$-module $L_\mathbf{p}$ as follows:

- $L_\mathbf{p}(f) := \begin{cases} \mathscr{S}_\mathbf{p}, & \text{if } f \text{ is of type } \lambda; \\ 0, & \text{if } f \text{ is not of type } \lambda. \end{cases}$
- For any f, g of type λ, any $\pi \in \mathscr{G}_{(\ell,d)}(f, g)$ and any $v \in \mathscr{S}_\mathbf{p}$, we set

$$L_\mathbf{p}(\pi) \cdot v := \sigma_{g,f_\lambda} \pi \sigma_{f_\lambda, f}(v).$$

and extend this linearly to an action of the whole of $\mathbb{C}\mathscr{G}_{(\ell,d)}$.

Proposition 3 *We have the following:*

(i) For $\mathbf{p} \in \mathbf{T}_\lambda$, the functor $L_\mathbf{p}$ defined above is a simple $\mathbb{C}\mathscr{G}_{(\ell,d)}$-module.
(ii) The set

$$\coprod_{\lambda \in \Lambda(\ell,d)} \{L_\mathbf{p} \mid \mathbf{p} \in \mathbf{T}_\lambda\}$$

is a cross section of isomorphism classes of simple $\mathbb{C}\mathscr{G}_{(\ell,d)}$-modules.

Proof The functoriality of $L_\mathbf{p}$ follows directly from the definitions and the observation that $\sigma_{f_\lambda, g} \sigma_{g, f_\lambda} = e_g$. The simplicity of $L_\mathbf{p}$ follows by construction from the facts that $\mathscr{G}_{(\ell,d)}$ is a groupoid and that $\mathscr{S}_\lambda$ is a simple $\mathscr{G}_{(\ell,d)}(f_\lambda, f_\lambda)$-module. This proves claim (i).

Claim (ii) follows from the facts that connected components of the groupoid $\mathscr{G}_{(\ell,d)}$ are indexed by $\lambda \in \Lambda(\ell, d)$ and that the set $\{\mathscr{S}_\mathbf{p} \mid \mathbf{p} \in \mathbf{T}_\lambda\}$ is a complete and irredundant set of representatives of isomorphism classes of simple $\mathbb{C}\mathscr{G}_{(\ell,d)}(f_\lambda, f_\lambda)$-modules (see, e.g., [30, Chapter 2]). This completes the proof. □

The vector space

$$A_{(\ell,d)} := \bigoplus_{f,g \in \mathscr{G}_{(\ell,d)}} \mathbb{C}\mathscr{G}_{(\ell,d)}(f, g)$$

inherits from $\mathbb{C}\mathscr{G}_{(\ell,d)}$ the structure of a finite-dimensional associative algebra over $\mathbb{C}$. As usual, there is a canonical equivalence of categories

$$\mathbb{C}\mathscr{G}_{(\ell,d)}\text{-mod} \cong A_{(\ell,d)}\text{-mod}, \tag{3}$$

where the right-hand side denotes the category of finite-dimensional $A_{(\ell,d)}$-modules. Taking this equivalence into account, Proposition 3 provides an explicit description of all simple $A_{(\ell,d)}$-modules. In what follows, we identify $A_{(\ell,d)}$-modules and $\mathbb{C}\mathscr{G}_{(\ell,d)}$-modules via this equivalence. We note that

$$\sum_{f \in \mathscr{G}_{(\ell,d)}} \dim(\mathrm{L}_{\mathbf{p}}(f)) = \frac{n!}{\lambda!} \dim \mathscr{S}_{\mathbf{p}}. \tag{4}$$

For $\mu \vdash n$, the Specht S_n-module $\mathscr{S}_\mu$ has a basis given by polytabloids corresponding to standard Young tableaux of shape μ, see [30, Section 2.5]. This extends in the evident way to a basis in $\mathscr{S}_{\mathbf{p}}$ and hence gives a basis in each $\mathrm{L}_{\mathbf{p}}(f)$, where f is of type λ. A more "natural" parameterization of the elements of this basis in the space $\mathrm{L}_{\mathbf{p}}(g)$ is obtained by applying $\sigma_{f_\lambda,g}$ to the entries of the corresponding standard tableaux.

2.5 Connection to $S(\ell, d)$

For $f, g \in \mathscr{G}_{(\ell,d)}$ and $\sigma \in \mathscr{G}_{(\ell,d)}(f, g)$, we will write $\sigma = \sigma_{(f,g)}$ to distinguish it from the same σ appearing as a morphism between another pair of objects. If $\sigma \notin \mathscr{G}_{(\ell,d)}(f, g)$, we write $\sigma_{(f,g)} = 0$ (viewing it as an element in $\mathbb{C}\mathscr{G}_{(\ell,d)}(f, g)$).

There is a unique linear map $\Phi : \mathbb{C}[S(\ell, d)] \to A_{(\ell,d)}$ such that

- $\Phi(\sigma) = \sum_{f,g} \sigma_{(f,g)}$ for $\sigma \in S_d$;
- $\Phi(s_0) = \sum_f \xi_\ell^{f(1)} e_f$.

Theorem 4 *The map* $\Phi : \mathbb{C}[S(\ell, d)] \to A_{(\ell,d)}$ *is an isomorphism of algebras.*

Proof Note that we defined the map on algebra generators. To prove that Φ is a well-defined homomorphism, it is enough to check that $\Phi(s_i)$, where $i = 0, 1, 2, \ldots, d-1$, satisfy the defining relations for $S(\ell, d)$ given in (2). All defining relations that do not involve s_0 are clear from the definition. That $\Phi(s_0)^\ell$ is the identity follows from the fact that $\xi_\ell^\ell = 1$. That $\Phi(s_0)\Phi(s_i) = \Phi(s_i)\Phi(s_0)$ for $i \neq 1$ is clear because the definition of $\Phi(s_0)$ only involves $f(1)$.

It remains to verify that $\Phi(s_0)\Phi(s_1)\Phi(s_0)\Phi(s_1) = \Phi(s_1)\Phi(s_0)\Phi(s_1)\Phi(s_0)$. It is straightforward to show that both sides of this equality are equal to the element $\sum_f \xi_\ell^{f(1)+f(2)} e_f$. This implies that Φ is a homomorphism.

Note that $\dim(\mathbb{C}S(\ell, d)) = \dim(A_{(\ell,d)})$. Therefore, to complete the proof, it is enough to check, say, surjectivity of Φ. Since each $\sigma_{(f,g)}$ appears in $\Phi(\sigma)$ with a nonzero coefficient, it is enough to check that the identity morphism e_f is in the image of Φ for each f.

Denote by B the subalgebra of $A_{(\ell,d)}$ generated by all e_f, where $f \in \mathscr{G}_{(\ell,d)}$. The algebra B is a commutative split semisimple $\mathbb{C}$-algebra of dimension l^d. We set $B' := \Phi(\mathbb{C}S(\ell, d)) \cap B$. Then, B' is a unital subalgebra of B and we need to show that $B' = B$. The left multiplication with the element $\Phi(s_0) \in B'$ on B has different eigenvalues $\xi_\ell^1, \xi_\ell^2, \ldots, \xi_\ell^\ell$. Therefore, by taking polynomials in $\Phi(s_0)$, we get that B' contains, for each $s = 1, 2, \ldots, \ell$, the element

$$x_s := \sum_{f : f(1) = s} e_f.$$

Claim Let now $s \in \underline{\ell}$ be fixed. Then, $e_f \in B'$ for each f with $f(1) = s$.

We prove the claim by downward induction on $m = |\{i \in \underline{d} \mid f(i) = s\}|$. Assume first that $m = d$, that is, $f = (s, s, \ldots, s)$. Note that

$$\Phi(\sigma^{-1} s_0 \sigma) = \Phi(\sigma^{-1})\Phi(s_0)\Phi(\sigma) \in B' \quad \text{for any} \quad \sigma \in S_d.$$

As x_s is a polynomial in $\Phi(s_0)$, we get $\Phi(\sigma^{-1}) x_s \Phi(\sigma) \in B'$ for each $\sigma \in S_d$. Since B' is a subalgebra of B, we have

$$\prod_{\sigma \in S_d} \Phi(\sigma^{-1}) x_s \Phi(\sigma) = \xi_\ell^a e_{(s,s,\ldots,s)} \in B'$$

for some $a \in \mathbb{Z}_{>0}$. This implies $e_{(s,s,\ldots,s)} \in B'$ and the basis of the induction is established.

Now we prove the induction step. Consider the set $X = \{i \in \underline{d} \mid f(i) = s\}$. Then, similar to the above, we have

$$\prod_{\sigma \in S(X)} \Phi(\sigma^{-1}) x_s \Phi(\sigma) = \sum_g \xi_\ell^{a_g} e_g \in B'$$

for some $a_g \in \mathbb{Z}_{>0}$, where the sum on the right-hand side is taken over all g such that $g(i) = s$ for each $i \in X$. If $g \neq f$, then

$$|\{i \in \underline{d} \mid g(i) = s\}| > |\{i \in \underline{d} \mid f(i) = s\}|,$$

and hence, by induction, $e_g \in B'$. Therefore, $e_f \in B'$ and the proof is complete. $\quad\square$

The isomorphism Φ from Theorem 4 induces an equivalence of categories

$$\overline{\Phi} : A_{(\ell,d)}\text{-mod} \rightarrow \mathbb{C}[S(\ell, d)]\text{-mod}. \tag{5}$$

Combined with Sect. 2.4, $\overline{\Phi}$ provides a very natural and neat description of simple $\mathbb{C}[S(\ell, d)]$-modules. For alternative descriptions of simple $\mathbb{C}[S(\ell, d)]$-modules we refer the reader to [5,20,27,29,38] and references therein.

2.6 Simple $\mathbb{C}[S(\ell, d)]$-modules via generalized Young subgroups

Here we connect our approach with the one in [29]. For $j = 1, 2, \ldots, n$, denote by $s_0^{(j)}$ the element $s_{j-1}s_{j-2} \ldots s_1 s_0 s_1 s_2 \ldots s_{j-1} \in S(\ell, d)$. This element is the diagonal $d \times d$ matrix in which the (j, j)-entry equals ξ_ℓ and all other diagonal entries are 1.

We have $s_0 = s_0^{(1)}$. It is easy to check that

$$\Phi\left(s_0^{(j)}\right) = \sum_{f \in \mathscr{G}_{(\ell,d)}} \xi_\ell^{f(j)} e_f. \tag{6}$$

Let $\lambda \in \Lambda(\ell, d)$ and $f \in \mathscr{G}_{(\ell,d)}^{\lambda}$. Then, we have the decomposition

$$\underline{n} = X_1^f \cup X_2^f \cup \cdots \cup X_\ell^f,$$

where $X_i^f := \{j \in \underline{n} \mid f(j) = i\}$ for $i = 1, 2, \ldots, \ell$. Denote by G_i^f the subgroup of $S(\ell, d)$ generated by all $s_0^{(j)}$, where $j \in X_i^f$, and also by all permutations of X_i^f which fix all points outside X_i^f. The subgroup G_i^f is isomorphic to $S(l, |X_i^f|)$ and the direct product

$$G^f := G_1^f \times G_2^f \times \cdots \times G_\ell^f$$

is, naturally, a subgroup of $S(\ell, d)$. The subgroup G^f is called a *generalized Young subgroup* of $S(l, n)$.

Lemma 5 *Let* $\lambda \in \Lambda(\ell, d)$, $\mathbf{p} \in \mathbf{T}_{\boldsymbol{\lambda}}$ *and* $f \in \mathscr{G}_{(\ell,d)}^{\boldsymbol{\lambda}}$.

(i) *The space* $\mathbf{L_p}(f)$ *inherits the structure of a simple* G^f-*module by restriction.*
(ii) *The* $S(l, n)$-*modules* $\overline{\Phi}(\mathbf{L_p})$ *and* $\mathrm{Ind}_{G^f}^{S(\ell,d)} \mathbf{L_p}(f)$ *are isomorphic.*

Proof That $\mathbf{L_p}(f)$ is stable under the action of all $s_0^{(j)}$ is clear from the definitions. Similarly, it is also clear that $\mathbf{L_p}(f)$ is stable under the action of all permutations which preserve colors. Claim (i) follows.

From claim (i), it follows by adjunction that $\mathrm{Ind}_{G^f}^{S(\ell,d)} \mathbf{L_p}(f)$ surjects onto $\overline{\Phi}(\mathbf{L_p})$. However, since the index of G^f in $S(\ell, d)$ equals $\frac{n!}{\lambda!}$, from (4) it follows that the modules $\overline{\Phi}(\mathbf{L_p})$ and $\mathrm{Ind}_{G^f}^{S(\ell,d)} \mathbf{L_p}(f)$ have the same dimension and thus are isomorphic. $\square$

Using this basis in each $\mathbf{L_p}(f)$, where f is of type $\boldsymbol{\lambda}$, described in Sect. 2.4 and the classical branching rule for the symmetric group, see [30, Section 2.8], one immediately recovers the branching rule for the restriction from $S(\ell, d)$ to $S(\ell, d-1)$ as described in [24]. Namely, the restriction of $\mathbf{L_p}$ to $S(\ell, d-1)$ is a multiplicity-free direct sum of $\mathbf{L_q}$ where $\mathbf{q}$ is obtained from $\mathbf{p}$ by removing one removable node from one of the parts of $\mathbf{p}$.

2.7 Gelfand model

Recall that a *Gelfand model* for a finite group G is a G-module isomorphic to a multiplicity-free direct sum of all simple G-modules. Similarly, one defines Gelfand models for semisimple algebras. Let $\mathcal{I}$ be the set of all involutions in S_d, that is, all

elements $w \in S_d$ satisfying $w^2 = e$. Define an S_d-module structure on $\mathbb{C}[\mathcal{I}]$, for $\sigma \in S_d$ and $w \in \mathcal{I}$, as follows:

$$\sigma \cdot w = (-1)^{\mathrm{inv}(\sigma,w)}(\sigma w \sigma^{-1}), \quad \text{where}$$
$$\mathrm{inv}(\sigma, w) := |\{(i, j) \mid i, j \in \underline{d}, \ i < j, \ w(i) = j, \ \sigma(i) > \sigma(j)\}|.$$

Proposition 6 [1,18] *The S_d-module $\mathbb{C}[\mathcal{I}]$ is a Gelfand model for S_d.*

This model (sometimes referred to as the *involutive Gelfand model*) was generalized to wreath products in [2,6], to inverse semigroups in [22] and to general diagram algebras in [15,25], see also references in these paper for other generalizations. An alternative approach to Gelfand models for certain classes of groups can be found in [7].

   In our setup, it is fairly straightforward to combine the above model with the construction used in [22,25] to produce a Gelfand model for $\mathscr{G}_{(\ell,d)}$ (significantly simplifying arguments from [2]). For each $f \in \mathscr{G}_{(\ell,d)}$, denote by $\mathcal{I}^f$ the set of all involutions in $\mathscr{G}_{(\ell,d)}(f, f)$. Define a representation Gelfand of $\mathbb{C}\mathscr{G}_{(\ell,d)}$ as follows:

- Set $\mathrm{Gelfand}(f) := \mathbb{C}[\mathcal{I}^f]$.
- For $f, g \in \mathscr{G}_{(\ell,d)}, \sigma \in \mathscr{G}_{(\ell,d)}(f, g)$ and $w \in \mathcal{I}^f$ set

$$\mathrm{Gelfand}(\sigma) \cdot w = (-1)^{\mathrm{inv}(\sigma,w)}(\sigma w \sigma^{-1}).$$

- Extend this to the whole of $\mathbb{C}\mathscr{G}_{(\ell,d)}$ by linearity.

Corollary 7 *The $\mathbb{C}\mathscr{G}_{(\ell,d)}$-module Gelfand is a Gelfand model for $\mathbb{C}\mathscr{G}_{(\ell,d)}$.*

Proof The fact that Gelfand is a $\mathbb{C}\mathscr{G}_{(\ell,d)}$-module follows directly from our definitions and the construction of the Gelfand model for S_d in [1]. Taking into account the classification of simple $\mathbb{C}\mathscr{G}_{(\ell,d)}$-modules in Sect. 2.4, to prove that Gelfand is a Gelfand model, we need to prove that for each $f \in \mathscr{G}_{(\ell,d)}$ the space $\mathrm{Gelfand}(f)$ is a Gelfand model for $\mathscr{G}_{(\ell,d)}(f, f)$. This again follows directly from the definitions and the main result of [1]. □

3 Schur–Weyl dualities for $S(\ell, d)$

3.1 Classical Schur–Weyl duality

For $n \in \mathbb{Z}_{>0}$, consider the (infinite!) group $\mathbf{GL}_n = \mathbf{GL}_n(\mathbb{C})$ and its *natural* representation $V := \mathbb{C}^n$ with standard basis $\mathbf{v} := (v_1, v_2, \dots, v_n)$. For $d \in \mathbb{Z}_{>0}$, consider the d-th tensor power $V^{\otimes d}$ with the usual diagonal coproduct action of $\mathbf{GL}_n$. The symmetric group S_d acts on $V^{\otimes d}$ by permuting the components of the tensor product. This action clearly commutes with the action of $\mathbf{GL}_n$. Moreover, these two actions have the *double centralizer property* in the sense that every linear operator on $V^{\otimes d}$ which commutes with the action of $\mathbb{C}[\mathbf{GL}_n]$ is given by the action of $\mathbb{C}[S_d]$ and vice versa, depicted as follows:

$$\mathrm{GL}_n \curvearrowright V^{\otimes d} \curvearrowleft S_d \tag{7}$$

This is the classical *Schur–Weyl duality* from [33,34,39].

The action of $\mathbb{C}[\mathbf{GL}_n]$ on $V^{\otimes d}$ is certainly never faithful. The action of $\mathbb{C}[S_d]$ on $V^{\otimes d}$ is faithful if and only if $n \geq d$. If $n < d$, then the kernel of this action is given by the ideal in $\mathbb{C}[S_d]$ corresponding to all Specht S_d-modules $\mathscr{S}_\mu$, where $\mu \vdash d$ has more than n rows, see [12, Theorem 9.1.2].

3.2 Splitting the left action

Let now $l, n \in \mathbb{Z}_{>0}$ with $l \leq n$. Fix a composition $\mathbf{k} = (k_1, k_2, \ldots, k_\ell) \in \Lambda(l, n)$ in which all $k_i > 0$. Consider the block-diagonal subgroup $\mathbf{GL}_\mathbf{k} \cong \mathbf{GL}_{k_1} \times \mathbf{GL}_{k_2} \times \cdots \times \mathbf{GL}_{k_\ell}$ in $\mathbf{GL}_n$ given by all matrices of the following form:

$\mathbf{GL}_{k_1}$	0	$\cdots$	0
0	$\mathbf{GL}_{k_2}$	$\cdots$	0
$\vdots$	$\vdots$	$\ddots$	$\vdots$
0	0	$\cdots$	$\mathbf{GL}_{k_\ell}$

By restriction, the $\mathbf{GL}_n$-module V from the previous subsection becomes a $\mathbf{GL}_\mathbf{k}$-module and V decomposes as $V = V_1 \oplus V_2 \oplus \cdots \oplus V_\ell$, where V_i the subspace of V spanned by all v_j, where

$$j \in \{k_1 + k_2 + \cdots + k_{i-1} + 1, k_1 + k_2 + \cdots + k_{i-1} + 2, \ldots, k_1 + k_2 + \cdots + k_i\}.$$

In particular, we have for $i, j \in \{1, 2, \ldots, \ell\}$ that V_i is the natural $\mathbf{GL}_{k_i}$-module, while the action of $\mathbf{GL}_{k_i}$ on V_j is trivial whenever $j \neq i$.

3.3 $\mathscr{G}_{(\ell,d)}$-action on $V^{\otimes d}$

The space $V^{\otimes d}$ has the structure of a $\mathbb{C}\mathscr{G}_{(\ell,d)}$-module G defined as follows:

- For $f \in \mathscr{G}_{(\ell,d)}$, we set $G(f) := V_{f(1)} \otimes V_{f(2)} \otimes \cdots \otimes V_{f(d)}$.
- For $f, g \in \mathscr{G}_{(\ell,d)}$ of the same type and $\sigma \in \mathscr{G}_{(\ell,d)}(f, g)$, the linear map $G(\sigma)$ acts by permuting factors of the tensor product, namely,

$$\sigma(w_1 \otimes w_2 \otimes \cdots \otimes w_d) := w_{\sigma^{-1}(1)} \otimes w_{\sigma^{-1}(2)} \otimes \cdots \otimes w_{\sigma^{-1}(d)} \qquad (8)$$

 where $w_i \in V_{f(i)}$ for all i [note that permutation of components induces the *opposite* action on indices of the components, which explains the appearance of σ^{-1} in (8)].

It is straightforward to check that this gives a well-defined $\mathbb{C}\mathscr{G}_{(\ell,d)}$-module. Using the equivalence (3), this defines on $V^{\otimes d}$ the structure of an $A_{(\ell,d)}$-module.

Lemma 8 *(i) The action of $\mathbf{GL}_\mathbf{k}$ preserves $G(f)$ for each $f \in \mathbb{C}\mathscr{G}_{(\ell,d)}$.*
(ii) The action of $A_{(\ell,d)}$ on $V^{\otimes d}$ commutes with the action of $\mathbf{GL}_\mathbf{k}$.

Proof Let $w_1 \otimes w_2 \otimes \cdots \otimes w_d \in V_{f(1)} \otimes V_{f(2)} \otimes \cdots \otimes V_{f(d)}$. Choose any $x_i \in \mathbf{GL}_{k_i}$ for all i and let $x = \mathrm{diag}(x_1, x_2, \ldots, x_\ell)$ be the corresponding element in $\mathbf{GL_k}$. Then, using the definitions and the fact that $\mathbf{GL}_{k_i}$ acts trivially on V_j for $i \neq j$, we have

$$x \cdot (w_1 \otimes w_2 \otimes \cdots \otimes w_d) = (x_{f(1)} \cdot w_1) \otimes (x_{f(2)} \cdot w_2) \otimes \cdots \otimes (x_{f(d)} \cdot w_d). \quad (9)$$

It follows that the action of $\mathbf{GL_k}$ preserves each $\mathrm{G}(f)$. This proves claim (i). Moreover, this also implies that to prove claim (ii) it is enough to show that the action of $\mathbf{GL_k}$ commutes with the action of each $\sigma \in \mathscr{G}_{(\ell,d)}(f, g)$.

Applying σ to (9), we get

$$\sigma \circ x \cdot (w_1 \otimes w_2 \otimes \cdots \otimes w_d)$$
$$= (x_{f(\sigma^{-1}(1))} \cdot w_{\sigma^{-1}(1)}) \otimes (x_{f(\sigma^{-1}(2))} \cdot w_{\sigma^{-1}(2)}) \otimes \cdots \otimes (x_{f(\sigma^{-1}(d))} \cdot w_{\sigma^{-1}(d)}). \quad (10)$$

Similar to (9), acting by x on (8), we get

$$x \circ \sigma \cdot (w_1 \otimes w_2 \otimes \cdots \otimes w_d)$$
$$= (x_{g(1)} \cdot w_{\sigma^{-1}(1)}) \otimes (x_{g(2)} \cdot w_{\sigma^{-1}(2)}) \otimes \cdots \otimes (x_{g(d)} \cdot w_{\sigma^{-1}(d)}).$$

The latter coincides with (10) since $f(i) = g(\sigma(i))$ for all i, and therefore, also $f(\sigma^{-1}(i)) = g(i)$ for all i. This proves claim (ii) and thus completes the proof. $\quad\square$

3.4 Schur–Weyl duality for $S(\ell, d)$

The Hecke algebra version of the next theorem appears in [16,31,36], see also further cases in [3,28].

Theorem 9 *The actions of $A_{(\ell,d)}$ and $\mathbf{GL_k}$ on $V^{\otimes d}$ have the double centralizer property*

$$\mathbf{GL_k} \overset{}{\curvearrowright} V^{\otimes d} \overset{}{\curvearrowleft} A_{(\ell,d)} \quad (11)$$

in the sense that they generate each others centralizers.

Proof We consider the action of $\mathbb{C}\mathscr{G}_{(\ell,d)}$ instead of the action of $A_{(\ell,d)}$. By Lemma 8(i), each $\mathrm{G}(f)$ is invariant under the action of $\mathbf{GL_k}$. We claim that $A_{(\ell,d)}$ surjects onto the space of $\mathbf{GL_k}$-intertwiners between the $\mathbf{GL_k}$-modules $\mathrm{G}(f)$ and $\mathrm{G}(g)$, where f and g are of the same type λ. Indeed, the action of $\mathbf{GL_k}$ on both $\mathrm{G}(f)$ and $\mathrm{G}(g)$ can be computed using (9). It follows that, as $\mathbf{GL_k}$-modules, both $\mathrm{G}(f)$ and $\mathrm{G}(g)$ can be identified with the external tensor product of the λ_i-th tensor powers $V_i^{\otimes \lambda_i}$ of the natural $\mathbf{GL}_{k_i}$-modules V_i, where $i = 1, 2, \ldots, \ell$. By construction, the "local" action of $\mathbb{C}\mathscr{G}_{(\ell,d)}$ on $V_i^{\otimes \lambda_i}$ is given by S_{λ_i} which acts by permuting components of the tensor product. This is exactly the setup of the classical Schur–Weyl duality (7), and hence, it follows that, locally, the action of $\mathbb{C}\mathscr{G}_{(\ell,d)}$ does generate the whole centralizer of the $\mathbf{GL}_{k_i}$-module $V_i^{\otimes \lambda_i}$. Note that this local action of $\mathbb{C}\mathscr{G}_{(\ell,d)}$ is independent of all other components.

Since the (external) tensor product of simple finite-dimensional $\mathbb{C}$-algebras is a simple $\mathbb{C}$-algebra (as $\mathbb{C}$ is algebraically closed), by comparing dimension it follows that $\mathbb{C}\mathscr{G}_{(\ell,d)}$ does generate all $\mathbf{GL_k}$-intertwiners between $G(f)$ and $G(g)$. Summing up over all f and g we get that the action of $A_{(\ell,d)}$ generates the centralizer of the $\mathbf{GL_k}$-action on $V^{\otimes d}$.

As $\mathbf{GL_k}$ is reductive, its action on $V^{\otimes d}$ is semisimple. The group algebra of the finite group $A_{(\ell,d)}$ is clearly semisimple and is the centralizer of the action of $\mathbf{GL_k}$ by the above. Therefore, the desired double centralizer property follows from the Double Centralizer Theorem, see [21, Subsection 3.2]. □

3.5 An extremal example: the symmetric inverse semigroup

Recall that for $d \in \mathbb{Z}_{\geq 0}$, the symmetric inverse semigroup IS_d (a.k.a. the rook monoid R_d) is the monoid of all bijections between subsets of $\underline{d}$, see [11, Section 2.5] for details. The symmetric group S_d is the group of invertible elements in IS_d. The monoid IS_d is generated by S_d and the (idempotent) identity transformation ε_1 on the subset $\{2, 3, \ldots, n\}$ of $\underline{d}$ (for a "Coxeter-like" presentation of IS_d we refer the reader to [23, Remark 4.13]).

If $l = 2, k_1 = n - 1$ and $k_2 = 1$, then the action on the left-hand side of the Schur–Weyl duality from Theorem 9 can be viewed as the action of $\mathbf{GL}_{n-1}$, with the trivial action on V_2. This is exactly the setup of Solomon's version of Schur–Weyl duality for IS_d established in [37]. The action of $\mathbb{C}[IS_d]$ is faithful for $d < n$.

Put together with Theorem 4, this gives a surjection $\mathbb{C}[S(2, d)] \twoheadrightarrow \mathbb{C}[IS_d]$ which was already observed on the level of the corresponding Hecke algebras in [14]. For convenience, we give here an explicit formula for this surjection in terms of Coxeter generators of the type B Weyl group $S(2, d)$:

Lemma 10 *There is a unique epimorphism of algebras* $\Psi : \mathbb{C}[S(2, d)] \twoheadrightarrow \mathbb{C}[IS_d]$ *such that*

$$\Psi(s_i) = s_i, \quad i = 1, 2, \ldots, d - 1; \quad \Psi(s_0) = 2\varepsilon_1 - e.$$

Proof Surjectivity is directly clear as $\mathbb{C}[IS_d]$ is generated by S_d and ε_1. To verify that this is a homomorphism, the only nontrivial relations to check are those involving the image of s_0 with itself and with the image of s_1. For the first one, using the fact that ε_1 is an idempotent, we have

$$(2\varepsilon_1 - e)^2 = 4\varepsilon_1^2 - 4\varepsilon_1 + e = 4\varepsilon_1 - 4\varepsilon_1 + e = e.$$

It remains to check that $(2\varepsilon_1 - e)s_1(2\varepsilon_1 - e)s_1 = s_1(2\varepsilon_1 - e)s_1(2\varepsilon_1 - e)$. Opening the brackets and canceling the obvious equal summands, this reduces (up to a nonzero scalar) to a desired equality $\varepsilon_1 s_1 \varepsilon_1 s_1 = s_1 \varepsilon_1 s_1 \varepsilon_1$. It is straightforward to check that both sides of the latter are, in fact, equal to the identity transformation on the subset $\{3, 4, \ldots, d\}$ of $\underline{d}$. □

The homomorphism Ψ allows us to view simple $\mathbb{C}[IS_d]$-modules as simple $\mathbb{C}[S(2, d)]$-modules. Combining Solomon's version of the Schur–Weyl duality

mentioned above and the results of Sect. 3.6 below, one gets that the simple $\mathbb{C}[S(2, d)]$-modules obtained in this way are exactly the modules $L_{(\mu_1, \mu_2)}$, where the partition μ_2 has one part. This parametrization is used, for instance, in [10]. In particular, our remarks on the basis in simple modules from the end of Sect. 2.4 correspond in this case to the main result in [13].

3.6 The kernel of the $A_{(\ell, d)}$-action

The action of $\mathbf{GL_k}$ on $V^{\otimes d}$ is, of course, never faithful. The action of $A_{(\ell, d)}$ on $V^{\otimes d}$ is usually not faithful either. The kernel of the latter action (or rather of the action of $\mathbb{C}\mathscr{G}_{(\ell, d)}$) can be described in terms similar to the description of the kernel of the action of $\mathbb{C}[S_d]$ in the classical Schur–Weyl duality.

Lemma 11 *The kernel of the action of $\mathbb{C}\mathscr{G}_{(\ell, d)}$ on $V^{\otimes d}$ is given by the ideal in $\mathbb{C}\mathscr{G}_{(\ell, d)}$ corresponding to all $\mathbb{C}\mathscr{G}_{(\ell, d)}$-modules $L_{\mathbf{p}}$ for $\mathbf{p} \in T_\lambda$ with $\lambda \in \Lambda(\ell, d)$, such that for some $i \in \{1, 2, \ldots, \ell\}$ the partition $p_i \vdash \lambda_i$ has more than k_i rows.*

Proof It is enough to determine the part of the kernel inside $\mathbb{C}\mathscr{G}_{(\ell, d)}(f_\lambda, f_\lambda)$. The latter is an algebra acting on $G(f_\lambda)$ and for that the statement follows from the classical Schur–Weyl duality, see Sect. 3.1.   □

For example, in the case $\mathbf{G} = \mathbf{GL_1} \times \mathbf{GL_1} \times \cdots \times \mathbf{GL_1}$, all simple $\mathbb{C}\mathscr{G}_{(\ell, d)}$-modules which are not indexed by 1-row multi-partitions are killed.

Remark 12 An important example here is the case where $\ell = 2$ and where we have the action of $\mathbf{G} := \mathbf{GL_n} \times \mathbf{GL_n} \subset \mathbf{GL_{2n}}$. Then, the group algebra $\mathbb{C}\mathscr{G}_{(2, d)}$ of the Weyl group of type B_d acts faithfully as endomorphisms of $V^{\otimes d}$ as long as $n \geq d$. A quantized version of this special case appears in [32], see also [4,9]. In [9] in fact a quantized version of Theorem 19 below appears. (Note that in there it is shown that the action of the coideal subalgebra, denoted $\mathcal{H}$ in [9], on the tensor space $V^{\otimes d}$ agrees with the action of the quantum group attached to the Lie algebra of $\mathbf{G}$).

3.7 Wreath product with an arbitrary abelian group

Let A be an abelian group with ℓ elements. Then, we have an isomorphism $\mathbb{C}[A] \cong \mathbb{C}[\mathbf{C}_\ell]$. Choose some $\mathbf{k} = (k_1, k_2, \ldots, k_\ell)$ such that all $k_i > d$. Then, the endomorphism algebra of V from Sect. 3.2 is isomorphic to both $\mathbb{C}[A]$ and $\mathbb{C}[\mathbf{C}_\ell]$. Going to the other side of the Schur–Weyl duality in Theorem 9 and taking Lemma 11 into account, we obtain $\mathbb{C}[A \wr S_d] \cong \mathbb{C}[\mathbf{C}_\ell \wr S_d]$. Therefore, all results of this paper can be reformulated (for an appropriately defined action of A) for the wreath product $A \wr S_d$.

4 Extension to $G(\ell, k, d)$

4.1 Complex reflection groups $G(\ell, k, d)$

For the rest of the article, fix $k \in \mathbb{Z}_{>0}$ such that $k | \ell$. The realization of $S(\ell, d)$ as the group of $d \times d$ matrices as described in Sect. 2.1 contains an index k subgroup

given by all matrices in $S(\ell, d)$ with determinant in $\mathbf{C}_{l/k}$. It is the complex reflection group $G(\ell, k, d)$. In this section, we generalize most of the above results to the case of $G(\ell, k, d)$. Our approach is motivated by [17].

4.2 The quotient groupoid $\mathscr{G}_{(\ell,d)}^{k}$

Let $\theta : \underline{\ell} \to \underline{\ell}$ denote the permutation sending i to $i + 1$ for $i < \ell$ and ℓ to 1. Then, θ is a cycle of order ℓ. Consider the permutation $\theta_k := \theta^{\ell/k}$, and let H_k be the group generated by θ_k, in particular $|H_k| = k$, the order of θ_k.

The group H_k acts by automorphisms on $\mathscr{G}_{(\ell,d)}$ permuting the colors. More explicitly, this action is described as follows:

- For $f \in \mathscr{G}_{(\ell,d)}$ with $f = (f(1), f(2), \ldots, f(d))$, we have

$$\theta_k(f) = (\theta_k(f(1)), \theta_k(f(2)), \ldots, \theta_k(f(d))).$$

- For $f, g \in \mathscr{G}_{(\ell,d)}$ and $\sigma \in \mathscr{G}_{(\ell,d)}(f, g)$ we have $\theta_k(\sigma) = \sigma$.

Note that the action of H_k on $\underline{\ell}$ is *free* in the sense that the stabilizer of each element is trivial. This implies that the action of H_k on $\mathscr{G}_{(\ell,d)}$ is free as well.

Denote by $\mathscr{G}_{(\ell,d)}^{k}$ the *quotient groupoid* $\mathscr{G}_{(\ell,d)}/H_k$ defined as follows:

- Objects of $\mathscr{G}_{(\ell,d)}^{k}$ are orbits of the action of H_k on objects of $\mathscr{G}_{(\ell,d)}$. For $f \in \mathscr{G}_{(\ell,d)}$, we will denote the H_k-orbit of f by $f^{(k)}$.
- For $f, g \in \mathscr{G}_{(\ell,d)}$, elements in the set $\mathscr{G}_{(\ell,d)}^{k}(f^{(k)}, g^{(k)})$ are orbits of H_k on the H_k-invariant set

$$\coprod_{f' \in f^{(k)}, g' \in g^{(k)}} \mathscr{G}_{(\ell,d)}(f', g').$$

For $\sigma \in \mathscr{G}_{(\ell,d)}(f, g)$, we will denote the H_k-orbit of σ by $\sigma^{(k)}$.

- For $f \in \mathscr{G}_{(\ell,d)}$, the identity morphism in $\mathscr{G}_{(\ell,d)}^{k}(f^{(k)}, f^{(k)})$ is $e_f^{(k)}$.
- The composition in $\mathscr{G}_{(\ell,d)}^{k}$ is the induced composition from $\mathscr{G}_{(\ell,d)}$.

Example 13 Let $d = \ell = 2$ (the setup of Example 1) and take $k = 2$. Then, $\theta = \theta_k$ swaps the two colors. The quotient groupoid $\mathscr{G}_{(\ell,d)}^{k}$ has two objects:

$$o_1 = \{ \circ \quad \circ \qquad \qquad \bullet \quad \bullet \} \qquad \text{and} \qquad o_2 = \{ \circ \quad \bullet \qquad \qquad \bullet \quad \circ \}$$

There are no morphisms between these objects. Both endomorphism sets have two elements. For the object o_1, we have endomorphisms

$$\qquad \text{and} \qquad$$

which form a group isomorphic to S_2. For the object o_2, we have endomorphisms

which form a group isomorphic to $H_2 = \langle \theta \rangle$ (which is also isomorphic to S_2 by a coincidence).

We denote by $\mathbb{C}\mathscr{G}^k_{(\ell,d)}$ the linearization of $\mathscr{G}^k_{(\ell,d)}$ constructed similar to the construction of $\mathbb{C}\mathscr{G}_{(\ell,d)}$ from $\mathscr{G}_{(\ell,d)}$ in Sect. 2.3.

4.3 Endomorphisms in $\mathscr{G}^k_{(\ell,d)}$

The group H_k acts naturally on the set $\Lambda(\ell, d)$ of all types by permuting the indices (i.e., colors) of a type. We consider the set $\Lambda(\ell, d)/H_k$ of the corresponding orbits. For $\lambda \in \Lambda(\ell, d)$, we denote by H^λ_k the stabilizer of λ in H_k and by $\lambda^{(k)}$ the orbit of λ with respect to the action of H_k. Thus, in Example 13, the stabilizer of the type $(2, 0)$ is trivial and the orbit of this type contains one more element, namely $(0, 2)$, while the stabilizer of the type $(1, 1)$ coincides with H_2 and the orbit of $(1, 1)$ contains no other elements.

As connected components of $\mathscr{G}_{(\ell,d)}$ are in bijection with $\Lambda(\ell, d)$, connected components of $\mathscr{G}^k_{(\ell,d)}$ are in bijection with $\Lambda(\ell, d)/H_k$. To determine simple $\mathbb{C}\mathscr{G}^k_{(\ell,d)}$-modules, we need to determine the endomorphism groups of objects in $\mathscr{G}^k_{(\ell,d)}$.

Let $\lambda \in \Lambda(\ell, d)$ and $f \in \mathscr{G}_{(\ell,d)}$ be of type λ. The assignment $\sigma \rightarrow \sigma^{(k)}$ defines a homomorphism from $\mathscr{G}_{(\ell,d)}(f, f)$ to $\mathscr{G}^k_{(\ell,d)}(f^{(k)}, f^{(k)})$ which is injective, since the action of H^λ_k is free. Hence, we can view $\mathscr{G}_{(\ell,d)}(f, f)$ as a subgroup of $\mathscr{G}^k_{(\ell,d)}(f^{(k)}, f^{(k)})$.

For a generator $h \in H^\lambda_k$, consider the cyclic subgroup of $\mathscr{G}^k_{(\ell,d)}(f^{(k)}, f^{(k)})$ generated by $\sigma^{(k)}_{(f,h(f))}$. Since the action of H^λ_k is free, all orbits of h have order $|H^\lambda_k|$. It follows that the order of $\sigma^{(k)}_{(f,h(f))}$ equals $|H^\lambda_k|$, and hence, H^λ_k can be identified with the subgroup generated by $\sigma^{(k)}_{(f,h(f))}$ by mapping $h \in H^\lambda_k$ to $\sigma^{(k)}_{(f,h(f))}$.

Lemma 14 Let $\lambda \in \Lambda(\ell, d)$ and $f \in \mathscr{G}_{(\ell,d)}$ be of type λ. Then,

$$\mathscr{G}^k_{(\ell,d)}(f^{(k)}, f^{(k)}) \cong H^\lambda_k \ltimes \mathscr{G}_{(\ell,d)}(f, f). \tag{12}$$

Proof That both H^λ_k and $\mathscr{G}_{(\ell,d)}(f, f)$ are subgroups of $\mathscr{G}^k_{(\ell,d)}(f^{(k)}, f^{(k)})$ with trivial intersection follows from the construction. It is easy to check that $\mathscr{G}_{(\ell,d)}(f, f)$ is normal. It remains to show that both sides of (12) have the same cardinality. Clearly,

$$|H^\lambda_k \ltimes \mathscr{G}_{(\ell,d)}(f, f)| = |H^\lambda_k| \cdot |\mathscr{G}_{(\ell,d)}(f, f)|. \tag{13}$$

From the definition $\mathscr{G}^k_{(\ell,d)}$, we have that elements in $\mathscr{G}^k_{(\ell,d)}(f^{(k)}, f^{(k)})$ are identified with H^λ_k-orbits on the set

$$\coprod_{g',g\in H_k^\lambda f} \mathscr{G}_{(\ell,d)}(g',g).$$

Using H_k^λ, we can move g' to f, which means that elements in $\mathscr{G}_{(\ell,d)}^k(f^{(k)}, f^{(k)})$ can be identified with elements in

$$\coprod_{g\in H_k^\lambda f} \mathscr{G}_{(\ell,d)}(f,g)$$

and this set has cardinality exactly given by (13). The claim follows. $\qquad\square$

4.4 Connection to $G(\ell, k, d)$

The vector space

$$A_{(\ell,k,d)} := \bigoplus_{f^{(k)},g^{(k)}\in\mathscr{G}_{(\ell,d)}^k} \mathbb{C}\mathscr{G}_{(\ell,d)}^k(f^{(k)}, g^{(k)})$$

inherits from $\mathbb{C}\mathscr{G}_{(\ell,d)}^k$ the structure of a finite-dimensional associative algebra over $\mathbb{C}$ and there is a canonical equivalence of categories

$$\mathbb{C}\mathscr{G}_{(\ell,d)}^k\text{-mod} \cong A_{(\ell,k,d)}\text{-mod}. \tag{14}$$

Lemma 15 *There is a unique homomorphism $\Psi : A_{(\ell,k,d)} \to A_{(\ell,d)}$ of algebras such that for $f \in \mathscr{G}_{(\ell,d)}$ we have*

$$\Psi(f^{(k)}) = \sum_{g\in f^{(k)}} g.$$

Moreover, this homomorphism is injective.

Proof Uniqueness is clear as $\{f^{(k)} \mid f \in \mathscr{G}_{(\ell,d)}\}$ forms a basis of $A_{(\ell,k,d)}$. Injectivity is clear as $\{f \mid f \in \mathscr{G}_{(\ell,d)}\}$ forms a basis of $A_{(\ell,d)}$, and thus, Ψ sends linear independent elements to linear independent elements. To check that Ψ extends to a homomorphism, it is enough to check that Ψ is compatible with composition in $\mathscr{G}_{(\ell,d)}^k$. This follows directly from the definitions. $\qquad\square$

Our main observation in this section is the following.

Theorem 16 *The image of $\Phi^{-1}\Psi$ coincides with the subalgebra $\mathbb{C}[G(\ell, k, d)]$ in $\mathbb{C}[S(\ell, d)]$. In particular, the algebras $A_{(\ell,k,d)}$ and $\mathbb{C}[G(\ell, k, d)]$ are isomorphic.*

Proof As the dimensions of $A_{(\ell,k,d)}$ and $\mathbb{C}[G(\ell, k, d)]$ agree by (12), we only need to check that each element in $\mathbb{C}[G(\ell, k, d)]$ belongs to the image of $\Phi^{-1}\Psi$. For elements in S_d, this follows directly from the definitions. Therefore, it remains to check the statement for elements inside the subgroup $K \subset S(\ell, d)$, which corresponds to all diagonal matrices. This subgroup has cardinality ℓ^d and is generated by $s_0^{(j)}$, where $j = 1, 2, \ldots, d$. From (6), for $a_j \in \mathbb{Z}$, we have

$$\Phi : \prod_{j=1}^{d} \left(s_0^{(j)} \right)^{a_j} \mapsto \sum_f \xi_\ell^{\sum_j a_j f(j)} e_f .$$

The effect of the action of θ_k (which maps the color s to $s + \frac{\ell}{k}$) on the latter is

$$\sum_f \xi_\ell^{\sum_j a_j (f(j) + \frac{\ell}{k})} e_f .$$

Note that the diagonal matrix with entries $\xi_\ell^{a_j}$, where $j = 1, 2, \ldots, d$, belongs to $G(\ell, k, d)$ if and only if k divides $\sum_j a_j$. In the latter case, ℓ divides $\frac{\ell}{k} \sum_j a_j$, which implies that $\Phi(x)$ is H_k-invariant for every $x \in K \cap G(\ell, k, d)$. Note that the group $K \cap G(\ell, k, d)$ has order $\frac{\ell^d}{k}$.

Let D be the subalgebra of $A_{(\ell,d)}$ generated by all e_f, where $f \in \mathscr{G}_{(\ell,d)}$. Then, the action of H_k preserves D and is free on D. Therefore, the algebra D^{H_k} of all H_k-invariant elements in D has dimension $\frac{l^d}{k}$. Comparing this with the previous paragraph, we thus get $\Phi(\mathbb{C}[K]) = D^{H_k}$. The statement of the theorem follows.   □

4.5 Simple $\mathscr{G}_{(\ell,d)}^k$-modules versus simple $\mathscr{G}_{(\ell,d)}$-modules

Consider the set $\Lambda(\ell, d)/H_k$ and let $\Gamma \subset \Lambda(\ell, d)$ be a cross section of the H_k-orbits. Choose any $\boldsymbol{\lambda} \in \Gamma$ and $\mathbf{p} \in \mathbf{T}_\lambda$. Then, for each $z \in H_k$ we have $z \cdot \boldsymbol{\lambda} \in \Lambda(\ell, d)$ and $z \cdot \mathbf{p} \in \mathbf{T}_{z \cdot \lambda}$. Consider the $\mathscr{G}_{(\ell,d)}$-module $Q_{\mathbf{p}}$ defined as follows:

- $Q_{\mathbf{p}}(f) := \begin{cases} \mathscr{S}_{z \cdot \mathbf{p}}, & \text{if } \boldsymbol{\lambda}_f = z \cdot \boldsymbol{\lambda} \text{ for some } z \in H_k; \\ 0, & \text{if } \boldsymbol{\lambda}_f \notin \boldsymbol{\lambda}^{(k)}. \end{cases}$
- For any f, g such that $\boldsymbol{\lambda}_f = \boldsymbol{\lambda}_g = z \cdot \boldsymbol{\lambda}$ for some $z \in H_k$, any $\pi \in \mathscr{G}_{(\ell,d)}(f, g)$ and any $v \in \mathscr{S}_{z \cdot \mathbf{p}}$, we set

$$L_{\mathbf{p}}(\pi) \cdot v := \sigma_{g, z \cdot f_\lambda} \pi \sigma_{z \cdot f_\lambda, f}(v).$$

- Extend this action to the whole of $\mathbb{C}\mathscr{G}_{(\ell,d)}$ by linearity.

Comparing this with the definition of $L_{\theta_k^i(\mathbf{p})}$ in Sect. 2.4, we get an isomorphism of $\mathscr{G}_{(\ell,d)}$-modules

$$Q_{\mathbf{p}} \cong \bigoplus_{i=1}^{[H_k : H_k^\lambda]} L_{\theta_k^i(\mathbf{p})}.$$

The group H_k acts on $Q_{\mathbf{p}}$ by permuting colors and twisting the action of $\mathscr{G}_{(\ell,d)}$ accordingly. This action of H_k is free and induces a free action of H_k on the corresponding $A_{(\ell,d)}$-module (denoted with the same symbol)

$$Q_{\mathbf{p}} := \bigoplus_{f \in \mathscr{G}_{(\ell,d)}} Q_{\mathbf{p}}(f).$$

Then, H_k acts by $A_{(\ell,k,d)}$-automorphism on $Q_{\mathbf{p}}$ since $A_{(\ell,k,d)}$ is exactly the set of fixed points in $A_{(\ell,d)}$ with respect to the action of H_k.

For $m \in \underline{k}$, consider the ξ_ℓ^m-eigenspace for θ_k:

$$Q_{\mathbf{p}}^m := \{v \in Q_{\mathbf{p}} \mid \theta_k \cdot v = \xi_\ell^m v\}.$$

Since the action of H_k commutes with the action of $A_{(\ell,k,d)}$, the space $Q_{\mathbf{p}}^m$ is an $A_{(\ell,k,d)}$-submodule of $Q_{\mathbf{p}}$.

Assume that f and g are of the same type. Then, thanks to Lemma 14, every element $\sigma^{(k)} \in \mathscr{G}_{(\ell,d)}^k(f^{(k)}, g^{(k)})$ can be uniquely written in the form

$$\sigma^{(k)} = (\sigma_{(f,h(f))}^{(k)})^t \cdot \pi^{(k)} \tag{15}$$

for some $t \in \{1, 2, \ldots, |H_k^\lambda|\}$ and $\pi \in \mathscr{G}_{(\ell,d)}(f, f)$.

Translating from $A_{(\ell,k,d)}$ to $\mathscr{G}_{(\ell,d)}^{(k)}$ via the equivalence (14), we get that the $A_{(\ell,k,d)}$-module $Q_{\mathbf{p}}^m$ corresponds to the $\mathscr{G}_{(\ell,d)}^{(k)}$-module $\mathrm{L}_{(\mathbf{p},m)}$ defined as follows:

- $\mathrm{L}_{(\mathbf{p},m)}(f^{(k)}) := \begin{cases} \mathscr{S}_{\mathbf{p}}, & \text{if } f \in \boldsymbol{\lambda}^{(k)}; \\ 0, & \text{if } f \notin \boldsymbol{\lambda}^{(k)}. \end{cases}$
- $\mathrm{L}_{(\mathbf{p},m)}(f^{(k)}) := \mathscr{S}_{\mathbf{p}}$ if $f \in \boldsymbol{\lambda}^{(k)}$.
- For any f, g of type $\boldsymbol{\lambda}$, any $\sigma \in \mathscr{G}_{(\ell,d)}(f, g)$ and any $v \in \mathscr{S}_{\mathbf{p}}$ write $\sigma^{(k)}$ in the form (15) and set

$$\mathrm{L}_{(\mathbf{p},z)}(\sigma^{(k)}) \cdot v := \xi_\ell^{\frac{\ell}{|H_k^\lambda|} \cdot t \cdot m} \cdot \pi(v).$$

- Extend the action to the whole of $\mathscr{G}_{(\ell,d)}^{(k)}$ by linearity.

For $\boldsymbol{\lambda} \in \Gamma$, denote by $\mathbf{T}_{\boldsymbol{\lambda}}^{(k)}$ the set of all pairs $(\mathbf{p}, m)$, where $\mathbf{p} \vdash \boldsymbol{\lambda}$ is a multi-partition and $m \in \underline{|H_k^\lambda|}$. Our main result here is the following:

Theorem 17 *The set of all* $\mathrm{L}_{(\mathbf{p},m)}$, *where*

$$(\mathbf{p}, m) \in \bigcup_{\boldsymbol{\lambda} \in \Gamma} \mathbf{T}_{\mathbf{p}}^{(k)},$$

is a cross section of isomorphism classes of simple $\mathscr{G}_{(\ell,d)}^{(k)}$-*modules.*

Proof The fact that each $\mathrm{L}_{(\mathbf{p},m)}$ is simple is clear as $\mathrm{L}_{(\mathbf{p},m)}$ takes nonzero values on a connected component of the groupoid $\mathscr{G}_{(\ell,d)}^{(k)}$, and each nonzero $\mathrm{L}_{(\mathbf{p},m)}(f)$ is a simple module already over $\mathscr{G}_{(\ell,d)}(f, f)$, which is a subgroup of $\mathscr{G}_{(\ell,d)}^{(k)}(f, f)$.

From Theorem 16, we know that simple $\mathscr{G}_{(\ell,d)}^{(k)}$-modules correspond to simple $A_{(\ell,k,d)}$-modules and the latter can be obtained by restricting simple $A_{(\ell,d)}$-modules. The latter are classified by Proposition 3(ii). This and our construction of $\mathrm{L}_{(\mathbf{p},m)}$ above

imply that the set of all $L_{(\mathbf{p},m)}$ (for all $\mathbf{p}$ and m) contains all simple $\mathscr{G}_{(\ell,d)}^{(k)}$-module (up to isomorphism).

Finally, directly from the definition we see that $L_{(\mathbf{p},m)} \cong L_{(\mathbf{p},m+|H_k^\lambda|)}$. From Lemma 14 it follows that $L_{(\mathbf{p},m)}(f)$ for $\mathbf{p} \vdash \lambda_f$ and $m \in \underline{|H_k^{\lambda_f}|}$ is a correct indexing set for simple $\mathscr{G}_{(\ell,d)}^{(k)}(f^{(k)}, f^{(k)})$-modules (see, e.g., [35, Subsection 8.2]). This completes the proof.   □

For a general exposition of representation theory of wreath products, see [8,19,35].

Corollary 18 *Let $\lambda \in \Lambda(\ell, d)$ and $\mathbf{p} \in \mathbf{T}_\lambda$. Then,*

$$\mathrm{Res}_{A_{(\ell,k,d)}}^{A_{(\ell,d)}} L_{\mathbf{p}} \cong \bigoplus_{m \in \underline{|H_k^\lambda|}} L_{(\mathbf{p},m)}.$$

Proof This follows directly from the construction of $L_{(\mathbf{p},m)}$.   □

4.6 Schur–Weyl duality for $G(\ell, k, d)$

For $n = \ell m$ with $m > d$, take $\mathbf{p} = (m, m, \ldots, m) \in \mathbb{Z}_{>0}^\ell$ and consider the action of $\mathbf{GL_p}$ on $V = \mathbb{C}^n$ as in Sect. 3.2. Consider further the free product $\mathbf{GL_p} * \mathbb{Z}$ and let the generator 1 of $\mathbb{Z}$ act on $\mathbb{C}^n$ by mapping v_i to $v_{i+\frac{\ell}{k}m}$, with the convention that $v_s = v_{s-n}$ for $s > n$. Our main observation in this subsection is the following result (for $k = \ell$, the quantum version of this result appears in [17]).

Theorem 19 *The action of $\mathbf{GL_p} * \mathbb{Z}$ and the restricted action of $A_{(\ell,k,d)}$ on $V^{\otimes d}$ have the double centralizer property*

$$\mathrm{GL_p} * \mathbb{Z} \;\curvearrowright\; V^{\otimes d} \;\curvearrowleft\; A_{(\ell,k,d)}$$
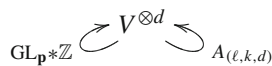
in the sense that they generate each others centralizers.

Proof Consider the action of $\mathscr{G}_{(l,k)}$ on $V^{\otimes d}$ as described in Sect. 3.3. The left action of the additional group $\mathbb{Z}$ corresponds to the permutation of colors as given by the group H_k. Therefore, the subalgebra $A_{(\ell,k,d)}$ of H_k-invariants in $A_{(\ell,d)}$ centralizes the action of $\mathbf{GL_p} * \mathbb{Z}$. Since the action of H_k is free and the action of $A_{(\ell,d)}$, and hence also of $A_{(\ell,k,d)}$, is faithful (see Sect. 3.6), the double centralizer property follows by locally comparing dimensions (as the left action increased by a factor of k, while the right action decreased by the same factor).   □

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