Cuspidal $\mathfrak{sl}_n$-modules and deformations of certain Brauer tree algebras

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Abstract

We show that the algebras describing blocks of the category of cuspidal weight (respectively generalized weight) $\mathfrak{sl}_n$-modules are one-parameter (respectively multi-parameter) deformations of certain Brauer tree algebras. We explicitly determine these deformations both graded and ungraded. The algebras we deform also appear as special centralizer subalgebras of Temperley-Lieb algebras or as generalized Khovanov algebras. They show up in the context of highest weight representations of the Virasoro algebra, in the context of rational representations of the general linear group and Schur algebras and in the study of the Milnor fiber of Kleinian singularities.

Contents

1 Introduction and description of the results

Weight modules, or, more generally, generalized weight modules, play an important role in the representation theory of semisimple complex Lie algebras. One reason for their importance is the fact that if we further assume the (generalized) weight spaces to be finite dimensional, then simple modules can be classified. No doubt, this classification theorem (finally completed by O. Mathieu in [Mat]) marks a major breakthrough in the field. Already from the results of S. Fernando ([Fe]) and V. Futorny ([Fu1, Fu2]) it was known that simple weight modules with finite dimensional weight spaces fall into two types:

- the so-called *cuspidal* modules, that is the ones which are not parabolically induced modules or equivalently on which all root vectors of the Lie algebra act bijectively ([Mat, Cor 1.4, Cor 1.5]); and
- the simple quotients of generalized Verma modules, parabolically induced from cuspidal modules.

The second type forms the bulk of simple weight modules (and also of the literature on weight modules); they are easy to classify, and their structure and Kazhdan-Lusztig type combinatorics is now relatively well understood, see [MS1], [BFL], [Maz4] and references therein. From [Fe] (see also [Mat, Prop. 1.6]) it is known that cuspidal modules only exist for the Lie algebras $\mathfrak{sl}_n$ (type A) and $\mathfrak{sp}_{2n}$ (type C), and it is the classification of simple cuspidal modules for these two series of Lie algebras, which was completed by Mathieu in [Mat].

The next natural step is thus to describe and understand the category of cuspidal (generalized) weight modules. In [BKLM] it was shown that for the algebra $\mathfrak{sp}_{2n}$ the category of cuspidal weight modules is semi-simple, hence completely understood. The very interesting rather recent paper [GS] deals with the remaining cases of weight modules and describes blocks of the category of cuspidal weight modules for the algebra $\mathfrak{sl}_n$ in terms of modules over the path algebra of a quiver with relations.

This latter paper has been the inspiration and motivation for our work, but we want to go much further. We first reprove the main result from [GS] by completely different methods, which also allow us to extend it to the category of all generalized weight modules. Moreover, we relate the associative algebras appearing in our description to (multi-parameter) deformations of certain self-injective symmetric algebras. In the case of weight modules we obtain the universal one-parameter deformation of the centralizer subalgebra A^{n-1} for a basic projective-injective module in a block of parabolic category $\mathcal{O}$ for $\mathfrak{sl}_n$.

We would like to emphasize that these algebras A^n show up in various other contexts, for instance as subalgebras of Temperley-Lieb algebras, as special examples of generalized Khovanov algebras ([BS1]), as special instances of Brauer tree algebras ([Ho1]), in the context of highest weight representations of the Virasoro algebra ([BNW]), in the context of rational representations of the general linear group and Schur algebras ([Xi]), in the study of the Milnor fiber of Kleinian singularities ([KhSe]), as a convolution algebra ([SW]) etc. It would be interesting to know which role our deformations play in these different contexts.

Our main results are then the following: let $\mathcal{C}$, $\hat{\mathcal{C}}$ be the category of finitely generated, cuspidal, weight (respectively generalized weight) $\mathfrak{sl}_n$-modules (for $n \geq 2$ fixed).

Theorem 1. *(i) Every non-integral or singular block of $\mathcal{C}$ is equivalent to the category of finite dimensional $\mathbb{C}[[x]]$-modules.*

(ii) Every non-integral or singular block of $\hat{\mathcal{C}}$ is equivalent to the category of finite dimensional $\mathbb{C}[[x_1, x_2, \dots, x_n]]$-modules.

(iii) For $n > 2$ every integral regular block of $\mathcal{C}$ is equivalent to the category of finite dimensional modules over a flat one-parameter deformation of A^{n-1} which is non-trivial as infinitesimal deformation, where A^{n-1} is the path algebra of the following quiver with $n-1$ vertices

$$\begin{tikzcd} \bullet \arrow[r, bend left, "a_1"] & \bullet \arrow[l, bend left, "b_1"] \arrow[r, bend left, "a_2"] & \bullet \arrow[l, bend left, "b_2"] \arrow[r, bend left, "a_3"] & \cdots \arrow[l, bend left, "b_3"] \arrow[r, bend left, "a_{n-2}"] & \bullet \arrow[l, bend left, "b_{n-2}"] \end{tikzcd}$$

modulo the relations $a_{i+1}a_i = 0 = b_i b_{i+1}$ and $b_i a_i = a_{i-1}b_{i-1}$ (whenever the expression makes sense) in the case $n > 3$ and $a_1 b_1 a_1 = 0 = b_1 a_1 b_1$ in the case $n = 3$. The path length induces a non-negative $\mathbb{Z}$-grading on A^{n-1}. Then the deformation in question is the unique (up to rescaling of the deformation parameter) non-trivial graded one-parameter deformation of A^{n-1}. This deformation is the completion of a Koszul algebra with respect to the graded radical.

(iv) For $n > 2$ every integral regular block of $\hat{\mathcal{C}}$ is equivalent to the category of finite dimensional modules over a flat n-parameter deformation of A^{n-1}. The associative algebra of this deformation is isomorphic to the completed tensor product of the deformation described in the previous claim (iii) *and the algebra $\mathbb{C}[[x_2, x_3, \dots, x_n]]$.*

A description of blocks of the category of cuspidal (generalized) weight modules for the Lie algebra $\mathfrak{sp}_{2n}$ was obtained recently in [MS2]. This and Theorem 1 complete the description of cuspidal generalized weight modules for semisimple Lie algebras.

The paper is organized as follows: Section 2 contains preliminaries on cuspidal modules. In Section 3 we prove the first two assertions of Theorem 1, that is we describe all singular and non-integral blocks of the category of cuspidal (generalized) weight modules. In Section 4 we show that regular blocks are described by deformations of the algebra A^{n-1} over the algebra describing the singular blocks. Finally, in Section 5 we describe the (graded and ungraded) deformation theory of these associative algebras in detail. We first give a summary of basic results on multi-parameter deformations of associative algebras (Subsections 5.1-5.5, this part is completely independent from the Lie theory) and then use it in Subsections 5.6 to complete the proof of the main theorem.

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2 Preliminaries on cuspidal modules

All algebras and categories in this paper are assumed to be over the field $\mathbb{C}$ of complex numbers. All functors are additive and $\mathbb{C}$-linear. All unspecified homomorphisms and tensor products are assumed to be over $\mathbb{C}$. We denote by $\mathbb{N}$ and $\mathbb{Z}_+$ the set of positive and nonnegative integers, respectively.

In this section we collect all necessary preliminaries on (generalized) weight modules and cuspidal modules. Note that our setup of generalized weight modules is more general than the setup of [GS] which does not allow us to refer to the setup of [GS] in a straightforward way (despite of the fact that the intersection of this introductory chapter with the introductory chapter of [GS] is substantial). Moreover, our arguments require some other techniques, in particular, that of Gelfand-Zetlin modules, which is described at the end of this section.

2.1 Weight and generalized weight $\mathfrak{sl}_n$-modules

Fix $n \in \{2, 3, \dots\}$ and consider the complex Lie algebra $\mathfrak{gl}_n$ spanned by the matrix units $e_{i,j}$, $i, j = 1, 2, \dots, n$. Let $\mathfrak{g} := \mathfrak{sl}_n$ be the Lie subalgebra of matrices with trace zero. We have the standard triangular decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ of $\mathfrak{g}$, i.e. the decomposition into a sum of strictly lower triangular, diagonal and strictly upper triangular matrices. Let $\Delta \subset \mathfrak{h}^*$ be the set of roots of $\mathfrak{g}$ and ρ the half-sum of all positive roots. If $\alpha \in \Delta$ then $\mathfrak{g}_\alpha$ denotes the root space of $\mathfrak{g}$ corresponding to α. Let W denote the Weyl group of Δ. In the following W will act on $\mathfrak{h}^*$ via the *dot action* defined by $w \cdot \lambda = w(\lambda + \rho) - \rho$.

For $i = 1, \dots, n-1$ let α_i be the simple (positive) root with the coroot $h_i = e_{i,i} - e_{i+1,i+1} \in \mathfrak{h}^*$, and let s_i be the corresponding simple reflection in W. For $\lambda \in \mathfrak{h}^*$ and $i \in \{1, 2, \dots, n-1\}$ set $\lambda_i = \lambda(h_i)$. An element $\lambda \in \mathfrak{h}^*$ is called a *weight*. It is *integral* provided that all λ_i's are integers. It is *singular*, if the cardinality of the stabilizer W_λ of λ is at least two; it is *regular* otherwise.

A $\mathfrak{g}$-module M is called a *weight module* or a *generalized weight module* if $M = \oplus_{\lambda \in \mathfrak{h}^*} M_\lambda$ or $M = \oplus_{\lambda \in \mathfrak{h}^*} M^\lambda$, respectively, where

$$M_\lambda := \{v \in M \mid hv = \lambda(h)v \text{ for all } h \in \mathfrak{h}\},$$
$$M^\lambda := \{v \in M \mid (h - \lambda(h))^k v = 0 \text{ for all } h \in \mathfrak{h} \text{ and } k \gg 0\}.$$

Here λ is called a *weight of* M with *weight space* M_λ and *generalized weight space* M^λ. Obviously, every weight module is a generalized weight module, whereas the converse need not to be true. However, from the defining relations for $\mathfrak{g}$ it follows that every simple generalized weight module is in fact a weight module. Note that $M_\lambda \subset M^\lambda$, moreover, $M_\lambda \neq 0$ if and only if $M^\lambda \neq 0$. The set $\mathrm{supp}(M) := \{\lambda \in \mathfrak{h}^* \mid M_\lambda \neq 0\}$ is called the *support* of M. A weight module M is called *pointed* provided that $\dim M_\lambda = 1$ for some λ (for indecomposable cuspidal modules the latter is equivalent to the requirement $\dim M_\lambda \leq 1$ for all λ). In the following we want to assume (following [Mat]) that all (generalized) weight spaces are finite dimensional. For examples of simple weight modules with infinite dimensional weight spaces we refer to [DFO].

From now on: all generalized weight spaces are assumed to be finite dimensional.

If M is a simple weight $\mathfrak{g}$-module, then there exists $\lambda \in \mathfrak{h}^*$ such that $\mathrm{supp}(M) \subset \lambda + \mathbb{Z}\Delta$ (in fact, for the latter it is enough to assume that M is indecomposable). If $\mathrm{supp}(M) = \lambda + \mathbb{Z}\Delta$, the M is called a *dense* module. By [DMP], if M is a simple dense module, then each nonzero element of any $\mathfrak{g}_\alpha$, $\alpha \in \Delta$, acts bijectively on M. Hence any simple dense module is cuspidal (in the sense as defined in the introduction). Obviously, every simple cuspidal $\mathfrak{g}$-module is dense.

Both categories $\mathcal{C}$ and $\hat{\mathcal{C}}$, defined in the introduction, are abelian categories and $\mathcal{C}$ is a full subcategory of $\hat{\mathcal{C}}$. Moreover, $\mathcal{C}$ and $\hat{\mathcal{C}}$ have the same simple objects, namely simple cuspidal modules, which we are going to describe in the next subsection.

2.2 Classification of simple cuspidal (dense) $\mathfrak{sl}_n$-modules

Let $U = U(\mathfrak{g})$ denote the universal enveloping algebra of $\mathfrak{g}$ with center Z. For $\lambda \in \mathfrak{h}^*$ let $L(\lambda)$ be the simple highest weight module with highest weight λ, and $\chi_\lambda : Z \to \mathbb{C}$ the algebra homomorphism (i.e. the *central character*), which defines the (scalar) action of Z on $L(\lambda)$ (see e.g. [Di, Chapter 7]). Denote by C_{dom} the closure of the dominant Weyl chamber with respect to the dot-action. Then there is a natural bijection between elements in C_{dom} and algebra homomorphisms $\chi : Z \to \mathbb{C}$, given by $C_{\mathrm{dom}} \ni \lambda \mapsto \chi_\lambda$. The action of Z preserves all (generalized) weight spaces and hence is locally finite on all modules $M \in \hat{\mathcal{C}}$. This gives a natural decomposition

$$\hat{\mathcal{C}} = \bigoplus_{\lambda \in C_{\mathrm{dom}}} \hat{\mathcal{C}}_\lambda,$$

where $\hat{\mathcal{C}}_\lambda$ denotes the full subcategory of $\hat{\mathcal{C}}$, consisting of all modules M such that $(z - \chi_\lambda(z))^k m = 0$ for all $m \in M$, $z \in Z$ and all $k \gg 0$. This decomposition induces the decomposition $\mathcal{C} = \oplus_{\lambda \in C_{\mathrm{dom}}} \mathcal{C}_\lambda$, where $\mathcal{C}_\lambda = \mathcal{C} \cap \hat{\mathcal{C}}_\lambda$. By [Mat, Section 8], the category $\hat{\mathcal{C}}_\lambda$ is nonzero in precisely the following cases:

(I) λ is regular, non-integral, and the set $\{j \mid \lambda_j \notin \mathbb{Z}_+\}$ coincides either with $\{1\}$ or $\{n-1\}$ or with $\{i, i+1\}$ for some $i \in \{1, 2, \dots, n-2\}$, moreover $\lambda_i + \lambda_{i+1} \in \{-1, 0, 1, 2, \dots\}$ in the latter case;

(II) λ is integral and singular and its stabilizer with respect to the dot action is $\langle s_i \rangle$ for some $i \in \{1, 2, \dots, n-1\}$;

(III) λ is regular and integral.

To describe simple cuspidal modules in each case we need an additional tool from [Mat]. Let $R \subset \Delta$ be a nonempty set of roots such that

$$\alpha, \beta \in R \Rightarrow \alpha + \beta \notin \Delta \cup \{0\}.$$

For every $\alpha \in R$ we fix a non-zero root vector $X_\alpha \in \mathfrak{g}_\alpha$. Note that $X_\alpha X_\beta = X_\beta X_\alpha$ for all $\alpha, \beta \in R$. Consider the multiplicative subset $S(R)$ of U, consisting of all elements of the form $\prod_{\alpha \in R} X_\alpha^{m_\alpha}$, where all $m_\alpha \in \mathbb{Z}_+$. The set $S(R)$ is an Ore subset of U and hence we have the corresponding Ore localization $U_{S(R)}$ (note that the localization is independent, up to isomorphism, of the choice of the root vectors). By [Mat, Lemma 4.3] there is a unique polynomial map on $\mathbb{C}^{|R|}$ with values in algebra automorphisms of $U_{S(R)}$, $\mathbf{x} = (x_\alpha)_{\alpha \in R} \mapsto \Phi_\mathbf{x}^R$, such that in case $\mathbf{x}$ is a tuple of integers we have

$$\Phi_\mathbf{x}^R(u) = \prod_{\alpha \in R} X_\alpha^{-x_\alpha} \cdot u \cdot \prod_{\alpha \in R} X_\alpha^{x_\alpha} \quad \text{for all} \quad u \in U_{S(R)}.$$

If φ is an automorphism of some Ring $\mathcal{R}$ and M an $\mathcal{R}$-module we denote by M^φ the module M with the $\mathcal{R}$-module action twisted by φ. Note that every cuspidal module is automatically a $U_{S(R)}$-module. Now the main classification result of [Mat] is the following:

Theorem 2 (Classification theorem)**.** *In all cases* (I)-(III)*, any simple module in $\hat{\mathcal{C}}_\lambda$ is isomorphic to a module of the form $\left(U_{S(R)} \otimes_U L(\mu)\right)^{\Phi_\mathbf{x}^R}$ for appropriate μ, $\mathbf{x}$, R. In cases* (I) *and* (II) *it is $\mu = \lambda$, whereas in case* (III) *we have $\mu = w \cdot \lambda$ with $w \in \{s_1, s_1 s_2, \dots, s_1 s_2 \cdots s_{n-2} s_{n-1}\} =: W^{\text{short}}$, the set of shortest coset representatives in $S_{n-1} \backslash S_n$ different from the identity.*

In each case the category $\hat{\mathcal{C}}_\lambda$ has infinitely many isomorphism classes of simple objects (as the module $L(\mu)^{\Phi_\mathbf{x}^R}$ is generically simple and there are infinitely many $\mathbf{x}$). To get rid of this problem one can decompose further: for $\xi \in \mathfrak{h}^*/\mathbb{Z}\Delta$ let $\hat{\mathcal{C}}_{\lambda,\xi}$ denote the full subcategory of $\hat{\mathcal{C}}_\lambda$, which consists of all M such that $\mathrm{supp}(M) = \xi$. Then we have the decomposition $\hat{\mathcal{C}}_\lambda = \oplus_\xi \hat{\mathcal{C}}_{\lambda,\xi}$ and the induced decomposition $\mathcal{C}_\lambda = \oplus_\xi \mathcal{C}_{\lambda,\xi}$. From [Mat, Section 11] and Theorem 2 one obtains the following:

Corollary 3. *(i) In cases* (I) *and* (II)*, the category $\hat{\mathcal{C}}_{\lambda,\xi}$ has at most one isomorphism class of simple modules.*

(ii) In case (III) *every nonzero $\hat{\mathcal{C}}_{\lambda,\xi}$ (and $\mathcal{C}_{\lambda,\xi}$) is indecomposable and has exactly $n-1$ isomorphism classes of simple modules, naturally in bijection with W^{short}.*

2.3 Reduction to special blocks

As shown in [Mat, Section11], Corollary 3 describes and indexes *blocks* (i.e. indecomposable direct summands) of the categories $\hat{\mathcal{C}}$ and $\mathcal{C}$. The goal of this paper is to describe associative algebras, whose categories of finite dimensional modules are equivalent to $\hat{\mathcal{C}}_{\lambda,\xi}$ or $\mathcal{C}_{\lambda,\xi}$. In this subsection we reduce this problem to some very special blocks. We would like to start with the following easy but very important observation, which gets rid of the parameter ξ:

Proposition 4. *Assume we are in one of the cases* (I)–(III) *and* $\xi_1, \xi_2 \in \mathfrak{h}^*/\mathbb{Z}\Delta$. *If* $\hat{\mathcal{C}}_{\lambda,\xi_k}$ *is nonzero for* $k = 1, 2$ *then we have equivalences of categories*

$$\hat{\mathcal{C}}_{\lambda,\xi_1} \cong \hat{\mathcal{C}}_{\lambda,\xi_2}, \qquad \mathcal{C}_{\lambda,\xi_1} \cong \mathcal{C}_{\lambda,\xi_2}.$$

Proof. Let R be as given by Theorem 2, then R generates $\mathfrak{h}^*$. Choose arbitrary $\nu_i \in \xi_i$, $i = 1, 2$. From the polynomiality of $\Phi^R_{\mathbf{x}}$ it follows that there exists $\mathbf{x}$ such that every $h \in \mathfrak{h}$ is mapped by $\Phi^R_{\mathbf{x}}$ to $h + (\nu_2 - \nu_1)(h)$. Indeed, one directly verifies the formula

$$\Phi^R_{\mathbf{x}}(h) = \prod_{\alpha \in R} X_\alpha^{-x_\alpha}\, h \prod_{\alpha \in R} X_\alpha^{x_\alpha} = h + \sum_{\alpha \in R} x_\alpha \alpha(h)$$

and then chooses $\mathbf{x}$ accordingly (which is possible as R generates $\mathfrak{h}^*$). Such $\Phi^R_{\mathbf{x}}$ thus maps $\hat{\mathcal{C}}_{\lambda,\xi_1}$ to $\hat{\mathcal{C}}_{\lambda,\xi_2}$ and is an equivalence with inverse $\Phi^R_{-\mathbf{x}}$. As $\Phi^R_{\mathbf{x}}$ maps linear polynomials over $\mathfrak{h}$ to linear polynomials, it restricts to an equivalence from $\mathcal{C}_{\lambda,\xi_1}$ to $\mathcal{C}_{\lambda,\xi_2}$. $\square$

Further reduction is given by the following statement, which, in particular, extends [GS, Lemma 2.8] to the situation of generalized weight modules:

Proposition 5. *Suppose we are given* λ_k, $k = 1, 2$, *in the situation of either of the cases* (I)-(III). *If* $\lambda_1 - \lambda_2$ *is integral, then there are equivalences of categories*

$$\hat{\mathcal{C}}_{\lambda_1} \cong \hat{\mathcal{C}}_{\lambda_2}, \qquad \mathcal{C}_{\lambda_1} \cong \mathcal{C}_{\lambda_2}.$$

Proof. Assume that we are in case (III) or in case (I). Then the desired equivalence is well-known: it is given by the projective functor (see [BG]) (=translation functor in the sense of [Ja]) $\mathrm{T}_{\lambda_1}^{\lambda_2}$, viewed as a functor from $\mathcal{C}_{\lambda_1}$ to $\mathcal{C}_{\lambda_2}$ or as a functor from $\hat{\mathcal{C}}_{\lambda_1}$ to $\hat{\mathcal{C}}_{\lambda_2}$, respectively. The same argument applies in case (II) provided that λ_1 has the same stabilizer as λ_2. In case the stabilizers are different, we may assume that they are generated by s_{j_1} and s_{j_2} for some j_1 and j_2 from $\{1, 2, \dots, n-1\}$. In case $j_2 = j_1 \pm 1$, the equivalence is given by projective functors as in [Ja, 5.9] (see Subsection 4.3 for more details). The general case follows by composing $|j_2 - j_1|$ such equivalences. $\square$

Corollary 6. *Assume we are given* $\mathcal{C}_{\lambda_1}$ *of type as in case* (I) *or* (II). *Then one can find* $\mathcal{C}_{\lambda_2}$ *of the same type such that*

(a) $\hat{\mathcal{C}}_{\lambda_1} \cong \hat{\mathcal{C}}_{\lambda_2}$ *and* $\mathcal{C}_{\lambda_1} \cong \mathcal{C}_{\lambda_2}$, *and*

(b) all simple modules in $\hat{\mathcal{C}}_{\lambda_2}$ *are pointed.*

Proof. By [BL2, Theorem 4.2], every simple cuspidal $\mathfrak{g}$-module is a submodule of a tensor product of a pointed cuspidal module and a finite dimensional module. This implies the existence of some λ_2 such that simple modules in $\mathcal{C}_{\lambda_2}$ are pointed and the difference $\lambda_1 - \lambda_2$ is integral. Then the claim follows from Proposition 5. $\square$

2.4 Realization of pointed cuspidal modules

Here we present a realization of pointed cuspidal modules from [BL1]. For $\mathbf{a} = (a_1, a_2, \ldots, a_n) \in \mathbb{C}^n$ consider the complex vector space $N(\mathbf{a})$ with the formal basis

$$\{\mathbf{v_b} = x_1^{a_1+b_1} x_2^{a_2+b_2} \cdots x_n^{a_n+b_n} \mid b_i \in \mathbb{Z} \text{ for all } i, \text{ and } b_1 + b_2 + \cdots + b_n = 0\}.$$

Then $N(\mathbf{a})$ can be turned into a $\mathfrak{g}$-module by restricting the natural $\mathfrak{gl}_n$-action in which the matrix unit $e_{i,j}$ acts as the differential operator $x_i \frac{\partial}{\partial x_j}$. More precisely, if $\{\varepsilon_i : i = 1, 2, \ldots, n\}$ denotes the standard basis of $\mathbb{C}^n$, then clearly $\frac{\partial}{\partial x_j} \mathbf{v_b} = (a_j + b_j)\mathbf{v}_{\mathbf{b}-\varepsilon_j}$, and

$$e_{i,j} \cdot \mathbf{v_b} = (a_j + b_j)\mathbf{v}_{\mathbf{b}+\varepsilon_i-\varepsilon_j}. \tag{1}$$

These modules exhaust the simple pointed cuspidal ones:

Theorem 7 ([BL1])**.** *Every simple pointed cuspidal $\mathfrak{g}$-module is isomorphic to $N(\mathbf{a})$ for some $\mathbf{a}$ as above.*

Note that (1) defines a cuspidal module if and only if $a_i \notin \mathbb{Z}$ for all i.

2.5 Connection to associative algebras

The categories $\hat{\mathcal{C}}_{\lambda,\xi}$ and $\mathcal{C}_{\lambda,\xi}$ are length categories containing finitely many isomorphism classes of simple objects. Hence by abstract nonsense (see for example [Ga, Section 7]), these categories are equivalent to categories of finite length modules over some complete associative algebras $\hat{D}_{\lambda,\xi}$ and $D_{\lambda,\xi}$, respectively. The latter algebra is a quotient of the former, since $\mathcal{C}_{\lambda,\xi}$ is a subcategory of $\hat{\mathcal{C}}_{\lambda,\xi}$. We will show that these categories are Ext-finite in the sense that $\mathrm{Ext}^1(M, N)$ is finite dimensional for any objects M and N. For $k \in \mathbb{N}$ we denote by $\hat{\mathcal{C}}^k_{\lambda,\xi}$ the full subcategory of $\hat{\mathcal{C}}_{\lambda,\xi}$ given by all objects with Loewy lengths at most k. Analogously we define $\mathcal{C}^k_{\lambda,\xi}$. These categories contain then enough projectives and so are equivalent to module categories over some finite dimensional algebras (the opposite of the endomorphism algebras of a minimal projective generator). The algebras $\hat{D}_{\lambda,\xi}$ and $D_{\lambda,\xi}$ are then limits of these finite-dimensional algebras corresponding to the directed systems given by the natural inclusions $\hat{\mathcal{C}}^k_{\lambda,\xi} \subset \hat{\mathcal{C}}^{k+1}_{\lambda,\xi}$ and $\mathcal{C}^k_{\lambda,\xi} \subset \mathcal{C}^{k+1}_{\lambda,\xi}$, respectively.

2.6 Gelfand-Zetlin realization

In this subsection we recall the realization of simple cuspidal modules from [Maz2], which uses the Gelfand-Zetlin approach. For $i \in \{2, 3, \dots\}$ set $\mathfrak{g}_i = \mathfrak{sl}_i$, $U_i = U(\mathfrak{g}_i)$ and let Z_i denote the center of U_i. We consider $\mathfrak{g}_i$ as a subalgebra of $\mathfrak{g}_{i+1}$ with respect to the embedding into the left upper corner. Let $\Gamma = \Gamma_n$ denote the subalgebra of $U = U_n$, generated by $\mathfrak{h}$ and all Z_i, $i \leq n$. The algebra Γ is called the *Gelfand-Zetlin* subalgebra of U ([DFO]). The algebra Γ is a maximal commutative subalgebra of U and is isomorphic to the polynomial algebra in $\frac{n(n+1)}{2} - 1$ variables (as generators one could use any basis of $\mathfrak{h}$ and any set of generators for each Z_i, which, in turn, is a polynomial algebra in $i-1$ variables). By [Ov, Lemma 3.2], U is free both as left and as right Γ-module.

For a $\mathfrak{g}$-module M and a homomorphism $\chi : \Gamma \to \mathbb{C}$ set

$$M_\chi = \{v \in M | (g - \chi(g))^k v = 0 \text{ for all } g \in \Gamma \text{ and } k \gg 0\}.$$

The module M is called a *Gelfand-Zetlin* module provided that $M = \oplus_\chi M_\chi$ and all M_χ are finite dimensional. As Γ is commutative and contains $\mathfrak{h}$, any generalized weight module with finite-dimensional weight spaces, in particular, any module from $\hat{\mathcal{C}}$, is obviously a Gelfand-Zetlin module. Conversely, any Gelfand-Zetlin module is weight, but with infinite dimensional weight spaces in general (see [DFO] for details).

A character $\chi : \Gamma \to \mathbb{C}$ is usually described by the corresponding *Gelfand-Zetlin tableau*, that means an equivalence class of patterns of the form

$$\begin{array}{cccccc}
\boxed{m_1} & \boxed{m_2} & \boxed{m_3} & \boxed{m_4} & \boxed{\cdots} & \boxed{m_n} \\
\boxed{y_{n\text{-}1}} & \boxed{a_2^{n\text{-}1}} & \boxed{a_3^{n\text{-}1}} & \boxed{\cdots} & \boxed{a_{n\text{-}1}^{n\text{-}1}} & \\
\boxed{y_{n\text{-}2}} & \boxed{a_2^{n\text{-}2}} & \boxed{\cdots} & \boxed{a_{n\text{-}2}^{n\text{-}2}} & & \\
\cdots & \cdots & & & & \\
\boxed{y_2} & \boxed{a_2^2} & & & & \\
\boxed{y_1} & & & & &
\end{array} \tag{2}$$

where entries are complex numbers. Two patterns are in the same equivalence class if they differ by a permutation which only permutes entries inside rows (hence keeps the multiset of entries in each row fixed). Given such a tableau, the corresponding values of χ on standard generators of Γ are computed as certain shifted symmetric functions in the entries of the tableau, see [DFO] for details. We will not need these explicit formulae. The number of entries in a tableau exceeds the number of generators of Γ by one. This is due to the fact that the Gelfand-Zetlin combinatorics is usually used for the Lie algebra $\mathfrak{gl}_n$ (insted of $\mathfrak{sl}_n$) where we have an extra central element.

One of the main advantages of Gelfand-Zetlin modules is that many Gelfand-Zetlin modules admit a so-called *tableau realization*, that is an explicit combinatorial construction in which a basis of the module is indexed by a set of tableaux (defined by imposing some conditions on the entries) and the action of the generators $e_{i,i+1}$ and $e_{i+1,i}$ of $\mathfrak{g}$ is given by the so-called *Gelfand-Zetlin formulae*, see [DFO], [Maz1], [Maz2] for details. In such a realization the tableaux represents a basis of common eigenvectors for Γ (as mentioned above, the corresponding eigenvalues are computed as certain symmetric functions), and the action of $e_{i,i+1}$ (resp. $e_{i+1,i}$) maps a basis vector corresponding to a tableau t to a linear combination (with some coefficients) of basis vectors corresponding to tableaux obtained from t by adding (resp. subtracting) the complex number 1 to one of the entries in the i-th row from the bottom. The coefficients can be expressed as certain rational functions in the entries of the i-th and $(i + 1)$-st (resp. the i-th and $(i - 1)$-st) rows. Again, we will not need these explicit formulae and refer to [DFO] and [Maz2] for details. The Gelfand-Zetlin formulae have denominators in which all possible differences between entries in the same row (up to row $n-1$) occur. Hence a necessary condition for the existence of a tableau realization is that we only use tableaux without multiple entries in the rows 2 to $n-1$.

Examples of modules that have such a realization include almost all simple cuspidal modules. Choose arbitrary $\mathbf{m} = (m_1, m_2, \dots, m_n) \in \mathbb{C}^n$ and $\mathbf{x} = (x_1, x_2, \dots, x_{n-1}) \in \mathbb{C}^{n-1}$ such that $m_i - m_{i+1} \in \mathbb{N}$ for all $i > 1$, $x_i - x_{i+1} \notin \mathbb{Z}$ for all i, $x_{n-1} - m_1 \notin \mathbb{Z}$ and $x_i - m_2 \notin \mathbb{Z}$ for all i. Let $T(\mathbf{m}, \mathbf{x})$ consist of all tableaux of the form (2) such that the following conditions are satisfied: $y_i - x_i \in \mathbb{Z}$, $a^i_j - a^{i-1}_j \in \mathbb{Z}_+$, $a^{i-1}_j - a^i_{j+1} \in \mathbb{N}$, $m_i - a^{n-1}_i \in \mathbb{Z}_+$ and $a^{n-1}_i - m_{i+1} \in \mathbb{N}$ whenever the expression makes sense. Let $M(\mathbf{m}, \mathbf{x})$ denote the vector space with basis $T(\mathbf{m}, \mathbf{x})$. Then Gelfand-Zetlin formulae define on $M(\mathbf{m}, \mathbf{x})$ the structure of a cuspidal $\mathfrak{g}$-module ([Maz2]). The vector $\mathbf{m}$ corresponds to λ. If λ is non-integral (i.e. $m_1 - m_2 \notin \mathbb{Z}$) or if λ is singular (i.e. $m_1 = m_s$ for some $s > 1$), then the module $M(\mathbf{m}, \mathbf{x})$ is simple. It is pinned provided that $m_i = m_{i+1} + 1$ for all $i > 1$.

For integral regular λ (i.e. $m_1 - m_2 \in \mathbb{Z}$, $m_1 \neq m_s$ for all $s > 1$) the situation is slightly more complicated. In this case the module $M(\mathbf{m}, \mathbf{x})$ is always indecomposable. It is simple if only if $m_1 - m_2 \in \mathbb{N}$ or $m_n - m_1 \in \mathbb{N}$. In all other cases this module has length two. Its unique proper submodule is the linear span of all tableaux satisfying the additional condition as follows: Let $s \in \{2, 3, \dots, n-1\}$ be such that $m_s - m_1 \in \mathbb{N}$ and $m_1 - m_{s+1} \in \mathbb{N}$. Set $k_1 = m_s$, $k_2 = m_2$, $k_3 = m_3, \dots$, $k_{s-1} = m_{s-1}$, $k_s = m_1$, $k_{s+1} = m_{s+1}, \dots$, $k_n = m_n$. In this notation the additional condition reads: $k_i - a^{n-1}_i \in \mathbb{Z}_+$ and $a^{n-1}_i - k_{i+1} \in \mathbb{N}$ whenever the expression makes sense. The quotient has a basis given by all other tableaux.

3 Blocks in cases (I) and (II)

Our main result in this section is the following statement:

Theorem 8. *Let λ be as in case* (I) *or* (II) *and ξ be such that $\hat{\mathcal{C}}_{\lambda,\xi}$ is nonzero.*

(i) The category $\hat{\mathcal{C}}_{\lambda,\xi}$ is equivalent to the category of finite dimensional $\mathbb{C}[[x_1, x_2, \dots, x_n]]$-modules.

(ii) The category $\mathcal{C}_{\lambda,\xi}$ is equivalent to the category of finite dimensional $\mathbb{C}[[x]]$-modules.

Theorem 8 says that $\hat{D}_{\lambda,\xi} \cong \mathbb{C}[[x_1, x_2, \dots, x_n]]$ and $D_{\lambda,\xi} \cong \mathbb{C}[[x]]$. The second statement of Theorem 8 is contained in [GS, Section 5]. For $n = 2$ both statements are true for all λ (even for case (III)). This special case is at least partly an unpublished result of P. Gabriel (see [Di, 7.8.16]), and is completely contained in [Dr] (see e.g. [Maz3, Chapter 3] for detailed proofs).

The categories $\hat{\mathcal{C}}_{\lambda,\xi}$ (and then also $\mathcal{C}_{\lambda,\xi}$) appearing in Theorem 8 contain a unique (up to isomorphism) simple object thanks to Corollary 3. By Corollary 6, we may assume (for the rest of this section) this simple module to be pointed, and so isomorphic to $N(\mathbf{a})$ for some $\mathbf{a} \in \mathbb{C}^n$, as described in Subsection 2.4. We are going to prove claim (i) and then deduce claim (ii) by restriction. We first need some preparation, the actual proof will then finally appear in Subsection 3.3. As an immedeate corollary we obtain the following statement which reproves (by completely different arguments) and strengthens [GS, Theorem 4.12].

Corollary 9. *(i) Let L is a simple object in $\hat{\mathcal{C}}^n_{\lambda,\xi}$ then $\dim \mathrm{Ext}^1_{\mathfrak{g}}(L,L) = n$, in particular, this space is finite dimensional.*

(ii) For any $M, N \in \hat{\mathcal{C}}_{\lambda,\xi}$ we have $\mathrm{Ext}^1_{\mathfrak{g}}(M,N) < \infty$.

Proof. As $\hat{\mathcal{C}}_{\lambda,\xi}$ is extension closed in $\mathfrak{g}$-mod, claim (i) follows directly from Theorem 8 and the corresponding claim for $\mathbb{C}[[x_1, x_2, \dots, x_n]]$. Claim (ii) follows from claim (i) by standard induction on the lengths of M and N. $\square$

3.1 A functor F from $\mathbb{C}[[x_1, x_2, \dots, x_n]]$-modules to $\hat{\mathcal{C}}_{\lambda,\xi}$

Let V be a finite dimensional $\mathbb{C}[[x_1, x_2, \dots, x_n]]$-module, hence x_i acts via some endomorphism which we call X_i. For every $\mathbf{b} \in \mathbb{Z}^n$, $b_1 + b_2 + \dots + b_n = 0$, fix a copy, $V_{\mathbf{b}}$, of V and consider the vector space $\mathrm{F}V := \oplus_{\mathbf{b}} V_{\mathbf{b}}$.

Lemma 10. *The space $\mathrm{F}V$ can be turned into a $\mathfrak{g}$-module, where $e_{i,j}$ acts via $v \mapsto (X_j + (a_j + b_j)\mathrm{Id}_V)v \in V_{\mathbf{b}+\varepsilon_i-\varepsilon_j}$, and h_i acts via*

$$v \mapsto (X_i - X_{i-1} + (a_i + b_i - a_{i+1} - b_{i+1})\mathrm{Id}_V)v \in V_{\mathbf{b}}$$

for $v \in V_{\mathbf{b}}$. Obviously, every $v \in V_{\mathbf{b}}$ annihilated by all X_i's is a weight vector.

Proof. Consider the $\mathfrak{g}$-module $N(\mathbf{a})$ for $\mathbf{a}$ as above. Then, for every $\mathbf{b}$ the defining relations of $\mathfrak{g}$ (in generators $e_{i,i\pm1}$), applied to $\mathbf{v}_{\mathbf{b}}$, can be written as some polynomial equations in the a_i's. Since (1) defines a $\mathfrak{g}$-module for any $\mathbf{a}$, these equations hold for any $\mathbf{a}$, that is they are actually formal identities. Write now $X_j + (a_j + b_j)\mathrm{Id}_V = A_j + B_j$, a sum of matrices, where $A_j = X_j + a_j\mathrm{Id}_V$ and $B_j = b_j\mathrm{Id}_V$. Note that A_j and B_j commute. Now the defining relations for $\mathfrak{g}$ on $\mathrm{F}V$ reduce to our formal identities and hence are satisfied. The formula for the action of h_i is obtained by a direct computation. $\square$

Let V and V' be two finite dimensional $\mathbb{C}[[x_1, x_2, \dots, x_n]]$-modules and let $f : V \to V'$ be a homomorphism. Then f extends diagonally to a $\mathbb{C}$-linear map $\mathrm{F}f : \mathrm{F}V \to \mathrm{F}V'$. As f commutes with all X_i, the map $\mathrm{F}f$ commutes with all $e_{i,j}$ and hence defines a homomorphism of $\mathfrak{g}$-modules. As a consequence, F becomes a functor from the category of finite dimensional $\mathbb{C}[[x_1, x_2, \dots, x_n]]$-modules to the category of $\mathfrak{g}$-modules. By construction, F is exact and faithful. Furthermore, it sends the simple one-dimensional $\mathbb{C}[[x_1, x_2, \dots, x_n]]$-module to $N(\mathbf{a})$, which is an object of $\hat{\mathcal{C}}_{\lambda,\xi}$. As $\hat{\mathcal{C}}_{\lambda,\xi}$ is extension closed in $\mathfrak{g}$-mod, we get that the image of F belongs to $\hat{\mathcal{C}}_{\lambda,\xi}$. Hence we have a functor from the category of finite dimensional $\mathbb{C}[[x_1, x_2, \dots, x_n]]$-modules to $\hat{\mathcal{C}}_{\lambda,\xi}$.

Let $\mu_{\mathbf{b}}$ denote the weight of $\mathbf{v}_{\mathbf{b}}$. Note that $\mathbf{b} \neq \mathbf{b}'$ implies $\mu_{\mathbf{b}} \neq \mu_{\mathbf{b}'}$. It follows that $V_{\mathbf{b}}$ is the generalized weight space of $\mathrm{F}V$ of weight $\mu_{\mathbf{b}}$. If $\varphi : \mathrm{F}V \to \mathrm{F}V'$ is a $\mathfrak{g}$-homomorphism, it must preserve the weight spaces and hence it induces a linear map $f : V \to V'$ (on the component with $\mathbf{b} = \mathbf{0}$). As φ commutes with all h_i, the map f commutes with all operators $X_i - X_{i+1}$. As φ commutes with the element $C = (h_1 + 1)^2 + 4e_{21}e_{12}$, the map f commutes with the operator $((a_1 + a_2 + 1)\mathrm{Id}_V + X_1 + X_2)^2$ and thus with all polynomials in this operator. If $a_1 + a_2 \notin \mathbb{Z}$ (which we may assume by Proposition 4), then the latter operator is invertible and we get that f commutes with the polynomial square root of it and hence with $X_1 + X_2$. This implies that f commutes with all X_i. Therefore $\varphi = \mathrm{F}f$ and thus the functor F is full.

To prove that F is an equivalence, we are left to show that it is dense (i.e. essentially surjective). We first consider the case $n = 3$.

3.2 Commutativity of $D_{\lambda,\xi}$ and $\hat{D}_{\lambda,\xi}$ for $\mathfrak{sl}_3$

In this subsection we assume that $\mathfrak{g} = \mathfrak{sl}_3$ and prove the following statements:

Proposition 11. *The algebras $D_{\lambda,\xi}$ and $\hat{D}_{\lambda,\xi}$ are commutative.*

Corollary 12. *Let U_0 denote the centralizer of $\mathfrak{h}$ in U, $x, y \in U_0$, $M \in \hat{\mathcal{C}}_{\lambda,\xi}$ and $m \in M$. Then we have $x(y(m)) = y(x(m))$.*

Corollary 13. *Let L be a simple object in $\hat{\mathcal{C}}_{\lambda,\xi}$ then* $\dim \mathrm{Ext}^1_{\mathfrak{g}}(L, L) < \infty$.

Consider the subalgebra Γ of U as defined in Subsection 2.6. If M is a weight $\mathfrak{g}$-module, then every M^μ becomes a finite dimensional Γ-module. For the category Γ-mod of finite dimensional Γ-modules we have the usual decomposition

$$\Gamma\text{-mod} \cong \bigoplus_{\chi:\Gamma\to\mathbb{C}} \Gamma_\chi\text{-mod},$$

where Γ_χ-mod denotes the full subcategory of Γ-mod, which consists of all M annihilated by some power of $\mathrm{Ker}(\chi)$. The category Γ_χ-mod is equivalent to the category of finite dimensional modules over the completion Γ_χ of Γ with respect to the kernel $\ker(\chi)$ of χ.

As U is free as a right Γ-module, the induction functor

$$\mathrm{Ind}_\Gamma^U := U \otimes_\Gamma \_ : \Gamma\text{-mod} \to U\text{-mod},$$

from the category Γ-mod to the category U-mod of finitely generated U-modules is exact and sends nonzero modules to nonzero modules. From [Ov] it even follows that any module in the image of Ind_Γ^U has finite length.

Fix some $\mu \in \xi$. If L is a simple object in $\hat{\mathcal{C}}_{\lambda,\xi}$, then L_μ is one-dimensional (as we assume that L is pointed). Since $\Gamma L_\mu \subset L_\mu$, it follows that L_μ is a simple Γ-module, which corresponds to some character $\chi = \chi_\mu : \Gamma \to \mathbb{C}$. All other nonzero weight spaces of L are also simple Γ-modules, but not isomorphic to L_μ (as Γ contains $\mathfrak{h}$). As any object in $\hat{\mathcal{C}}_{\lambda,\xi}$ has a filtration with subquotients isomorphic to L, it follows that the assignment

$$\hat{\mathcal{C}}_{\lambda,\xi} \ni M \mapsto M_\mu \in \Gamma_\chi\text{-mod}$$

defines a functor G_χ from $\hat{\mathcal{C}}_{\lambda,\xi}$ to Γ_χ-mod. The functor G_χ is by definition exact and sends simple modules to simple modules. From the classical adjunction between induction and restriction, the left adjoint of G_χ is the functor F_χ, which is the composition of Ind_Γ^U followed by taking the maximal quotient, which belongs to $\hat{\mathcal{C}}_{\lambda,\xi}$ (the latter is well-defined as the induced module has finite length and $\hat{\mathcal{C}}_{\lambda,\xi}$ is extension closed in U-mod).

Lemma 14. *The functor F_χ sends simple modules to simple modules.*

Proof. Let $L \in \hat{\mathcal{C}}_{\lambda,\xi}$ be simple. Then L_μ is a simple Γ-module and we have

$$\mathrm{Hom}_{\mathfrak{g}}(\mathrm{Ind}_\Gamma^U L_\mu, L) \cong \mathrm{Hom}_\Gamma(L_\mu, L) \cong \mathbb{C}. \tag{3}$$

Moreover, we claim that the multiplicity of L in $\mathrm{Ind}_\Gamma^U L_\mu$ is one, which then implies $\mathrm{F}_\chi L_\mu \cong L$ and completes the proof. Because of (3) we only have to show that this multiplicity is at most one. The verification of this claim is a technical computation using the Gelfand-Zetlin combinatorics for $\mathfrak{sl}_3$. Our argument does not generalize to $\mathfrak{sl}_n$, $n > 3$.

As explained in Subsection 2.6, the character $\chi = \chi_\mu$ is represented by a tableau of the following form:

$$\begin{array}{|c|c|c|}
\hline
a & b & b-1 \\
\hline
\end{array}$$
$$\begin{array}{|c|c|}
\hline
x & b \\
\hline
\end{array}$$
$$\begin{array}{|c|}
\hline
z \\
\hline
\end{array} \tag{4}$$

where $a, b, x, y \in \mathbb{C}$ and $z - x, z - b, x - a, x - b \notin \mathbb{Z}$. For any $\nu \in \mathrm{supp}(L)$ the corresponding χ_ν is represented by a tableau τ' with the same a, b and some x', z' such that $x - x' \in \mathbb{Z}$ and $z - z' \in \mathbb{Z}$.

Simple Γ-subquotients appearing in the module $\mathrm{Ind}_\Gamma^U L_\mu$ (considered as Γ-module) are indexed (with the corresponding multiplicities) by tableaux obtained from τ by adding arbitrary integers in the two bottom rows, that is to x, b and z (see [Ov, Section 4]). As $x - b \notin \mathbb{Z}$, it follows that τ itself appears in this set only once (namely when we add zeros). This implies that the multiplicity of L in $\mathrm{Ind}_\Gamma^U L_\mu$ is at most one (note that the latter is not true for $n > 3$). □

Lemma 15. *The adjunction morphisms*

$$\mathrm{adj} : \mathrm{F}_\chi \mathrm{G}_\chi \to \mathrm{ID}_{\hat{\mathcal{C}}_{\lambda,\xi}} \quad \textit{and} \quad \mathrm{adj}' : \mathrm{ID}_{\Gamma_\chi\text{-mod}} \to \mathrm{G}_\chi \mathrm{F}_\chi$$

are surjective.

Proof. If L is a simple object of $\hat{\mathcal{C}}_{\lambda,\xi}$, then $\mathrm{G}_\chi L$ is simple, in particular, nonzero. Therefore, adj_L is nonzero (as the image of the nonzero identity morphism on $\mathrm{G}_\chi L$ under the adjunction isomorphism), hence surjective. The claim about adj follows now by induction on the length of a module and the Five-Lemma. The second statement follows completely analogously using Lemma 14. □

For every $k \in \mathbb{N}$ denote by $\Gamma_\chi\text{-mod}^k$ and $\hat{\mathcal{C}}^k_{\lambda,\xi}$ the full subcategories of Γ_χ-mod and $\hat{\mathcal{C}}_{\lambda,\xi}$, respectively, which consist of all modules of Loewy length at most k. Then $\Gamma_\chi\text{-mod}^k$ is equivalent to the category of finite dimensional modules over the finite dimensional algebra $\Gamma_\chi/(\ker(\chi))^k$. The category $\hat{\mathcal{C}}^k_{\lambda,\xi}$ is an abelian length category.

Corollary 16. *(i) The pair $(\mathrm{F}_\chi, \mathrm{G}_\chi)$ restricts to an adjoint pair of functors between $\Gamma_\chi\text{-mod}^k$ and $\hat{\mathcal{C}}^k_{\lambda,\xi}$.*

(ii) The functor F_χ sends projective modules from $\Gamma_\chi\text{-mod}^k$ to projective modules in $\hat{\mathcal{C}}^k_{\lambda,\xi}$.

Proof. The exactness of G_χ implies that $\hat{\mathcal{C}}^k_{\lambda,\xi}$ gets mapped to Γ_χ-mod^k. That F_χ maps Γ_χ-mod^k to $\hat{\mathcal{C}}^k_{\lambda,\xi}$ follows from Lemma 14. This implies claim (i). Then the functor F_χ is left adjoint to an exact functor and hence sends projective modules to projective modules or zero. Since F_χ does not annihilate simple modules, claim (ii) follows. □

Corollary 17. *The functor* F_χ *is full on projective modules. In particular,* $\hat{D}_{\lambda,\xi}$ *is a quotient of* Γ_χ*.*

Proof. Let $P \in \Gamma_\chi$-mod be projective and $\varphi \in \mathrm{Hom}_{\mathfrak{g}}(\mathrm{F}_\chi P, \mathrm{F}_\chi P)$. Thanks to Lemma 15 and the projectivity of P, the morphism $\mathrm{G}_\chi\varphi$ fits into the commutative diagram

$$\begin{tikzcd}
P \arrow[r, two heads, "\mathrm{adj}'_P"] \arrow[d, dotted, "\psi"'] & \mathrm{G}_\chi\mathrm{F}_\chi P \arrow[d, "\mathrm{G}_\chi\varphi"] \\
P \arrow[r, two heads, "\mathrm{adj}'_P"] & \mathrm{G}_\chi\mathrm{F}_\chi P
\end{tikzcd} \tag{5}$$

for some $\psi : P \to P$. Applying F_χ gives the square on the left hand side of the commutative diagram:

$$\begin{tikzcd}
\mathrm{F}_\chi P \arrow[r, two heads, "\mathrm{F}_\chi\mathrm{adj}'_P"] \arrow[d, dotted, "\mathrm{F}_\chi\psi"'] & \mathrm{F}_\chi\mathrm{G}_\chi\mathrm{F}_\chi P \arrow[r, two heads, "\mathrm{adj}_{\mathrm{F}_\chi P}"] \arrow[d, "\mathrm{F}_\chi\mathrm{G}_\chi\varphi"] & \mathrm{F}_\chi P \arrow[d, "\varphi"] \\
\mathrm{F}_\chi P \arrow[r, two heads, "\mathrm{F}_\chi\mathrm{adj}'_P"] & \mathrm{F}_\chi\mathrm{G}_\chi\mathrm{F}_\chi P \arrow[r, two heads, "\mathrm{adj}_{\mathrm{F}_\chi P}"] & \mathrm{F}_\chi P.
\end{tikzcd} \tag{6}$$

By the adjointness property, the compositions in each row are the identity maps. Hence $\varphi = \mathrm{F}_\chi\psi$ and the claim follows. □

Proof of Proposition 11: For the algebra $\hat{D}_{\lambda,\xi}$ the claim follows from Corollary 17 and the fact that the algebra Γ_χ is commutative. Thus the algebra $D_{\lambda,\xi}$ is commutative as it is a quotient of the commutative algebra $\hat{D}_{\lambda,\xi}$. □

Proof of Corollary 12: It is enough to prove the statement under the assumption that $m \in M_\mu$, $\mu \in \xi$. Define $I := \bigcap_{N\in\hat{\mathcal{C}}_{\lambda,\xi}} \mathrm{Ann}_{U_0} N_\mu$, which is an ideal of U_0. Let L be a simple object of $\hat{\mathcal{C}}_{\lambda,\xi}$ and $\pi : U_0 \to \mathbb{C}$ be the homomorphism defining the simple U_0-module L_μ. Then $\pi(I) = 0$ and hence we have the induced morphism $\overline{\pi} : U_0/I \to \mathbb{C}$. By the PBW Theorem the algebra U is free both as a left and as a right U_0-module. Hence the usual induction and restriction define an equivalence between $\hat{\mathcal{C}}_{\lambda,\xi}$ and the category of finite-dimensional modules over the completion $\mathtt{U}$ of U_0/I with respect to the kernel of $\overline{\pi}$. By Proposition 11, the algebra $\mathtt{U}$ is commutative. The claim follows. □

Proof of Corollary 13: As mentioned in Subseection2.6, the algebra Γ is a polynomial algebra in five variables (see [DFO]). Hence the quotient $\Gamma_\chi/(\ker(\chi))^2$ is finite-dimensional. As $\hat{D}_{\lambda,\xi}$ is a quotient of Γ_χ (Corollary 17), it follows that $\dim \mathrm{Ext}^1_{\mathfrak{g}}(L, L)$ does not exceed the dimension of $\Gamma_\chi/(\ker(\chi))^2$. □

3.3 Density of the functor F (proof of Theorem 8)

In this subsection we complete the proof of Theorem 8. For this we are left to show that the functor F is dense (i.e. essentially surjective). We will prove this by induction on n. By Proposition 4 we might restrict to blocks $\hat{\mathcal{C}}_{\lambda,\xi}$, where $\xi \in \mathfrak{h}^*/\mathbb{Z}$ is represented by an element μ such that $\mu(h_i) = a_i - a_{i+1}$ with

$$a_i + a_{i+1} \notin \mathbb{Z} \quad \text{for all } i. \tag{7}$$

In the following we will only consider such blocks, hence assume the functor F has values in a block for which the condition (7) is satisfied. We begin with the starting point of our induction:

Lemma 18. *Let $n = 2$ and λ be as in case* (I). *Then the functor* F *is dense.*

Remark 19. As stated in the paragraph after Theorem 8, this theorem is known to be true in case $n = 2$, hence we do not need to prove Theorem 8 in this case. However, Lemma 18 is a stronger statement than just Theorem 8, it gives us some information about the functor F, which we will need later on to prove the general case. Our assumption (7) excludes case (II) and we do not know if the functor F is dense in that case. For our induction argument it is enough to have the density as formulated in Lemma 18.

Proof. Let $M \in \hat{\mathcal{C}}_{\lambda,\xi}$. Then $\mu = a_1 - a_2 \in \xi$. Set $V = M_\mu$. Let Y_1 and Y be the linear operators on V representing the actions of the element h_1 and the $\mathfrak{sl}_2$-Casimir element C (see Subsection 3.1), respectively. As we are in case (I), the element λ is regular and we have that the (unique) eigenvalue of Y is nonzero, in particular, Y is invertible. Let Y' denote any square root of Y, which is a polynomial in Y. Then Y' commutes with Y_1 and Y. Set

$$X_1 := \frac{Y_1 + Y' - \mathrm{Id}_V}{2} - a_1\mathrm{Id}_V, \qquad X_2 := \frac{Y' - Y_1 - \mathrm{Id}_V}{2} - a_2\mathrm{Id}_V. \tag{8}$$

Then both X_1 and X_2 are polynomials in Y_1 and Y, in particular, X_1 and X_2 commute. A direct calculation, using the definition of the functor F, shows that the action of h_1 on $(\mathrm{F}V)_\mu$ is given by Y_1 and the action of C on $(\mathrm{F}V)_\mu$ is given by Y. Since any module in $\hat{\mathcal{C}}_{\lambda,\xi}$ is uniquely determined (up to isomorphism) by the actions of h_1 and C (see for example [Maz3, Chapter 3]), it follows that $\mathrm{F}V \cong M$. This completes the proof. □

Let $M \in \hat{\mathcal{C}}_{\lambda,\xi}$, let Y be the linear operator representing the action of C on $V = M_\mu$, and $Y_1, Y_2, \ldots, Y_n$ be the linear operators representing the actions of $h_1, h_2, \ldots, h_n$ on M_μ, respectively. Define X_1 and X_2 by (8) and then inductively define $X_{i+1} = X_i - Y_i - (a_i - a_{i+1})\mathrm{Id}_V$, $i = 2, 3, \ldots, n-1$. Then $X_1, X_2, \ldots, X_n$ are commuting nilpotent linear operators on V.

The $U(\mathfrak{sl}_{n-1})$-module $M' = U(\mathfrak{sl}_{n-1})M_\mu$ is cuspidal. Now either $n = 3$ and $a_1 + a_2 \notin \mathbb{Z}$, then the $U(\mathfrak{sl}_2)$-module $U(\mathfrak{sl}_2)M_\mu$ has non-integral central character, which allows us to use Lemma 18, or $n > 3$ where we can use the induction hypothesis to conclude that the module M' is in the image of the functor F for the algebra $U(\mathfrak{sl}_{n-1})$. By definition of the functor F, we have $M' \cong N' := \oplus_{\mathbf{b}} V_{\mathbf{b}}$, where $\mathbf{b}$ runs through all elements $(b_1, b_2, \ldots b_{n-1}, 0) \in \mathbb{Z}^n$ such that $b_1 + b_2 + \cdots + b_{n-1} = 0$; and according to Lemma 10 the element $e_{i,j}$, $i, j < n$, $i \neq j$, acts on N' by mapping $v \in V_{\mathbf{b}}$ to $(X_j + (a_j + b_j)\mathrm{Id}_V)v \in V_{\mathbf{b}+\varepsilon_i - \varepsilon_j}$. To complete the proof it is enough to show the following unique extension property (since from the uniqueness it will then follow that this extension is isomorphic to either of M and $\mathrm{F}V$, implying $M \cong \mathrm{F}V$):

Proposition 20. *The pair (N', X_n)* **uniquely** *extends to a cuspidal $\mathfrak{g}$-module. That is, under the assumption* (7) *there is a unique up to isomorphism cuspidal $\mathfrak{g}$-module N such that $N = UN_\mu$, $U(\mathfrak{sl}_{n-1})N_\mu = N'$ and which gives the linear operator X_n when computed as above.*

Proof. Since $a_n \notin \mathbb{Z}$, the endomorphism $X_n + (a_n + b_n)\mathrm{Id}_V$ is invertible for all $b_m \in \mathbb{Z}$. As the action of $e_{n-1,n}$ on N is bijective, we may fix bases for all weight spaces of N such that the $U(\mathfrak{sl}_{n-1})$-action on N' is as described above and the action of $e_{n-1,n}$ is given by the definition of $\mathrm{F}V$. Now we have to define actions of all $e_{i,i+1}$, $i = 1, 2, \dots, n-2$, and all $e_{i+1,i}$, $i = 1, 2, \dots, n-1$ on N. If $i < n-2$, then $e_{i,i+1}$ and $e_{n-1,n}$ commute and hence the action of $e_{i,i+1}$ extends uniquely from N' to the rest of N. Similarly with the action of all $e_{i+1,i}$, $i = 1, 2, \dots, n-2$. Pictorially this can be illustrated as follows:

$$\begin{tikzcd}
 & \bullet \arrow[r, dotted, bend left] & \bullet \arrow[l, dotted, bend left] \arrow[r, dotted, bend left] & \bullet \arrow[l, dotted, bend left] \arrow[r, dotted, bend left] & \bullet \arrow[l, dotted, bend left] \\
N' : & \bullet \arrow[u] \arrow[r, bend left] & \bullet \arrow[u] \arrow[l, bend left] \arrow[r, bend left] & \bullet \arrow[u] \arrow[l, bend left] \arrow[r, bend left] & \bullet \arrow[u] \arrow[l, bend left]
\end{tikzcd} \tag{9}$$

The $\bullet$'s represent weight spaces with fixed basis vectors, with N' represented at the bottom. The arrows pointing up indicate the action of $e_{n-1,n}$, right and left arrows represent the actions of $e_{i,i+1}$ and $e_{i+1,i}$, respectively. Solid arrows represent already known actions, and dotted indicate the actions for which we show that they are uniquely defined.

This leaves us with the elements $e_{n-2,n-1}$ and $e_{n,n-1}$. That the action of these elements is uniquely defined, follows from Lemmata 21 and 22 below. $\square$

Lemma 21. *Under the assumption* (7) *there is a unique way to define the action of $e_{n-2,n-1}$ on N.*

Proof. Consider the following picture:

$$\begin{tikzcd}
\bullet \arrow[r, dotted, bend left, "x"] & \bullet \arrow[l, bend left, "a+1"] \arrow[r, dotted, bend left, "y"] & \bullet \arrow[l, bend left, "a+2"] \\
\bullet \arrow[u, "c"] \arrow[r, bend left, "b"] & \bullet \arrow[u, "c"] \arrow[l, bend left, "a+1"] \arrow[r, bend left, "b-1"] & \bullet \arrow[u, "c"] \arrow[l, bend left, "a+2"]
\end{tikzcd} \tag{10}$$

Arrows pointing up indicate again the action of $e_{n-1,n}$ (the labels are the corresponding linear maps), but right and left arrows represent now the actions of $e_{n-2,n-1}$ and $e_{n-1,n-2}$, respectively. Solid arrows represent already known actions, and dotted indicate the actions for which we would like to show that they are uniquely defined. There are $k, l, m \in \mathbb{Z}$ such that the coefficient a, b and c are of the form $X_{n-2} + (a_{n-2} + k)\mathrm{Id}_V$, $X_{n-1} + (a_{n-1} + l)\mathrm{Id}_V$ and $X_n + (a_n + m)\mathrm{Id}_V$, respectively. By Corollary 12, the action of the $\mathfrak{sl}_2$-Casimir element $(h_{n-1}+1)^2 + 4e_{nn-1}e_{n-1n}$ commutes with the action of all Cartan elements and with the action of the $\mathfrak{sl}_2$-Casimir element C from above. This implies that either of $x(a+1)$ and $y(a+2)$ commutes with any X_i. As both, $a+1$ and $a+2$, are invertible, x and y commute with all X_i's. From the relation

$$e_{n-2,n-1}e_{n-1,n-2} - e_{n-1,n-2}e_{n-2,n-1} = e_{n-2,n-2} - e_{n-1,n-1}$$

we obtain the equation

$$x(a+1) - y(a+2) = a - b + 1. \tag{11}$$

From the Serre relation

$$e^2_{n-2,n-1}e_{n-1,n} - 2e_{n-2,n-1}e_{n-1,n}e_{n-2,n-1} + e_{n-1,n}e^2_{n-2,n-1} = 0,$$

taking into account that c is invertible, we obtain the equation

$$xy - 2yb + b(b-1) = 0. \tag{12}$$

Multiplying (12) by the invertible element $(a+1)$ and inserting $x(a+1)$ from (11) we get:

$$y\big(y(a+2) + a - b + 1\big) - 2yb(a+1) + b(b-1)(a+1) = 0.$$

The latter factorizes as

$$\big(y(a+2) - (b-1)(a+1)\big)(y-b) = 0. \tag{13}$$

In the case of a simple module we know that $y = b$. Note that the equality $b = (b-1)(a+1)/(a+2)$ implies $a+b \in \mathbb{Z}$. As we have assumed that $a_i + a_j \notin \mathbb{Z}$ for all i, j, it follows that the element $(b-1)(a+1)/(a+2)$ acts nonzero on the corresponding weight space of a simple module, hence invertible on the corresponding weight space of any module. This implies that $y = b$, that is y is uniquely defined. This fact defines inductively the action of $e_{n-2,n-1}$ on the vector space $(e_{n-1,n})^k N'$ for all $k \in \mathbb{N}$ uniquely.

To determine the action of $e_{n-2,n-1}$ on the vector space $e^{-k}_{n-1,n}N'$ for all $k \in \mathbb{N}$ we consider the following picture with the same notation as in (9):

$$\begin{tikzcd}
\bullet \arrow[r, bend left, "b+1"] & \bullet \arrow[l, bend left, "a+1"] \\
\bullet \arrow[u, bend left, "c"] \arrow[r, bend left, "b"] & \bullet \arrow[u, bend left, "c"] \arrow[l, bend left, "a+1"] \\
\bullet \arrow[u, bend left, "c+1"] \arrow[r, bend left, dotted, "x"] & \bullet \arrow[u, bend left, "c+1"] \arrow[l, bend left, "a+1"]
\end{tikzcd}$$

From the Serre relation

$$e_{n-1,n}^2 e_{n-2,n-1} - 2e_{n-1,n}e_{n-2,n-1}e_{n-1,n} + e_{n-2,n-1}e_{n-1,n}^2 = 0,$$

using the invertibility of c and $c+1$ we get $x - 2b + (b+1) = 0$. Hence there is a unique solution $x = b - 1$. Inductively one defines the only possible action of $e_{n-2,n-1}$ on the vector space $e_{n-1,n}^{-k}N'$ for all $k \in \mathbb{N}$. This completes the proof. □

Lemma 22. *There is a unique way to define the action of* $e_{n,n-1}$ *on* N*.*

Proof. To determine this action of $e_{n,n-1}$ on N we consider the following picture with the same notation as in (9):

$$\begin{tikzcd}
\bullet \arrow[r, bend left, "b+1"] \arrow[d, bend left, dotted, "u"] & \bullet \arrow[l, bend left, "a+1"] \arrow[d, bend left, dotted, "v"] \\
\bullet \arrow[u, bend left, "c"] \arrow[r, bend left, "b"] \arrow[d, bend left, dotted, "x"] & \bullet \arrow[u, bend left, "c"] \arrow[l, bend left, "a+1"] \arrow[d, bend left, dotted, "y"] \\
\bullet \arrow[u, bend left, "c+1"] \arrow[r, bend left, "b-1"] & \bullet \arrow[u, bend left, "c+1"] \arrow[l, bend left, "a+1"]
\end{tikzcd}$$

Here all right arrows are now determined (in particular, by Lemma 21) and we have to figure out the down arrows, representing the action of $e_{n,n-1}$. Similarly to the arguments above we obtain that the elements x, y, u and v commute with all X_i's. As the elements $e_{n,n-1}$ and $e_{n-2,n-1}$, acting in the lower square, must commute, we obtain $y = x(b-1)b^{-1}$. From the relation

$$e_{n-1,n}e_{n,n-1} - e_{n,n-1}e_{n-1,n} = e_{n-1,n-1} - e_{n,n}$$

we obtain $u = xc^{-1}(c+1) - bc^{-1} + 1$, and $v = yc^{-1}(c+1) - bc^{-1} + 1 + c^{-1}$. As the elements $e_{n,n-1}$ and $e_{n-2,n-1}$ acting in the upper square must commute, we have $bu = v(c+1)$, which gives a linear equation on x with nonzero coefficients. This equation has a unique solution (which is easily verified to be $x = b$). □

We proved in fact the following:

Corollary 23. *Assume that* $n > 2$ *and that* $a_i + a_j \notin \mathbb{Z}$ *for all* i, j*. Then the functor* F *is an equivalence. In particular, Theorem 8*(i) *holds.*

Proof of Theorem 8(ii). The category $\mathcal{C}_{\lambda,\xi}$ is defined inside $\hat{\mathcal{C}}_{\lambda,\xi}$ by the condition that the action of $\mathfrak{h}$ is diagonalizable. On modules in the image of F, the action of h_i, $1 \leq i \leq n-1$, is given by the linear operators

$$(a_i - a_{i+1})\mathrm{Id}_V + X_i - X_{i+1}.$$

Hence such module is an object of $\mathcal{C}_{\lambda,\xi}$ if and only if the matrices $X_i - X_{i+1}$ are zero. This means that $D_{\lambda,\xi}$ is the quotient of $\mathbb{C}[[x_1, x_2, \ldots, x_n]]$ modulo the ideal generated by the elements $x_i - x_{i+1}$. This proves the claim of Theorem 8(ii) for $n > 2$. For $n = 2$ it is known anyway. □

4 The regular case as a deformation over the singular case

The aim of this section is to make a first step in the proof of Theorem 1(iii) and (iv). Here we will show that regular integral blocks of the category of cuspidal modules can be considered as deformations of certain finite dimensional algebras over singular blocks.

4.1 The algebras A^k

Thanks to Theorem 8 we have a complete and explicit description of the categories of (generalized) weight modules in the Cases (I) and (II). In case (III), the classification theorem (Corollary 3) suggests a connection with the representation theory of highest weight modules for $\mathfrak{sl}_n$. Namely the modules $L(w \cdot \lambda)$ with $w \in W^{\text{short}}$ are precisely the non-trivial simple modules in the regular integral block $\mathcal{O}^{\mathfrak{p}}_{\lambda}$ of the $\mathfrak{p}$-parabolic category $\mathcal{O}$ for $\mathfrak{sl}_n$ with respect to the standard parabolic subalgebra with Levi factor $\mathfrak{sl}_{n-1}$. The category $\mathcal{O}^{\mathfrak{p}}_{\lambda}$ is equivalent to the category of finite dimensional modules over the algebra $\tilde{A}^k$ for $k = n-1$, where $\tilde{A}^k$ is defined as the quotient of the path algebra of the following quiver with $k+1$ vertices:

$$
\begin{tikzcd}
0 \arrow[r, bend left, "a_0"] & 1 \arrow[l, bend left, "b_0"] \arrow[r, bend left, "a_1"] & 2 \arrow[l, bend left, "b_1"] \arrow[r, bend left, "a_2"] & 3 \arrow[l, bend left, "b_2"] \arrow[r, bend left, "a_3"] & \cdots \arrow[l, bend left, "b_3"] \arrow[r, bend left, "a_{k-1}"] & k \arrow[l, bend left, "b_{k-1}"]
\end{tikzcd}
$$

modulo the relations $a_{i+1}a_i = 0 = b_i b_{i+1}$ and $b_i a_i = a_{i-1} b_{i-1}$ (whenever the expression makes sense) and the additional relation $b_0 a_0 = 0$. The algebra $\tilde{A}^k$ is a Koszul quasi-hereditary algebra with a simple preserving duality (see e.g. [BS2] where this algebra is called K_Λ, where Λ is the block containing the finite weights with one up and $k-1$ downs). This algebra also describes blocks of the Temperley-Lieb algebra ([Mar], [We], [KX2]), blocks of the category $\mathcal{O}$ for the Virasoro algebra ([BNW]) and appears in the theory of rational representations for the algebraic group GL_2 ([MT], [Xi]).

Let $e_i \in \tilde{A}^k$ denote the trivial path at the vertex $i \in \{0, 1, \ldots, k\}$ (these are the primitive idempotents). We are interested in the subalgebra $A^k = e\tilde{A}^k e$, where $e = \sum_{i=1}^{k} e_i$. In other words, for $k > 1$ we denote by A^k the quotient of the path algebra of the following quiver with k vertices:

$$
\begin{tikzcd}
1 \arrow[r, bend left, "a_1"] & 2 \arrow[l, bend left, "b_1"] \arrow[r, bend left, "a_2"] & 3 \arrow[l, bend left, "b_2"] \arrow[r, bend left, "a_3"] & \cdots \arrow[l, bend left, "b_3"] \arrow[r, bend left, "a_{k-1}"] & k \arrow[l, bend left, "b_{k-1}"]
\end{tikzcd}
$$

modulo the relations $a_{i+1}a_i = 0 = b_i b_{i+1}$ and $b_i a_i = a_{i-1} b_{i-1}$ (whenever the expression makes sense) in case $k > 2$ and $a_1 b_1 a_1 = 0 = b_1 a_1 b_1$ in case $k = 2$. Set $A^1 = \mathbb{C}[x]/(x^2)$.

Note that the algebra A^k is self-injective and even symmetric. As a centralizer subalgebra of a quasi-hereditary algebra with duality, it is cellular ([KX1, Proposition 4.3], see also [BS1] for an explicit graded cellular basis). The algebra A^k belongs to the family of symmetric cellular algebras called generalized Khovanov algebras in [BS1]. A realization as a convolution algebra using Springer fibres can be found in [SW]. Replacing the double arrows in the above quiver by simple edges, one obtains a tree and hence A^k can be realized as the corresponding Brauer tree algebra (with $k-1$ edges and no exceptional vertex in the classification of [Ri, 4.2]).

Let now $k \geq 2$ be fixed. We will need a few basic properties of A^k which we recall now: For $1 \leq i \leq k$ let $P_i = A^k e_i$ denote the indecomposable projective module corresponding to the i-th vertex. Then for $1 \leq i, j \leq k$ the following holds:

$$\dim \operatorname{Hom}_{A^k}(P_i, P_j) = \begin{cases} 2, & \text{if } i = j; \\ 1, & \text{if } i = j \pm 1; \\ 0, & \text{otherwise.} \end{cases}$$

In particular, for $k > 1$ the endomorphism algebra of the projective generator $P = P_1 \oplus P_2 \oplus \cdots \oplus P_k$ of A^k-mod, which is naturally identified with $A^k \cong (A^k)^{\mathrm{op}}$, is generated by unique (up to nonzero scalars) homomorphisms from P_i to $P_{i\pm1}$ (whenever this makes sense). As a consequence, for $k > 1$ the modules P_1 and P_k have length three while all other indecomposable projectives have length four; all indecomposable projectives have Loewy length three. Moreover the following is a basis of A^k:

$$\mathbf{B}_k = \{e_j, a_i, b_i, a_i b_i, b_1 a_1\}$$

(where $1 \leq i \leq k-1$, $1 \leq j \leq k$). In particular, $\dim A^k = 4k - 2$.

4.2 Deformations

For a general overview on deformation theory and related topics see for instance [Et, Section 3] and [Gr, Section II.1].

Let A be a finite dimensional algebra. Given a finite dimensional vector space U we denote by $\mathbb{C}_U = \mathbb{C}[[U]]$ the algebra of $\mathbb{C}$-valued formal functions on U. If $u_1, u_2, \ldots, u_m$ is a basis of U, then $\mathbb{C}_U$ is naturally identified with the algebra of formal power series $\mathbb{C}[[u_1, u_2, \ldots, u_m]]$. This algebra is local and we denote by $\mathfrak{m}$ its maximal ideal. A *flat deformation* (or just a *deformation*) of A over $\mathbb{C}_U$ is a $\mathbb{C}_U$-algebra A_U which is topologically free as $\mathbb{C}_U$-module (i.e. isomorphic to some $V[[U]]$ for some vector space V) together with an isomorphism $\varphi : A_U/\mathfrak{m}A_U \to A$ of algebras. In particular $A_U \cong A[[U]]$ as a $\mathbb{C}_U$-module. If U has dimension m then A_U is an *m-parameter deformation* of A. Two deformations A_U and A'_U are isomorphic if there is a $\mathbb{C}[[U]]$-isomorphism of algebras which is the identity modulo $\mathfrak{m}$. A deformation A_U is *trivial* if it is isomorphic to $A[[U]]$ with the ordinary multiplication of formal power series. Similarly one defines k-order deformations by replacing $\mathbb{C}_U$ with $\mathbb{C}_U/\mathfrak{m}^{k+1}$. First order deformations are also called *infinitesimal* deformations.

4.3 Case (III) as a deformation of A^{n-1} over case (II)

The main result in this subsection is the following claim:

Theorem 24. *Let λ be as in case* (III) *and ξ be such that $\hat{\mathcal{C}}_{\lambda,\xi}$ is nonzero. Then $\hat{D}_{\lambda,\xi}$ is a deformation of A^{n-1} over $\mathbb{C}[[x_1, x_2, \ldots, x_n]]$; and $D_{\lambda,\xi}$ is a deformation of A^{n-1} over $\mathbb{C}[[x]]$.*

In case $n = 2$, the algebras $\hat{D}_{\lambda,\xi}$ and $D_{\lambda,\xi}$ are well-known (see e.g. [Dr] or [Maz3, Chapter 3]) and the result is straightforward. Hence in what follows we assume $n > 2$. Fix some $\mathbf{x}$ and R as in Theorem 2. Recall the set W^{short} indexing the simple objects and set

$$\hat{L}_i := \big(U_{S(R)} \otimes_U L((s_1 s_2 \cdots s_{i-1} s_i) \cdot \lambda)\big)^{\Phi_{\mathbf{x}}^R}, \quad 1 \le i \le n-1.$$

Fix some integral dominant singular $\lambda_i \in \mathfrak{h}^*$ with stabilizer $\langle s_i \rangle$ for the dot-action of W. Let χ_λ, χ_{λ_i} be the (under the Harish-Chandra isomorphism) corresponding central characters and denote by $\mathcal{M}_\lambda$ and $\mathcal{M}_{\lambda_i}$ the full subcategories of $\mathfrak{g}$-mod consisting of modules on which $\mathrm{Ker}(\chi_\lambda)$ and $\mathrm{Ker}(\chi_{\lambda_i})$ act locally nilpotently, respectively. Let

$$\theta_i^{\mathrm{on}}: \quad \mathcal{M}_\lambda \to \mathcal{M}_{\lambda_i} \quad \text{and} \quad \theta_i^{\mathrm{out}}: \quad \mathcal{M}_{\lambda_i} \to \mathcal{M}_\lambda$$

denote the corresponding projective functors *translation on the wall* and *translation out of the wall*, respectively (see [BG], [St1] for details). These functors are both left and right adjoint to each other and preserve the category of cuspidal modules as well as the category of weight modules. Recall ([Ja, 4.12], [St1, Lemma 5.3, Lemma A1]) that $\theta_i^{\mathrm{on}} L\big((s_1 s_2 \cdots s_{j-1} s_j) \cdot \lambda\big)$ is zero if $i \neq j$ and is a simple module if $i = j$. From this it follows by standard arguments that $\theta_i^{\mathrm{on}} \hat{L}_j$ is zero if $i \neq j$ and is a simple module if $i = j$. We denote the latter module by L_i (this is the simple object of $\hat{\mathcal{C}}_{\lambda_i,\xi_i}$).

Set $\xi_i = \mathrm{supp}(L_i)$. Then our functors restrict to biadjoint functors as follows:

$$\begin{tikzcd} \hat{\mathcal{C}}_{\lambda,\xi} \arrow[r, shift left, "\theta_i^{\mathrm{on}}"] & \hat{\mathcal{C}}_{\lambda_i,\xi_i} \arrow[l, shift left, "\theta_i^{\mathrm{out}}"] \end{tikzcd}, \qquad \begin{tikzcd} \mathcal{C}_{\lambda,\xi} \arrow[r, shift left, "\theta_i^{\mathrm{on}}"] & \mathcal{C}_{\lambda_i,\xi_i} \arrow[l, shift left, "\theta_i^{\mathrm{out}}"] \end{tikzcd}.$$

Furthermore, for any $i, i+1$ the composition $\theta_{i+1}^{\mathrm{on}} \theta_i^{\mathrm{out}}$ is an equivalence with inverse $\theta_i^{\mathrm{on}} \theta_{i+1}^{\mathrm{out}}$ (this follows from [Ja, 5.9], see [St1, (6.1),(6.2)]). By [BG] for every i we also have isomorphisms of functors

$$\theta_i^{\mathrm{on}} \theta_i^{\mathrm{out}} \cong \mathrm{Id}_{\hat{\mathcal{C}}_{\lambda_i,\xi_i}} \oplus \mathrm{Id}_{\hat{\mathcal{C}}_{\lambda_i,\xi_i}}. \tag{14}$$

From now on we fix such an isomorphism and let p_1 and p_2 denote the projection onto the first and second summands, respectively.

For $m \in \mathbb{N}$ consider the categories $\hat{\mathcal{C}}^m_{\lambda_i,\xi_i}$ and $\mathcal{C}^m_{\lambda_i,\xi_i}$ as in Subsection 2.5. By Theorem 8, the categories $\hat{\mathcal{C}}_{\lambda_i,\xi_i}$ and $\mathcal{C}_{\lambda_i,\xi_i}$ are equivalent to the categories of finite dimensional modules over some local ring. In either case let $\mathfrak{m}$ denote the maximal ideal.

Lemma 25. *The categories $\hat{\mathcal{C}}^m_{\lambda_i,\xi_i}$ and $\mathcal{C}^m_{\lambda_i,\xi_i}$ coincide with the full subcategories of $\hat{\mathcal{C}}_{\lambda_i,\xi_i}$ and $\mathcal{C}_{\lambda_i,\xi_i}$, respectively, consisting of modules annihilated by $\mathfrak{m}^m$.*

Proof. This follows directly from the definitions. $\square$

Denote by $\hat{\mathcal{C}}^{(m)}_{\lambda,\xi}$ and $\mathcal{C}^{(m)}_{\lambda,\xi}$ the full subcategories of $\hat{\mathcal{C}}_{\lambda,\xi}$ and $\mathcal{C}_{\lambda,\xi}$ which for every i are mapped to $\hat{\mathcal{C}}^m_{\lambda_i,\xi_i}$ and $\mathcal{C}^m_{\lambda_i,\xi_i}$ by the corresponding translation to the wall, respectively. The category $\hat{\mathcal{C}}^{(m)}_{\lambda,\xi}$ is abelian and the translation functors restrict, thanks to (14), to (biadjoint, exact) functors between $\hat{\mathcal{C}}^{(m)}_{\lambda,\xi}$ and $\hat{\mathcal{C}}^{m}_{\lambda_i,\xi_i}$. These categories then have enough projectives, and θ_i^{out} (as left adjoint to an exact functor) maps projective objects to projective objects. In particular, if we denote by $\hat{R}(m)$ the indecomposable projective module in $\hat{\mathcal{C}}^{m}_{\lambda_1,\xi_1}$, and set

$$\hat{P}_i(m) := \mathrm{F}_i\hat{R}(m), \quad \text{where} \quad \mathrm{F}_i = \theta_i^{\mathrm{out}}\theta_i^{\mathrm{on}}\theta_{i-1}^{\mathrm{out}}\cdots\theta_2^{\mathrm{out}}\theta_2^{\mathrm{on}}\theta_1^{\mathrm{out}},$$

then $\hat{P}(m) := \bigoplus_{i=1}^{n-1}\hat{P}_i(m)$ is a (minimal) projective generator of $\hat{\mathcal{C}}^{(m)}_{\lambda,\xi}$. Let $\hat{E}^{(m)}$ be its endomorphism ring and $\hat{E}_i^{(m)}$ the endomorphism ring of $\hat{P}_i(m)$. By Section 2.5 we have $\hat{R} := \hat{D}_{\lambda_1,\xi_1} = \varprojlim \mathrm{End}_{\mathfrak{g}}(\hat{R}(m))$ and $\hat{D}_{\lambda,\xi} = \varprojlim \hat{E}^{(m)}$. Let $\hat{P}_i(\infty) = \varprojlim \hat{P}_i(m)$, $\hat{P}(\infty) = \varprojlim \hat{P}(m)$ and $\hat{R}^{(m)} = \mathrm{End}_{\mathfrak{g}}(\hat{R}(m))$. Define algebra homomorphisms

$$\Psi_i : \quad \hat{R} \to \varprojlim \hat{E}_i^{(m)}, \qquad f \mapsto F_i(f). \tag{15}$$

and $\Psi = \bigoplus_{i=1}^{n}\Psi_i : \hat{R} \to \hat{D}_{\lambda,\xi}$.

For $1 \leq i \leq n-1$ let α_i be the adjunction morphism from the identity functor to the composition $\theta_{i+1}^{\mathrm{out}}\theta_{i+1}^{\mathrm{on}}$ and β_i the adjunction morphism in the opposite direction.

Proposition 26. *(i) For $1 \leq i \leq n-1$ the map Ψ_i is an inclusion. It turns $\varprojlim E_i^{(m)}$ into an $\hat{R}$-module which is free of rank 2 with basis $\mathrm{id} = \mathrm{id}_i$, and the compositions $\alpha_{i-1}\beta_{i-1}$ of adjunction morphisms for $i > 1$ and $\beta_1\alpha_1$ otherwise.*

(ii) The map Ψ is an inclusion. It turns $\hat{D}_{\lambda,\xi}$ into a free left and right $\hat{R}$-module of rank $4n-2$. A basis of this module is given by the elements id_j, α_i, β_i, $\alpha_i\beta_i$, $\beta_1\alpha_1$, for $1 \leq i \leq n-2$ and $1 \leq j \leq n-1$.

Proof. Since F_i is the composition of θ_i^{out} and equivalences, it is enough to prove the first statement for $i = 1$. For all other i the proof is similar. To show injectivity we assume $\Psi_1(f) = \Psi_1(f')$, that is $\mathrm{F}_1(f) = \mathrm{F}_1(f')$. Then $\theta_1^{\mathrm{on}}\theta_1^{\mathrm{out}}(f) = \theta_1^{\mathrm{on}}\theta_1^{\mathrm{out}}(f')$, hence $f = f'$ by (14). This implies that Ψ_1 is injective.

The space $\varprojlim E_1^{(m)}$ becomes an $\hat{R}$-module by setting $f.g = \theta_1^{\mathrm{out}}(f) \circ g$ for $g \in \varprojlim E_i^{(m)}$, $f \in \hat{R}$. Moreover, Ψ_1 becomes an $\hat{R}$-module morphism. We claim that the map $g \mapsto p_1\theta_1^{\mathrm{on}}(g)$ defines a split S of Ψ_1. Since

$$\begin{aligned} S(f.g) &= p_1\theta_1^{\mathrm{on}}(\theta_1^{\mathrm{out}}(f) \circ g) \\ &= p_1(\theta_1^{\mathrm{on}}\theta_1^{\mathrm{out}}(f) \circ \theta_1^{\mathrm{on}}(g)) \\ &= p_1((f \oplus f) \circ \theta_1^{\mathrm{on}} g) \\ &= f \circ p_1\theta_1^{\mathrm{on}}(g), \end{aligned}$$

this map is an $\hat{R}$-module homomorphism and obviously $S \circ \Psi_1$ is the identity on $\hat{R}$. It is now easy to verify that the map $g \mapsto p_2\theta_1^{\mathrm{on}}(g)$ defines a complement and so $\varprojlim E_1^{(m)} \cong \hat{R} \oplus \hat{R}$. By direct calculations one verifies that the composition $\beta_1\alpha_1$ can be chosen as a second basis vector. Claim (i) follows.

Using again adjunctions and the fact that $\theta_{i+1}^{\mathrm{on}}\theta_i^{\mathrm{out}}$ is an equivalence we obtain from Theorem 8 the following:

$$\begin{aligned} \mathrm{Hom}(\hat{P}_i(\infty), \hat{P}_{i+1}(\infty)) &= \mathrm{Hom}(\hat{P}_i(\infty), \theta_{i+1}^{\mathrm{out}}\theta_{i+1}^{\mathrm{on}}\hat{P}_i(\infty)) \\ &= \mathrm{Hom}(\theta_{i+1}^{\mathrm{on}}\hat{P}_i(\infty), \theta_{i+1}^{\mathrm{on}}\hat{P}_i(\infty)) \\ &\cong \hat{R}. \end{aligned}$$

Hence, under the identification $\theta^{\mathrm{out}}_{i+1}\theta^{\mathrm{on}}_{i+1}\hat{P}_i(\infty) = \hat{P}_{i+1}(\infty)$ the map

$$\hat{R} \to \mathrm{Hom}_{\mathfrak{g}}(\hat{P}_i(\infty), \hat{P}_{i+1}(\infty)), \quad f \mapsto \alpha_i \circ f$$

defines an isomorphism of $\hat{R}$-modules, where $f.g = \theta^{\mathrm{out}}_{i+1}\theta^{\mathrm{on}}_{i+1}(\Psi_i(f))\circ g$ for $f \in \hat{R}$, $g \in \mathrm{Hom}_{\mathfrak{g}}(\hat{P}_i(\infty), \hat{P}_{i+1}(\infty))$, since the natural transformation α_i satisfies

$$\theta^{\mathrm{out}}_{i+1}\theta^{\mathrm{on}}_{i+1}(\Psi_i(f)) \circ \alpha_i \circ g = \alpha_i \circ \Psi_i(f) \circ g = \alpha_i \circ (f.g).$$

Therefore, $\mathrm{Hom}_{\mathfrak{g}}(\hat{P}_i(\infty), \hat{P}_{i+1}(\infty))$ is a free right $\hat{R}$-module of rank one with basis α_i. Similarly, $\mathrm{Hom}_{\mathfrak{g}}(\hat{P}_{i+1}(\infty), \hat{P}_i(\infty))$ is a free left $\hat{R}$-module of rank one with basis β_i. Finally we claim that

$$\mathrm{Hom}_{\mathfrak{g}}(\hat{P}_i(\infty), \hat{P}_j(\infty)) = 0 \quad \text{if } |i-j| > 1. \tag{16}$$

By the standard properties of translation out of the wall, the module $\theta^{\mathrm{out}}_i L_i$ has simple top and simple socle, both isomorphic to $\hat{L}_i$. By adjunction,

$$\mathrm{Hom}_{\mathfrak{g}}(\theta^{\mathrm{out}}_i L_i, \theta^{\mathrm{out}}_{i\pm1} L_{i\pm1}) = \mathrm{Hom}_{\mathfrak{g}}(L_i, \theta^{\mathrm{on}}_i \theta^{\mathrm{out}}_{i\pm1} L_{i\pm1}) \neq 0$$

and hence $\theta^{\mathrm{out}}_i L_i$ has at least one composition factor isomorphic to $\hat{L}_{i\pm1}$, whenever $i \pm 1$ makes sense. Comparing the character of $\theta^{\mathrm{out}}_i L_i$ with its $\Phi^R_{\mathbf{x}}$-twisted character, we conclude that $\theta^{\mathrm{out}}_i L_i$ has length three if $i = 1, n-1$ and length four otherwise and the mentioned above simple subquotients are all simple subquotients of $\theta^{\mathrm{out}}_i L_i$. Since θ^{out}_i is exact, Theorem 8 implies that there is no simple composition factor of the form $\hat{L}_i$ appearing in $\hat{P}_j(\infty)$, hence the claim (16) follows, since $\hat{P}_i(\infty)$ is a limit of projective covers of $\hat{L}_i$. The problem with left and right $\hat{R}$-module structures is dealt with in Lemma 27 below. Hence $\hat{D}_{\lambda,\xi}$ is a free left $\hat{R}$-module of rank $4n-2$ with a basis as stated in the proposition. □

Lemma 27. *The image I of Ψ is a central subalgebra.*

Proof. By the proof of Proposition 26 it is enough to show that any $f \in \hat{R}$ satisfies $\Psi(f) \circ \alpha_i = \alpha_i \circ \Psi(f)$ and $\Psi(f) \circ \beta_i = \beta_i \circ \Psi(f)$. From the definition of a natural transformation we have $\theta^{\mathrm{out}}_{i+1}\theta^{\mathrm{on}}_{i+1}(g) \circ \alpha_i = \alpha_i \circ g$ for any morphism g, in particular $\theta^{\mathrm{out}}_{i+1}\theta^{\mathrm{on}}_{i+1}(\theta^{\mathrm{out}}_i(g)) \circ \alpha_i = \alpha_i \circ \theta^{\mathrm{out}}_i(g)$. From the definition of Ψ it follows that

$$\Psi_{i+1}(f) \circ \alpha_i = \alpha_i \circ \Psi_i(f)$$

and then $\Psi(f)\circ\alpha_i = \alpha_i\circ\Psi(f)$. Analogously one obtains $\Psi(f)\circ\beta_i = \beta_i\circ\Psi(f)$. □

Proof of Theorem 24. By Theorem 1(ii), Proposition 26 and Lemma 27, the algebra $\hat{D}_{\lambda,\xi}$ has the structure of a $\mathbb{C}[[x_1, x_2, \ldots, x_n]]$-algebra with the basis as described in Proposition 26(ii). Restricting the above arguments to $\mathcal{C}_{\lambda,\xi}$ and using Theorem 1(i), we get that the algebra $D_{\lambda,\xi}$ has the structure of a $\mathbb{C}[[x]]$-algebra with the basis as described in Proposition 26(ii). Hence there are obvious isomorphisms $\hat{D}_{\lambda,\xi} \cong A^{n-1}[[x_1, x_2, \ldots, x_n]]$ and $D_{\lambda,\xi} \cong A^{n-1}[[x]]$ of $\mathbb{C}[[x_1, x_2, \ldots, x_n]]$- and $\mathbb{C}[[x]]$-modules, respectively, sending α_i to a_i and β_i to b_i.

Reducing modulo $\mathfrak{m}$ we get that the algebra $\hat{D}^{(1)}_{\lambda,\xi} \cong D^{(1)}_{\lambda,\xi}$ is generated by the images of α_i and β_i (since $n > 2$). Thanks to (16), we have the relations $\alpha_{i+1}\alpha_i = 0$ and $\beta_{i-1}\beta_i = 0$.

The compositions $\beta_i\alpha_i$ and $\alpha_i\beta_i$ send the top of $\theta_i^{\mathrm{out}}L_i$ to the socle. This implies $\alpha_i\beta_i\alpha_i = 0$ and $\beta_i\alpha_i\beta_i = 0$ for all i. Therefore in case $n = 3$ we get $D_{\lambda,\xi}^{(1)} \cong A^2$.

Since $\theta_i^{\mathrm{out}}L_i$ has simple socle, for $n > 3$ we obtain $\beta_i\alpha_i = c_i\alpha_i\beta_i$ for some $c_i \in \mathbb{C}$. That $c_i \neq 0$ can easily be verified by a direct calculation. Rescaling α_{i-1}, if necessary, we may assume $c_i = 1$ for all i. It follows that $D_{\lambda,\xi}^{(1)} \cong A^{n-1}$ for all n, which completes the proof of Theorem 24. □

5 Explicit description of deformations

5.1 Deformations and Hochschild cohomology

We use the notation from Subsection 4.2. Identify $\mathbb{C}_U$ with $\mathbb{C}[[u_1, u_2, \dots, u_m]]$, where $u_1, u_2, \dots, u_m$ is a basis of U. Then a deformation A_U of A over U is nothing else than the vector space $A[[u_1, u_2, \dots u_m]]$ together with a $k[[\mathbf{u}]]$-linear associative star product

$$a \star b = \sum \mu_{\mathbf{d}}(a, b)\mathbf{u}^{\mathbf{d}}, \tag{17}$$

where the sum runs over all multi-indices $\mathbf{d} = (d_1, d_2, \dots, d_m) \in \mathbb{Z}_+^m$, $\mathbf{u} = (u_1, u_2, \dots, u_m)$, $\mathbf{u}^{\mathbf{d}} = u_1^{d_1}u_2^{d_2}\cdots u_m^{d_m}$, $\mu_{\mathbf{d}} : A \otimes A \to A$ is a linear map and $\mu_{\mathbf{0}}(a, b) = ab$ for all $a, b \in A$. In particular, if $m = 1$ and $u = u_1$, then

$$a \star b = ab + \mu_1(a, b)u + \mu_2(a, b)u^2 + \cdots . \tag{18}$$

A classical result of Gerstenhaber ([Ge]) says that infinitesimal one-parameter deformations are classified by the second Hochschild cohomology $\mathbf{HH}^2(A, A)$ of A with values in A in the sense that the associativity of the star product implies that $\mu_1 : A \otimes A \to A$ is always a 2-cocycle and the isomorphism classes (of deformations) are exactly given by the coboundaries. The analogous statement for m-parameter deformations reads as follows (see [BeGi, Theorem 1.1.5]):

Proposition 28. *Isomorphism classes of infinitesimal m-parameter deformations of A are in bijection with the space*

$$\mathrm{Hom}\big(\mathrm{Hom}(\mathfrak{m}/\mathfrak{m}^2, \mathbb{C}), \mathbf{HH}^2(A, A)\big).$$

Proof. Let $B = A_U$ be an m-parameter deformation of A. Let K be the kernel of the multiplication map $B \otimes B \to B$. Then we have a short exact sequence

$$0 \to K \to B \otimes B \to B \to 0.$$

Note that the kernel K is a free right B-module, hence we may reduce modulo $\mathfrak{m}$ from the right hand side and obtain a short exact sequence

$$0 \to K \otimes_{\mathbb{C}_U} \mathbb{C}_U/\mathfrak{m} \to B \otimes B \otimes_{\mathbb{C}_U} \mathbb{C}_U/\mathfrak{m} \to B \otimes_{\mathbb{C}_U} \mathbb{C}_U/\mathfrak{m} \to 0.$$

By reducing the latter modulo $\mathfrak{m}$ from the left hand side we obtain an exact sequence

$$0 \to A \otimes \mathfrak{m}/\mathfrak{m}^2 \to \mathbb{C}_U/\mathfrak{m} \otimes_{\mathbb{C}_U} K \otimes_{\mathbb{C}_U} \mathbb{C}_U/\mathfrak{m} \to A \otimes A \to A \to 0, \tag{19}$$

using the identification $\mathbb{C}_U/\mathfrak{m}\otimes_{\mathbb{C}_U} A\otimes A\otimes_{\mathbb{C}_U}\mathbb{C}_U/\mathfrak{m} = A\otimes A$ and the identification $\mathbb{C}_U/\mathfrak{m}\otimes_{\mathbb{C}_U} A\otimes_{\mathbb{C}_U}\mathbb{C}_U/\mathfrak{m} = A$, the fact that $A\otimes A$ is a free left A-module, and finally the isomorphism

$$\operatorname{Tor}_1(\mathbb{C}_U/\mathfrak{m}, A) \cong A\otimes \operatorname{Tor}_1(\mathbb{C}_U/\mathfrak{m}, \mathbb{C}_U/\mathfrak{m}) = A\otimes \mathfrak{m}/\mathfrak{m}^2.$$

In particular, (19) defines an element in

$$\operatorname{Ext}^2_{A-A}(A, A\otimes \mathfrak{m}/\mathfrak{m}^2) = \operatorname{Hom}\bigl(\operatorname{Hom}(\mathfrak{m}/\mathfrak{m}^2, \mathbb{C}), \mathbf{HH}^2(A,A)\bigr).$$

One can check that this defines the required isomorphism. $\square$

5.2 Flat deformations associated with associative Hochschild 2-cocycles

Consider the natural partial order $\leq$ on $\mathbb{Z}_+^m$, defined as follows: $(d_1', d_2', \dots, d_m') \leq (d_1, d_2, \dots, d_m)$ provided that $d_i' \leq d_i$ for all i. Note that for $\mathbf{d}', \mathbf{d} \in \mathbb{Z}_+^m$ we have $\mathbf{d}' \leq \mathbf{d}$ if and only if $\mathbf{d} - \mathbf{d}' \in \mathbb{Z}_+^m$. Recall that $\{\varepsilon_i | i = 1, 2, \dots, m\}$ is the standard basis of $\mathbb{Z}_+^m$. Set $\mathbf{0} = (0, 0, \dots, 0)$.

In the notation of (17), the associativity equation $(a\star b)\star c = a\star(b\star c)$ reduces to the following system of equations, where $\mathbf{d}\in\mathbb{Z}_+^m$:

$$\sum_{\mathbf{d}'\leq\mathbf{d}} \mu_{\mathbf{d}'}(\mu_{\mathbf{d}-\mathbf{d}'}(a,b),c) = \sum_{\mathbf{d}'\leq\mathbf{d}} \mu_{\mathbf{d}'}(a,\mu_{\mathbf{d}-\mathbf{d}'}(b,c)). \tag{20}$$

In particular, in the notation of (18) this reduces to the following:

$$\mu_k(a,b)c + \mu_k(ab,c) - a\mu_k(b,c) - \mu_k(a,bc) = \\ = \sum_{i=1}^{k-1} \bigl(\mu_i(a,\mu_{k-1}(b,c)) - \mu_i(\mu_{k-1}(a,b),c)\bigr). \tag{21}$$

The right hand side of (21) is a Hochschild 3-cocycle and the equation is solvable if and only if it is a coboundary. Hence obstructions to extend infinitesimal deformations to flat deformations are given by $\mathbf{HH}^3(A,A)$. We will need the following easy observation:

Proposition 29. *Let ν be a Hochschild 2-cocycle of A. For any $\mathbf{d}\in\mathbb{Z}_+^m\setminus\{\mathbf{0}\}$ choose $c_{\mathbf{d}}\in\mathbb{C}$. Assume that ν is associative, that is $\nu(\nu(a,b),c) = \nu(a,\nu(b,c))$. Then setting $\mu_{\mathbf{d}} = c_{\mathbf{d}}\nu$ for all $\mathbf{d}\in\mathbb{Z}_+^m\setminus\{\mathbf{0}\}$ defines a flat deformation of A over $\mathbb{C}[[u_1, u_2, \dots, u_m]]$.*

Proof. For any $\mathbf{0} < \mathbf{d}' < \mathbf{d}$ we have

$$\mu_{\mathbf{d}'}(\mu_{\mathbf{d}-\mathbf{d}'}(a,b),c) = c_{\mathbf{d}'}c_{\mathbf{d}-\mathbf{d}'}\nu(\nu(a,b),c) = \\ c_{\mathbf{d}'}c_{\mathbf{d}-\mathbf{d}'}\nu(a,\nu(b,c)) = \mu_{\mathbf{d}'}(a,\mu_{\mathbf{d}-\mathbf{d}'}(b,c))$$

because of the associativity of ν. Taking into account that ν is a Hochschild 2-cocycle, we also have

$$\mu_{\mathbf{d}}(a,b)c + \mu_{\mathbf{d}}(ab,c) - a\mu_{\mathbf{d}}(b,c) - \mu_{\mathbf{d}}(a,bc) = \\ c_{\mathbf{d}}(\nu_{\mathbf{d}}(a,b)c + \nu_{\mathbf{d}}(ab,c) - a\nu_{\mathbf{d}}(b,c) - \nu_{\mathbf{d}}(a,bc)) = 0.$$

This implies that all equations in the system (20) are, in fact, identities. The claim follows. $\square$

Corollary 30. *Assume that $\mathbf{HH}^2(A,A)$ is one-dimensional and that there exists a nontrivial Hochschild 2-cocycle ν, which is associative (as in Proposition 29). Then any two m-parameter flat deformations B_i, $i = 1, 2$, of A, which are nontrivial deformations when reduced modulo $\mathfrak{m}^2$, are isomorphic as associative algebras (but not necessarily as deformations).*

Proof. Choose $c_{\mathbf{d}} = 0$ for all $\mathbf{d} \in \mathbb{Z}_+^m \setminus \{\mathbf{0}\}$ except of $c_{\varepsilon_1} = 1$. Then we have the corresponding flat deformation B of A, given by Proposition 29. To prove the corollary we may assume $B = B_2$.

Assume that B_1 is a deformation of A over $\mathbb{C}[[h_1, h_2, \dots, h_m]]$ with the star product given by (17). From Proposition 28 it follows that up to isomorphism of deformations we may assume that $\mu_{\varepsilon_i} = c'_{\varepsilon_i}\nu$ for some $c'_{\varepsilon_i} \in \mathbb{C}$ for all i. As B_1 is nontrivial, when reduced modulo $\mathfrak{m}^2$, at least one of c'_{ε_i} is nonzero.

Now we prove, by induction with respect to the order $\leq$, that $\mu_{\mathbf{d}} = c'_{\mathbf{d}}\nu$ for some $c'_{\mathbf{d}} \in \mathbb{C}$ for all $\mathbf{d} \in \mathbb{Z}_+^m \setminus \{\mathbf{0}\}$. The basis of the induction is proved in the previous paragraph. To prove the induction step we take some $\mathbf{d} \in \mathbb{Z}_+^m \setminus \{\mathbf{0}\}$. Using the inductive assumption, the corresponding equation from (20) reduces to

$$\mu_{\mathbf{d}}(a,b)c + \mu_{\mathbf{d}}(ab,c) - a\mu_{\mathbf{d}}(b,c) - \mu_{\mathbf{d}}(a,bc) = 0.$$

This means that $\mu_{\mathbf{d}}$ must be a Hochschild 2-cocycle. As $\mathbf{HH}^2(A,A)$ is one-dimensional, up to isomorphism of deformations we thus have $\mu_{\mathbf{d}} = c'_{\mathbf{d}}\nu$ for some $c'_{\mathbf{d}} \in \mathbb{C}$.

It follows that the star product for B_1 looks as follows:

$$a \star b = ab + \sum_{\mathbf{d}\neq\mathbf{0}} c'_{\mathbf{d}}\nu_{\mathbf{d}}(a,b)\mathbf{h}^{\mathbf{d}} = ab + \nu_{\mathbf{d}}(a,b)\left(\sum_{\mathbf{d}\neq\mathbf{0}} c'_{\mathbf{d}}\mathbf{h}^{\mathbf{d}}\right).$$

Without loss of generality we may assume that c'_{ε_1} is nonzero. Then there is an isomorphism from $\mathbb{C}[[u_1, u_2, \dots, u_m]]$ to $\mathbb{C}[[h_1, h_2, \dots, h_m]]$ which maps u_1 to $\sum_{\mathbf{d}\neq\mathbf{0}} c'_{\mathbf{d}}\mathbf{h}^{\mathbf{d}}$ and u_i to h_i for all $i > 1$. This induces a ring isomorphism from B_2 to B_1. The statement follows. □

5.3 Hochschild cohomology of A^k

The even Hochschild cohomology of the algebras A^k was first described in [Ho1]. Thanks to [Ri], [EH], [Ho1], there exists moreover an explicit description of the full Hochschild cohomology ring. We will not need these detailed descriptions and just recall the result about the dimensions:

Proposition 31. *We have*

$$\dim \mathbf{HH}^i(A^k) = \begin{cases} k+1, & i = 0; \\ 1, & i > 0. \end{cases}$$

Proof. Thanks to [Ri, Theorem 4.2], every Brauer tree algebra with k simple modules and exceptional multiplicity m is derived equivalent to a path algebra B_k^m of a quiver which is a single oriented cycle of length k modulo the ideal generated by all pathes of length $mk + 1$ (the latter is again a Brauer tree algebra). The algebras A^k are such Brauer tree algebras with $m = 1$. Since Hochschild cohomology is an invariant of the derived category, it is enough to determine the Hochschild cohomology of B^1_k. Then the result is a special case of [Ho1, Theorem 8.1] and [EH, Theorem 5.19]. $\square$

In particular, we have $\dim \mathbf{HH}^2(A^k) = 1$. The corresponding nontrivial Hochschild 2-cocycle is given by the following:

Lemma 32. *In the basis $\mathbf{B}_k$ of A^k, the assignment*

$$\begin{array}{cccc} \mu: & A^k \otimes A^k & \longrightarrow & A^k \\ & a_s \otimes b_s & \longmapsto & (-1)^{s+1} e_{s+1}, \\ & b_1 \otimes a_1 & \longmapsto & e_1, \\ & a_s b_s \otimes a_s b_s & \longmapsto & (-1)^s a_s b_s, \\ & a_s \otimes a_{s-1} b_{s-1} & \longmapsto & (-1)^{s-1} a_s \\ & a_{s-1} b_{s-1} \otimes b_s & \longmapsto & (-1)^{s-1} b_s, \end{array} \tag{22}$$

extended by zero to all other basis vectors, is a nontrivial associative Hochschild 2-cocycle.

Proof. That μ is an associative Hochschild 2-cocycle is checked by a straightforward computation. The element e_1 does not belong to the radical of A^k, while both b_1 and a_1 do. Hence $b_1 \otimes a_1 \mapsto e_1$ is not possible for any coboundary, which implies that μ is nontrivial. The claim follows. $\square$

5.4 One-parameter deformations of A^k

Let us start with an explicit example, illustrating Proposition 31 and Corollary 30.

Example 33. Consider the algebra $A^1 = \mathbb{C}[X]/(X^2)$ and the bimodule $R := A^1 \otimes A^1$. Let $f, g : R \to A^1$ be the A^1-bimodule maps given by $f(1 \otimes 1) = X \otimes 1 - 1 \otimes X$ and $g(1 \otimes 1) = X \otimes 1 + 1 \otimes X$. Then

$$\cdots \xrightarrow{g} R \xrightarrow{f} R \xrightarrow{g} R \xrightarrow{f} R \xrightarrow{\text{mult}} A^1_0 \to 0$$

is a free R-module resolution of A^1. A direct computation gives $\mathbf{HH}^0(A^1, A^1) = A^1$ (as A^1 is commutative) and $\mathbf{HH}^i(A^1, A^1) \cong \mathbb{C}$ for all $i > 0$ (as claimed by Proposition 31). In particular, $\mathbf{HH}^2(A^1, A^1) = \mathbb{C}$, hence there is a unique nontrivial infinitesimal one-parameter deformation for which we can take the corresponding nontrivial associative Hochschild 2-cocycle μ_1 as follows: $\mu_1(a, b) = 0$ for $(a, b) \in \{(1, 1), (1, X), (X, 1)\}$ and $\mu_1(X, X) = 1$. Hence, by Corollary 30 we have a unique, up to isomorphism, associative algebra, which is an m-parameter deformation of A^1, nontrivial when reduced modulo $\mathfrak{m}^2$. If $m = 1$, this is realized as follows: consider $A^1[[t]]$ with $X \star X = t$ and therefore isomorphic to $\mathbb{C}[[u]]$ by sending X to u. If $m > 1$, we similarly get the polynomial algebra $\mathbb{C}[[u_1, u_2, \ldots, u_m]]$.

This example can be generalized to all A^k as follows: For $k > 1$ consider the quotient $\hat{B}^k$ of the path algebra of the following quiver (in case k is even or odd, respectively):

$$\begin{tikzcd} 1 \arrow[loop left, "y_1"] \arrow[r, bend left, "x_1"] & 2 \arrow[l, bend left, "x_2"] \arrow[r, bend left, "y_2"] & 3 \arrow[l, bend left, "y_3"] \arrow[r, bend left, "x_3"] & \cdots \arrow[l, bend left, "x_4"] \arrow[r, bend left, "x_{k-1}"] & k \arrow[l, bend left, "x_k"] \arrow[loop right, "y_k"] \end{tikzcd}$$

$$\begin{tikzcd} 1 \arrow[loop left, "y_1"] \arrow[r, bend left, "x_1"] & 2 \arrow[l, bend left, "x_2"] \arrow[r, bend left, "y_2"] & 3 \arrow[l, bend left, "y_3"] \arrow[r, bend left, "x_3"] & \cdots \arrow[l, bend left, "x_4"] \arrow[r, bend left, "y_{k-1}"] & k \arrow[l, bend left, "y_k"] \arrow[loop right, "x_k"] \end{tikzcd}$$

modulo the relations $x_i y_j = 0 = y_j x_i$ whenever the expression makes sense. Let B^k denote the completion of $\hat{B}^k$ with respect to the ideal generated by all arrows.

Consider also the deformation $A^k[[t]]$ of A^k in which the product $\star$ is given by $x \star y = xy + \mu(x,y)t$, where μ is as in Lemma 32. Associativity of this product follows from Proposition 29. Obviously, $A^k[[t]]$ is nontrivial when reduced modulo $\mathfrak{m}^2$.

Theorem 34. *(i) Putting the x's and y's in degree one, turns the algebra* $\hat{B}^k$ *into a Koszul algebra.*

(ii) The algebra B^k *has the structure of a one-parameter flat deformation of* A^k*, isomorphic, as a deformation, to the deformation* $A^k[[t]]$ *described above.*

Proof. The algebra $\hat{B}^k$ is by definition quadratic with monomial relations, hence Koszul, see [PP, Corollary II.4.3] (it is in fact straightforward to write down projective resolutions of simple modules and see that they are linear). This proves (i). To prove the second statement we will have to check several things.

For every point i of the quiver set $t_i = x(i) - y(i)$, where $x(i)$ is the shortest x-loop starting in i and $y(i)$ is the shortest y-loop starting in i (for example, $y(1) = y_1$, $x(1) = x_2x_1$, $x(2) = x_1x_2$, $y(2) = y_2$ if $k = 2$ and $y(2) = y_3y_2$ if $k > 2$ and so on). Define $t(k)$ to be the sum of all t_i's. The element $t(k)$ is a sum of loops and hence commutes with all idempotents of the path algebra. As all x-monomials of $t(k)$ have coefficient 1 and the product of any x and any y is zero, it follows that $t(k)$ commutes with all x_i. As all y-monomials of $t(k)$ have coefficient -1 and the product of any x and any y is zero, it follows that $t(k)$ commutes with all y_j. Hence $t(k)$ is a central element of $\hat{B}^k$.

Let I denote the ideal of $\hat{B}^k$, generated by all arrows and J denote the ideal of $\hat{B}^k$, generated by $t(k)$. Then $J \subset I$. At the same time, I^4 is generated by elements of the form $x_i x_{i\pm1} x_i x_{i\pm1}$ and $y_i y_{i\pm1} y_i y_{i\pm1}$, which both belong to J^2. Hence $I^4 \subset J^2$, which implies that the completions of $\hat{B}^k$ with respect to I and J coincide, and hence both are isomorphic to B^k.

Next we claim that $X := \hat{B}^k/J \cong A^k$. Indeed, using the relation $x_i y_j = 0 = y_j x_i$ it is easy to see that there is a unique algebra homomorphism $\varphi : \hat{B}^k \to A^k$, which maps y_1 to b_1a_1, x_1 to a_1, x_2 to b_1, y_2 to a_2, y_3 to b_3 and so on (i.e. all non-loop arrows are mapped to the corresponding non-loop arrows, and loops at the ends are mapped to length two loops at the ends). From the definition we have that φ is surjective and maps $t(k)$ to 0. Therefore it induces a surjective homomorphism $\overline{\varphi} : \hat{B}^k/J \twoheadrightarrow A^k$. At the same time it is easy to see that the images of non-loop arrows in $\hat{B}^k/J$ satisfy the defining relations of A^k. Therefore $\overline{\varphi}$ is injective and hence an isomorphism.

It is easy to see that the algebra $\hat{B}^k$ has a $\mathbb{C}$-basis $\hat{\mathbf{B}}_k$, which consists of all trivial paths, all monomials in x_i's and all monomials in y_j's which make sense. We claim that $\hat{B}^k$ is free over $\mathbb{C}[t(k)]$. The map φ maps elements from $\hat{\mathbf{B}}_k$ to elements from $\mathbf{B}_k$ or zero. For every element from $\mathbf{B}_k$ fix its tautological preimage under φ (for example e_i is the preimage of e_i, x_1 is the preimage of a_1, x_2x_1 is the preimage of b_1a_2 and so on). Call the latter set $\underline{\hat{\mathbf{B}}}_k$. There is an obvious bijection between the elements in $\hat{\mathbf{B}}_k$ and elements $zt(k)^i$, where $z \in \underline{\hat{\mathbf{B}}}_k$ and $i \in \mathbb{Z}_+$. It follows that $\underline{\hat{\mathbf{B}}}_k$ is a free $\mathbb{C}[t(k)]$-basis of $\hat{B}^k$. The latter yields that B^k is free over $\mathbb{C}[[t(k)]]$ and hence is a flat deformation of A^k.

It is easy to check that $x_1x_2x_1 \neq 0$ in B^k/J^2, which yields that this deformation is nontrivial when reduced modulo J^2. Therefore (ii) follows from Corollary 30, completing the proof of the theorem. $\square$

Remark 35. One easily checks that the map

$$\begin{array}{ccc}
\Psi : A^m[[t]] & \longrightarrow & B^m, \\
e_i & \mapsto & e_i, \\
a_s & \mapsto & \begin{cases} x_s & \text{if } s \text{ odd} \\ y_s & \text{if } s \text{ even} \end{cases} \\
b_s & \mapsto & \begin{cases} x_{s+1} & \text{if } s \text{ odd}, \\ y_{s+1} & \text{if } s \text{ even}, \end{cases} \\
b_1a_1 & \mapsto & y_1, \\
a_{k-1}b_{k-1} & \mapsto & \begin{cases} y_k, & \text{if } k \text{ even}, \\ x_k, & \text{if } k \text{ odd}, \end{cases}, \\
a_sb_s & \mapsto & \begin{cases} x_sx_{s+1}, & \text{if } s \text{ even}, \\ y_sy_{s+1}, & \text{if } t \text{ odd}, \end{cases} \\
t & \mapsto & t(k)
\end{array}$$

for i and s for which the expression makes sense, is an isomorphism of deformations.

The algebras $\hat{B}^k$ from Theorem 34 are studied in [GP], [Kh]. They belong to the class of special biserial algebras ([WW]), in particular, they are tame. A complete description of indecomposable modules over these algebras can be found in [Kh], [WW], [GS]. The algebra $\mathbb{C}[[x, y]]$ is a classical wild algebra. Therefore from Theorem 1(ii) and Theorem 24 it follows that all nonzero categories $\hat{\mathcal{C}}_{\lambda,\xi}$ are wild.

Remark 36. By Theorem 34, the algebra $\hat{B}^k$ is Koszul, as well as A^1. This is however not the case for A^k if $k > 1$ (as can easily be seen by writing down explicit projective resolutions, for $k = 2$ the algebra is not even quadratic).

5.5 One-parameter graded deformations of A^k

The algebra A^k can be equipped with two different natural nonnegative $\mathbb{Z}$-gradings $A^k = \oplus_{j\geq 0} A^k(j)$ by either putting the a_i's and the b_i's all in degree one or by putting the a_i's in degree one and the b_i's in degree zero, respectively.

In the first case we get a positive grading in the sense of [MOS], which is natural in the study of Koszul algebras.

Remark 37. In [BLPPW] universal Koszul deformations were studied. This construction leads to different deformations than the one obtained in this paper as universal Koszul deformations behave badly with respect to taking the centralizer subalgebras. For example, as mentioned earlier, the algebra $\mathbb{C}[X]/(X^2) \cong A^1$ is the centralizer subalgebra of the Koszul quasi-hereditary algebra $\tilde{A}^1$ (the latter one is graded in the usual way by path length). Let $\tilde{A}^1[t]$ be the universal Koszul deformation in the sense of [BLPPW]. Then one might hope that the corresponding centralizer subalgebra in $\tilde{A}^1[t]$ is isomorphic to the deformation of A^1 described in Example 33. This is however not the case since the first one is isomorphic to $\mathbb{C}[X] \otimes_{\mathbb{C}[X^2]} \mathbb{C}[X]$, whereas the second one is isomorphic to $\mathbb{C}[X]$. Analogous phenomenon appears for all A^k.

Remark 38. The 0-th Hochschild cohomology of A^k, i.e. the center of the algebra A^k, or, more generally, of generalized Khovanov algebras has a geometric interpretation via the cohomology ring of Springer fibres ([Br], [St2]). In [St2], the Koszul deformation (in the sense of Remark 37) was used as an essential tool to obtain the result. The center of this deformed algebra is in fact the T-equivariant cohomology ring ([St2], [GM]), for a one-dimensional torus T.

Despite of Remark 36, the algebra B^k can be understood using the graded picture. The algebra $\hat{B}^k$ is graded in the natural way by putting all non-loop arrows in degree one and loops in degree two (in case $m = 1$ the only loop is put in degree one). The following was established during the proof of Theorem 34: The algebra $\hat{B}^k$ is free over the graded central subalgebra $\mathbb{C}[t(k)]$ and the quotient of $\hat{B}^k$ over the homogeneous ideal generated by $t(k)$ is isomorphic to A^k as a graded algebra. This means that $\hat{B}^k$ is a graded flat one-parameter deformation of A^k. The algebra B^k is the completion of the positively graded algebra $\hat{B}^k = \oplus_{j\geq 0}\hat{B}^k(j)$ with respect to the graded radical, that is the ideal $\oplus_{j>0}\hat{B}^k(j)$. The graded version of the deformation theory is described in [BrGa]. Using these results we obtain:

Corollary 39. *Let $k \in \mathbb{N}$. Consider A^k as a graded algebra by assuming that all arrows have degree one (except $k = 1$ when the only loop is assumed to have degree two). Then $\hat{B}^k$ is the unique (up to rescaling of the deformation parameter) nontrivial graded flat one-parameter deformation of A^k.*

Proof. The algebra $\hat{B}^k$ has the structure of a nontrivial graded flat one-parameter deformation of A^k given by Theorem 34. In particular, A^k has a nontrivial graded infinitesimal deformation, which is unique up to rescaling of the deformation parameter as $\mathbf{HH}^2(A^k, A^k)$ is one-dimensional. The existence of $\hat{B}^k$ says that this deformation extends to a nontrivial flat graded deformation of A^k by Theorem 34. By [BrGa, Proposition 1.5(c)], the freedom of extending our infinitesimal deformation to a flat deformation is given by the graded pieces of $\mathbf{HH}^2(A^k, A^k)$ in higher degrees. These spaces are however zero as $\mathbf{HH}^2(A^k, A^k)$ is one-dimensional and hence is concentrated in the degree which produces the non-trivial infinitesimal deformation. □

Consider now the second grading on A^k (obtained by putting the a_i's in degree 1 and the b_i's in degree zero, respectively). This grading is no longer positive in the sense of [MOS]. To distinguish the obtained graded algebra from the previous graded algebra, we denote it by $\tilde{A}^k$. Similarly we define a grading on $\hat{B}^k$ by putting all right arrows and end loops in degree one and all left arrows in degree zero. Let $\tilde{B}^k$ be the resulting algebra.

Proposition 40. *Let $k \in \{2, 3, \dots\}$. Then the algebra $\tilde{B}^k$ is the unique (up to rescaling of the deformation parameter) non-trivial graded flat one-parameter deformation of $\tilde{A}^k$.*

Proof. Mutatis mutandis that of Corollary 39. $\square$

Remark 41. The graded algebra $\tilde{A}^k$ is a subalgebra of the algebra studied in [KhSe] in the context of Floer cohomology of Lagrangian intersections in the symplectic manifolds which are Milnor fibres of simple singularities of type A_k.

5.6 Proof of Theorem 1(iii) and (iv)

Let λ be as in case (III) and ξ be such that $\hat{\mathcal{C}}_{\lambda,\xi}$ is nontrivial. We will use the notation from Subsection 4.3. Theorem 24 gives the algebra $\hat{D}_{\lambda,\xi}$ the structure of a deformation of A^{n-1} over $\mathbb{C}[[x_1, x_2, \dots, x_n]]$ and the algebra $D_{\lambda,\xi}$ the structure of a deformation of A^{n-1} over $\mathbb{C}[[x]]$.

Lemma 42. *There is a uniserial module $M \in D_{\lambda,\xi}$ of Loewy length four such that the layers of the radical filtration of M are $\hat{L}_1$, $\hat{L}_2$, $\hat{L}_1$ and $\hat{L}_2$.*

Proof. This follows from [GS, Lemma 6.4]. Alternatively one can construct this module using the Gelfand-Zetlin approach, described in [Maz2]. Let us briefly outline the latter. Let V be a two dimensional vector space and X be a nonzero nilpotent operator on V.

As explained in Subsection 2.6, the element λ corresponds to some $(m_1, m_2, m_3, \dots, m_n) \in \mathbb{C}^n$ such that $m_2 - m_1$, $m_1 - m_3$, $m_3 - m_4, \dots, m_{n-1} - m_n \in \mathbb{N}$. By Proposition 4, we may arbitrarily change ξ, so we choose $x_1, x_2, \dots, x_{n-1} \in \mathbb{C}$ such that $m_1 - x_i \notin \mathbb{Z}$ and $x_i - x_j \notin \mathbb{Z}$ for all i, j. Consider the set $T(\mathbf{m}, \mathbf{x})$ and the corresponding module $M(\mathbf{m}, \mathbf{x})$ as defined in Subsection 2.6. Note that we are in case (III) and $m_2 - m_1, m_1 - m_3 \in \mathbb{N}$. Hence, by construction, the module $M(\mathbf{m}, \mathbf{x})$ is a uniserial module of length two with subquotients of the radical filtration being $\hat{L}_2$, $\hat{L}_1$. Now we would like to construct a nontrivial selfextension of this module module using X.

For every $t \in T(\mathbf{m}, \mathbf{x})$ let V_t denote a copy of V. Consider the vector space $M = \oplus_{t \in T(\mathbf{m},\mathbf{x})} V_t$. For $i \in \{1, 2, \dots, n-1\}$ define on M the action of $e_{i,i+1}$ and $e_{i+1,i}$ by the classical Gelfand-Zetlin formulae ([DFO], [Maz2]) in which we treat all entries of our tableaux as the corresponding scalar operators on V with the exception of m_1, which should be understood as $m_1 \mathrm{Id}_V + X$, and y_i, $i = 1, 2, \dots, n-1$, which should be understood as $y_i \mathrm{Id}_V + X$. Similarly to the proof of Lemma 10 one shows that this defines on M the structure of a $\mathfrak{g}$-module for any X. It is easy to check that this module is uniserial of length four with subquotients of the radical filtration being $\hat{L}_2$, $\hat{L}_1$, $\hat{L}_2$, $\hat{L}_1$. Taking the restricted dual, we obtain a module which satisfies all the requirements of the lemma. $\square$

Lemma 43. *Both $\hat{D}_{\lambda,\xi}$ and $D_{\lambda,\xi}$ are nontrivial deformations when reduced modulo $\mathfrak{m}^2$.*

Proof. As $\mathbb{C}[[x]]$ is a quotient of $\mathbb{C}[[x_1, x_2, \dots, x_n]]$, it is enough to prove the statement for the smaller algebra $D_{\lambda,\xi}$. In case $n = 2$ the claim follows directly from the well-known description of $D_{\lambda,\xi}$ (see [Maz3, Chapter 3]). Hence we may assume $n > 2$. One could observe that the claim follows from the main result of [GS] and our Theorem 34. However, we would like to give an independent argument based on our description of $D_{\lambda,\xi}$ as a deformation, given by Theorem 24.

Consider the element $x = \beta_1\alpha_1 \in D_{\lambda_1,\xi_1}$. We know that the image of x in A^{n-1} (that is, when we further reduce modulo $\mathfrak{m}$) is a nonzero element $\overline{x}$ satisfying $\overline{x}^2 = 0$. Hence, to prove the statement of the lemma it is enough to show that x^2 is nonzero when reduced modulo $\mathfrak{m}^2$. For this it is enough to show that x^2 does not annihilate some module from $\mathcal{C}^{(2)}_{\lambda,\xi}$. This is a straightforward computation, which we describe below.

Set $R := \mathcal{C}_{\lambda_1,\xi_1} \cong \mathbb{C}[[x]]$. The projective module $R/\mathfrak{m}^2$ has length two with simple top and simple socle isomorphic to L_1. In particular, it is also injective in $\mathcal{C}^{(2)}_{\lambda_1,\xi_1}$. Applying the exact selfadjoint functor F_1 to the short exact sequence

$$0 \to L_1 \to R/\mathfrak{m}^2 \to L_1 \to 0$$

we get the exact sequence

$$0 \to M_1 \to P_1(2) \to M_2 \to 0,$$

where $P_1(2)$ is the projective cover of $\hat{L}_1$ and $M_1 \cong M_2 \cong \mathrm{F}_1 L_1$. The module $\mathrm{F}_1 L_1$ is uniserial of length three, with simple top and simple socle isomorphic to $\hat{L}_1$ and the intermediate layer isomorphic to $\hat{L}_2$. We assert that x^2 does not annihilate $P_1(2)$.

Similarly to the above one shows that for the projective-injective module $P_2(2)$ there is an exact sequence

$$0 \to N_1 \to P_2(2) \to N_2 \to 0,$$

where $N_1 \cong N_2$ has Loewy length three, with simple top and simple socle isomorphic to $\hat{L}_2$, and the middle layer of the radical (=socle) filtration isomorphic to $\hat{L}_1 \oplus \hat{L}_3$ (the summand $\hat{L}_3$ should be omitted in case $n = 3$).

We want to show that the composition

$$P_1(2) \overset{\alpha_1}{\to} P_2(2) \overset{\beta_1}{\to} P_1(2) \overset{\alpha_1}{\to} P_2(2) \overset{\beta_1}{\to} P_1(2) \tag{23}$$

is nonzero. Note that the kernels of the morphisms α_1 and β_1 contain only simple subquotients of the form $\hat{L}_1$ and $\hat{L}_2$, respectively (since we know that this is the case when we reduce modulo $\mathfrak{m}$).

Since $P_1(2)$ has a filtration with subquotients M_1 and M_2, it has at most one submodule N of length two, both composition subquotients of which are isomorphic to $\hat{L}_1$. At the same time the module M from Lemma 42 must be a quotient of $P_1(2)$. Therefore N as above exists, is unique, and $\alpha_1 P_1(2) \cong P_1(2)/N \cong X$ is uniserial with layers of the radical filtration as described in Lemma 42.

As the kernel of β_1 contains only simple subquotients of the form $\hat{L}_2$, it follows that $Y = \beta_1 X$ is uniserial of length three with the layers of the radical filtration being $\hat{L}_1$, $\hat{L}_2$, $\hat{L}_1$. As the kernel of α_1 contains only simple subquotients of the form $\hat{L}_1$, it follows that $Z = \alpha_1 Y$ is uniserial of length two with the layers of the radical filtration being $\hat{L}_1$, $\hat{L}_2$. As the kernel of β_1 contains only simple subquotients of the form $\hat{L}_2$, it follows that $\beta_1 Z \cong \hat{L}_1$. Therefore $\alpha_1\beta_1\alpha_1\beta_1 \neq 0$ and the claim of the lemma follows. $\square$

Proof of Theorem 1(iii) *and Theorem 1*(iv). By Theorem 24, the algebra $D_{\lambda,\xi}$ is a deformation of A^{n-1} over $\mathbb{C}[[x]]$. By Lemma 43, this deformation is nontrivial when reduced modulo $\mathfrak{m}^2$. By Theorem 34, the algebra described in Theorem 1(iii) (resp. (iv)) is also a deformation of A^{n-1} over $\mathbb{C}[[x]]$ (resp. $\mathbb{C}[[x_1, x_2, \ldots, x_n]]$), nontrivial when reduced modulo $\mathfrak{m}^2$. Hence the claim of Theorem 1(iii) (resp. (iv)) follows from Corollary 30. $\square$

References

[BG] J. Bernstein, S. Gelfand; Tensor products of finite- and infinite-dimensional representations of semisimple Lie algebras. Compositio Math. **41** (1980), no. 2, 245–285.

[BeGi] R. Bezrukavnikov, V. Ginzburg; On deformations of associative algebras. Ann. of Math. (2) **166** (2007), no. 2, 533–548.

[BNW] B. Boe, D. Nakano, E. Wiesner; Category $\mathcal{O}$ for the Virasoro algebra: cohomology and Koszulity. Pacific J. Math. **234** (2008), no. 1, 1–21.

[BLPPW] T. Braden, A. Licata, C. Phan, N. Proudfoot, B. Webster; Localization algebras and deformations of Koszul algebras. Preprint arXiv:0905.1335. To appear in Selecta Mathematica.

[BrGa] A. Braverman, D. Gaitsgory; Poincaré-Birkhoff-Witt theorem for quadratic algebras of Koszul type. J. Algebra **181** (1996), no. 2, 315–328.

[BFL] D. Britten, V. Futorny, F. Lemire; Submodule lattice of generalized Verma modules. Comm. Algebra **31** (2003), no. 12, 6175–6197.

[BKLM] D. Britten, O. Khomenko, F. Lemire, V. Mazorchuk; Complete reducibility of torsion free C_n-modules of finite degree. J. Algebra **276** (2004), no. 1, 129–142.

[BL1] D. Britten, F. Lemire; A classification of simple Lie modules having a 1-dimensional weight space. Trans. Amer. Math. Soc. **299** (1987), no. 2, 683–697.

[BL2] D. Britten, F. Lemire; Tensor product realizations of simple torsion free modules. Canad. J. Math. **53** (2001), no. 2, 225–243.

[Br] J. Brundan; Centers of degenerate cyclotomic Hecke algebras and parabolic category $\mathcal{O}$. Represent. Theory **12** (2008), 236–259.

[BS1] J. Brundan, C. Stroppel; Highest weight categories arising from Khovanov's diagram algebra I: cellularity. Preprint arXiv:0806.1532. To appear in Mosc. Math. Journal.

[BS2] J. Brundan, C. Stroppel; Highest weight categories arising from Khovanov's diagram algebra II: Koszulity. Transform. Groups **15** (2010), no. 1, 1–45.

[DMP] I. Dimitrov, O. Mathieu, I. Penkov; On the structure of weight modules. Trans. Amer. Math. Soc. **352** (2000), no. 6, 2857–2869.

[Di] J. Dixmier; Enveloping algebras. Graduate Studies in Mathematics, **11**. American Mathematical Society, Providence, RI, 1996.

[Dr] Yu. Drozd; Representations of Lie algebras $\mathfrak{sl}(2)$. Visnyk Kyiv. Univ. Ser. Mat. Mekh. No. **25** (1983), 70–77.

[DFO] Yu. Drozd, V. Futorny, S. Ovsienko; Harish-Chandra subalgebras and Gelfand-Zetlin modules. Finite-dimensional algebras and related topics (Ottawa, ON, 1992), 79–93, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., **424**, Kluwer Acad. Publ., Dordrecht, 1994.

[EH] K. Erdmann, T. Holm; Twisted bimodules and Hochschild cohomology for self-injective algebras of class A_n. Forum Math. **11** (1999), no. 2, 177–201.

[Et] P. Etingof; Calogero-Moser systems and representation theory. Zurich Lectures in Advanced Mathematics. European Mathematical Society (EMS), Zürich, 2007.

[Fe] S. Fernando; Lie algebra modules with finite-dimensional weight spaces. I. Trans. Amer. Math. Soc. **322** (1990), no. 2, 757–781.

[Fu1] V. Futorny; Weight representations of semisimple finite dimensional Lie algebras. Ph.D. Thesis, Kyiv University, Kyiv, 1987.

[Fu2] V. Futorny; The weight representations of semisimple finite-dimensional Lie algebras, in: *Algebraic structures and applications*. Kiev University, 1988, 142–155.

[Ga] P. Gabriel; Indecomposable representations. II. Symposia Mathematica, Vol. XI (Convegno di Algebra Commutativa, INDAM, Rome, 1971), pp. 81–104. Academic Press, London, 1973.

[GP] I. Gelfand, V. Ponomarev; Indecomposable representations of the Lorentz group. Uspehi Mat. Nauk **23** (1968), no. 2, 3–60.

[Ge] M. Gerstenhaber; On the deformation of rings and algebras, Ann. of Math. (2) **79** (1964), 59–103.

[GM] M. Goresky, R. MacPherson; On the Spectrum of the Equivariant Cohomology Ring. Canad. J. Math. **16** (2010), no. 4, 262–283.

[GS] D. Grantcharov, V. Serganova; Cuspidal representations of $\mathfrak{sl}(n+1)$. Adv. Math. **224** (2010), no. 4, 1517–1547.

[Gr] G.-M. Greuel, C. Lossen, E. Shustin; Introduction to singularities and deformations. Springer Monographs in Mathematics. Springer, Berlin, 2007.

[Ho1] T. Holm; Hochschild cohomology of Brauer tree algebras, Comm. Algebra **26** (1998), no. 11, 3625–3646.

[Ho2] T. Holm; Hochschild cohomology rings of algebras $k[X]/(f)$, Beiträge Algebra Geom. **41** (2000), no. 1, 291–301.

[Ja] J. Jantzen; Einhüllende Algebren halbeinfacher Lie-Algebren. Ergebnisse der Mathematik und ihrer Grenzgebiete (3), **3**. Springer-Verlag, Berlin, 1983.

[Kh] S. Khoroshkin; Indecomposable representations of Lorentz groups. Funktsional. Anal. i Prilozhen. **15** (1981), no. 2, 50–60.

[KhSe] M. Khovanov, P. Seidel; Quivers, Floer cohomology, and braid group actions. J. Amer. Math. Soc. **15** (2002), no. 1, 203–271.

[KX1] S. König, C. Xi; On the structure of cellular algebras. Algebras and modules, II (Geiranger, 1996), 365–386, CMS Conf. Proc., 24, Amer. Math. Soc., Providence, RI, 1998.

[KX2] S. König, C. Xi; Strong symmetry defined by twisting modules, applied to quasi-hereditary algebras with triangular decomposition and vanishing radical cube. Comm. Math. Phys. **197** (1998), no. 2, 427–441.

[Mar] P. Martin; Potts models and related problems in statistical mechanics. Series on Advances in Statistical Mechanics, **5**. World Scientific Publishing Co., Inc., Teaneck, NJ, 1991.

[Mat] O. Mathieu; Classification of irreducible weight modules. Ann. Inst. Fourier (Grenoble) **50** (2000), no. 2, 537–592.

[Maz1] V. Mazorchuk; Tableaux realization of generalized Verma modules. Canad. J. Math. **50** (1998), no. 4, 816–828.

[Maz2] V. Mazorchuk; Quantum deformation and tableaux realization of simple dense $\mathfrak{gl}(n,\mathbb{C})$-modules. J. Algebra Appl. **2** (2003), no. 1, 1–20.

[Maz3] V. Mazorchuk; Lectures on $\mathfrak{sl}_2$-modules. Imperial College Press, 2010.

[Maz4] V. Mazorchuk; Stratified algebras arising in Lie theory. in: Representations of finite dimensional algebras and related topics in Lie theory and geometry, 245–260, Fields Inst. Commun., **40**, Amer. Math. Soc., Providence, RI, 2004.

[MOS] V. Mazorchuk, S. Ovsienko, C. Stroppel; Quadratic duals, Koszul dual functors, and applications. Trans. Amer. Math. Soc. **361** (2009), no. 3, 1129–1172.

[MS1] V. Mazorchuk, C. Stroppel; Categorification of (induced) cell modules and the rough structure of generalised Verma modules. Adv. Math. **219** (2008), no. 4, 1363–1426.

[MS2] V. Mazorchuk, C. Stroppel; Blocks of the category of cuspidal $\mathfrak{sp}_{2n}$-modules. Preprint arXiv:1004.1168. To appear in Pacific J. Math.

[MT] V. Miemietz, W. Turner; Homotopy, homology, and GL_2, Proc. Lond. Math. Soc. (3) **100** (2010), no. 2, 585–606.

[Ov] S. Ovsienko; Finiteness statements for Gelfand-Zetlin modules. Third International Algebraic Conference in the Ukraine, 323–338, Natsional. Akad. Nauk Ukrainy, Inst. Mat., Kiev, 2002.

[PP] A. Polishchuk, L. Positselski; Quadratic algebras. University Lecture Series **37**, American Mathematical Society, Providence, RI, 2005.

[Ri] J. Rickard; Derived categories and stable equivalence. J. Pure Appl. Algebra **61** (1989), no. 3, 303–317.

[St1] C. Stroppel; Categorification of the Temperley-Lieb category, tangles, and cobordisms via projective functors. Duke Math. J. **126** (2005), no. 3, 547–596.

[St2] C. Stroppel; Parabolic category $\mathcal{O}$, perverse sheaves on Grassmannians, Springer fibres and Khovanov homology. Compos. Math. **145** (2009), no. 4, 954–992.

[SW] C. Stroppel, B. Webster; 2-block Springer fibers: convolution algebras and coherent sheaves. Preprint arXiv:0802.1943. To appear in Comm. Math. Helvetici.

[WW] B. Wald, J. Waschbüsch; Tame biserial algebras. J. Algebra **95** (1985), no. 2, 480–500.

[We] B. Westbury; The representation theory of the Temperley-Lieb algebras. Math. Z. **219** (1995), no. 4, 539–565.

[Xi] C.C. Xi; The structure of Schur algebras $S_k(n,p)$ for $n \geq p$. Canad. J. Math. **44** (1992), no. 3, 665–672.

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