FUSION RINGS FOR QUANTUM GROUPS

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Abstract. We study the fusion rings of tilting modules for a quantum group at a root of unity modulo the tensor ideal of negligible tilting modules. We identify them in type A with the combinatorial rings from [14] and give a similar description of the $\mathfrak{sp}_{2n}$-fusion ring in terms of non-commutative symmetric functions. Moreover we give a presentation of all fusion rings in classical types as quotients of polynomial rings. Finally we also compute the fusion rings for type G_2.

Contents

1. Introduction

Fusion rings associated with semisimple Lie algebras have been studied from different perspectives, see e.g. [5], [6], [8], [11], [12], [14], [21]. In this paper we approach them via tilting modules for quantum groups $U_q(\mathfrak{g})$ at complex roots of unity and relate them to previously obtained descriptions obtained in loc. cit. In particular, we give an alternative way of proving the combinatorial fusion rule for type A established by C. Korff and the second author in [14] and also obtain an analogous formula in the type C case.

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Our approach gives for all types an easy method to realize the fusion rings as quotients of polynomial rings. In particular, it provides a set of generators of the fusion ideal. In the type A and C cases this leads to minimal sets of generators identical to those found in the above mentioned papers and suitable for nice combinatorics. The other cases are more involved and the naturally arising sets of generators are in general not minimal, as our case by case analysis reveals.

As a special feature of our approach we obtain two different fusion rings in each type depending on whether our quantum parameter has even or odd order. In type G_2 it moreover depends on whether 3 divides this order or not, see Section 7; we avoid this special case for the rest of the introduction. To obtain the above mentioned fusion rings from the literature it is enough to restrict to quantum groups where the order of the root of unity is even. The "odd" fusion rings however do not seem to have been studied previously (except implicitly in [2]). Finding a minimal set of generators for these fusion ideals seem to be even more challenging and so far no satisfactory answer is known. Based on our presentations we could verify the intriguing conjectural presentations of [6] for small ranks by computer calculations, but unfortunately are not able to prove or disprove them in general. We expect that an identification of the two presentations yields interesting new identities inside the character ring.

Let us explain our approach and results in a little more detail. We work with a simple complex Lie algebra $\mathfrak{g}$ with root system R. We choose a set of positive roots R^+ and then have inside the set X of integral weights the cone $X^+ \subset X$ of dominant weights. Let $q \in \mathbb{C}$ be a root of unity of order ℓ and denote by $U_q(\mathfrak{g})$ the quantum group associated with $\mathfrak{g}$. If ℓ is even we set $\ell' = \ell/2$.

Inside the category of finite dimensional $U_q(\mathfrak{g})$-modules there is the additive category $\mathcal{T}_q$ given by the *tilting modules*. This is a tensor category and its (split) Grothendieck ring K has a $\mathbb{Z}$-basis consisting of the images of all isomorphism classes of indecomposable tilting modules. The latter are classified; for each $\lambda \in X^+$ there is a unique indecomposable tilting module $T_q(\lambda)$ with highest weight λ. Inside $\mathcal{T}_q$ we consider the tensor ideal $\mathcal{N}_q$, cf. [1], consisting of all *negligible tilting modules*, i.e. those tilting modules Q for which $\mathrm{Tr}_q(f) = 0$ for all $f \in \mathrm{End}_{\mathcal{T}_q}(Q)$, where Tr_q denotes the *quantum trace*. The corresponding *fusion ring* for $U_q(\mathfrak{g})$ is then the ring $\mathcal{F}_q = K/I_\ell$, where I_ℓ denotes the ideal in K corresponding to $\mathcal{N}_q$.

To describe this fusion ring we relate it with the corresponding polynomial ring of characters. Let W denote the Weyl group for $\mathfrak{g}$. The character ring $\mathbb{Z}[X]^W$ has a $\mathbb{Z}$-basis given by the $\chi(\lambda)_{\lambda \in X^+}$, where for any $\lambda \in X$ we denote by $\chi(\lambda)$ the *Weyl character* at λ. These characters are linked for $w \in W$ by the formula $\chi(w \cdot \lambda) = (-1)^{\ell(w)}\chi(\lambda)$ with the *"dot"-action* given by $w \cdot \lambda = w(\lambda + \rho) - \rho$ where ρ is the half-sum of the positive roots. Consider the *fundamental alcove*

$$\mathcal{A}_\ell = \{\lambda \in X \mid 0 < \langle \lambda + \rho, \alpha^\vee \rangle < \ell \text{ for all } \alpha \in R^+\},$$

if ℓ is odd and with ℓ replaced by $\ell' = \ell/2$ if ℓ is even. Then the tensor ideal $\mathcal{N}_q$ consists of those tilting modules which have no summands $T_q(\lambda)$ with $\lambda \in \mathcal{A}_\ell$. Note that this means $K/I_\ell = 0$ when $\ell, \ell' < h$, where h denotes the Coxeter number for R. Hence we assume always $\ell, \ell' \geq h$. In that case it is proved in [2] that I_ℓ may be described as

$$\left\{ \sum a_\lambda \chi(\lambda) \in \mathbb{Z}[X]^W \mid a_\lambda = a_{s \cdot \lambda} \text{ for some reflection } s \text{ in a face of } \mathcal{A}_\ell \right\}.$$

Based on this observation we use the representation theory and combinatorics of affine Weyl groups to obtain explicit descriptions of the fusion ideal for our fusion rings arising from quantum groups which makes it possible to connect them with existing descriptions in the literature (where ℓ is often replaced by the "level" $k = \ell - h$, respectively $k = \ell' - h'$ (see the case by case treatments below) and the dot action by the ordinary action).

The paper is organized as follows. In Section 2 we establish, based on [2], the multiplication rules in the fusion ring $\mathcal{F}_q$ for $U_q(\mathfrak{g})$ and give a general recipe how to produce a set of generators for the tensor ideal $\mathcal{N}_q$ (and hence for the fusion ideal I_ℓ). We apply this for type A in Section 3 to show how our formulas translate into the combinatorics in [14] and demonstrate how our results easily provide minimal sets of generators for I_ℓ. Then we introduce in Section 4 some analogous "combinatorics" for type C and in the even case we are again able to cut our set of generators down to a minimal such set. In Sections 5-7 we give presentations for the types D, B and G_2, respectively.

2. The fusion rule for q-groups at roots of unity

In this section we recall the main definitions and facts about fusion rings for quantum groups at complex roots of unity. Our presentation is based on [1] and [2].

Suppose $\mathfrak{g}$ is a complex simple Lie algebra. For an indeterminate v denote by U_v the quantum group over $\mathbb{Q}(v)$ corresponding $\mathfrak{g}$. This is the $\mathbb{Q}(v)$-algebra with generators $E_i, F_i, K_i^{\pm 1}$, $i = 1, 2, \dots, n = \mathrm{rank}(\mathfrak{g})$ and relations as given e.g. in [13, Chapter 5].

Set $A = \mathbb{Z}[v, v^{-1}]$. Then A contains the quantum numbers $[r]_d = \frac{v^{dr} - v^{-rd}}{v^d - v^{-d}}$ for any $r, d \in \mathbb{Z}$, $d \neq 0$, as well as the corresponding q binomials $\begin{bmatrix} m \\ t \end{bmatrix}_d$, $m \in \mathbb{Z}, t \in \mathbb{N}$ defined via the factorials $[r]_d! = [r]_d[r-1]_d \cdots [1]_d$ for $r \geq 0$. In case $d = 1$ they become the usual binomial numbers and so we will often omit the d from the notation.

Let C be the Cartan matrix associated with $\mathfrak{g}$. We denote by D a diagonal matrix whose entries are relatively prime natural numbers d_i with the property that DC is symmetric. Then we set $E_i^{(r)} = E_i^r/[r]_{d_i}!$. With a similar expression for $F_i^{(r)}$ we define now the A-form U_A of U_v to be the A-subalgebra of U_v generated by the elements $E_i^{(r)}, F_i^{(r)}, K_i^{\pm 1}$, $i = 1, \ldots, n$, $r \geq 0$. This is the Lusztig *divided power quantum group.*

Fix now a root of unity $q \in \mathbb{C}$ of order ℓ. The corresponding quantum group is the specialization $U_q = U_A \otimes_A \mathbb{C}$ where $\mathbb{C}$ is considered as an A-module via $v \mapsto q$, cf. [16], [17]. We abuse notation and write $E_i^{(r)}$ also for the element $E_i^{(r)} \otimes 1 \in U_q$ and similarly for $F_i^{(r)}$.

We have a triangular decomposition $U_q = U_q^- U_q^0 U_q^+$ with U_q^- and U_q^+ being the subalgebra generated by $F_i^{(r)}$ or $E_i^{(r)}$, $i = 1, \ldots, n$, $r \geq 0$, respectively. The "Cartan part" U_q^0 is the subalgebra generated by $K_i^{\pm 1}$ and $\left[\begin{smallmatrix} K_i \\ t \end{smallmatrix}\right]$, $i = 1, \ldots, n$, $t \geq 0$, where

$$\begin{bmatrix} K_i \\ t \end{bmatrix} = \prod_{j=1}^{t} \frac{K_i v^{d_i(1-j)} - K_i v^{-d_i(1-j)}}{v^{d_i j} - v^{-d_i j}}.$$

We denote the "Borel subalgebra" $U_q^0 U_q^+$ by B_q.

Recall that U_v is a Hopf algebra, for explicit formulas for the comultiplication Δ, counit ϵ and antipode S see [13, 4.11]. It is easy to see that their restrictions give U_A the structure of a Hopf algebra over A. Then U_q also gets an induced Hopf algebra structure.

Let X be the lattice of integral weights with the positive cone X^+ of integral dominant weights. Then each $\lambda \in X$ defines in a natural way a 1-dimensional B_q-module, also denoted λ, and we set

$$\nabla_q(\lambda) = \operatorname{Ind}_{B_q}^{U_q} \lambda := F(\operatorname{Hom}_{B_q}(U_q, \lambda)), \tag{2.1}$$

where F is the functor which sends a U_q-module into its maximal integrable submodule, cf. 2.8 in [3].

Then $\nabla_q(\lambda) \neq 0$ iff $\lambda \in X^+$ and for such λ it contains a unique simple U_q-submodule which we denote $L_q(\lambda)$. The modules $L_q(\lambda)$, $\lambda \in X^+$ form a complete set of pairwise non-isomorphic representatives for the isomorphism classes of simple U_q-modules of type $\mathbf{1}$.

2.1. **The category $\mathcal{C}_q$.** Let $\mathcal{C}_q$ be the category of finite dimensional U_q-modules of type $\mathbf{1}$, and denote by $\operatorname{Gr}(\mathcal{C}_q)$ its Grothendieck group, i.e. the abelian group generated by the isomorphism classes $[M]$ of objects M in $\mathcal{C}_q$ modulo the relation $[C] = [A] + [B]$ for any short exact sequence $0 \to A \to C \to B \to 0$ in $\mathcal{C}_q$. The sets $\{[L_q(\lambda)]\}_{\lambda \in X^+}$ and $\{[\nabla_q(\lambda)]\}_{\lambda \in X^+}$ are two $\mathbb{Z}$-bases of $\operatorname{Gr}(\mathcal{C}_q)$. For any element $f \in \operatorname{Gr}(\mathcal{C}_q)$ we denote by $[f : \nabla_q(\lambda)]$ the coefficient of f when expressed in the basis $\{[\nabla_q(\lambda)]\}_{\lambda\in X^+}$, i.e.

$$f = \sum_{\lambda\in X^+} [f : \nabla_q(\lambda)][\nabla_q(\lambda)]. \tag{2.2}$$

If $f = [M]$ with $M \in \mathcal{C}_q$ we write $[M : \nabla_q(\lambda)]$ instead of $[[M] : \nabla_q(\lambda)]$. The comultiplication of U_q turns $\mathcal{C}_q$ into a tensor category and gives $\mathrm{Gr}(\mathcal{C}_q)$ a natural ring structure, the *Grothendieck ring* of $\mathrm{Gr}(\mathcal{C}_q)$.

Special elements in $\mathrm{Gr}(\mathcal{C}_q)$ are the *Euler characters* $\chi(N)$ of finite dimensional B_q-modules N defined as follows:

$$\chi(N) = \sum_{i\geq 0}(-1)^i[\mathcal{R}^i \operatorname{Ind}_{B_q}^{U_q} N], \tag{2.3}$$

where $\mathcal{R}^i \operatorname{Ind}_{B_q}^{U_q}$ denotes the ith right derived functor of the left exact functor $\operatorname{Ind}_{B_q}^{U_q}$. Then χ is additive with respect to short exact sequences. Moreover, on the 1-dimensional B_q-module given by a character $\lambda \in X^+$ we have $\chi(\lambda) = [\nabla_q(\lambda)]$, by the q-version of Kempf's vanishing theorem, [19]. More generally,

$$\chi(\mu) = (-1)^{l(w)}\chi(w\cdot\mu) \tag{2.4}$$

for all $\mu \in X$ and $w \in W$.

The B_q-modules we want to consider are finite dimensional and split into weight spaces $M = \oplus_\mu M_\mu$ as U_q^0-modules. The corresponding B_q-filtration of M gives via the additivity of χ

$$\chi(M) = \sum_{\mu}(\dim M_\mu)\chi(\mu). \tag{2.5}$$

The tensor identity $\mathcal{R}^i \operatorname{Ind}_{B_q}^{U_q}(M\otimes\mu) \simeq M\otimes \mathcal{R}^i \operatorname{Ind}_{B_q}^{U_q}\mu$ for all $i \in \mathbb{N}$ and $M \in \mathcal{C}_q$ implies

$$\chi(M\otimes\mu) = [M]\chi(\mu). \tag{2.6}$$

The additivity of χ then gives for each $\lambda \in X^+$

$$[M][\nabla_q(\lambda)] = \chi(M\otimes\lambda) = \sum_{\nu\in X}(\dim M_\nu)\chi(\lambda+\nu). \tag{2.7}$$

Using (2.4) we may rewrite this as

$$[M][\nabla_q(\lambda)] = \sum_{\nu\in X^+}\left(\sum_{w\in W}(-1)^{l(w)}\dim M_{w\cdot\nu-\lambda}\right)[\nabla_q(\nu)]. \tag{2.8}$$

This formula can then also be written as

$$[M\otimes\nabla_q(\lambda) : \nabla_q(\nu)] = \sum_{w\in W}(-1)^{l(w)}\dim M_{w\cdot\nu-\lambda}. \tag{2.9}$$

If all weights of $M\otimes\lambda$ are dominant after adding ρ, then simplifies to

$$[M\otimes\nabla_q(\lambda) : \nabla_q(\nu)] = \dim M_{\nu-\lambda}. \tag{2.10}$$

The antipode on U_q gives the linear dual M^* of a U_q-module M a U_q-module structure. For each $\lambda \in X^+$ we define $\Delta_q(\lambda)$ as the dual of $\nabla_q(-w_0\lambda)$. Then $\Delta_q(\lambda)$ has the same character as $\nabla_q(\lambda)$ and it has $L_q(\lambda)$ as its unique simple quotient (it is the Weyl module with highest weight λ). A module $Q \in \mathcal{C}_q$ is called *tilting* if it has both a ∇_q- and a ∇_q-filtration. For each $\lambda \in X^+$ there exists a unique indecomposable tilting module $T_q(\lambda)$ which has highest weight λ. Moreover, the set $\{T_q(\lambda)\}_{\lambda \in X^+}$ is, up to isomorphisms, a complete list of indecomposable tilting modules in $\mathcal{C}_q$, see e.g. [18], [7], [1].

We denote by $\mathcal{T}_q$ the subcategory of $\mathcal{C}_q$ consisting of all tilting modules. This is an additive (but not abelian) subcategory and since the collection of modules with a ∇_q-filtration is stable under tensor products, we see that $\mathcal{T}_q$ is an additive tensor category, see [1].

Set $K = \mathrm{Gr}(\mathcal{T}_q)$, the Grothendieck ring of $\mathcal{T}_q$. Since all short exact sequences consisting of tilting modules split, K is defined by the relations $[Q] = [Q_1] + [Q_2]$ whenever $Q = Q_1 \oplus Q_2$ in $\mathcal{T}_q$. The multiplicative structure on K comes from the tensor product on $\mathcal{T}_q$. As a $\mathbb{Z}$-module K is free with basis $\{[T_q(\lambda)]\}_{\lambda \in X^+}$. Note that we again write $[Q]$ for the image in K of a module $Q \in \mathcal{T}_q$. We shall denote by $(f : T_q(\lambda))$ the coefficient of $f \in K$ when expressed in the above basis. Thus if $Q \in \mathcal{T}_q$, then $(Q : T_q(\lambda))$ is the multiplicity of $T_q(\lambda)$ as a direct summand of Q.

We denote by $\mathcal{A}_\ell$ the *fundamental alcove* in X^+, see [1] which is defined, excluding the special type G_2 treated later, as follows. When ℓ is odd then

$$\mathcal{A}_\ell = \{\lambda \in X^+ \mid \langle \lambda + \rho, \alpha_0^\vee \rangle < \ell\}$$

where α_0 denotes the maximal short root. The closure of $\mathcal{A}_\ell$ is

$$\bar{\mathcal{A}}_\ell = \{\lambda \in X \mid \langle \lambda+\rho, \alpha_i^\vee \rangle \geq 0 \text{ for all simple roots } \alpha_i \text{ and } \langle \lambda+\rho, \alpha_0^\vee \rangle \leq \ell\}.$$

If ℓ is even, then we set $\ell' = \ell/2$ and

$$\mathcal{A}_\ell = \{\lambda \in X^+ \mid \langle \lambda + \rho, \beta_0^\vee \rangle < \ell'\}, \tag{2.11}$$

where β_0 denotes the maximal (long) root. Let $\bar{\mathcal{A}}_\ell$ be its closure. In both the odd and even case the *affine Weyl group* W_ℓ is the group generated by the reflections in the walls of $\mathcal{A}_\ell$. A weight is called ℓ-*singular* if it lies on a wall for W_ℓ. In case G_2 the shape of $\mathcal{A}_\ell$ depends additionally on whether 3 divides ℓ or not, see Section 7.

2.1.1. *Warning.* Note that in classical types for odd ℓ (respectively for ℓ prime to 3 when the type is G_2) our affine Weyl group W_ℓ is actually the affine Weyl group for the dual root system in the Bourbaki convention, cf. [4, Chapter VI, §2]. In the case of even ℓ (respectively for ℓ not prime to 3 if the type is G_2) our affine Weyl group W_ℓ corresponds to the affine Weyl group for the root system itself.

If $\lambda \in \bar{\mathcal{A}}_\ell \cap X^+$, then the linkage principle ([2]) gives

$$L_q(\lambda) = \nabla_q(\lambda) = T_q(\lambda). \tag{2.12}$$

We have the following formula, cf [2], [11]

Theorem 2.1. *Let $Q \in \mathcal{T}_q$ and $\lambda \in \mathcal{A}_\ell$. Then*

$$(Q : T_q(\lambda)) = \sum_w (-1)^{l(w)} [Q : \nabla_q(w \cdot \lambda)],$$

where the sum runs over those $w \in W_\ell$ for which $w \cdot \lambda \in X^+$.

Proof *(Sketch, cf.* [2]*):* Since both sides are additive in Q we may assume that $Q = T_q(\nu)$ for some $\nu \in X^+$. Then the left hand side is $\delta_{\lambda,\nu}$. When $\nu = \lambda$ we see from (2.12) that the right hand side is 1. The linkage principle [2] gives that it is 0 unless $\nu \in W_\ell \cdot \lambda$ and so we are left to show that it vanishes when $\nu = w \cdot \lambda$ for some non-trivial element $w \in W_\ell$. This follows from the inductive construction of $T_q(\nu)$ via translation functors, see [1]. □

By the above we can apply this formula to $Q = L_q(\lambda) \otimes L_q(\mu)$ when $\lambda, \mu \in \bar{\mathcal{A}}_\ell$ and obtain via formula (2.9) the following *fusion rule*.

Corollary 2.2. *Suppose $\lambda, \mu \in \bar{\mathcal{A}}_\ell \cap X^+$. Then for $\nu \in \mathcal{A}_\ell$ we have*

$$(L_q(\lambda) \otimes L_q(\mu) : T_q(\nu)) = \sum_{w \in W_\ell} (-1)^{l(w)} \dim \nabla_q(\lambda)_{w \cdot \nu - \mu}.$$

Note that the right hand side is a known integer because the weight spaces of $\nabla_q(\lambda)$ are determined e.g. by the Weyl character formula. A simple special case of this is the following (compare (2.10)).

Corollary 2.3. *Suppose $\lambda, \mu \in \bar{\mathcal{A}}_\ell \cap X^+$ are such that $\eta + \mu \in \bar{\mathcal{A}}_\ell$ for all weights η of $\nabla_q(\lambda)$. Then for any $\nu \in \mathcal{A}_\ell$ we have*

$$(L_q(\lambda) \otimes L_q(\mu) : T_q(\nu)) = \dim \nabla_q(\lambda)_{\nu - \mu}.$$

2.2. **The fusion rings $\mathcal{F}_q = \mathcal{F}_q(\mathfrak{g}, \ell)$.** Let $\mathcal{N}_q$ denote the subcategory of $\mathcal{T}_q$ consisting of those tilting modules which have no summands $T_q(\lambda)$ with $\lambda \in \mathcal{A}_\ell$. These are the *negligible tilting modules* which all have quantum dimension 0, cf. [2, §3]. It is also proved in [2] that $\mathcal{N}_q$ is a tensor ideal in $\mathcal{T}_q$ and the quotient $\mathcal{T}_q/\mathcal{N}_q$ is semisimple.

We define the *fusion ring* $\mathcal{F}_q = \mathcal{F}_q(\mathfrak{g}, \ell)$ to be the corresponding quotient of K. For $\lambda \in \mathcal{A}_\ell$ we denote the image of $[T_q(\lambda)]$ in $\mathcal{F}_q$ by $[\lambda]$. Then the set $\{[\lambda]\}_{\lambda \in \mathcal{A}_\ell}$ constitutes a $\mathbb{Z}$-basis for $\mathcal{F}_q$. Note that $\mathcal{F}_q$ is a commutative, associative ring. The commutativity follows from the fact that a tilting module is determined, up to isomorphisms, by its character, so that the tensor product on $\mathcal{T}_q$ is commutative.

By Corollary 2.2 the multiplication in $\mathcal{F}_q$ is given in basis vectors by

$$\begin{aligned}[\lambda][\mu] &= \sum_{\nu\in\mathcal{A}_\ell}(L_q(\lambda)\otimes L_q(\mu):T_q(\nu))[\nu] \\ &= \sum_{\nu\in\mathcal{A}_\ell}(\sum_{w\in W_\ell}(-1)^{l(w)}\dim\nabla_q(\lambda)_{w\cdot\nu-\mu})[\nu]. \end{aligned} \tag{2.13}$$

We extend the notation $[\lambda]\in\mathcal{F}_q$ to all $\lambda\in X$ in the following way: If $\lambda\in X$ is ℓ-singular, then we set $[\lambda]=0$. If λ is not ℓ-singular, then there exists a unique $w\in W_\ell$ with $w\cdot\lambda\in\mathcal{A}_\ell$ and we define $[\lambda]=(-1)^{l(w)}[w\cdot\lambda]$. In this notation we can also formulate (2.13) as

$$[\lambda][\mu]=\sum_{\nu\in X}\dim L_q(\lambda)_\nu[\mu+\nu] \tag{2.14}$$

for all $\lambda,\mu\in\mathcal{A}_\ell$.

2.2.1. *Generators of $\mathcal{N}_q$.* Suppose $\omega_i\in\mathcal{A}_\ell$ for all i. For classical types this means: If ℓ is odd, then $\ell\geq n+1$ for type A_n, $\ell\geq 2n+1$ for types B_n and C_n, and $\ell\geq 2n-1$ for type D_n. If ℓ is even, then $\ell\geq 2n+2$ for types A_n, B_n and C_n, whereas $\ell\geq 4n-2$ for type D_n.

The linkage principle shows that this assumption implies that $L_q(\omega_i)=T_q(\omega_i)$ for all i. Here the ω_i's are the fundamental weights in X. Hence the $[L_q(\omega_i)]$'s generate K.

Let us denote by $\leq'$ the ordering on X defined by $\lambda\leq'\mu$ iff $\mu=\lambda+\sum_i m_i\omega_i$ for some $m_i\in\mathbb{Z}_{\geq 0}$. Note that this ordering is not the same as the usual ordering $\leq$ on X defined by the positive roots. We do have, however, that if $\lambda\leq'\mu$, then $\mu-\lambda$ is a rational linear combination of positive roots (with denominators at worst equal to the index of connection for our root system).

Proposition 2.4. *The tensor ideal $\mathcal{N}_q$ in $\mathcal{T}_q$ is generated by the set $\mathcal{G}=\{T_q(\mu)\mid\mu$ minimal in $X^+\setminus\mathcal{A}_\ell$ with respect to $\leq'\}$.*

Proof. By definition of $\mathcal{N}_q$ it contains the ideal I generated by $\mathcal{G}$. To show the converse we need to check that $T_q(\lambda)\in I$ for all $\lambda\in X^+\setminus\mathcal{A}_\ell$. To see this pick $\gamma\in\sum_i\mathbb{Z}\alpha_i^\vee$ such that $\langle\alpha_j,\gamma\rangle\in\mathbb{Z}_{>0}$ for all simple roots α_j. Multiplying γ if necessary by a large enough integer we can ensure that also $\langle\omega_j,\gamma\rangle\in\mathbb{Z}_{>0}$ for all j. Suppose now that there exists $\lambda\in X^+\setminus\mathcal{A}_\ell$ with $T_q(\lambda)\notin I$. Then we choose such a λ for which $\langle\lambda,\gamma\rangle$ is minimal. Now this λ cannot be minimal with respect to the ordering $\leq'$ because then $T_q(\lambda)$ would be one of the generators for I. Hence there exists i such that $\lambda-\omega_i\in X^+\setminus\mathcal{A}_\ell$. As $\langle\lambda-\omega_i,\gamma\rangle<\langle\lambda,\gamma\rangle$ our assumption on λ implies that $T_q(\lambda-\omega_i)\in I$. But then I also contains $T_q(\lambda-\omega_i)\otimes L_q(\omega_i)$. Now this tilting module decomposes as $T_q(\lambda)\oplus T'$ with $T'=\bigoplus_\nu a_\nu T_q(\nu)$, where $a_\nu\neq 0$ implies $\nu<\lambda$ and $\nu\in X^+\setminus\mathcal{A}_\ell$. In particular $T'\in I$ because all its summands $T_q(\nu)$ satisfy $\langle\nu,\gamma\rangle<\langle\lambda,\gamma\rangle$. Hence we have a contradiction. □

Remark. In type A we always have that the set of minimal elements in $X^+ \backslash \mathcal{A}_\ell$ are those which belong to the upper wall of $\mathcal{A}_\ell$. Case by case considerations reveals that the same is also true for type C whereas for types B and D it is only so when ℓ is odd.

3. Three descriptions in Type A

We shall connect our fusion ring directly with the description of the Verlinde algebra for $\hat{\mathfrak{sl}}_n$ at a fixed level k given by Korff and the second author, [14], [21], and extend the combinatorial techniques developed there to the case $\mathfrak{g} = \mathfrak{sp}_{2n}$ in the next section. Note that these two fusion rings play a special role, since their explicit ring structure is known and they are reduced complete intersection rings, see [6].

In this section we consider first $\mathfrak{gl}_n$ and then $\mathfrak{sl}_n$. We assume for convenience that ℓ is odd (except in the last subsection where we explain the easy transition to the even case). The corresponding level k is then given by $k = \ell - n$.

3.1. **Fusion algebras for $\mathfrak{gl}_n$.** We shall now study the fusion ring in the case where $\mathfrak{g} = \mathfrak{gl}_n$. In this case we identify $X = \mathbb{Z}^n$ by choosing the standard basis $\{\epsilon_1, \epsilon_2, \cdots, \epsilon_n\}$ and write $\lambda \in X$ as $\lambda = \sum_{i=1}^n \lambda_i \epsilon_i$ or alternatively $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n)$. The positive roots for $\mathfrak{gl}_n$ are $\epsilon_i - \epsilon_j$ with $1 \le i < j \le n$. Hence λ is dominant iff $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n$.

In this terminology the alcove $\mathcal{A}_\ell$ is determined by

$$\mathcal{A}_\ell = \{\lambda \in X^+ \mid \lambda_1 - \lambda_n \le \ell - n\}. \tag{3.1}$$

In particular $\mathcal{A}_\ell = \emptyset$ if $\ell < n$. So we assume $0 \le \ell - n =: k$ in the following and call k the *level*.

We set $\omega_i = \epsilon_1 + \epsilon_2 + \cdots + \epsilon_i$. This is the i-th fundamental weight. The weights of $\nabla_q(\omega_i)$ are $\{\epsilon_{j_1} + \epsilon_{j_2} + \cdots + \epsilon_{j_i} \mid 1 \le j_1 < j_2 < \cdots < j_i \le n\}$, all occurring with multiplicity 1 (the ω_i's are minuscule). In particular, we observe that if $\mu \in \mathcal{A}_\ell$, then $\eta + \mu \in \bar{\mathcal{A}}_\ell$ for all weights η of $\nabla_q(\omega_i)$. This means that in the fusion ring $\mathcal{F}_q(\mathfrak{gl}_n, \ell)$ for $U_q(\mathfrak{gl}_n)$ we have (cf. Corollary 2.3 or (2.14)) the Pieri type rule

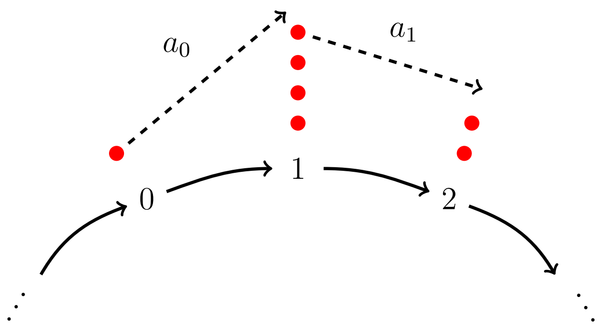
$$[\omega_i][\mu] = \sum [\epsilon_{j_1} + \epsilon_{j_2} + \cdots + \epsilon_{j_i} + \mu], \tag{3.2}$$

where the sum runs over those i-tuples $1 \le j_1 < j_2 < \cdots < j_i \le n$ for which $\epsilon_{j_1} + \epsilon_{j_2} + \cdots + \epsilon_{j_i} + \mu \in \mathcal{A}_\ell$.

Let $0 \le i \le n-1$ and consider the operator $\mathbf{a}_i$ on $\mathcal{A}_\ell$ given by

$$\mathbf{a}_i(\lambda) = \begin{cases} \lambda + \epsilon_{i+1} & \text{if } \lambda + \epsilon_{i+1} \in \mathcal{A}_\ell, \\ 0 & \text{otherwise.} \end{cases} \tag{3.3}$$

We may represent an arbitrary $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n) \in X$ by n rows of boxes in which the i-th row is infinite to the left and stops at the number λ_i (note that λ_i may be negative). In this configuration the rows are non-increasing and there are at most k more boxes in the first row than in the last. The operator $\mathbf{a}_0$ adds a box in the first row if there are less than k more boxes in this row than in the last. Otherwise, $\mathbf{a}_0$ kills λ. Similarly, if $i > 0$, then $\mathbf{a}_i$ adds a box in the $i+1$'th row if this results in a non-increasing configuration and maps to 0 otherwise. The differences $m_i = \lambda_i - \lambda_{i-1}$ encode λ in the basis of fundamental weights, $\lambda = \sum_i m_i \omega_i$. Then $\lambda \in \mathcal{A}_\ell$ if and only if $m = \sum_{i=1}^{n-1} m_i \leq k$. Setting $m_0 = k - m$ as in [14], weights can be viewed as a configuration of k particles on a circle with n marked points with m_i particles at place i (viewed as an extended Dynkin diagram of type $\tilde{A}_{n-1}$) and these operators can be thought of as *"particle hopping"* from one point to the next in clockwise direction. If there are no particles at the point from where the operator makes a particle hop, the operator kills the configuration instead:



Here the action of $\mathbf{a}_1$ and $\mathbf{a}_0$ on $\lambda = 4\omega_1 + 2\omega_2$ is illustrated in the case $k = 7$.

Set $\underline{\mathbf{a}} = (\mathbf{a}_0, \mathbf{a}_1, \cdots, \mathbf{a}_{n-1})$ and define the non-commutative elementary symmetric polynomials $\mathbf{e}_1(\underline{\mathbf{a}}), \mathbf{e}_2(\underline{\mathbf{a}}), \cdots, \mathbf{e}_{n-1}(\underline{\mathbf{a}})$ as in [14] by

$$\mathbf{e}_j(\underline{\mathbf{a}}) = \sum_I \underline{\mathbf{a}}_I, \tag{3.4}$$

where I runs through all subsets of $\{0, 1, \cdots, n-1\}$ consisting of exactly j elements and $\underline{\mathbf{a}}_I$ is the product over I of its elements in anticlockwise cyclical order, see [14, §5]. By convention $\mathbf{e}_j(\underline{\mathbf{a}}) = 0$ if either $j < 0$ or $j > n$ and $\mathbf{e}_0(\underline{\mathbf{a}}) = \mathbf{e}_n(\underline{\mathbf{a}})$ is the identity. Following [14, §6] we define the *non-commutative Schur polynomial*

$$s_\lambda(\underline{\mathbf{a}}) = \det(\mathbf{e}_{\lambda_i^t - i + j}(\underline{\mathbf{a}})), \tag{3.5}$$

where λ^t denotes the partition with m_i rows of length i and then the *combinatorial fusion ring* $\mathcal{F}_{comb}(\mathfrak{gl}_n, \ell)$ as the free $\mathbb{Z}$-module with basis $\{\lambda\}_{\lambda \in \mathcal{A}_\ell}$ equipped with the following multiplication

$$\lambda \star \mu = s_\lambda(\underline{\mathbf{a}})\mu. \tag{3.6}$$

It is not obvious that the definition (3.5) makes sense and that (3.6) is commutative. However the proof of the following result establishes this implicitly (see [14] for a completely different proof in that setup using Bethe Ansatz techniques).

Theorem 3.1. *The map* $\Phi : \mathcal{F}_q(\mathfrak{gl}_n, \ell) \to \mathcal{F}_{comb}(\mathfrak{gl}_n, \ell)$ *taking each basis element* $[\lambda] \in \mathcal{F}_q(\mathfrak{gl}_n, \ell)$ *to the basis element* $\lambda \in \mathcal{F}_{comb}(\mathfrak{gl}_n, \ell)$ *is a ring isomorphism.*

Proof: We shall prove that $\Phi([\lambda][\mu]) = \lambda \star \mu$ by induction on $m = \lambda_1 - \lambda_n$. This is clear if $m = 0$. If $m = 1$, then, up to a multiple of $\epsilon_1 + \epsilon_2 + \cdots + \epsilon_n$, we have $\lambda = \omega_i$ for some i. In that case $s_{\omega_i}(\underline{\mathbf{a}}) = \mathbf{e}_i(\underline{\mathbf{a}})$ operates by a formula identical to (3.2). In particular, the operators $\mathbf{e}_i(\underline{\mathbf{a}})$ commute, since $\mathbf{e}_i(\underline{\mathbf{a}})$ corresponds to multiplication by $[\omega_i]$ in the commutative fusion ring $\mathcal{F}_q(\mathfrak{gl}_n, \ell)$. Therefore it makes sense to form the determinant in (3.5).

So suppose $m > 1$. We may write $\lambda = \lambda' + \omega_i$ for some i and some $\lambda' \in \mathcal{A}_\ell$. Then we have

$$[\lambda'][\omega_i] = [\lambda] + \sum_\eta [\eta] \tag{3.7}$$

with the sum running over certain $\eta \in \mathcal{A}_\ell$ with $\eta_1 - \eta_n < m$. Hence, by the above combined with the induction hypothesis, we get

$$\begin{aligned} \Phi([\lambda][\mu]) &= \Phi([\lambda'][\omega_i][\mu]) - \textstyle\sum_\eta \Phi([\eta][\mu]) \\ &= \Phi([\lambda'])\Phi([\omega_i][\mu]) - \textstyle\sum_\eta \eta \star \mu = \lambda' \star \omega_i \star \mu - \textstyle\sum_\eta \eta \star \mu = \lambda \star \mu. \end{aligned}$$

Here the last equality comes from the fact that $\lambda' \star \omega_i = \lambda + \sum_\eta \eta$ where the sum ranges over the same η's as in (3.7). □

3.2. **Fusion algebras for $\mathfrak{sl}_n$.** Preserving the notation from above we set $\epsilon = \epsilon_1 + \epsilon_2 + \cdots + \epsilon_n$ and write $X_0 = \{\lambda \in X \mid \lambda_n = 0\}$. Then any $\lambda \in X$ equals a unique element in X_0 modulo a multiple of ϵ. We set $X_0^+ = X_0 \cap X^+$ and $A_{0,\ell} = X_0 \cap \mathcal{A}_\ell$. Note that $A_{0,\ell}$ is finite.

Restriction from $\mathfrak{gl}_n$ to $\mathfrak{sl}_n$ gives a natural surjection from the fusion ring $\mathcal{F}_q(\mathfrak{gl}_n, \ell)$ to the corresponding fusion ring $\overline{\mathcal{F}_q(\mathfrak{gl}_n, \ell)} = \mathcal{F}_q(\mathfrak{sl}_n, \ell)$ for $\mathfrak{sl}_n$. Similarly, we get a surjection from $\mathcal{F}_{comb}(n, \ell)$ to the corresponding construction $\overline{\mathcal{F}_{comb}(\mathfrak{gl}_n, \ell)}$ for $A_{0,\ell}$. We obtain

Theorem 3.2. *We have ring isomorphisms*

$$\overline{\mathcal{F}_q(\mathfrak{gl}_n, \ell)} \simeq \mathcal{F}_q(\mathfrak{gl}_n, \ell)/([\epsilon] - 1)$$

and

$$\overline{\mathcal{F}_{comb}(\mathfrak{gl}_n, \ell)} \simeq \mathcal{F}_{comb}(\mathfrak{gl}_n, \ell)/(\epsilon - 1),$$

so that the isomorphism Φ from Theorem 3.1 induces an isomorphism

$$\overline{\mathcal{F}_q(\mathfrak{gl}_n, \ell)} \simeq \overline{\mathcal{F}_{comb}(\mathfrak{gl}_n, \ell)}.$$

3.2.1. ℓ *even.* In case ℓ is even we set $\ell' = \ell/2$. Since the roots in type A have the same length the only change we have to make when describing the fusion rules in this case is to replace ℓ by ℓ', see [2]. We can then argue exactly as before to see that all the above statements remain true. Note in particular that we need $\ell \geq 2n$ now as otherwise the fundamental alcove $A_{\ell'}$ is empty.

3.3. **Commutative presentation.** The following connects our approach with well-known presentations of the fusion ring, [6], [5], [8].

Theorem 3.3. *There are isomorphisms of commutative rings*

$$\begin{aligned}\mathcal{F}_{comb}(\mathfrak{gl}_n, \ell) &\simeq \mathbb{Z}[\chi(\omega_1), \dots, \chi(\omega_{n-1})]/I \\ &\simeq \mathbb{Z}[\chi(\omega_1), \dots, \chi(\omega_{n-1})]/J\end{aligned}$$

with $I = \langle \chi(s\omega_1) \mid k{+}1 \leq s < k{+}n \rangle$, $J = \langle \chi(k\omega_1 + \omega_i) \mid 1 \leq i \leq n{-}1 \rangle$.

Proof. The generators given by Proposition 2.4 of the defining ideal are the $\chi(\lambda)$'s with $\lambda = \sum_{i=1}^{n-1} m_i\omega_i$ satisfying $\sum_{i=1}^{n-1} m_i = k+1$. For such λ's the Jacobi-Trudy identity [9, (A.5)] expresses $\chi(\lambda)$ as the determinant of a matrix whose first row contains entries equal to the complete symmetric polynomials $\chi((k+1)\omega_1), \cdots, \chi((k+n-1)\omega_1)$. Hence expanding the determinant along this row we obtain $\chi(\lambda)$ as a $\mathbb{Z}[\chi(\omega_1), \cdots, \chi(\omega_{n-1})]$-linear combination of $\chi(s\omega_1), s = k+1, \cdots, k+n-1$. This proves that our defining ideal is contained in the ideal I. The reversed inclusion is clear once we observe that for s in the given range we have $L_q(s\omega_1) = \nabla_q(s\omega_1) = T_q(s\omega_1)$, see [20, Theorem 3.5]. For the induction step we use the Pieri formula [9, (A.7)]

$$\chi(r\omega_1)\chi(\omega_i) = \chi(r\omega_1 + \omega_i) + \chi((r-1)\omega_1 + \omega_{i+1})$$

for $r \geq 0$ and $1 \leq i \leq n$. This gives for $k+1 \leq s \leq k+n-2$

$$\begin{aligned}\chi(s\omega_1)\chi(\omega_1) &= \chi((s+1)\omega_1) + \chi((s-1)\omega_1 + \omega_2) \\ \chi((s-1)\omega_1)\chi(\omega_2) &= \chi((s-1)\omega_1 + \omega_2) + \chi((s-2)\omega_1 + \omega_3) \\ &\vdots \\ \chi((k+1)\omega_1)\chi(\omega_{s-k}) &= \chi((k+1)\omega_1 + \omega_{s-k}) + \chi(k\omega_1 + \omega_{s-k+1}).\end{aligned}$$

Reading from bottom to top we see that the outer terms are in J, hence the middle term as well and the claim follows. □

4. Three descriptions in Type C

In this section we assume that $\mathfrak{g} = \mathfrak{sp}_{2n}$.

4.1. **The classical case.** We still have $X = \mathbb{Z}^n$ with basis $\epsilon_1, \epsilon_2, \cdots, \epsilon_n$. The positive roots are now $\{\epsilon_i - \epsilon_j, \epsilon_i + \epsilon_j \mid 1 \leq i < j \leq n\} \cup \{2\epsilon_i \mid 1 \leq i \leq n\}$ and the simple roots are $\alpha_i = \epsilon_i - \epsilon_{i+1}, i = 1, \cdots, n-1$ together with $\alpha_n = 2\epsilon_n$. The corresponding fundamental weights are $\omega_i = \epsilon_1 + \epsilon_2 + \cdots + \epsilon_i$, $i = 1, \cdots, n$. A weight $\lambda \in X$ can be expressed in both the ϵ-basis, say $\lambda = \sum_{i=1}^n \lambda_i \epsilon_i$, and in the ω-basis, say $\lambda = \sum_{i=1}^n m_i \omega_i$. The dictionary between the λ_i's and the m_i's is $m_i = \lambda_i - \lambda_{i+1}$, $i = 1, \cdots, n$ (with the convention that $\lambda_{n+1} = 0$) and $\lambda_i = m_i + m_{i+1} + \cdots + m_n$. We have that λ is dominant iff $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$ iff $m_i \geq 0$ for all i.

The highest short and and the highest long root are respectively

$$\begin{aligned} \alpha_0 &= \alpha_1 + 2\alpha_2 + 2\alpha_3 + \cdots + 2\alpha_{n-1} + \alpha_n = \epsilon_1 + \epsilon_2 \\ \beta_0 &= 2\alpha_1 + 2\alpha_2 + \cdots 2\alpha_{n-1} + \alpha_n = 2\epsilon_1. \end{aligned}$$

We have $\langle \omega_1, \alpha_0^\vee \rangle = 1$ whereas $\langle \omega_i, \alpha_0^\vee \rangle = 2$ for $i > 1$. On the other hand, $\langle \omega_i, \beta_0^\vee \rangle = 1$ for all i. Hence $\langle \rho, \alpha_0^\vee \rangle = 2n - 1$ and $\langle \rho, \beta_0^\vee \rangle = n$.

Let now V denote the natural $2n$-dimensional module for $\mathfrak{sp}_{2n}(\mathbb{C})$ with its standard basis $v_1, \cdots, v_{2n}$. Setting $\epsilon_{n+i} = -\epsilon_i$, $i = 1, 2, \cdots, n$ we have that v_i has weight ϵ_i, $i = 1, 2, \cdots, 2n$.

Letting $L_\mathbb{C}(\lambda)$ denote the irreducible $\mathfrak{sp}_{2n}(\mathbb{C})$-module with highest weight $\lambda \in X^+$ and let $\chi(\lambda)$ be its character. We have clearly $L_\mathbb{C}(0) = \mathbb{C}$ and $L_\mathbb{C}(\omega_1) = V$. If $i > 1$, then $L_\mathbb{C}(\omega_i)$ fits into a short exact sequence

$$0 \to L_\mathbb{C}(\omega_i) \to \Lambda^i V \xrightarrow{f} \Lambda^{i-2} V \to 0, \tag{4.1}$$

where f comes from the symplectic form on V, [9, Theorem 17.5].

In analogy with the $\mathfrak{gl}_n$-case in Section 3.1, the weights of $\Lambda^i V$ are $\epsilon_J = \epsilon_{j_1} + \epsilon_{j_2} + \cdots + \epsilon_{j_i}$ with $J = j_1 < j_2 < \cdots < j_i$ running through all increasing sequences in $\{1, 2, \cdots, 2n\}$ consisting of i elements. Note however, that since $\epsilon_{n+i} = -\epsilon_i$ two different such sequences may well lead to the same weight. For instance when $i = 2$ the zero weight equals ϵ_{J_i} for all $J_i = \{i, n+i\}$, $i = 1, 2, \cdots, n$ and so also $\epsilon_J = \epsilon_{J \cup J_i}$ whenever $J \cap J_i = \emptyset$.

4.2. **The quantum case.** Let now $U_q = U_q(\mathfrak{sp}_{2n})$ denote the quantum group for $\mathfrak{sp}_{2n}$ at a complex root of unity q. We denote by ℓ the order of q and assume $\ell > 2n$. When ℓ is odd we have

$$\begin{aligned} \mathcal{A}_\ell &= \{\lambda \in X^+ \mid \langle \lambda + \rho, \alpha_0^\vee \rangle < \ell\} \\ &= \{\lambda \in X^+ \mid m_1 + 2m_2 + 2m_3 + \cdots + 2m_n < \ell - 2n + 1\} \end{aligned}$$

In the case when ℓ is even we set $\ell' = \ell/2$ and have

$$\begin{aligned} \mathcal{A}_\ell &= \{\lambda \in X^+ \mid \langle \lambda + \rho, \beta_0^\vee \rangle < \ell'\} \\ &= \{\lambda \in X^+ \mid m_1 + m_2 + \cdots + m_n < \ell' - n\}. \end{aligned}$$

In either case we have $\mathcal{A}_\ell \neq \emptyset$ is equivalent to our assumption $\ell > 2n$.

Recall that for each $\lambda \in X^+$ the dual Weyl module $\nabla_q(\lambda)$ is a module for U_q whose weights are the same as for the classical module $L_{\mathbb{C}}(\lambda)$. So if for $i \leq n$ we set

$$V_q^i = \begin{cases} \nabla_q(0) \oplus \nabla_q(\omega_2) \oplus \cdots \oplus \nabla_q(\omega_i) & \text{if } i \text{ is even;} \\ \nabla_q(\omega_1) \oplus \nabla_q(\omega_3) \oplus \cdots \oplus \nabla_q(\omega_i) & \text{if } i \text{ is odd;} \end{cases} \tag{4.2}$$

then by (4.1) V_q^i has the same weights as $\Lambda^i V$.

Our assumptions on ℓ and ℓ' ensure that all the $\nabla_q(\omega_i)$ are irreducible tilting modules because all ω_i belong to the closure of $\mathcal{A}_\ell$. In the following we write therefore $L_q(\omega_i)$ instead of $\nabla_q(\omega_i)$. The V_q^i are therefore also tilting modules and formula (2.14) gives

$$[V_q^i][\mu] = \sum_{|J|=i} [\mu + \epsilon_J], \tag{4.3}$$

where the J's appearing in the sum are subsets of $\{1, 2, \cdots, 2n\}$. The formulas (4.1) then imply again Pieri rules

$$[\omega_i][\mu] = \begin{cases} \sum_{j=1}^{2n} [\mu + \epsilon_j], & \text{if } i = 1, \\ \sum_{|J|=i} [\mu + \epsilon_J] - \sum_{|J|=i-2} [\mu + \epsilon_J], & \text{if } i \geq 2. \end{cases} \tag{4.4}$$

4.3. **Combinatorics.** We shall now describe the product formulas (4.4) combinatorially, in analogy with the type A case considered in Section 3.1. This will then allow us to deduce the general fusion rule in terms of combinatorially described operators.

Consider the free $\mathbb{Z}$-modules $\mathbb{Z}[X]$ and $\mathbb{Z}[\mathcal{A}_\ell]$ with bases e^λ for $\lambda \in X$ and $\lambda \in \mathcal{A}_\ell$, respectively. We define a $\mathbb{Z}$-linear map $\pi_\ell : \mathbb{Z}[X] \to \mathbb{Z}[\mathcal{A}_\ell]$ by the recipe

$$\pi_\ell(e^\lambda) \quad = \quad \begin{cases} (-1)^{l(w)} e^{w \cdot \lambda}, & \text{if there exists } w \in W_\ell \text{ with } w \cdot \lambda \in \mathcal{A}_\ell, \\ 0, & \text{otherwise.} \end{cases}$$

For $j \in \{1, 2, \cdots, 2n\}$ we define $\mathbb{Z}$-linear endomorphisms $\mathbf{a}_j$ of $\mathbb{Z}[X]$ by

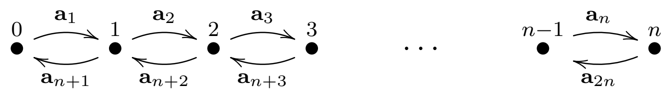
$$\mathbf{a}_j(e^\lambda) = e^{\lambda + \epsilon_j}, \ \lambda \in X. \tag{4.5}$$

For each subset $J = \{j_1 < j_2 < \cdots < j_i\} \subset \{1, 2, \cdots, 2n\}$ these operators give rise to the operator $\mathbf{a}_J = \mathbf{a}_{j_1} \circ \mathbf{a}_{j_2} \circ \cdots \circ \mathbf{a}_{j_i}$ on $\mathbb{Z}[X]$. Then for any $i \leq n$ we define $\mathbf{e}_i : \mathbb{Z}[\mathcal{A}_\ell] \to \mathbb{Z}[\mathcal{A}_\ell]$ by

$$\mathbf{e}_i(e^\lambda) = \sum_{|J|=i} \pi_\ell(\mathbf{a}_J(e^\lambda)), \ \lambda \in \mathcal{A}_\ell. \tag{4.6}$$

4.3.1. *Case ℓ even.* Let ℓ be even. If $\lambda = \sum_{i=1}^n m_i \omega_i \in X$ we set $m_0 = \ell' - n - 1 - \sum_{i=1}^n m_i$ and we identify λ with the particle configuration on the extended Dynkin diagram of type $\tilde{C}_n$ with m_i particles if $m_i \geq 0$, respectively $-m_i$ antiparticles if $m_i < 0$, placed at node i, $i = 0, 1, \cdots, n$. Counted with signs (+ for particles and − for antiparticles) we have a total of $k := \ell' - n - 1$ particles. This number k is preserved by the above operators $\mathbf{a}_i$ and called the *level*. For $1 \le i \le n$ we may describe $\mathbf{a}_i$ as particle hopping from node $i-1$ to node i. If there are no particles at node $i-1$, then we first add a pair consisting of a particle and an antiparticle to this node and then make this particle hop. If at node i we have only antiparticles, then the particle added to this node annihilates one of the antiparticles. The inverse $\mathbf{a}_{n+i}$ is hopping in the reverse direction:

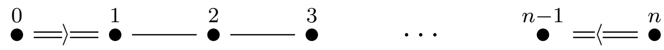
(4.7)



Let $\lambda \in \mathcal{A}_\ell$. This means that the particle configuration corresponding to λ contains no antiparticles. There are $k = l' - n - 1$ particles in this configuration. Let J be a subset of $\{1, 2, \cdots, 2n\}$ with $|J| \le n$. Note that $\langle \epsilon_J, \alpha_i^\vee \rangle \in \{0, \pm 1, \pm 2\}$ for all $i = 1, \cdots, n-1$ whereas $\langle \epsilon_J, \alpha_n^\vee \rangle, \langle \epsilon_J, \beta_0^\vee \rangle \in \{0, \pm 1\}$. Hence $\lambda + \epsilon_J$ contains at most 2 antiparticles at a given node. This node cannot be an end node (0 or n) and if node i contains 2 antiparticles, then no antiparticles are positioned at the adjacent nodes $i-1$ and $i+1$. When we apply π_ℓ we get 0 if there is exactly 1 antiparticle at some node, because then the weight is ℓ-singular. If there are 2 antiparticles at node i, then we replace the configuration by minus the one where we have removed 1 particle from each of the two nodes $i-1$ and $i+1$ and placed both of them at node i (thus annihilating the 2 antiparticles there). If the resulting configuration still contains antiparticles, we repeat the above process. The end result corresponds then to $\pi_\ell(e^{\lambda + \epsilon_J})$. Note that there are at most as many annihilation steps as there are nodes with 2 antiparticles.

The action of W_ℓ which is applied to remove 2 antiparticles can be visualized on the extended Dynkin diagram of type $\tilde{C}_n$:

(4.8)



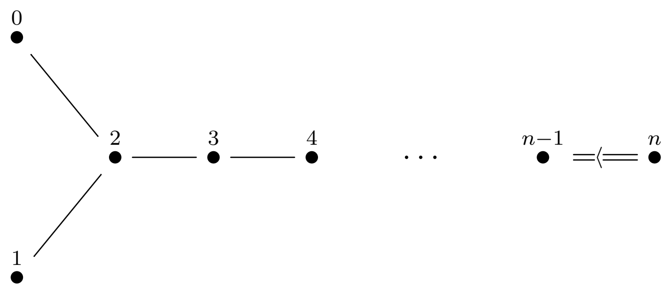
Note that particles are always taken from neighboring vertices, one for each edge; for the special vertices 0 and n two particles get moved along a double edge against the direction of the arrow.

4.3.2. *Case ℓ odd.* Suppose that ℓ is odd. If $\lambda = \sum_{i=1}^n m_i \omega_i \in X$, we set this time $m_0 = \ell - 2n - m_1 - 2m_2 - 2m_3 - \cdots - 2m_n$. We may again interpret λ as a configuration of particles and antiparticles placed at the nodes $0, 1, \cdots, n$. If $m_i \ge 0$, there are m_i particles at node i and if $m_i < 0$, there are $-m_i$ antiparticles at node i. This time we count particles and antiparticles at the nodes $2, 3, \cdots, n$ double, i.e. we have a total of $k = \ell - 2n$ particles. This number is preserved by the $\mathbf{a}_i$'s which we can describe as follows: For $i \le n$ the operator $\mathbf{a}_i$ moves a particle from node $i-1$ to node i if $i \ne 2$, while $\mathbf{a}_2$ removes 1 particle from each of the nodes 0 and 1 and fuse these into 1 particle placed at node 2. Again $\mathbf{a}_{n+i}$ is the inverse of $\mathbf{a}_i$. So for instance $\mathbf{a}_{n+2}$ removes a particle from node 2, splits it into 2 and places these at nodes 0 and 1. Hence it could be thought of as moving particles one step to the left resp. right along the diagram (4.9) below (except for $\mathbf{a_1}$ and $\mathbf{a_{n+1}}$ which moves between nodes 0 and 1).

Let $\lambda \in \mathcal{A}_\ell$. This means that the corresponding particle configuration contains no antiparticles and k particles. For any $J \subset \{1, 2, \cdots, 2n\}$ the configuration for $\lambda + \epsilon_J$ contains at most 1 antiparticle at node n and at most 2 antiparticles at any other node. Moreover, if node i has 2 antiparticles, there are no antiparticles at nodes $i-1$ and $i+1$. If there is a node with exactly 1 antiparticle, then $\pi_\ell(e^{\lambda+\epsilon_J}) = 0$, since the corresponding weight is ℓ-singular. If there are 2 antiparticles at node 0 or 1 take a particle at node 2 (which counts double) and make it annihilate the 2 antiparticles at the given node. If there are 2 antiparticles at node 2, we annihilate them by taking a particle from each of the nodes $0, 1$, and 3 (the 2 particles from nodes 0 and 1 need to combine in order to annihilate one of the antiparticles from node 2). Finally, 2 antiparticles at any node i with $2 < i < n$ are annihilated by taking a particle from each of the two adjacent nodes. If after this process the result still contains antiparticles, we repeat it until this is no longer the case. This end configuration is then, up to signs, equal to $\pi_\ell(e^{\lambda+\epsilon_J})$. In case it is non-zero, the sign is positive if we have used an even number of annihilation steps and negative otherwise. Again the number of steps is at most equal to the number of nodes with 2 antiparticles.

In this case the action of W_ℓ which is applied to remove 2 antiparticles can be visualized using the extended Dynkin diagram of type $\tilde{B}_n^t$:

(4.9)



Again particles are always taken from neighboring vertices, one for each single edge; for the special vertex n two particles get moved along a double edge against the arrow.

4.4. **Combinatorial fusion rule.** Let $\mathbf{e}_i$ be the operators on $\mathbb{Z}[\mathcal{A}_\ell]$ defined above. The definitions of $\mathcal{A}_\ell$ and of $\mathbf{e}_i$ are different in the odd and even cases. However, in both cases we have

Proposition 4.1. *The operators $\mathbf{e}_i$ and $\mathbf{e}_j$ commute for $1 \le i, j \le n$.*

Proof: Note that as $\mathbb{Z}$-modules we have an isomorphism $\mathcal{F}_q \simeq \mathbb{Z}[\mathcal{A}_\ell]$ such that the operator $\mathbf{e}_i$ corresponds to multiplication by V_q^i in $\mathcal{F}_q$. But $\mathcal{F}_q$ is a commutative and associative ring, therefore multiplications by V_q^i and V_q^j commute. □

Set now $\mathbf{e}'_0 = 1$, $\mathbf{e}'_1 = \mathbf{e}_1$ and $\mathbf{e}'_i = \mathbf{e}_i - \mathbf{e}_{i-2}$ for $2 \le i \le n$. If $\lambda \in \mathcal{A}_\ell$ we define in analogy with (3.5) the operator $\mathbf{s}'_\lambda$ on $\mathbb{Z}[\mathcal{A}_\ell]$ by

$$\mathbf{s}'_\lambda = \det(\mathbf{e}'_{\lambda_i^t - i + j}). \tag{4.10}$$

By the above proposition the $\mathbf{e}'_i$'s clearly commute so that it makes sense to form this determinant. By [9, Appendix A.3, Corollary 24.24] the operator $\mathbf{s}'_\lambda$ corresponds to multiplication by $L_q(\lambda)$ in $\mathcal{F}_q$. Hence we define multiplication on $\mathbb{Z}[\mathcal{A}_\ell]$ (as before we denote the basis vector corresponding to $\lambda \in \mathcal{A}_\ell$ by λ) as follows

$$\lambda \star \mu = \mathbf{s}'_\lambda(e^\mu) \tag{4.11}$$

and denote the resulting ring by $\mathcal{F}_{comb}(\mathfrak{sp}_{2n}, \ell)$. This is then a commutative, associative ring with unit $1 = e^0$ and underlying vector space canonically identified with $\mathcal{F}_q(\mathfrak{sp}_{2n}, \ell)$ via $\lambda \mapsto [\lambda]$. Moreover, we obtain

Theorem 4.2. *For any $\ell > 2n$ we have an isomorphism of rings*

$$\mathcal{F}_q(\mathfrak{sp}_{2n}, \ell) = \mathcal{F}_{comb}(\mathfrak{sp}_{2n}, \ell)$$

taking each basis element $[\lambda]$ to the basis element $\lambda \in \mathcal{F}_{comb}(\mathfrak{sp}_{2n}, \ell)$.

4.5. **Commutative presentation.** Finally we connect our combinatorial description with well-known presentations of the fusion ring, [12], [5]. For the analogous statement in type A see [14, Theorem 6.20].

Theorem 4.3. *Let ℓ be even. There is an isomorphism of commutative associative rings*

$$\Psi : \mathcal{F}_{comb}(\mathfrak{sp}_{2n}, \ell) \simeq \mathbb{Z}[\chi(\omega_1), \dots, \chi(\omega_n)]/\langle \chi(k\omega_1 + \omega_i) \mid 1 \le i \le n \rangle$$

which sends λ to the character $\chi(\lambda)$.

Note that the number of generators in the ideal is independent of the level in this case.

Proof. The fusion ring is a quotient of $\mathbb{Z}[\chi(\omega_1), \dots, \chi(\omega_n)]$ by some ideal I_ℓ whose generators are given by Proposition 2.4. These are the $[T_q(\lambda)]$ with λ minimal such that $\langle \lambda, \beta_0^\vee \rangle > k$. In our case this is precisely $\langle \lambda, \beta_0^\vee \rangle = k + 1$, hence $\lambda_1 = k + 1$. Since $[T_q(\lambda)] = [L_q(\lambda)]$ by (2.12), Theorem 4.2 implies that I_ℓ is generated by all $\chi(\lambda)$ where $\lambda_1 = k+1$. Now using the Jacobi-Trudi type determinant formula [10, 3.9] for symplectic characters, expanded along the first row, implies that $\chi(\lambda)$ is a $\mathbb{Z}[\chi(\omega_1), \dots, \chi(\omega_n)]$-linear combination of the elements $\chi((k+1)\omega_1)$ and $\chi((k+i+1)\omega_1 + \chi(k+1-i)\omega_1)$ for $1 \le i \le n-1$. Hence I_ℓ is contained in the ideal generated by these and they are obviously also contained in this ideal. Expanding the determinant formulas for

$\chi(k\omega_1+\omega_i)$, we get $\chi(k\omega_1+\omega_i) = \mp\chi((k+i)\omega_1)+R$ for some linear combination of $\chi(\lambda)$'s with λ smaller in the dominance ordering of partitions. Therefore, our ideal is also generated by the $\chi(k\omega_1+\omega_i)$ for $1 \le i \le n$. □

For odd ℓ the presentation is not as explicit, but at least we have

Theorem 4.4. *Let ℓ be odd. There is an isomorphism of commutative, associative rings*

$$\Psi : \mathcal{F}_{comb}(\mathfrak{sp}_{2n}, \ell) \simeq \mathbb{Z}[\chi(\omega_1), \dots, \chi(\omega_n)]/I_\ell$$

which sends λ to the character $\chi(\lambda)$. Here I_ℓ is generated by all characters $\chi(\lambda)$ where $\lambda_1+\lambda_2 = k+1$.

Proof. We again use Proposition 2.4 which shows that I_ℓ is generated by all $[T(\lambda)]$ where λ satisfies

(1) $m_1 + 2\sum_{i=2}^n m_i = k+1 = \ell-2n+1$ or
(2) $m_1 = 0$ and $2\sum_{i=2}^n m_i = k+2 = \ell-2n+2$.

Note that k, and then also $k+2$, is odd, hence (2) has no solutions. For the first set of generators we have again by (2.12) the equality $[T_q(\lambda)] = [L_q(\lambda)]$ and $\lambda_1+\lambda_2 = k+1$. □

5. Even and odd fusion rings for Type D

In this section we assume that $\mathfrak{g} = \mathfrak{so}_{2n}$. The simple roots are now $\alpha_i = \epsilon_i - \epsilon_{i+1}, i = 1, \cdots, n-1$ together with $\alpha_n = \epsilon_{n-1}+\epsilon_n$. The corresponding fundamental weights are $\omega_i = \epsilon_1+\epsilon_2+\cdots+\epsilon_i$, $i = 1, \cdots, n-2$ and $\omega_{n-1} = \frac{1}{2}(\epsilon_1 + \cdots \epsilon_{n-1} - \epsilon_n)$ and $\omega_n = \frac{1}{2}(\epsilon_1 + \cdots \epsilon_{n-1} + \epsilon_n)$. The highest root is

$$\alpha_0 \quad = \quad \alpha_1 + 2\alpha_2 + 2\alpha_3 + \cdots + \alpha_{n-1} + \alpha_n = \epsilon_1+\epsilon_2 = \omega_2.$$

Set $k = \ell - (2n-2)$ in case ℓ is odd and $\frac{\ell}{2} - (2n-2)$ if ℓ is even. Then $\mathcal{A}_\ell$ consists of all dominant weights $\sum_{i=1}^n m_i\omega_i$ such that

$$m_1 + 2m_2 + \cdots + 2m_{n-2} + m_{n-1} + m_n \quad \le \quad k.$$

Proposition 2.4 implies

Theorem 5.1. *Let ℓ be odd or $\ell \equiv 2 \mod 4$. Then there is an isomorphism of commutative, associative rings*

$$\Psi : \mathcal{F}_q(\mathfrak{so}_{2n}, \ell) \simeq \mathbb{Z}[\chi(\omega_1), \dots, \chi(\omega_n)]/I_\ell$$

which sends $[\lambda]$ to the character $\chi(\lambda)$. Here I_ℓ is generated by all characters $\chi(\lambda)$ where $\lambda_1+\lambda_2 = k+1$. In case $\ell \equiv 0 \mod 4$ the ideal I_ℓ is generated by all characters $\chi(\lambda)$ where $\lambda_1+\lambda_2 = k+1$ together with the characters of all $T(\lambda)$, where $2(\lambda_2 - \lambda_{n-1}) = k+2$.

Proof. Note that $m_1+2m_2+\cdots+2m_{n-2}+m_{n-1}+m_n = \lambda_1+\lambda_2$. If ℓ is odd or $\ell \equiv 2 \mod 4$, then k and $k+2$ are odd and there is no minimal $\lambda \in X/\mathcal{A}_\ell$ with $\lambda_1+\lambda_2 = k+2$. In case $\ell \equiv 0 \mod 4$, k is even and the elements λ with $2(\lambda_2 - \lambda_{n-1}) = 2m_2+\cdots+2m_{n-2} = k+2$ are minimal. □

6. Even and odd fusion rings for Type B

In this section we assume that $\mathfrak{g} = \mathfrak{so}_{2n+1}$. The simple roots are now $\alpha_i = \epsilon_i - \epsilon_{i+1}, i = 1, \cdots, n-1$ together with $\alpha_n = \epsilon_n$. The corresponding fundamental weights are $\omega_i = \epsilon_1+\epsilon_2+\cdots+\epsilon_i$, $i = 1,\cdots, n-1$ and $\omega_n = \frac{1}{2}(\epsilon_1+\cdots\epsilon_{n-1}+\epsilon_n)$. The highest short respectively long roots are

$$\begin{aligned}\alpha_0 &= \alpha_1+\alpha_2+\alpha_3+\cdots+\alpha_n = \epsilon_1 = \omega_1\\ \beta_0 &= \alpha_1+2\alpha_2+2\alpha_3+\cdots+2\alpha_n = \epsilon_1+\epsilon_2 = \omega_2.\end{aligned}$$

Then $\mathcal{A}_\ell$ consists of all dominant weights $\sum_{i=1}^n m_i\omega_i$ such that

$$\begin{aligned}2m_1+2m_2+2m_3+\cdots+2m_{n-1}+m_n &\leq \ell-2n =: k\\ m_1+2m_2+2m_3+\cdots 2m_{n-2}+2m_{n-1}+m_n &\leq \ell/2-2n+1 =: k\end{aligned}$$

in case ℓ is odd respectively even.

Theorem 6.1. *Let $\ell \geq 2n$ if ℓ is odd and $\ell \geq 4n-4$ if ℓ is even. Then there is an isomorphism of commutative, associative rings*

$$\Psi : \mathcal{F}_q(\mathfrak{so}_{2n+1}, \ell) \simeq \mathbb{Z}[\chi(\omega_1), \ldots, \chi(\omega_n)]/I_\ell$$

which sends $[\lambda]$ to the character $\chi(\lambda)$. Here I_ℓ is

(1) $I_\ell = \langle \chi(\lambda) \mid 2\lambda_1 = k+1\rangle = \langle\{\chi(\frac{k-1}{2}\omega_1+\omega_i) \mid 1 \leq i \leq n-1\} \cup \{\chi(\frac{k-1}{2}\omega_1+2\omega_n)\}\rangle$ in case ℓ is odd,
(2) $I_\ell = \langle \chi(\lambda) \mid \lambda_1+\lambda_2 = k+1\rangle$ in case $\ell \equiv 0 \mod 4$,
(3) $I_\ell = \langle \chi(\lambda), [T(\mu)] \mid \lambda_1+\lambda_2 = k+1, \mu_1+\mu_2 = k+2\rangle$ in case $\ell \equiv 2 \mod 4$.

Proof. The first equality in each case is a direct consequence from Proposition 2.4. To see the second equality in case ℓ is odd we abbreviate $m = \frac{k+1}{2}$ and $\chi^s := \chi(\frac{k-1}{2}\omega_1+\omega_s)$ and set $a_s = h_{m-1+s} - h_{m-1-s}$ for $1 \leq s \leq n$. Then

Lemma 6.2. $a_s = (-1)^{s+1}(\chi^s - h_1\chi^{s-1} + h_2\chi^{s-2} - h_3\chi^{i-3} + \cdots \mp h_s\chi^1)$.

Proof. We use the following determinant formula from [9, Ex. 24.46]

$$\chi(\lambda) = |h_{\lambda_j-i+j} - h_{\lambda_i-i-j}|_{1\leq i,j\leq n}$$

Then χ^1 is given by a lower diagonal matrix with diagonal entries all 1 except of the top entry which is $h_m - h_{m-2} = a_1$. The matrix for χ^2

is of the form

$$\begin{pmatrix} h_m - h_{m-2} & 1 & 0 & 0 & 0 & \cdots \\ h_{m+1} - h_{m-3} & h_1 & 1 & 0 & 0 & \cdots \\ h_{m+2} - h_{m-4} & h_2 & h_1 & 0 & 0 & \cdots \\ h_{m+3} - h_{m-5} & h_3 & h_2 & 1 & 0 & \cdots \\ & \vdots & & & & \end{pmatrix}$$

Expanding along the second row gives $-\chi^2 = a_2 - h_1 a_1$, hence $a_2 = -\chi^2 + h_1 a_1$. The result follows inductively by computing χ^s by expanding along the sth row. □

Expanding the determinant formula for $\chi(\lambda)$ with $2\lambda_1 = k+1$ shows that the a_i generate I_ℓ. Since obviously $\langle \chi(\frac{k-1}{2}\omega_1 + \omega_i) \mid 1 \leq i \leq n \rangle \subset I_\ell$, the equality follows. □

7. An exceptional example: G_2

Let $\mathfrak{g}$ be of type G_2. In Bourbaki notation the positive roots are α_1, α_2, $\alpha_1 + \alpha_2$, $2\alpha_1 + \alpha_2 = \alpha_0 = \omega_1$, $3\alpha_1 + \alpha_2$, $3\alpha_1 + 2\alpha_2 = \beta_0 = \omega_2$ with $\check{\alpha}_0 = 2\check{\alpha}_1 + 3\check{\alpha}_2$, and $\check{\beta}_0 = \check{\alpha}_1 + 2\check{\alpha}_2$. Note that $\langle \rho, \check{\beta}_0 \rangle = 3$ and $\langle \lambda + \rho, \check{\alpha}_0 \rangle = 5$. The definition of the *level* k depends on the parity of ℓ and whether 3 divides ℓ or not. If 3 does not divide ℓ, set $k = \ell - 6$ for ℓ odd and $k = \ell/2 - 6$ for ℓ even. In case $3|\ell$ set $\ell = 3\ell'$ and define $k = l' - 4$ if ℓ is odd and $k = \ell'/2 - 4$ if ℓ is even. Then the fundamental alcove $\mathcal{A}_\ell$ equals

$$\mathcal{A}_\ell = \begin{cases} \{\lambda \in X^+ \mid \langle \lambda, \check{\beta}_0 \rangle \leq k\} = \{(a,b) \mid a + 2b \leq k\}, & \text{if } 3|\ell, \\ \{\lambda \in X^+ \mid \langle \lambda, \check{\alpha}_0 \rangle \leq k\} = \{(a,b) \mid 2a + 3b \leq k\}, & \text{otherwise,} \end{cases}$$

where we abbreviate $(a,b) = a\omega_1 + b\omega_2$.

Theorem 7.1. *The G_2 fusion ring is isomorphic to $\mathbb{Z}[\chi(\omega_1), \chi(\omega_2)]/I_\ell$, where I_ℓ is generated by*

$$\begin{cases} \chi(0, \frac{k+1}{2}), \chi(2, \frac{k-1}{2}), \chi(4, \frac{k-3}{2}), & \textit{if } 3|l \textit{ and } k \textit{ odd,} \\ \chi(0, \frac{k}{2}+1) + \chi(0, \frac{k}{2}), \chi(1, \frac{k}{2}), \chi(3, \frac{k}{2} - 1), & \textit{if } 3|\ell,\ k \textit{ even,} \\ \chi(\lambda), \chi(\mu) + \chi(\mu - \omega_1), & \textit{if } 3 \not|\ell,\ k \equiv 2 \bmod 3, \\ \chi((0, \frac{k+2}{3}), \chi(\lambda), \chi(\mu) + \chi(\mu - \omega_1), & \textit{if } 3 \not|\ell,\ k \equiv 1 \bmod 3. \end{cases}$$

Here λ runs through $\Lambda := \{(a,b) \mid 2a + 3b = k+1\}$ and $\mu \in \Lambda' := \{(a,b) \mid a \neq 0, 2a + 3b = k+2\}$.

Proof. Abbreviate $L_i = [L(\omega_i)]$ for $i = 1,2$ and let J denote the ideal generated by the proposed generators of I_ℓ. Assume $3|\ell$. Let first k be odd. The minimal elements in the sense of Proposition (2.4) are the tilting modules of highest weights (a,b) where $a + 2b = k + 1$. All weights belong to the upper closure of $\mathcal{A}_\ell$ so that $T((a,b)) = \Delta((a,b)) = L((a,b))$. Hence it is enough to show that $g_j := [\Delta(2j, \frac{k+1}{2} -$

$j)] \in J$ for $0 \leq j \leq k$. From the linkage principle we obtain the classes of tilting modules $t_i := [T(2i+1, \frac{k+1}{2} - i)] = [\Delta(2i+1, \frac{k+1}{2} - i)] + [\Delta(2i+1, \frac{k+1}{2} - i - 1)]$ for $0 \leq i \leq k$ and $s_i = [T(2i, \frac{k+3}{2} - i)] = [\Delta(2i, \frac{k+3}{2} - i)] + [\Delta(2i, \frac{k+3}{2} - i - 2)]$ for $0 \leq j \leq k+1$. Set $t_i = g_i = s_i = 0$ if $i < 0$. Using (2.9) or alternatively the type G_2 Littlewood-Richardson rule, [15], we obtain

$$g_r L_1 = \begin{cases} t_0 + g_1, & \text{if } r = 0, \\ g_{r-1} + t_{r-1} + g_r + t_r + g_{r+1}, & \text{if } r > 0; \end{cases} \tag{7.1}$$

$$g_r L_2 = \begin{cases} s_0 + g_0 + g_1 + t_1, & \text{if } r = 0, \\ t_{r-2} + g_{r-1} + t_{r-1} + s_r + 2g_r + t_r + g_{r+1} + t_{r+1}, & \text{if } r > 0; \end{cases} \tag{7.2}$$

$$t_r L_1 = t_{r-1} + s_r + 2g_r + t_r + s_{r+1} + 2g_{r+1} + t_{r+1}, \qquad \text{for } r \geq 0. \tag{7.3}$$

Using (7.1) twice (7.2), (7.3), (7.2), (7.3) we obtain $t_0, t_1, s_0, s_1, t_2, s_2 \in J$. Repeatedly applying (7.1), (7.2), (7.3) we obtain that $g_a, t_b, s_b \in J$ for $a \geq 3$, $b \geq 1$ and the first case of the theorem follows.

For even k we have an additional minimal generator in the sense of Proposition (2.4), namely the class of $T(\lambda)$ where $\lambda = (0, \frac{k+2}{2})$ which belongs to the second alcove $s_{\alpha_0,1}\mathcal{A}_\ell$. Hence $[T(\lambda)] = [\Delta(\lambda)] + [\Delta(\lambda - \omega_2)]$. Set $g_0 = s_0 = 0$ and otherwise $g_j := [\Delta(2j-1, \frac{k}{2} - j + 1)]$, $t_j := [T(2i, \frac{k}{2} - j + 1)] = [\Delta(2j, \frac{k}{2} - j + 1)] + [\Delta(2j, \frac{k}{2} - j)]$ and $s_j = [T(2i-1, \frac{k}{2} - i + 2)] = [\Delta(2i, \frac{k}{2} - i + 1)] + [\Delta(2i, \frac{k}{2} - i - 1)]$ for all $0 \leq j \leq \frac{k}{2} + 1$. It is enough to show that $g_j \in J$ for $0 \leq j \leq \frac{k}{2} + 1$. Formula (2.9) or alternatively the Littlewood-Richardson rules give us

$$g_r L_1 = g_{r-1} + t_{r-1} + g_r + t_r + g_{r+1}, \qquad \text{for } r \geq 0; \tag{7.4}$$

$$g_r L_2 = \begin{cases} s_1 + 2g_1 + t_1 + g_2 + t_2, & \text{if } r = 1, \\ t_{r-2} + g_{r-1} + t_{r-1} + 2g_r + s_r + t_r + g_{r+1} + t_{r+1}, & \text{if } r > 1; \end{cases} \tag{7.5}$$

$$t_r L_1 = \begin{cases} s_1 + 2g_1 + t_1, & \text{if } r = 0, \\ t_{r-1} + s_r + 2g_r + t_r + s_{r+1} + 2g_{r+1} + t_{r+1}, & \text{if } r > 0. \end{cases} \tag{7.6}$$

Using (7.4), (7.6), (7.5) repeatedly we obtain

$$t_1, s_1, t_2, g_3, s_2, t_3, \ldots g_a, s_{a-1}, t_a \ldots \in J = \langle t_0, g_1, g_2 \rangle. \tag{7.7}$$

Hence all g's are in J and the second case of the theorem follows.

Assume now that 3 does not divide ℓ. The minimal elements in the sense of Proposition (2.4) are the tilting modules of highest weight (a, b) where $(a, b) \in \Lambda \cup \Lambda'$ and additionally $(a, b) = (0, \frac{k+2}{2})$ if $k = 1$ mod 3. The weights from Λ belong to the upper closure of A_ℓ and hence we have $T(\lambda) = \Delta(\lambda) = L(\lambda)$ for $\lambda \in \Lambda$. The weights from

Λ' belong to the second alcove $s_{\alpha_0,1}(\mathcal{A}_\ell)$ and hence we have $[T(\mu)] = [\Delta(\mu)] + [\Delta(\mu - \omega_1)]$ for $\mu \in \Lambda'$ by the linkage principle. Since $(0, \frac{k+2}{2})$ is minimal in X^+ with respect to the strong linkage principle we must have $T(0, \frac{k+2}{3}) = \Delta(0, \frac{k+2}{3}) = L(0, \frac{k+2}{3})$. Hence the theorem follows from Proposition 2.4. □

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