GENERIC $\mathfrak{gl}_2$-FOAMS, WEB AND ARC ALGEBRAS

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Abstract. We define parameter dependent $\mathfrak{gl}_2$-foams and their associated web and arc algebras and verify that they specialize to several known $\mathfrak{sl}_2$ or $\mathfrak{gl}_2$ constructions related to higher link and tangle invariants. Moreover, we show that all these specializations are equivalent, and we deduce several applications, e.g. for the associated link and tangle invariants (and their functoriality).

Contents

1. Introduction

Let $\mathtt{P} = \{\boldsymbol{\alpha}, \boldsymbol{\tau}^{\pm 1}, \boldsymbol{\omega}_+^{\pm 1}, \boldsymbol{\omega}_-^{\pm 1}\}$ be a set of *generic parameters*. In this paper we introduce a $\mathtt{P}$-version of *singular topological quantum field theories (TQFTs)* which we use to define a 4-*parameter foam* 2-*category* $\mathfrak{F}[\mathtt{P}]$, that is a certain 2-category of topological origin. We obtain from $\mathfrak{F}[\mathtt{P}]$ several *specializations*. Among the specializations of this 4-parameter version one can find the main foam 2-categories studied in the context of higher link and tangle invariants:

- Khovanov/Bar-Natan's cobordisms (see [23] or [3]) can be obtained by specializing $\boldsymbol{\alpha} = 0$, $\boldsymbol{\tau} = 1$, $\boldsymbol{\omega}_+ = 1$, $\boldsymbol{\omega}_- = 1$,
- Caprau's "foams" (see [11]) by specializing $\boldsymbol{\alpha} = 0$, $\boldsymbol{\tau} = 1$, $\boldsymbol{\omega}_+ = i$, $\boldsymbol{\omega}_- = -i$,
- Clark-Morrison-Walker's disoriented cobordisms (see [15]) by specializing $\boldsymbol{\alpha} = 0$, $\boldsymbol{\tau} = 1$, $\boldsymbol{\omega}_+ = i$, $\boldsymbol{\omega}_- = -i$, and
- Blanchet's foams (see [4]) by specializing $\boldsymbol{\alpha} = 0$, $\boldsymbol{\tau} = -1$, $\boldsymbol{\omega}_+ = 1$, $\boldsymbol{\omega}_- = -1$.

We write for these theories **KBN**, **Ca**, **CMW** and **Bl** respectively.

We also study the *web algebra* $\mathfrak{W}[\mathtt{P}]$ corresponding to $\mathfrak{F}[\mathtt{P}]$, i.e. an algebra which has an associated 2-category of certain bimodules giving a (fully) faithful 2-representation of $\mathfrak{F}[\mathtt{P}]$. (Similarly for any specialization of $\mathtt{P}$.)

For $\mathtt{Q} = \{\boldsymbol{\alpha}, \boldsymbol{\varepsilon}, \boldsymbol{\omega}^{\pm 1}\}$ (obtained by specializing $\mathtt{P}$ via $\boldsymbol{\tau} = \boldsymbol{\varepsilon}\boldsymbol{\omega}^2$, $\boldsymbol{\omega}_+ = \boldsymbol{\omega}$ and $\boldsymbol{\omega}_- = \boldsymbol{\varepsilon}\boldsymbol{\omega}$ with $\boldsymbol{\varepsilon} = \pm 1$) we define an *algebraic model* simultaneously for $\mathfrak{F}[\mathtt{Q}]$ and $\mathfrak{W}[\mathtt{Q}]$, that is, an *arc algebra* $\mathfrak{A}[\mathtt{Q}]$ encoding algebraically/combinatorially the topological data coming from $\mathfrak{F}[\mathtt{Q}]$ and $\mathfrak{W}[\mathtt{Q}]$. The foam 2-category, web and arc algebra, called *signed 2-parameter* versions, still contain our four main examples as specializations. We call the $\boldsymbol{\varepsilon} = 1$ specializations the $\mathfrak{sl}_2$ *specializations* and the $\boldsymbol{\varepsilon} = -1$ specializations the $\mathfrak{gl}_2$ *specializations*, since they correspond to the web algebras describing the tensor categories of finite-dimensional representations of the respective complex Lie algebra (see in the second subsection below for more details).

Our main result is, surprisingly, that any two specializations of $\boldsymbol{\varepsilon}, \boldsymbol{\omega}^{\pm 1}$ to values in some ring R (with $\boldsymbol{\varepsilon} = \pm 1$) are *isomorphic/equivalent* (in fact, isomorphic/equivalent to the signed 2-parameter versions). More precisely, if we denote by $*$ any such specialization, then (we only extend scalars to get an isomorphism of Q-algebras):

Theorem. Let $Q = \mathbb{Z}[\boldsymbol{\alpha}, \boldsymbol{\varepsilon}, \boldsymbol{\omega}^{\pm 1}], \boldsymbol{\varepsilon} = \pm 1$. There are graded algebra isomorphisms

$$\Psi \colon \mathfrak{A}[\mathtt{Q}] \xrightarrow{\cong} \mathfrak{A}[*] = \mathfrak{A}_R[*] \otimes_{\mathbb{Z}} Q.$$

(Similarly for the corresponding web algebras.) ■

Additionally one can also specialize $\boldsymbol{\alpha}$. But in contrast to the other parameters involved this sometimes has to be done on both sides of the isomorphisms/equivalences. We will call this *simultaneous* specialization for short. From this we obtain:

Theorem. The isomorphisms from above induce isomorphisms of graded, Q-linear 2-categories (of certain graded bimodules)

$$\boldsymbol{\Psi} \colon \mathfrak{A}[\mathtt{Q}]\text{-}\mathbf{biMod}^p_{\mathrm{gr}} \xrightarrow{\cong} \mathfrak{A}[*]\text{-}\mathbf{biMod}^p_{\mathrm{gr}},$$

giving on the topological side equivalences of graded, Q-linear 2-categories

$$\mathfrak{A}[\mathtt{Q}]\text{-}\mathbf{biMod}^p_{\mathrm{gr}} \cong \mathfrak{F}[\mathtt{Q}] \cong \mathfrak{F}[*] \cong \mathfrak{A}[*]\text{-}\mathbf{biMod}^p_{\mathrm{gr}}.$$

(Similarly for any further simultaneous specialization of $\boldsymbol{\alpha}$.) ■

The (main ingredients for the) proofs of all these statements are less trivial than we expected. Since some of them are also rather lengthy, we have moved them into an extra section, see Section 6. An almost direct consequence of the above results is:

Corollary. As special cases: the **KBN**, **Ca**, **CMW** and **Bl** setups are all equivalent (when one works over the ground ring $\mathbb{Z}[i]$). □

Our results are even stronger since everything is *explicit.* As an application of our explicit isomorphisms/equivalences we discuss how one can obtain a "singular TQFT model" for the graded BGG parabolic category $\mathcal{O}$ for a certain two-block parabolic in type **A** (this is the category used in the Lie theoretical construction of Khovanov homology, see e.g. [42] or [44]). Another application is that the higher tangle invariants constructed from the various 2-categories are the same (they get identified by the above equivalence) and *not just* the associated link homologies. Moreover, the $\mathfrak{gl}_2$ specializations of these tend to be functorial with respect to link cobordisms, as e.g. the **Ca**, **CMW** and **Bl** specializations (see [11, Theorem 3.5], [15, Theorem 1.1] and [4, Theorem 5.1]), while the $\mathfrak{sl}_2$ versions are usually not, as e.g. the **KBN** specialization (see e.g. [20]). Using our explicit translation between these, we give a way to make the Khovanov complex associated to links (via the famous cube construction) functorial *without changing* its simple framework (keeping the linear structure fixed, but changing the bimodule structure instead).

Let us describe the content of the paper in more detail.

Historical background. Around 1999 Khovanov introduced in [23] his groundbreaking *categorification of the Jones polynomial*, i.e. a certain chain complex (attached to each link) whose homotopy class is an invariant of the link and whose graded Euler characteristic gives back the Jones polynomial. His original construction, nowadays known as *Khovanov homology*, used a certain graded TQFT, i.e. a functor from the category of two dimensional cobordisms into graded vector spaces.

In order to extend his higher invariant to tangles, Khovanov studied in [24] an algebra $H = \bigoplus_{b,t\in\mathbb{Z}_{\geq 0}} H_b^t$ cooked up from his original TQFT. In this setup, the higher invariant associated to a tangle with $2b$ bottom and $2t$ top boundary points is a *chain complex of graded H-bimodules* with $H_b^{\cdot}$ acting non-trivially from the bottom (left) and $H_{\cdot}^t$ acting non-trivially from the top (right). Khovanov showed in [24] that the chain homotopy equivalence class of this complex is an invariant of the tangle, and that, on the level of Grothendieck groups, it descends to the Kauffman bracket of the tangle. A related, but topological construction, is due to Bar-Natan [3] who constructed the higher invariant for tangles directly using a linearized 2-category of cobordisms modulo relations found "in the kernel of Khovanov's original construction". We denote the corresponding 2-category by $\mathfrak{F}_{\mathbb{Z}}[\mathbf{KBN}]$. We will also call H the **KBN** *web algebra* and denote it by $\mathfrak{W}_{\mathbb{Z}}[\mathbf{KBN}]$.

Note that the **KBN** framework has a slight flaw: it is *not functorial* with respect to link cobordisms. That is, their construction can be seen as a functor from the category of tangles to a suitable (2-)category of $\mathfrak{W}_{\mathbb{Z}}[\mathbf{KBN}]$-bimodules, but it does not extend to a 2-functor from the 2-category of tangles. It turns out that the extension to the 2-categorical setup *only works up to a sign*, see e.g. [20].

There are three main attempts to solve this issue: via certain 2-categories of cobordisms with extra data denoted here by $\mathfrak{F}_{\mathbb{Z}[i]}[\mathbf{Ca}]$, $\mathfrak{F}_{\mathbb{Z}[i]}[\mathbf{CMW}]$ and $\mathfrak{F}_{\mathbb{Z}}[\mathbf{Bl}]$ (the former two need a square root of -1). All of these are functorial with respect to link cobordisms, see [11, Theorem 3.5], [15, Theorem 1.1] and [4, Theorem 5.1].

The web algebra $\mathfrak{W}_{\mathbb{Z}}[\mathbf{KBN}]$ is not only of interest because of its connections to low-dimensional topology, it also appears in representation theory, geometry and combinatorics. For example, in a series of papers [6], [7], [8], [9] and [10], Brundan and the second author studied an algebraic version $\mathfrak{A}_{\mathbb{Z}}[\mathbf{KBN}]$ of $\mathfrak{W}_{\mathbb{Z}}[\mathbf{KBN}]$, called *arc algebra*, revealing that $\mathfrak{W}_{\mathbb{Z}}[\mathbf{KBN}]$ has, left aside its knot theoretical origin, interesting representation theoretical, algebraic geometrical and combinatorial properties. Since the arc algebra is algebraic in nature, using the combinatorics of arc diagrams, the algebra $\mathfrak{A}_{\mathbb{Z}}[\mathbf{KBN}]$ controls the topological information coming from Khovanov's original TQFT in an algebraic way and is accessible for explicit calculations. It has also other advantages, i.e. it provides an important link to the alternative versions of Khovanov homology arising from Lie theory, see e.g. [44], symplectic geometry, see e.g. [1] or [41] and geometric representation theory, see e.g. [45].

The introduction of $\mathfrak{A}_{\mathbb{Z}}[\mathbf{KBN}]$ has also influenced successive works. Nowadays there are many *variations and generalizations* of Khovanov's original formulation, e.g. an $\mathfrak{sl}_3$-variation considered in [34] and [46], and an $\mathfrak{sl}_n$-variation studied in [33] and [47], all of them having relations to (cyclotomic) Khovanov-Lauda and Rouquier (KL-R) algebras as defined in [26] or [39], and link homologies in the sense of Khovanov and Rozansky [27]. There is also the $\mathfrak{gl}_{1|1}$-variation developed in [40], which is related to the Alexander polynomial. And there is a type **D**-version introduced and studied in [16] and [17] with connections to the representation theory of Brauer's centralizer algebras and orthosymplectic Lie superalgebras.

$\mathfrak{sl}_2$ *versus* $\mathfrak{gl}_2$. The original cobordism 2-category $\mathfrak{F}_{\mathbb{Z}}[\mathbf{KBN}]$ and the associated web algebra *"categorify" the Temperley-Lieb category*, see [24, Proposition 23]. Similarly, for the arc algebra $\mathfrak{A}_{\mathbb{Z}}[\mathbf{KBN}]$, see [42, Section 6] (combined with [8, Theorem 1.2]). The Temperley-Lieb category gives a presentation of the category of (quantum) $\mathfrak{sl}_2$-intertwiners (as neatly explained, although not originally obtained, in [29]). In contrast, $\mathfrak{F}[\mathtt{P}]$ and its associated web and arc algebra *categorify* $\mathfrak{gl}_2$*-webs* (webs for short) which give a presentation of the category of (quantum) $\mathfrak{gl}_2$-intertwiners (these webs come from a Howe duality of $\mathrm{U}_q(\mathfrak{gl}_M)$ and $\mathrm{U}_q(\mathfrak{gl}_2)$, see [13] or more specifically [48, Remark 1.1]). Here certain "phantom edges" correspond to the determinant representation $\bigwedge_q^2\mathbb{C}_q^2$ of (quantum) $\mathfrak{gl}_2$ which is however trivial as a (quantum) $\mathfrak{sl}_2$-representation. For cases where $\bigwedge_q^2\mathbb{C}_q^2$ is not trivially categorified (e.g. for $\boldsymbol{\varepsilon}=-1$), we can say that such a categorification encodes $\mathfrak{gl}_2$ instead of $\mathfrak{sl}_2$.

The setup in details. The following questions arise:

(QI) Is there a *generic* foam 2-category and its associated web and arc algebra such that the frameworks of **KBN**, **Ca**, **CMW** and **Bl** are *specializations*?

(QII) If so, then how can these be related via the generic construction? What are the similarities and differences of these in the light of the generic setup?

The attempt to answer these questions is precisely the purpose of this paper.

Regarding (QI). To answer (QI) we generalize the construction from [18]. That is, we topologically construct a category of *singular cobordisms* obtained via a gluing procedure of two TQFTs, denoted $\mathfrak{T}_{\mathcal{A}_{\mathrm{o}}}$ and $\mathfrak{T}_{\mathcal{A}_{\mathrm{p}}}$, associated to the two Frobenius algebras $\mathcal{A}_{\mathrm{o}}=P[X]/(X^2-\boldsymbol{\alpha})$ and $\mathcal{A}_{\mathrm{p}}=P$ (here $P=\mathbb{Z}[\boldsymbol{\alpha},\boldsymbol{\tau}^{\pm 1},\boldsymbol{\omega}_+^{\pm 1},\boldsymbol{\omega}_-^{\pm 1}]$, which is also always the ground ring in case of $\mathtt{P}$). The index o stands for *"ordinary"* and p for *"phantom"* (the "non-existing" phantom parts should be thought of encoding the difference between $\mathfrak{gl}_2$ and $\mathfrak{sl}_2$). Here $\boldsymbol{\tau}$ is the parameter twisting the trace of $\mathcal{A}_{\mathrm{p}}$, while $\boldsymbol{\omega}_+,\boldsymbol{\omega}_-$ govern how the two theories are glued together, and we obtain a *singular TQFT* $\mathfrak{T}[\mathtt{P}]$. By linearization and by "analyzing the kernel of $\mathfrak{T}[\mathtt{P}]$", we obtain a 4*-parameter foam* 2*-category* $\mathfrak{F}[\mathtt{P}]$ and its associated *web algebra* $\mathfrak{W}[\mathtt{P}]$.

Regarding (QII). Specializing as above $\mathtt{P}$ further to $\mathtt{Q}$, we obtain $\mathfrak{F}[\mathtt{Q}]$ and $\mathfrak{W}[\mathtt{Q}]$, and an algebraic model $\mathfrak{A}[\mathtt{Q}]$ as well (in case of $\mathtt{Q}$ we work over $Q=\mathbb{Z}[\boldsymbol{\alpha},\boldsymbol{\omega}^{\pm 1}]$). Indeed, the further specialization is mild and we still get our four main examples as *special cases* of $\mathfrak{F}[\mathtt{Q}]$, $\mathfrak{W}[\mathtt{Q}]$ and $\mathfrak{A}[\mathtt{Q}]$.

In fact, the algebraic model $\mathfrak{A}[\mathtt{Q}]$ of the topological setup eases to work with the associated (more flexible, yet harder to control) topological 2-categories: using the algebraic description, we obtain as the answer to the first question of (QII) the surprising result that all of them are *isomorphic*, regardless of the specialization of the parameters $\boldsymbol{\varepsilon}$ (with $\boldsymbol{\varepsilon}=\pm 1$) and $\boldsymbol{\omega}$. We are even able to match the corresponding bimodules explicitly using isomorphisms which count certain weighted numbers of cups, caps and shifts. In particular, this shows that all of our four main examples are actually *isomorphic/equivalent*.

This gives us an answer to part of the second question of (QII). Namely, we can construct higher link and tangle invariants from $\mathfrak{F}[\mathtt{P}]$ and $\mathfrak{W}[\mathtt{P}]$. Using our explicit isomorphisms, we can show that these are all the same in case of $\mathtt{Q}$, *even for tangles* and not just for links. In case of links, the algebras acting are trivial and a weaker result is sufficient to show that the homologies agree (see e.g. [15, Theorem 4.1]). Moreover, using our explicit isomorphisms, one can *redefine* the original **KBN** complex, without changing its simple framework, to make it functorial. Another approach to functoriality is given by Vogel [49] (unfortunately we do not know how his results precisely relate to ours). It also follows that all of these describe the *same Lie theoretical instances*, i.e. they give a "singular TQFT model" of a certain 2-block parabolic version of category $\mathcal{O}$, and they all are 2-representations of the categorified quantum group in the sense of KL-R.

An open issue: more general singular TQFTs. Foam 2-categories are also known in higher ranks. For example, originated in the study of $\mathfrak{sl}_N$-versions of Khovanov homology, foam 2-categories coming from singular TQFTs were studied in the $\mathfrak{sl}_3$-case by Khovanov in [25] and in the $\mathfrak{sl}_N$-case by Mackaay, Stošić and Vaz in [35]. These topological 2-categories are algebraically and combinatorially very complicated for $N > 3$ and the authors of [35] needed the so-called Kapustin-Li formula from [21] to have some control. In contrast, Queffelec and Rose introduced in [36] (based on joint work of Queffelec and Rose with Lauda [30]) non-topological $\mathfrak{gl}_N$-foam 2-categories, i.e. without having a (singular) TQFT around. These 2-categories do not have a topological description or interpretation yet, but are built such that they fit into the KL-R 2-category picture (and are sufficient to study the associated link homologies, see e.g. [36, Section 4] or [38], but functoriality of these is not clear). Furthermore, for the "symmetric story" (that is, categorifying tensor products of symmetric powers of the vector representation instead of, as for all other cases above, exterior powers) or in other types foams are yet to be defined. For example, such a topological description is still missing for for symmetric type **A**-webs in the sense of [37] or [48], as well as the type **D**-version of the arc algebra studied in [17].

Our approach to define foams is topological in nature (i.e. via singular TQFTs). Indeed, we follow Khovanov's original construction of $\mathfrak{sl}_3$-foams [25] as well as the construction from [18], but generalize both. Our setup seems to be well adapted to generalization towards higher ranks, other types etc., and might lead to a "singular TQFT model" of parabolic category $\mathcal{O}$ attached to N-block parabolics.

Abstract reasons for the existence of our main isomorphisms. The idea that our theorems from above should be true grew out of the following.

In [34, Proposition 5.18] it was shown that an $\mathfrak{sl}_3$-analog of $\mathfrak{W}_{\mathbb{C}}[\mathbf{KBN}]$ is Morita equivalent to a certain KL-R algebra of level 3 (using $\mathbb{C}$ as a ground field). The two main ingredients in this proof were a categorification of an instance of q-Howe duality, see [34, Subsection 5.3] (showing that the $\mathfrak{sl}_3$-analog categorifies a certain highest weight module of the "Howe dual" quantum group), as well as Rouquier's universality theorem [39, Proposition 5.6 and Corollary 5.7] ("such categorifications are unique"). Now, Brundan and the second author showed in [8, Theorem 9.2] that another instance of q-Howe duality can be categorified using $\mathfrak{A}_{\mathbb{C}}[\mathbf{KBN}]$. Moreover, one can deduce from [30, Propositions 3.5 and 3.3] (in the light of Proposition 2.43) the same for $\mathfrak{W}_{\mathbb{Z}[i]}[\mathbf{CMW}]$ and $\mathfrak{W}_{\mathbb{Z}}[\mathbf{Bl}]$. Since $\mathfrak{A}_{\mathbb{C}}[\mathbf{KBN}]$ is constructed from $\mathfrak{W}_{\mathbb{C}}[\mathbf{KBN}]$, uniqueness of categorification in type **A** should yield our theorems.

We should stress here however that there is still work to do for this abstract approach since it is only known for $\mathfrak{A}_{\mathbb{C}}[\mathbf{KBN}]$ that it categorifies the corresponding highest weight module of the "Howe dual" quantum group (one needs to check the same for the **Ca**, **CMW**, **Bl** setups or the signed 2-parameter version as well).

Alternatively, our theorems should also follow from categorification results connected to highest weight categories. That is, in [8, Theorem 8.5] it was also shown that $\mathfrak{A}_{\mathbb{C}}[\mathbf{KBN}]$ appears as an endomorphism ring in a certain tensor product categorification. The results from [30] mentioned above could also be interpreted in this sense. Again, "uniqueness of such categorifications", see [32, Theorem A], should yield our theorems. As before, the statements are only known for $\mathfrak{A}_{\mathbb{C}}[\mathbf{KBN}]$ and one would still need to prove the same for the various parameter versions.

In contrast to these abstract reasons, our work is *completely explicit.* This has many advantages. For example one can not deduce just from the abstract existence of such isomorphisms any of our applications with respect to higher link and tangle invariants in Subsection 5.2 since such isomorphisms could be "uncontrollable" (e.g. on the bimodules used to define these invariants).

Conventions used throughout.

Convention 1.1. By a *ring* R we always understand a commutative, unital ring without zero divisors. By an *algebra* we always mean an R-algebra A (over which ring will be clear from the context). We do not assume that such A's are (locally) unital, associative or free over R and it will be a non-trivial fact that all A's which we consider are actually (locally) unital, associative and free (to be precise, they are direct sums of unital, associative, free algebras of finite rank). Given two algebras A and B, then an A-B-*bimodule* is a free R-module M with a left action of A and a right action of B in the usual compatible sense. If $\mathrm{A} = \mathrm{B}$, then we also write A-*bimodule* for short. We denote the category of A-bimodules which are free over R and of finite rank by A-**biMod**. Moreover, we call an A-B-bimodule M *biprojective*, if it is projective as a left A-module as well as a right B-module. ▲

Convention 1.2. Throughout the paper: *graded* should be read as $\mathbb{Z}$-*graded*. By a *graded algebra* we mean an algebra A which decomposes into graded pieces $\mathrm{A} = \bigoplus_{i\in\mathbb{Z}} \mathrm{A}_i$ such that $\mathrm{A}_i\mathrm{A}_j \subset \mathrm{A}_{i+j}$ for all $i, j \in \mathbb{Z}$. Given two graded algebras A and B, we study (and only consider) *graded* A-B-bimodules, i.e. A-B-bimodules $M = \bigoplus_{i\in\mathbb{Z}} M_i$ such that $\mathrm{A}_i M_j \mathrm{B}_k \subset M_{i+j+k}$ for all $i, j, k \in \mathbb{Z}$. We also set $M\{s\}_i = M_{i-s}$ for $s \in \mathbb{Z}$ (thus, positive integers shift up).

If A is a graded algebra and M is a graded A-bimodule, then $\overline{M}$ obtained from M by forgetting the grading is in A-**biMod**. Given such A-bimodules $\overline{M}, \overline{N}$, then

$$\mathrm{Hom}_{\text{A-}\mathbf{biMod}}(\overline{M}, \overline{N}) = \bigoplus_{s\in\mathbb{Z}} \mathrm{Hom}_0(M, N\{s\}). \tag{1}$$

Here Hom_0 means all *degree-preserving* A-homomorphisms, i.e. $\phi(M_i) \subset N_i$. ▲

Convention 1.3. We consider three diagrammatic calculi in this paper: webs, foams and arc diagrams. Our reading convention for all of these is from bottom to top and left to right. Furthermore, diagrammatic left respectively right actions will be given by acting on the bottom respectively on the top. Moreover, we often only illustrate local pieces. The corresponding diagram is meant to be the identity or arbitrary outside of the displayed part (which one will be clear from the context). ▲

Remark 1.4. We use colors in this paper. It is only necessary to distinguish colors for webs and foams. For the readers with a black-and-white version: we illustrate colored web edges using dashed lines, while colored foam facets appear shaded. ▲

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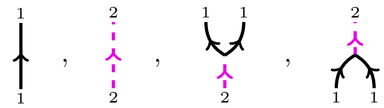
2. A family of singular TQFTs, foams and web algebras

In this section we introduce a 4-parameter version of singular TQFTs. We use these to define the 4-parameter foam 2-category $\mathfrak{F}[\mathtt{P}]$ and its web algebra $\mathfrak{W}[\mathtt{P}]$.

2.1. **Webs and pre-foams.** We start by recalling the definition of webs and of pre-foams (where we closely follow [18, Section 2]). For this we denote by $\mathbb{bl}$ the set of all vectors $\vec{k} = (k_i)_{i\in\mathbb{Z}} \in \{0, 1, 2\}^{\mathbb{Z}}$ with $k_i = 0$ for $|i| \gg 0$. Abusing notation, we also sometimes write $\vec{k} = (k_a, \ldots, k_b)$ for some fixed part of $\vec{k}$ (with $a < b \in \mathbb{Z}$) where it is to be understood that all non-displayed entries are zero. By convention, the empty vector $\emptyset \in \mathbb{bl}$ is the unique vector containing only zeros. We consider $\vec{k} \in \mathbb{bl}$ as a set of discrete labeled points in $\mathbb{R} \times \{\pm 1\}$ (or in $\mathbb{R} \times \{0\}$) by putting the symbols k_i at position $(i, \pm 1)$ (or $(i, 0)$). If not stated otherwise, then the first non-zero entry of such $\vec{k}$'s is assumed to be k_i for $i = 0$.

Definition 2.1. A *web* is a labeled, oriented, trivalent graph which can be obtained by gluing (whenever this makes sense and the labels fit) or juxtaposition of finitely many of the following pieces (we do not allow downwards pointing edges):



We assume that webs (the *empty web* $\emptyset$ is also a web) are embedded in $\mathbb{R} \times [-1, 1]$ such that each edge starts and ends in a trivalent vertex or at the boundary of the strip at the points $(i, \pm 1)$. We assume that the points at $(i, \pm 1)$ are labeled 0, 1 or 2. In particular, these webs have distinguished bottom $\vec{k}$ and top $\vec{l}$ boundary which we will throughout denote from left to right by $\vec{k} = (k_a, \ldots, k_b)$ and $\vec{l} = (l_{a'}, \ldots, l_{b'})$ where k_i is the label at $(i, -1)$ and l_i is the label at $(i, 1)$. Edges come in two different types, namely as *ordinary edges* which are only allowed to touch boundary points labeled 1, and *phantom edges* which are only allowed to touch those labeled 2. We draw phantom edges dashed (and colored); one should think of them as "non-existing". If we talk for instance about "*circles in webs*", we will always just ignore all phantom edges. ▲

By a *surface* we mean a marked, orientable, compact surface with possibly finitely many boundary components and with finitely many connected components. Additionally, by a *trivalent surface* we understand the same as in [25, Subsection 3.1], i.e. certain embedded, marked, singular cobordisms whose boundaries are webs.

Precisely, fix the following data denoted by $\boldsymbol{S}$:

(I) A surface S with connected components divided into two sets $S^{\mathrm{o}}_1, \ldots, S^{\mathrm{o}}_r$ and $S^{\mathrm{p}}_1, \ldots, S^{\mathrm{p}}_{r'}$. The former are called *ordinary surfaces* and the latter are called *phantom surfaces*.

(II) The boundary components of S are partitioned into triples $(C^{\mathrm{o}}_i, C^{\mathrm{o}}_j, C^{\mathrm{p}}_k)$ such that each triple contains precisely one boundary component C^{p}_k of a phantom surface.

(III) The three circles $C^{\mathrm{o}}_i, C^{\mathrm{o}}_j$ and C^{p}_k in each triple are identified via diffeomorphisms $\varphi_{ij} \colon C^{\mathrm{o}}_i \to C^{\mathrm{o}}_j$ and $\varphi_{jk} \colon C^{\mathrm{o}}_j \to C^{\mathrm{p}}_k$.

(IV) A finite (possible empty) set of markers per connected components $S^{\mathrm{o}}_1, \ldots, S^{\mathrm{o}}_r$ and $S^{\mathrm{p}}_1, \ldots, S^{\mathrm{p}}_{r'}$ that move freely around its connected component.

Definition 2.2. Let $\boldsymbol{S}$ be as above. The *closed, singular trivalent surface* $f_c = f_c^{\boldsymbol{S}}$ attached to $\boldsymbol{S}$ is the CW-complex obtained as the quotient of S by the identifications φ_{ij} and φ_{jk}. We call all such f_c's *closed pre-foams* (following [25]) and their markers *dots*. A triple $(C^{\mathrm{o}}_i, C^{\mathrm{o}}_j, C^{\mathrm{p}}_k)$ becomes one circle in f_c which we call a *singular seam*, while the interior of the connected components $S^{\mathrm{o}}_1, \ldots, S^{\mathrm{o}}_r$ and $S^{\mathrm{p}}_1, \ldots, S^{\mathrm{p}}_{r'}$ are facets of f_c, called *ordinary facets* and *phantom facets*. We only consider pre-foams which can be embedded into $\mathbb{R}^2 \times [-1, 1]$ such that the three annuli glued to a singular seam are consistently oriented (which induces an orientation on the singular seam, compare to (2)). We consider closed pre-foams modulo isotopies in $\mathbb{R}^2 \times [-1, 1]$. ▲

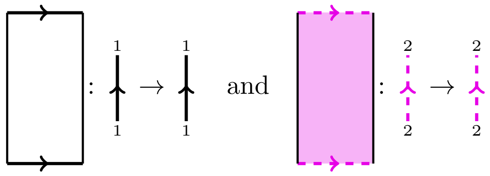
We color phantom facets in what follows. An example of our construction from Definition 2.2 is illustrated in [18, Example 2.3].

We need *not necessarily closed* pre-foams as well. Following [25, Subsection 3.3], we consider the xy-plane $\mathbb{R}^2 \subset \mathbb{R}^3$ and say that $\mathbb{R}^2$ *intersects* a closed pre-foam f_c *generically*, if $\mathbb{R}^2 \cap f_c$ is a web (forgetting the orientations).

Definition 2.3. A *(not necessarily closed) pre-foam* f is defined as the intersection of $\mathbb{R}^2 \times [-1,1]$ with some closed foam f_c such that $\mathbb{R}^2 \times \{\pm 1\}$ intersects f_c generically. We consider such pre-foams modulo isotopies in $\mathbb{R}^2 \times [-1,1]$ which fix the horizontal boundary. We see such a pre-foam f as a singular cobordism between $(\mathbb{R}^2 \times \{-1\}) \cap f_c$ (bottom, source) and $(\mathbb{R}^2 \times \{+1\}) \cap f_c$ (top, target) embedded in $\mathbb{R}^2 \times [-1,1]$. Moreover, there is an evident composition $g \circ f$ via gluing and rescaling (if the top boundary of f and the bottom boundary of g coincide). Similarly, we construct pre-foams embedded in $\mathbb{R} \times [-1,1] \times [-1,1]$ with vertical boundary components. These vertical boundary components should be the boundary of the webs at the bottom/top times $[-1,1]$. We finally take everything modulo isotopies that preserve the vertical boundary as well as the horizontal boundary. ▲

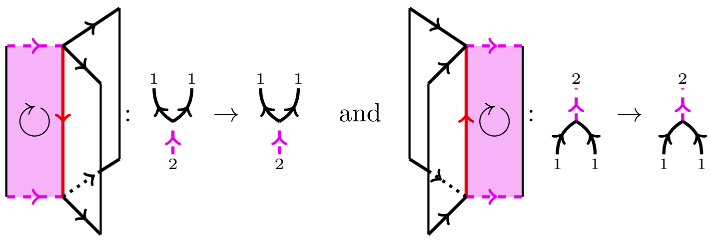
We call pre-foam parts *ordinary*, if they do not contain singular seams or phantom facets, and we call pre-foam parts *ghostly*, if they contain phantom facets only.

Example 2.4. Given the identity web on the object 1 or 2 as below, we have the following identity pre-foams:

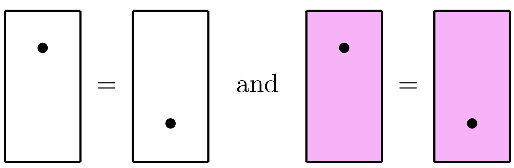


The facet on the left is an ordinary facet. Whereas the facet on the right is a phantom facet and the reader might think of it as "non-existing" (similar to a phantom edge). In general, pre-foams can be seen as singular surfaces (with oriented, singular seams) in $\mathbb{R} \times [-1,1] \times [-1,1]$ such that the bottom boundary and the top boundary are webs. Moreover, the only vertical boundary components of pre-foams f come from the boundary points of webs times $[-1,1]$. By definition, all generic slices of pre-foams are webs, and the singularities of f are all locally of the following form (where the other orientations of the facets/seams are also allowed):

(2)



Here we have only indicated the orientation of the phantom facet, since the other two orientations are determined by this choice. Note that it even suffices to indicate the orientation of the singular seams in what follows. Such pre-foams can carry dots that freely move around its facets:



▲

Remark 2.5. Pre-foams are considered modulo boundary preserving isotopies that do preserve the condition that each generic slice is a web. These isotopies form a finite list: isotopies coming from the two cobordism theories associated to the two different types of facets as explained below (see for example [28, Section 1.4]) and isotopies coming from isotopies of the singular seams seen as tangles in $\mathbb{R}^2 \times [-1,1]$. ▲

2.2. **Generic singular TQFTs.** We now define $\mathtt{P}$, $\mathtt{Q}$ and specializations.

Definition 2.6. Let $\mathtt{P} = \{\boldsymbol{\alpha}, \boldsymbol{\tau}^{\pm 1}, \boldsymbol{\omega}_+^{\pm 1}, \boldsymbol{\omega}_-^{\pm 1}\}$ (we always view $\mathtt{P}$ as being ordered in this way) a set of generic parameters. The associated ring which we consider is $P = \mathbb{Z}[\boldsymbol{\alpha}, \boldsymbol{\tau}^{\pm 1}, \boldsymbol{\omega}_+^{\pm 1}, \boldsymbol{\omega}_-^{\pm 1}]$. The second set of parameters we need is $\mathtt{Q} = \{\boldsymbol{\alpha}, \boldsymbol{\varepsilon}, \boldsymbol{\omega}^{\pm 1}\}$ with $\boldsymbol{\varepsilon} = \pm 1$. We will use this from Section 3 onwards. The corresponding ring which we consider is the ring $Q = \mathbb{Z}[\boldsymbol{\alpha}, \boldsymbol{\omega}^{\pm 1}]$. We denote any mathematical object X with underlying ring P respectively Q by $X[\mathtt{P}]$ respectively by $X[\mathtt{Q}]$. We consider P and Q to be graded with $\deg_P(\boldsymbol{\alpha}) = 4$ (and everything else of degree 0). ▲

Definition 2.7. Let $\mathtt{p}\colon P \to R$ be a ring homomorphism to some ring R. We denote by $X_R[\mathtt{p}(\boldsymbol{\alpha}), \mathtt{p}(\boldsymbol{\tau}), \mathtt{p}(\boldsymbol{\omega}_+), \mathtt{p}(\boldsymbol{\omega}_-)]$ the corresponding mathematical object obtained from $X[\mathtt{P}]$ by *specialization via* $\mathtt{p}$. Similarly, given a ring homomorphism $\mathtt{q}\colon Q \to R$, we denote by $X_R[\mathtt{q}(\boldsymbol{\alpha}), \mathtt{q}(\boldsymbol{\varepsilon}), \mathtt{q}(\boldsymbol{\omega})]$ the corresponding *specializing via* $\mathtt{q}$. Abusing notation, we will always use $\mathtt{p}$ respectively $\mathtt{q}$ as a symbol for any specialization of P respectively Q and sometimes even omit to write $\mathtt{p}(\cdot)$ respectively $\mathtt{q}(\cdot)$. For example, $X_{\mathbb{Z}}[0,1,1,1]$ will denote the specialization of $X[\mathtt{P}]$ via $\mathtt{p}(\boldsymbol{\alpha}) = 0 \in \mathbb{Z}$ and $\mathtt{p}(\boldsymbol{\tau}) = \mathtt{p}(\boldsymbol{\omega}_+) = \mathtt{p}(\boldsymbol{\omega}_-) = 1 \in \mathbb{Z}$, and $X_{\mathbb{Z}}[0,1,1]$ will denote the specialization of $X[\mathtt{Q}]$ via $\mathtt{q}(\boldsymbol{\alpha}) = 0 \in \mathbb{Z}$ and $\mathtt{q}(\boldsymbol{\varepsilon}) = \mathtt{q}(\boldsymbol{\omega}) = 1 \in \mathbb{Z}$.

The following specialization is very important for us:

$$\mathtt{p}\colon P \to Q, \quad \mathtt{p}(\boldsymbol{\alpha}) = \boldsymbol{\alpha}, \quad \mathtt{p}(\boldsymbol{\tau}) = \boldsymbol{\varepsilon}\boldsymbol{\omega}^2, \quad \mathtt{p}(\boldsymbol{\omega}_+) = \boldsymbol{\omega}, \quad \mathtt{p}(\boldsymbol{\omega}_-) = \boldsymbol{\varepsilon}\boldsymbol{\omega}. \tag{3}$$

Note that $\mathtt{p}$ from (3) is degree preserving. ▲

Convention 2.8. When we write formulations as "similarly for any specialization of $\mathtt{P}$" after some statement, then this is to be understood that the statement holds up to the grading part because some specializations do not preserve the grading (e.g. some will only preserve the filtration obtained from the grading as the "Lee specializations $\boldsymbol{\alpha} = 1$" - we will not elaborate on these filtered versions in this paper, but everything works analogously). ▲

To work with the 4-parameter foam 2-category it will be enough (for our purposes) to consider its image under a certain monoidal functors from the category of pre-foams to the category of free P-modules called *singular TQFTs*. To understand our construction, recall that equivalence classes of TQFTs for surfaces are in $1:1$ correspondence with isomorphism classes of associative, commutative Frobenius algebras (which are free P-modules of finite rank). The reader unfamiliar with this might consult Kock's book [28], which is our main source for these kind of TQFTs (in fact, Kock works over an arbitrary field, but his arguments work in our setup over P as well). Given such a Frobenius algebra F corresponding to a TQFT $\mathfrak{T}_{\mathrm{F}}$, then the association is as follows. To a disjoint union of m circles one associates the m-fold tensor product $\mathrm{F}^{\otimes m}$ (if not mentioned otherwise, $\otimes = \otimes_P$). To a cobordism Σ with distinguished incoming and outgoing boundary components consisting of m and m' circles, one assigns a P-linear map from $\mathrm{F}^{\otimes m}$ to $\mathrm{F}^{\otimes m'}$. Hereby the usual cup, cap and pants cobordisms correspond to the unit, counit, multiplication and comultiplication maps (given by the Frobenius structure). Then the TQFT assigns to a surface Σ a P-linear map $\mathfrak{T}_{\mathrm{F}}(\Sigma)\colon \mathrm{F}^{\otimes m} \to \mathrm{F}^{\otimes m'}$, which is obtained by decomposing Σ into basic pieces (i.e. cup, cap and pants cobordisms).

To get a singular TQFT we glue two such Frobenius algebras. The Frobenius algebras we use are (with the evident units and multiplications)

$$\mathrm{F}_{\mathrm{o}} = P[X]/(X^2 - \boldsymbol{\alpha}) \quad \text{and} \quad \mathrm{F}_{\mathrm{p}} = P. \tag{4}$$

Their counits $\varepsilon_i(\cdot)$ and comultiplications $\Delta_i(\cdot)$ (with $i = \mathrm{o}, \mathrm{p}$) are given as

$$\begin{aligned}\varepsilon_{\mathrm{o}}(1) &= 0, & \Delta_{\mathrm{o}}(1) &= 1\otimes X + X\otimes 1,\\ \varepsilon_{\mathrm{o}}(X) &= 1, & \Delta_{\mathrm{o}}(X) &= X\otimes X + \boldsymbol{\alpha}\cdot 1\otimes 1,\\ \varepsilon_{\mathrm{p}}(1) &= \boldsymbol{\tau}\cdot 1, & \Delta_{\mathrm{p}}(1) &= \boldsymbol{\tau}^{-1}\cdot 1\otimes 1.\end{aligned}$$

Thus, we have the traces

$$\mathrm{tr}_{\mathrm{o}}(1) = 0, \quad \mathrm{tr}_{\mathrm{o}}(X) = 1, \quad \mathrm{tr}_{\mathrm{p}}(1) = \boldsymbol{\tau}\cdot 1. \tag{5}$$

The following construction is inspired from [4] and [18], but generalizes both. That is, we want to construct a 4-parameter, singular TQFT $\mathfrak{T}[\mathtt{P}]$ on the category whose objects are webs as in Definition 2.1 above. To this end, let us denote by $p\mathcal{F}$ the monoidal category whose objects are webs and whose morphisms are pre-foams (composition is gluing of pre-foams and the monoidal product is given by placing pre-foams next to each other). We define for $a, b, c, d \in P$ gluing maps

$$\begin{gathered}\mathrm{gl}_{\mathrm{F_o}}\colon \mathrm{F_o}\otimes\mathrm{F_o}\to\mathrm{F_o},\ (a{+}bX)\otimes(c+dX)\mapsto(a+\boldsymbol{\omega}_{+}bX)(c+\boldsymbol{\omega}_{-}dX),\\ \mathrm{gl}_{\mathrm{F_p}}\colon \mathrm{F_p}\to\mathrm{F_o},\ 1\mapsto 1.\end{gathered} \tag{6}$$

Definition 2.9. Let $\mathfrak{T}_{\mathrm{F_o}}$ and $\mathfrak{T}_{\mathrm{F_p}}$ denote the TQFTs associated to $\mathrm{F_o}$ and $\mathrm{F_p}$ from (4). Given a closed pre-foam f_c, let $\dot{f}_c = f_{\mathrm{o}}\dot{\cup} f_{\mathrm{p}}$ be the pre-foam obtained by cutting f_c along the singular seams (of which we assume to have m in total). Here f_{o} is the surface which in f_c is attached to the ordinary parts and f_{p} is the surface which in f_c is attached to phantom parts. Note that the boundary of f_{o} splits into σ_i^+ and σ_i^- for each $i \in \{1,\dots,m\}$. Which one is which depends on the orientation of the singular seam: use the right-hand rule with the index finger pointing in the direction of the singular seam and the middle finger pointing in direction of the attached phantom facet, then the thumb points in direction of σ_i^+. In contrast, f_{p} has only boundary components σ_i for each $i \in \{1,\dots,m\}$. Now

$$\begin{aligned}\mathfrak{T}_{\mathrm{F_o}}(f_{\mathrm{o}}) &\in \bigotimes_{i=1}^{m}(\mathfrak{T}_{\mathrm{F_o}}(\sigma_i^+)\otimes\mathfrak{T}_{\mathrm{F_o}}(\sigma_i^-)) \cong (\mathrm{F_o}\otimes\mathrm{F_o})^{\otimes m},\\ \mathfrak{T}_{\mathrm{F_p}}(f_{\mathrm{p}}) &\in \bigotimes_{i=1}^{m}\mathfrak{T}_{\mathrm{F_p}}(\sigma_i) \cong \mathrm{F_p}^{\otimes m}.\end{aligned} \tag{7}$$

Let $\mathrm{tr_o}\colon \mathrm{F_o}\to P$ be as in (5), and let $\mathrm{gl}_{\mathrm{F_o}}, \mathrm{gl}_{\mathrm{F_p}}$ be as in (6).

Then we set $\mathfrak{T}[\mathtt{P}](f_c) = \mathrm{tr}_{\mathrm{o}}^{\otimes m}(m_{\mathrm{F_o}}(\mathrm{gl}_{\mathrm{F_o}}^{\otimes m}(\mathfrak{T}_{\mathrm{F_o}}(f_{\mathrm{o}}))\otimes\mathrm{gl}_{\mathrm{F_p}}^{\otimes m}(\mathfrak{T}_{\mathrm{F_p}}(f_{\mathrm{p}})))) \in P^{\otimes m}\cong P$. This procedure assigns to any pre-foam f_c a value $\mathfrak{T}[\mathtt{P}](f_c)\in P$. ▲

For an example we refer to the proof of Lemma 2.16 below.

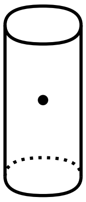
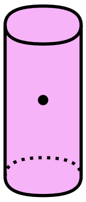
Theorem 2.10. The construction from Definition 2.9 with (7) can be extended to a monoidal functor $\mathfrak{T}[\mathtt{P}]\colon p\mathcal{F}\to P\text{-}\mathbf{Mod}_{\mathrm{free}}$. ■

Proof. This follows by the universal construction from [5]. □

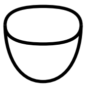
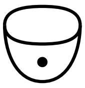
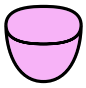
We call $\mathfrak{T}[\mathtt{P}]$ the 4-*parameter, singular TQFT*. Similarly, we call all such monoidal functors *singular TQFTs* (e.g. for any specialization of $\mathtt{P}$).

Note that $p\mathcal{F}$ has two important subcategories, i.e. those pre-foams with only ordinary parts and those with only phantom parts. We associate the Frobenius algebra $\mathrm{F_o}$ to the ordinary parts and the Frobenius algebra $\mathrm{F_p}$ to the phantom parts of a pre-foam f in the sense that $\mathfrak{T}_{\mathrm{F_o}}$ can be seen as a monoidal functor on the subcategory with only ordinary parts and $\mathfrak{T}_{\mathrm{F_p}}$ as a monoidal functor on the subcategory with only phantom parts (both coming from $\mathfrak{T}[\mathtt{P}]$ via restriction).

Example 2.11. In our context, dots correspond to multiplication by X or $\boldsymbol{\tau}^{-1}$:

 $\overset{\mathfrak{T}_{\mathrm{F_o}}}{\longmapsto} \cdot X \colon \mathrm{F_o} \to \mathrm{F_o}$,  $\overset{\mathfrak{T}_{\mathrm{F_p}}}{\longmapsto} \cdot\boldsymbol{\tau}^{-1} \colon \mathrm{F_p} \to \mathrm{F_p}$.

Moreover, if we view a P-linear map $\phi\colon P \to \mathrm{F}_i^{\otimes m}$ as $\phi(1) \in \mathrm{F}_i^{\otimes m}$, then

(8)  $\overset{\mathfrak{T}_{\mathrm{F_o}}}{\longmapsto} 1 \in \mathrm{F_o}$,  $\overset{\mathfrak{T}_{\mathrm{F_o}}}{\longmapsto} X \in \mathrm{F_o}$,  $\overset{\mathfrak{T}_{\mathrm{F_p}}}{\longmapsto} 1 \in \mathrm{F_p}$.

These are ι_{o}, $(\cdot X) \circ \iota_{\mathrm{o}}$ and ι_{p} as maps. The ι's are called *units*. The *counits* ε_i are obtained by flipping the pictures (and scaling by $\boldsymbol{\tau}$ in the phantom case). ▲

Note that the values of non-closed pre-foams can be determined by closing them in all possible ways using (8) and its dual.

Specializations 2.12. Using the specializations $\mathtt{p}\colon P \to \mathbb{Z}$ given by $\mathtt{p}(\boldsymbol{\alpha}) = 0$ respectively $\mathtt{p}(\boldsymbol{\alpha}) = 1$ (all other parameters are send to 1), we obtain

$$\mathrm{F} \cong \mathbb{Z}[X]/(X^2) \quad \text{respectively} \quad \mathrm{F}_{\mathbf{Lee}} \cong \mathbb{Z}[X]/(X^2 - 1),$$

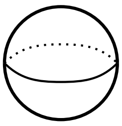
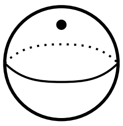
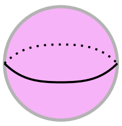
with the latter studied by Lee in her deformation of Khovanov's complex, see [31]. (Similarly for the TQFTs associated to F and $\mathrm{F}_{\mathbf{Lee}}$.)

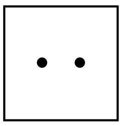
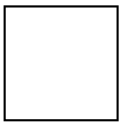
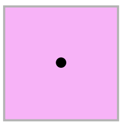
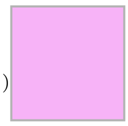
Moreover, specializing via $\mathtt{p}(\boldsymbol{\alpha}) = 0$, $\mathtt{p}(\boldsymbol{\tau}) = -1$, $\mathtt{p}(\boldsymbol{\omega}_+) = 1$ and $\mathtt{p}(\boldsymbol{\omega}_-) = -1$ we obtain the singular TQFT studied in [4] as well as in [18, Subsection 2.2]. ▲

Here and throughout, we say for short that a relation $a = b$ (where a, b are formal P-linear combinations of pre-foams) *lies in the kernel of a (singular) TQFT* $\mathfrak{T}$, if $\mathfrak{T}(a) = \mathfrak{T}(b)$ as P-linear maps.

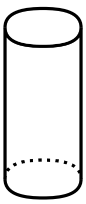
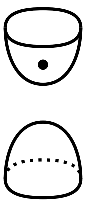
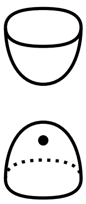
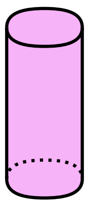
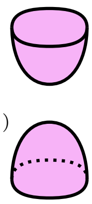
Remark 2.13. Later we are going to mostly use the specialization of $\mathtt{P}$ to $\mathtt{Q}$ from (3). For convenience, we also indicate in small print, with brackets and in gray the values of the relations in the kernel of (singular) TQFTs under the specialization to $\mathtt{Q}$. ▲

Lemma 2.14. The *ordinary and ghostly sphere relations* and the *dot removing relations* as displayed here

(9)  $= 0$,  $= 1$,  $= \underset{(\varepsilon\boldsymbol{\omega}^2)}{\boldsymbol{\tau}}$,

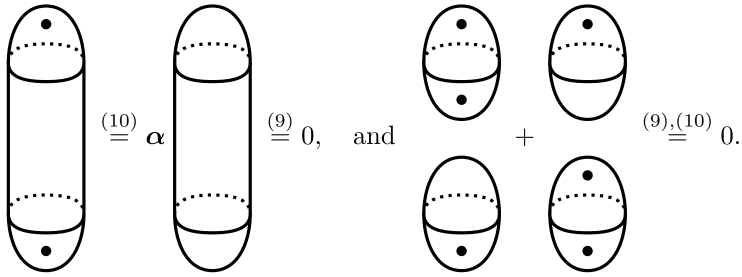
(10)  $= \boldsymbol{\alpha}$  and  $= \underset{(\varepsilon\boldsymbol{\omega}^{-2})}{\boldsymbol{\tau}^{-1}}$ 

as well as the *ordinary and ghostly neck cutting relations*

(11)  $=$  $+$  and  $= \underset{(\varepsilon\boldsymbol{\omega}^{-2})}{\boldsymbol{\tau}^{-1}}$ 

are in the kernel of $\mathfrak{T}_{\mathrm{F_o}}$ (ordinary) respectively of $\mathfrak{T}_{\mathrm{F_p}}$ (ghostly). ■

Proof. A direct computation. For example, the traces from (5) immediately give the sphere relations from (9). The remaining local relations can be shown by closing the local pictures in all possible ways, e.g. (we have indicated one possible closure)



□

The neck cutting relations (11) give a topological interpretation of dots as a shorthand notation for handles, see also [3, (4)].

By construction, the relations from (9), (10) and (11) are in the kernel of $\mathfrak{T}[\mathtt{P}]$. The following lemmas give some additional relations in its kernel.

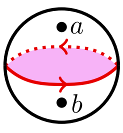
Lemma 2.15. Let $\tilde{f}$ be the pre-foam obtained from a pre-foam f by reversing the orientation of a singular seam. Then

$$f - \tilde{f}_{\boldsymbol{\omega}_+ \rightleftarrows \boldsymbol{\omega}_-} = 0 \quad (f - \tilde{f}_{1 \rightleftarrows \boldsymbol{\varepsilon}} = 0)$$

(changing the coefficients of $\tilde{f}$ by swapping $\boldsymbol{\omega}_+$ and $\boldsymbol{\omega}_-$) is in the kernel of $\mathfrak{T}[\mathtt{P}]$. ■

Proof. This follows because switching the orientation of a singular seam swaps the attached parts of σ_i^+ and σ_i^-. This changes $\boldsymbol{\omega}_+$ to $\boldsymbol{\omega}_-$ and vice versa because this swaps the two copies of $\mathrm{F_o}$ in the source of $\mathrm{gl}_{\mathrm{F_o}}$ from (6) (we note that the part $b = d = 0$ is killed by applying the trace ε_{o} in the formula for $\mathfrak{T}[\mathtt{P}](f_c)$). □

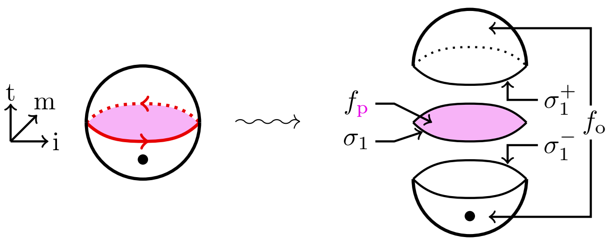
Lemma 2.16. The *sphere relations*, i.e.

(12) 

$$= \begin{cases} \underset{(\omega)}{\boldsymbol{\omega}_+}, & \text{if } a = 1, b = 0, \\ \underset{(\varepsilon\omega)}{\boldsymbol{\omega}_-}, & \text{if } a = 0, b = 1, \\ 0, & \text{otherwise,} \end{cases}$$

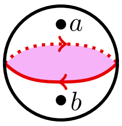
(with $a, b \in \mathbb{Z}_{\geq 0}$ dots), are in the kernel of $\mathfrak{T}[\mathtt{P}]$. ■

Proof. We prove the case $a = 0, b = 1$. The others are similar and omitted for brevity. Decompose the sphere f_c into (t=thumb, i=index finger, m=middle finger)



Now, because of the assignment in (8), we have $\mathfrak{T}_{\mathrm{F_o}}(f_{\mathrm{o}}) = 1 \otimes X$ and $\mathfrak{T}_{\mathrm{F_p}}(f_{\mathrm{p}}) = 1$. Thus, $\mathrm{gl}_{\mathrm{F_o}}(\mathfrak{T}_{\mathrm{F_o}}(f_{\mathrm{o}})) = \boldsymbol{\omega}_- \cdot X$ and $\mathrm{gl}_{\mathrm{F_p}}(\mathfrak{T}_{\mathrm{F_p}}(f_{\mathrm{p}})) = 1$, both considered in $\mathrm{F_o}$. Applying the trace $\mathrm{tr_o}$ to $(\boldsymbol{\omega}_- \cdot X) \otimes 1$ gives $\boldsymbol{\omega}_-$ as in (12). □

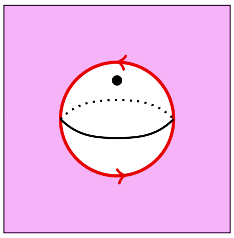
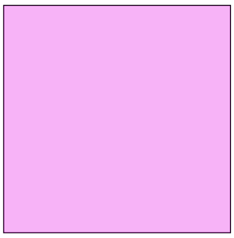
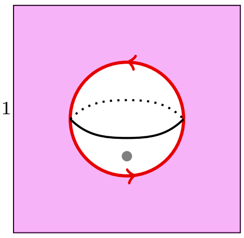
Remark 2.17. Combining Lemmas 2.15 and 2.16 we obtain (for $a, b \in \mathbb{Z}_{\geq 0}$ dots)

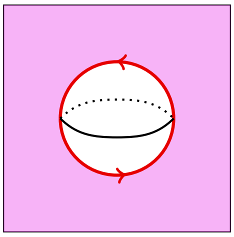
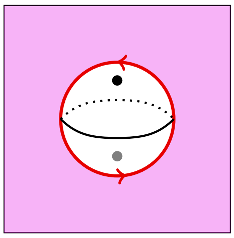


$$= \begin{cases} \underset{(\varepsilon\omega)}{\boldsymbol{\omega}_-}, & \text{if } a = 1, b = 0, \\ \underset{(\omega)}{\boldsymbol{\omega}_+}, & \text{if } a = 0, b = 1, \\ 0, & \text{otherwise.} \end{cases}$$

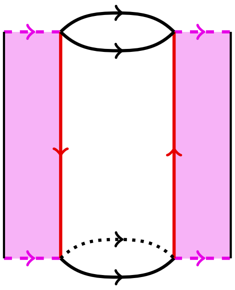
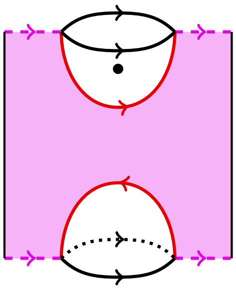
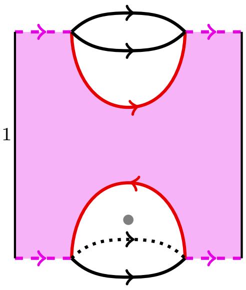
Analogously for all other relations below if one swaps an orientation of a singular seam. Therefore, we will fix an orientation for the singular seams, and the relations below are asymmetric in $\boldsymbol{\omega}_+$ and $\boldsymbol{\omega}_-$ only due to our choice of such orientations. ▲

Lemma 2.18. The *bubble removals* (where we have a "sphere" in a phantom plane, with the top dots on the front facets and the bottom dots on the back facets)

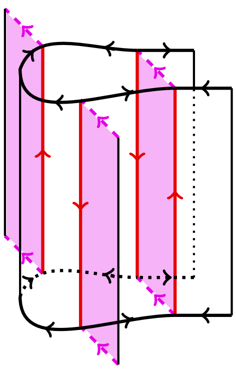
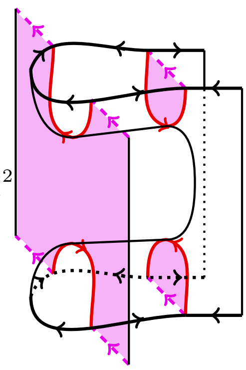
(13) $\underset{(\omega)}{\boldsymbol{\tau\omega}_-^{-1}}$  $=$  $= \underset{(\varepsilon\omega)}{\boldsymbol{\tau\omega}_+^{-1}}$ 

(14)  $= \; 0 \; =$ 

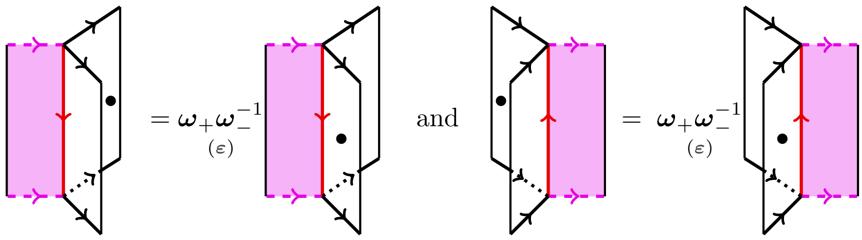
are in the kernel of $\mathfrak{T}[\mathsf{P}]$. The *(singular) neck cutting relation*

(15)  $= \underset{(\omega)}{\boldsymbol{\tau\omega}_-^{-1}}$  $+ \underset{(\varepsilon\omega)}{\boldsymbol{\tau\omega}_+^{-1}}$ 

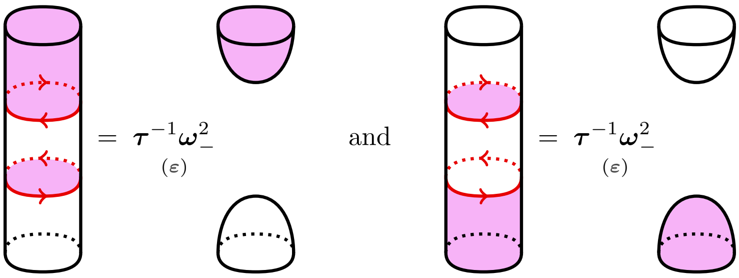
(with top dot on the front facet and bottom dot on the back facet) is also in the kernel of $\mathfrak{T}[\mathsf{P}]$. Furthermore, the *squeezing relation*

(16)  $= \underset{(\varepsilon)}{\boldsymbol{\tau\omega}_-^{-2}}$ 

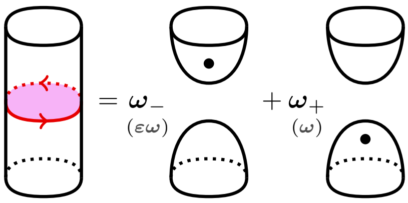
the *dot migration relations*

(17) 

and the *ordinary-to-phantom neck cutting relations* (in the leftmost picture the upper closed circle is an ordinary facet, while the lower closed circle is a phantom facet, and vice versa for the rightmost picture)

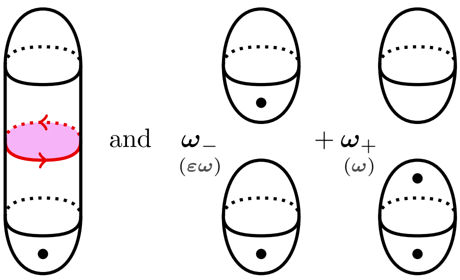
(18) 

as well as *the closed seam removal relations*

(19) 

are also in the kernel of $\mathfrak{T}[\mathtt{P}]$. ■

Proof. We only prove (19). The other relations are verified similarly. First note that we have to consider all possible ways to close the pre-foams on the left-hand and on the right-hand side of the equations. For (19) we consider the closure



A direct computation, using the relations (12) on the left-hand side and (9) on the right-hand side, shows that they agree for this closure (that is, both give $\boldsymbol{\omega}_-$ in the illustrated case). All other closures work in the same way (and are omitted for brevity) which shows that (19) is in the kernel of $\mathfrak{T}[\mathtt{P}]$. □

The leftmost situation in (15) is called a *cylinder* - as are all local parts of a pre-foam f which are cylinders after removing the phantom facets. Note that the squeezing relation (16) enables us to use the neck cutting (15) on more general cylinders (with possibly internal phantom facets).

If we define a grading on F_{o} by setting $\deg_{\mathrm{F}_{\mathrm{o}}}(1) = -1$ and $\deg_{\mathrm{F}_{\mathrm{o}}}(X) = 1$, then the TQFT $\mathfrak{T}_{\mathrm{F}_{\mathrm{o}}}$ respects the grading, where the degree of a cobordism Σ is given by

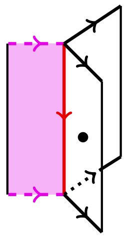
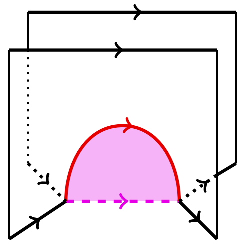
$\deg(\Sigma) = -\chi(\Sigma) + 2 \cdot \text{dots}$. Here $\chi(\Sigma)$ is the topological Euler characteristic of Σ, that is, the number of vertices minus the number of edges plus the number of faces of Σ seen as a CW complex, and "dots" is the number of dots. Additionally, we can see $\mathfrak{T}_{\mathrm{F_p}}$ as being trivially graded. Motivated by this we define the following.

Definition 2.19. Given a pre-foam f, let $\hat{f}$ be the CW complex obtained from it by removing the phantom edges and phantom facets. We define a *degree* of f via

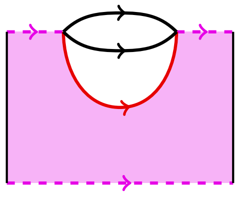
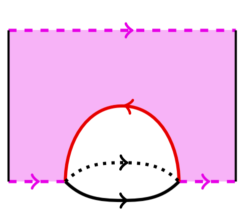
$$\deg(f) = -\chi(\hat{f}) + 2 \cdot \text{dots} + \tfrac{1}{2}\text{vbound},$$

where vbound is the total number of vertical boundary components. ▲

Example 2.20. If $\hat{f} = \emptyset$, then $\chi(\hat{f}) = 0$. Moreover, recalling that P is graded, we can see pre-foams now as forming a graded, free P-module. For example,

$\deg\left(\boldsymbol{\alpha} \cdot\right.$  $\left.\right) = \deg(f) + \deg_P(\boldsymbol{\alpha}) = 6$, $\deg\left(\right.$  $\left.\right) = 1.$

The pre-foam on the right is called a *saddle* (as well as its horizontal mirror). Furthermore, we have

$\deg\left(\right.$  $\left.\right) = -1 = \deg\left(\right.$  $\left.\right)$

for the pre-foams called *cup foam* respectively *cap foam*. ▲

2.3. **Foam 2-categories.** We like to study the following 2-category which we call the 4-*parameter foam* 2-*category*.

Definition 2.21. Let $\mathfrak{F}[\mathrm{P}]$ be the P-linear 2-category given by:

- The objects are all $\vec{k} \in \mathbb{bl}$ (which includes $\emptyset = (, \dots, 0, 0, 0, \dots,))$.
- The 1-morphisms spaces $\mathrm{Hom}_{\mathfrak{F}[\mathrm{P}]}(\vec{k}, \vec{l})$ consists of all webs whose bottom boundary is $\vec{k}$ and whose top boundary is $\vec{l}$ (which includes $\emptyset \in \mathrm{End}_{\mathfrak{F}[\mathrm{P}]}(\emptyset)$). We have $\mathrm{Hom}_{\mathfrak{F}[\mathrm{P}]}(\vec{k}, \vec{l}) = \varnothing$ iff $k_a + \cdots + k_b \neq l_{a'} + \cdots + l_{b'}$.
- The 2-morphisms spaces $2\mathrm{Hom}_{\mathfrak{F}[\mathrm{P}]}(u, v)$ are finite, formal P-linear combinations of pre-foams with bottom boundary u and top boundary v.
- Composition of webs $v \circ u = uv$ is stacking v on top of u, vertical composition $g \circ f$ of pre-foams is stacking g on top of f, horizontal composition $g \circ_h f$ is putting g to the right of f (whenever those operations make sense).
- Everything is taken modulo the relations (9), (10) and (11), as well as the relations from Lemmas 2.15, 2.16 and 2.18 (the relations "in the kernel").

With the definition of degree from Definition 2.19, the relations are homogeneous (which endows $2\mathrm{Hom}_{\mathfrak{F}[\mathrm{P}]}(u, v)$ with the structure of a graded P-module whose grading is additive under composition). Hence, $\mathfrak{F}[\mathrm{P}]$ is a graded, P-linear 2-category. ▲

We call the 2-morphisms in $\mathfrak{F}[\mathrm{P}]$ *foams*, and we adapt all notions we had for pre-foams to the setting of foams. Note now that, if one fixes a ring R and a specialization $\mathtt{p}\colon P \to R$, then there exists an induced *specialization 2-functor* and an induced *specialized 2-category* $\mathrm{Sp}_{\mathtt{p}}\colon \mathfrak{F}[\mathtt{P}] \to \mathfrak{F}_R[\mathtt{p}(\boldsymbol{\alpha}), \mathtt{p}(\boldsymbol{\tau}), \mathtt{p}(\boldsymbol{\omega}_+), \mathtt{p}(\boldsymbol{\omega}_-)]$. We keep on calling the 2-morphisms in such specializations foams.

Example 2.22. If we see R as trivially graded, then any specialization of $\mathfrak{F}[\mathtt{P}]$ with $\mathtt{p}(\boldsymbol{\alpha}) = 0$ respects the grading because the relation on the left in (10) will be a homogeneous relation while the others are clearly homogeneous. Thus, in this case, specializations of $\mathfrak{F}[\mathtt{P}]$ with $\mathtt{p}(\boldsymbol{\alpha}) = 0$ are graded, R-linear 2-categories. ▲

The following easy, yet important, lemma implies that 2-hom spaces of $\mathfrak{F}[\mathtt{P}]$ are free P-modules of finite rank (as we show below). Moreover, it also justifies to think of foams between webs which have only phantom edges as being "closed". To this end, let $\ell \in \mathbb{Z}_{\geq 0}$ and let $\omega_\ell = (1, \dots, 1, 0, \dots, 0)$ with ℓ numbers equal 1, and let $\mathbf{1}_{2\omega_\ell}$ denote the identity web on ω_ℓ.

Lemma 2.23. Let $\ell \in \mathbb{Z}_{\geq 0}$. Then $2\mathrm{End}_{\mathfrak{F}[\mathtt{P}]}(\mathbf{1}_{2\omega_\ell}) \cong P$. ■

Proof. This follows since $2\mathrm{End}_{\mathfrak{F}[\mathtt{P}]}(\emptyset) \cong P$ and the fact that one can close phantom facets only in one way (note that all closures are the same by the ghostly relations from (9) and (10)). Details can be found in [18, Lemma 4.2]. □

Let $\mathrm{CUP}(\vec{k}) = \mathrm{Hom}_{\mathfrak{F}[\mathtt{P}]}(2\omega_\ell, \vec{k})$ (its elements are called *cup webs*). We denote by * the involution which flips webs upside down and reverses their orientations. Next, we define a basis which we call the *cup foam basis*.

Definition 2.24. Fix cup webs $u, v \in \mathrm{CUP}(\vec{k})$ and consider $2\mathrm{Hom}_{\mathfrak{F}[\mathtt{P}]}(\mathbf{1}_{2\omega_\ell}, uv^*)$. Perform the following steps.

(I) Label each circle in uv^* by either "no dot" or "dot". Consider all possibilities of labeling the circles in such a way.

(II) For each such possibility we construct a foam $f\colon \mathbf{1}_{2\omega_\ell} \to uv^*$ via "cupping it off" (details can be found in [18, Definition 4.12]) with cup foams (as e.g. in Example 2.20) and then placing a dot on its rightmost facet iff the label was "dot" (note that this is ill-defined since there could be more than one rightmost facet, but one can choose any of them by (17) and the easy observation that one always has to pass a singular seam from left to right for each singular seam one passes from right to left while moving a dot from one rightmost facet to another).

(III) This procedure is to be performed repeatedly starting always with circles which do not have any other nested components.

From this we obtain a set ${}_u\mathbb{B}^\circ(\vec{k})_v$ called *cup foam basis.* Note that the definition of the cup foam basis goes clearly through for any specialization of $\mathtt{P}$ as well. ▲

Lemma 2.25. Let $u, v \in \mathrm{CUP}(\vec{k})$. The set ${}_u\mathbb{B}^\circ(\vec{k})_v$ is a homogeneous, P-linear basis of the space $2\mathrm{Hom}_{\mathfrak{F}[\mathtt{P}]}(\mathbf{1}_{2\omega_\ell}, uv^*)$. (Similarly for any specialization of $\mathtt{P}$.) ■

Proof. Almost word-by-word as in [18, Lemma 4.13] and left to the reader (the main ingredient is indeed Lemma 2.23). □

Corollary 2.26. All 2-hom spaces of $\mathfrak{F}[\mathtt{P}]$ are free P-modules of finite rank. Any specialization is of the same rank as for the 4-parameter version. ■

Proof. This follows from the P-module isomorphisms

$$2\mathrm{Hom}_{\mathfrak{F}[\mathtt{P}]}(\mathbf{1}_{2\omega_\ell}, uv^*) \cong 2\mathrm{Hom}_{\mathfrak{F}[\mathtt{P}]}(u^*, v^*) \cong 2\mathrm{Hom}_{\mathfrak{F}[\mathtt{P}]}(u, v)$$

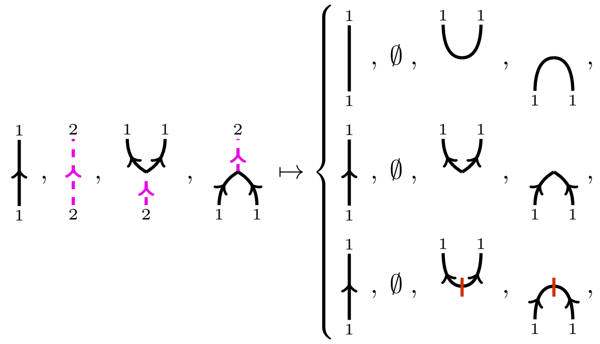
and Lemma 2.25. □

2.4. **Known specializations.** Several 2-categories which appear in the literature (i.e. our four main examples, but keeping $\boldsymbol{\alpha}$ generic) are specializations of $\mathfrak{F}[\mathtt{P}]$.

Definition 2.27. We define three 2-categories, denoted by $\mathfrak{F}_{\mathbb{Z}[\boldsymbol{\alpha}]}[\mathbf{KBN}]$, $\mathfrak{F}_{\mathbb{Z}[\boldsymbol{\alpha},i]}[\mathbf{Ca}]$ and $\mathfrak{F}_{\mathbb{Z}[\boldsymbol{\alpha},i]}[\mathbf{CMW}]$, as in Definition 2.21 except for the following differences.

- In all three cases: as objects one allows only $\vec{k} \in \mathbb{bl}$ without entries 2.
- As 1-morphisms one has "webs" generated by (we have already indicated the assignment for the 2-functors we define below)

(20)



in case **KBN**,

in case **Ca**,

in case **CMW**,

rather than webs in the sense of Definition 2.1.
- We introduce them on the 2-morphisms level in the proof of Proposition 2.29.

The 2-category $\mathfrak{F}_{\mathbb{Z}[\boldsymbol{\alpha}]}[\mathbf{Bl}]$ is defined as the specialization of the 2-category $\mathfrak{F}[\mathtt{P}]$ via $\mathtt{p}(\boldsymbol{\alpha}) = 0$, $\mathtt{p}(\boldsymbol{\tau}) = -1$, $\mathtt{p}(\boldsymbol{\omega}_+) = 1$ and $\mathtt{p}(\boldsymbol{\omega}_-) = -1$ (with values in $\mathbb{Z}[\boldsymbol{\alpha}]$). We consider all of them as graded, R-linear 2-categories (for R being either $\mathbb{Z}[\boldsymbol{\alpha}]$, in cases **KBN** and **Bl**, or $\mathbb{Z}[\boldsymbol{\alpha}, i]$, in cases **Ca** and **CMW**, with $\deg_R(\boldsymbol{\alpha}) = 4$). ▲

Remark 2.28. The 2-category $\mathfrak{F}_{\mathbb{Z}[\boldsymbol{\alpha}]}[\mathbf{KBN}]$ coincides with the 2-category studied in [23, Subsection 2.3] and also in [3, Subsection 11.2]. The 2-categories $\mathfrak{F}_{\mathbb{Z}[\boldsymbol{\alpha},i]}[\mathbf{Ca}]$, $\mathfrak{F}_{\mathbb{Z}[\boldsymbol{\alpha},i]}[\mathbf{CMW}]$ and $\mathfrak{F}_{\mathbb{Z}[\boldsymbol{\alpha},i]}[\mathbf{Bl}]$ are only 2-subcategories of the 2-categories considered in [11, Section 2], in [15, Subsection 2.2] and in [4, Section 1] respectively, since we only allow upwards pointing webs and also only allow disorientation lines coming from singular seams. These 2-subcategories however suffice for the construction of the Khovanov complex and the corresponding higher link and tangle invariants. ▲

Proposition 2.29. There are specializations of the parameters $\mathtt{P}$ and equivalences of graded, R-linear 2-categories (which are in fact isomorphisms)

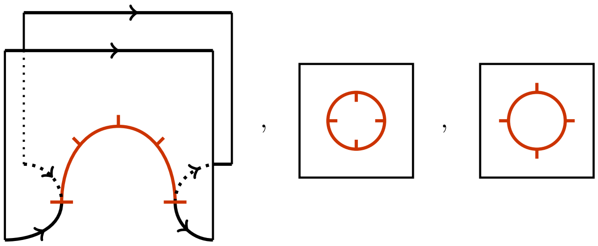
$$\mathfrak{F}_{\mathbb{Z}[\boldsymbol{\alpha}]}[\boldsymbol{\alpha}, 1, 1, 1] \xrightarrow{\cong} \mathfrak{F}_{\mathbb{Z}[\boldsymbol{\alpha}]}[\mathbf{KBN}],$$
$$\mathfrak{F}_{\mathbb{Z}[\boldsymbol{\alpha},i]}[\boldsymbol{\alpha}, 1, i, -i] \xrightarrow{\cong} \mathfrak{F}_{\mathbb{Z}[\boldsymbol{\alpha},i]}[\mathbf{Ca}],$$
$$\mathfrak{F}_{\mathbb{Z}[\boldsymbol{\alpha},i]}[\boldsymbol{\alpha}, 1, i, -i] \xrightarrow{\cong} \mathfrak{F}_{\mathbb{Z}[\boldsymbol{\alpha},i]}[\mathbf{CMW}],$$
$$\mathfrak{F}_{\mathbb{Z}[\boldsymbol{\alpha}]}[\boldsymbol{\alpha}, -1, 1, -1] \xrightarrow{\cong} \mathfrak{F}_{\mathbb{Z}[\boldsymbol{\alpha}]}[\mathbf{Bl}],$$

extending the assignment from (20). Here $R = \mathbb{Z}[\boldsymbol{\alpha}]$ in the first and fourth case and $R = \mathbb{Z}[\boldsymbol{\alpha}, i]$ in the other two cases. (Similarly for any further specialization of $\boldsymbol{\alpha}$.)■

From now on we will identify the various 2-categories and their specializations.

Proof. We first need to define the 2-categories in question on the level of 2-morphisms and then the 2-functors which provide the equivalences. The 2-morphisms of $\mathfrak{F}_{\mathbb{Z}[\boldsymbol{\alpha}]}[\mathbf{KBN}]$ are $\mathbb{Z}[\boldsymbol{\alpha}]$-linear combinations (modulo the relations below) of pre-foams with only ordinary parts. The 2-morphisms of $\mathfrak{F}_{\mathbb{Z}[\boldsymbol{\alpha},i]}[\mathbf{Ca}]$ are $\mathbb{Z}[\boldsymbol{\alpha}, i]$-linear combinations of the topological CW complexes obtained from pre-foams by removing the phantom edges and phantom facets (modulo the relations below). Moreover, the

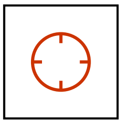
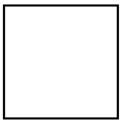
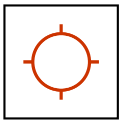
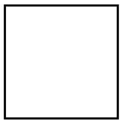
2-morphisms of $\mathfrak{F}_{\mathbb{Z}[\boldsymbol{\alpha},i]}[\mathbf{CMW}]$ are $\mathbb{Z}[\boldsymbol{\alpha},i]$-linear combinations of these with extra disorientation lines (modulo the relations below). For example



Here we assume that such disorientation lines all come from singular seams as explained below (see also Remark 2.28).

The relations for $\mathfrak{F}_{\mathbb{Z}[\boldsymbol{\alpha}]}[\mathbf{KBN}]$, $\mathfrak{F}_{\mathbb{Z}[\boldsymbol{\alpha},i]}[\mathbf{Ca}]$ and $\mathfrak{F}_{\mathbb{Z}[\boldsymbol{\alpha},i]}[\mathbf{CMW}]$ which are imposed upon the 2-morphisms are the ordinary sphere, dot removing and neck cutting relations from (9), (10) and (11).

For $\mathfrak{F}_{\mathbb{Z}[\boldsymbol{\alpha},i]}[\mathbf{Ca}]$ we additionally impose the relations from Lemmas 2.15, 2.16 and 2.18 for the specialization $\mathtt{p}(\boldsymbol{\tau})=1$, $\mathtt{p}(\boldsymbol{\omega}_+)=i$ and $\mathtt{p}(\boldsymbol{\omega}_-)=-i$ (and we remove the phantom facets); for $\mathfrak{F}_{\mathbb{Z}[\boldsymbol{\alpha},i]}[\mathbf{CMW}]$ we additionally impose the disorientation removals (and all local relations they induce by closing in all possible ways)

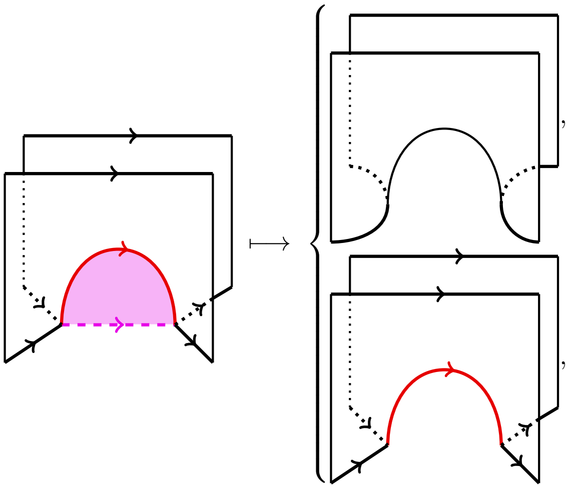
(21)  $= i \cdot$  ,  $= -i \cdot$ 

The grading in all cases is, as in Definition 2.19, induced by the topological Euler characteristic. In particular, disorientation lines do not change the degree.

Thus, using the dictionary given above, we have 2-functors

$$\Gamma_{\mathbf{KBN}}\colon \mathfrak{F}_{\mathbb{Z}[\boldsymbol{\alpha}]}[\boldsymbol{\alpha},1,1,1]\to\mathfrak{F}_{\mathbb{Z}[\boldsymbol{\alpha}]}[\mathbf{KBN}],\quad \Gamma_{\mathbf{Ca}}\colon \mathfrak{F}_{\mathbb{Z}[\boldsymbol{\alpha},i]}[\boldsymbol{\alpha},1,i,-i]\to\mathfrak{F}_{\mathbb{Z}[\boldsymbol{\alpha},i]}[\mathbf{Ca}]$$

given on objects by replacing every entry 2 in a given $\vec{k}$ by a 0, on 1-morphisms by (20) and on 2-morphisms by removing the phantom edges and phantom facets for $\Gamma_{\mathbf{Ca}}$ and additionally by removing the singular seams for $\Gamma_{\mathbf{KBN}}$. For example

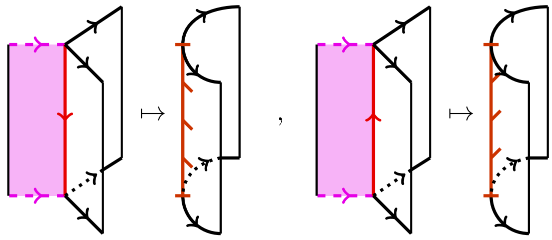


in case **KBN**,

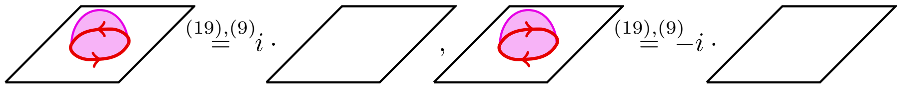
in case **Ca**.

That these 2-functors are well-defined, grading preserving $\mathbb{Z}[\boldsymbol{\alpha}]$-linear (respectively $\mathbb{Z}[\boldsymbol{\alpha},i]$-linear) follows directly by comparing the resulting specialized relations from (9), (10) and (11), and from Lemmas 2.15, 2.16 and 2.18. Clearly, $\Gamma_{\mathbf{KBN}}$ and $\Gamma_{\mathbf{Ca}}$, are essential surjective on objects and 1-morphisms and full on 2-morphisms. That they are faithful on 2-morphisms is evident (one can check this on the cup basis from Definition 2.24 which one easily writes down, mutatis mutandis, for $\Gamma_{\mathbf{KBN}}$ and $\Gamma_{\mathbf{Ca}}$ as well), which shows that they are equivalences as claimed.

We define a 2-functor $\Gamma_{\mathbf{CMW}}\colon \mathfrak{F}_{\mathbb{Z}[\boldsymbol{\alpha}]}[\boldsymbol{\alpha},1,i,-i] \to \mathfrak{F}_{\mathbb{Z}[\boldsymbol{\alpha}]}[\mathbf{CMW}]$ on objects and on 1-morphisms analogously to the two 2-functors from above (but using the third assignment in (20)). On 2-morphisms it is defined by removing all phantom edges and phantom facets and replacing singular seams by disorientation lines, where the orientation of the seam induces the direction of the disorientation line:



(with disorientation lines pointing out of and into the paper). One can check directly that the relations from $\mathfrak{F}_{\mathbb{Z}[\boldsymbol{\alpha},i]}[\boldsymbol{\alpha},1,i,-i]$ hold in the image of $\Gamma_{\mathbf{CMW}}$ which shows that $\Gamma_{\mathbf{CMW}}$ is well-defined. For example, the disorientation removals (21) on level of foams are (with $\mathtt{p}(\tau)=1, \mathtt{p}(\boldsymbol{\omega}_+)=i$ and $\mathtt{p}(\boldsymbol{\omega}_-)=-i$)



These imply that the relation [15, Figure 3] holds in the image of $\Gamma_{\mathbf{CMW}}$. As above, it follows also that $\Gamma_{\mathbf{CMW}}$ is a grading preserving, $\mathbb{Z}[\boldsymbol{\alpha},i]$-linear 2-functor which is essential surjective on objects and 1-morphisms, as well as fully faithful on 2-morphisms. This shows that $\Gamma_{\mathbf{CMW}}$ gives the claimed equivalence.

Last, $\mathfrak{F}_{\mathbb{Z}[\boldsymbol{\alpha}]}[\mathbf{Bl}]$ is defined precisely as in Definition 2.21, but with the choice of parameters $\mathtt{p}(\boldsymbol{\alpha})=0$, $\mathtt{p}(\boldsymbol{\tau})=-1$, $\mathtt{p}(\boldsymbol{\omega}_+)=1$ and $\mathtt{p}(\boldsymbol{\omega}_-)=-1$. Thus, the statement for this case follows directly from the definition (which formally uses Lemma 2.25 and Corollary 2.26 again).

Moreover, the cases with specialized $\boldsymbol{\alpha}$ work similarly and are omitted for brevity which finishes the proof. □

2.5. **Web algebras.** We define the following "algebraic" version $\mathfrak{W}[\mathtt{P}]$ of $\mathfrak{F}[\mathtt{P}]$. As we will see later in Proposition 4.34, when passing to $\mathtt{Q}$, the 2-category $\mathfrak{F}[\mathtt{P}]$ will be equivalent to a certain $\mathfrak{W}[\mathtt{P}]$-bimodule 2-category as defined in Definition 2.42.

Definition 2.30. Denote by $\mathbb{bl}^{\diamond} \subset \mathbb{bl}$ the set of all $\vec{k} \in \mathbb{bl}$ which have an even number of entries 1. We call elements of $\mathbb{bl}^{\diamond}$ *balanced*. ▲

Given $\vec{k} \in \mathbb{bl}^{\diamond}$ with $\sum_{i\in\mathbb{Z}} k_i = 2\ell$, define the *shift* $d(\vec{k}) = \ell - \sum_{i\in\mathbb{Z}} k_i(k_i-1)$. For example, for $\vec{k} \in \mathbb{bl}^{\diamond}$ with $k_i \neq 2$ we have $d(\vec{k})=\ell$.

We are now ready to define some of our main algebras under study.

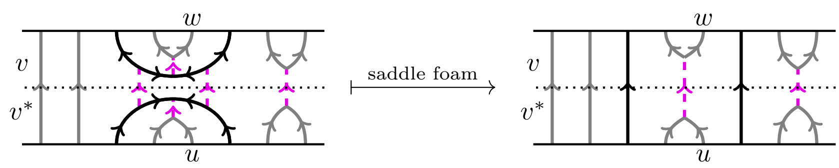
Definition 2.31. Let $\vec{k} \in \mathbb{bl}^{\diamond}, u,v \in \mathrm{CUP}(\vec{k})$. We denote by ${}_u(\mathfrak{W}[\mathtt{P}]_{\vec{k}})_v$ the space $2\mathrm{Hom}_{\mathfrak{F}}(\mathbf{1}_{2\omega_\ell}, uv^*)\{d(\vec{k})\}$. The 4-*parameter web algebra* $\mathfrak{W}[\mathtt{P}]_{\vec{k}}$ and the *(full) 4-parameter web algebra* $\mathfrak{W}[\mathtt{P}]$ are the P-modules

$$\mathfrak{W}[\mathtt{P}]_{\vec{k}} = \bigoplus_{u,v\in\mathrm{CUP}(\vec{k})} {}_u(\mathfrak{W}[\mathtt{P}]_{\vec{k}})_v, \qquad \mathfrak{W}[\mathtt{P}] = \bigoplus_{\vec{k}\in\mathbb{bl}^{\diamond}} \mathfrak{W}[\mathtt{P}]_{\vec{k}}.$$

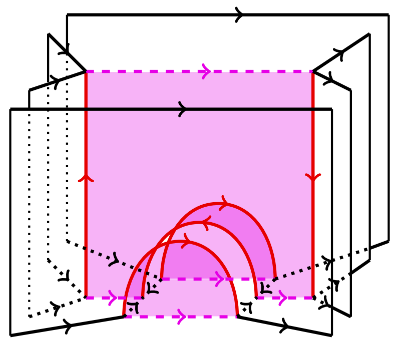
We consider these as graded P-modules by using the degree from Definition 2.19. Moreover, we endow them with a multiplication

$$\mathbf{Mult}\colon \mathfrak{W}[\mathtt{P}]_{\vec{k}} \otimes \mathfrak{W}[\mathtt{P}]_{\vec{k}} \to \mathfrak{W}[\mathtt{P}]_{\vec{k}},\ f \otimes g \mapsto \mathbf{Mult}(f,g) = fg \tag{22}$$

using *multiplication foams* as follows: to multiply $f \in {}_u(\mathfrak{W}_{\vec{k}})_v$ with $g \in {}_{\tilde{v}}(\mathfrak{W}_{\vec{k}})_w$ to obtain fg stack the diagram $\tilde{v}w^*$ on top of uv^* and obtain $uv^*\tilde{v}w^*$. Then $fg = 0$ if $v \neq \tilde{v}$. Otherwise, pick any cup-cap pair as below and perform a "surgery"

(23) 

where the saddle foam is locally of the following form (and the identity elsewhere)

(24) 

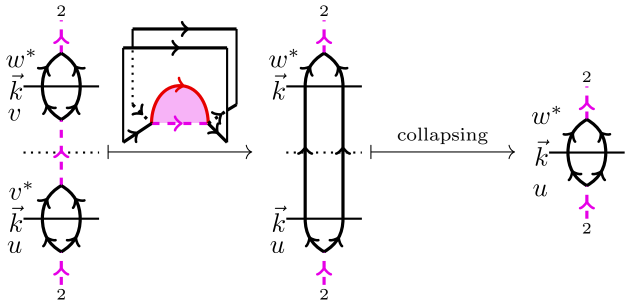
This process should be read as follows: start with a foam $f \in 2\mathrm{Hom}_{\mathfrak{F}[\mathtt{P}]}(\mathbf{1}_{2\omega_\ell}, uv^*vw)$ and stack on top of it a foam which is the identity at the bottom (u part) and top (w part) of the web and the saddle in between. Repeat until no cup-cap pair as above remains. This gives inductively rise to a multiplication foam (after the last surgery step we collapse the webs and foams as is easiest explained in an example, see Example 2.33). Compare also to [34, Definition 3.3]. ▲

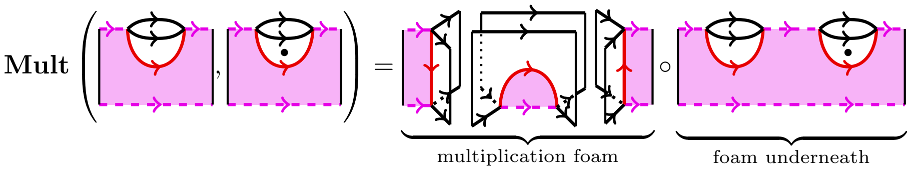
One nice feature of web algebras is that the following lemma is "obvious", since the web algebras are defined topologically via singular TQFTs.

Lemma 2.32. The map $\mathbf{Mult}\colon \mathfrak{W}[\mathtt{P}]_{\vec{k}} \otimes \mathfrak{W}[\mathtt{P}]_{\vec{k}} \to \mathfrak{W}[\mathtt{P}]_{\vec{k}}$ given above is degree preserving and independent of the order in which the surgeries are performed. This turns $\mathfrak{W}[\mathtt{P}]_{\vec{k}}$ into a graded, associative, unital algebra, which is a free P-module of finite rank. (Similarly for the locally unital algebra $\mathfrak{W}[\mathtt{P}]$.) ■

Proof. That they are free P-modules of (locally) finite rank follows from Corollary 2.26. Everything else follows by identifying the multiplication in the general web algebras with composition in $\mathfrak{F}[\mathtt{P}]$, see e.g. [18, Lemma 2.26] or [34, Lemma 3.7]. □

Example 2.33. An easy multiplication example for $u = v = w \in \mathrm{CAP}((1,1))$ is





where the reader should think about any foam $f\colon \mathbf{1}_{2\omega_1} \to uv^*vw^*$ sitting underneath (as illustrated in one case above). The rightmost step above is the collapsing step (and usually omitted from illustrations). The saddle is of degree 1 and thus, taking the shift $d((1,1)) = 1$ into account, the multiplication foam is of degree zero. ▲

Remark 2.34. Everything from this (sub)section goes through for $\mathtt{Q}$ or any other specialization as well. In particular, we have a graded Q-linear 2-category $\mathfrak{F}[\mathtt{Q}]$, called the *signed* 2-*parameter foam* 2-*category*, and graded algebras $\mathfrak{W}[\mathtt{Q}]_{\vec{k}}$ and $\mathfrak{W}[\mathtt{Q}]$ called *signed* 2-*parameter web algebras*. These still include our main examples. ▲

Specializations 2.35. By Proposition 2.29, the specialization $\mathfrak{W}_{\mathbb{Z}[\boldsymbol{\alpha}]}[\boldsymbol{\alpha}, 1, 1, 1]$ is graded isomorphic to the algebra $\mathfrak{W}_{\mathbb{Z}[\boldsymbol{\alpha}]}[\mathbf{KBN}]$. Similarly, $\mathfrak{W}_{\mathbb{Z}[\boldsymbol{\alpha}]}[\boldsymbol{\alpha}, -1, 1, -1]$ is graded isomorphic to $\mathfrak{W}_{\mathbb{Z}[\boldsymbol{\alpha}]}[\mathbf{Bl}]$. Moreover, we can view $\mathfrak{W}_{\mathbb{Z}[\boldsymbol{\alpha},i]}[\boldsymbol{\alpha}, 1, i, -i]$ as describing the setups of **Ca** or **CMW**, see also Specializations 2.45. ▲

2.6. **Web bimodules and foam 2-categories.** We still consider only $\vec{k}, \vec{l} \in \mathbb{bl}^{\diamond}$.

Definition 2.36. Given any web $u \in \mathrm{Hom}_{\mathfrak{F}[\mathtt{P}]}(\vec{k}, \vec{l})$ (with boundaries $\vec{k}$ and $\vec{l}$ summing up to 2ℓ), we consider the $\mathfrak{W}[\mathtt{P}]$-bimodule

$$\mathcal{W}[\mathtt{P}](u) = \bigoplus_{\substack{v \in \mathrm{CUP}(\vec{k}),\\ w \in \mathrm{CUP}(\vec{l})}} 2\mathrm{Hom}_{\mathfrak{F}[\mathtt{P}]}(\mathbf{1}_{2\omega_\ell}, vuw^*)$$

with left (bottom) and right (top) action of $\mathfrak{W}[\mathtt{P}]$ as in Definition 2.31. We call all such $\mathfrak{W}[\mathtt{P}]$-bimodules $\mathcal{W}[\mathtt{P}](u)$ *web bimodules*. ▲

Web bimodules also have a cup foam basis.

Definition 2.37. Given $u \in \mathrm{Hom}_{\mathfrak{F}}(\vec{k}, \vec{l})$, define a *cup foam basis* $\mathbb{B}^{\circ}(u)$ *of* $\mathcal{W}[\mathtt{P}](u)$ as in Definition 2.24 by considering all webs vuw^* for $v \in \mathrm{CUP}(\vec{k}), w \in \mathrm{CUP}(\vec{l})$. ▲

Lemma 2.38. Let $u \in \mathrm{Hom}_{\mathfrak{F}[\mathtt{P}]}(\vec{k}, \vec{l})$. The set $\mathbb{B}^{\circ}(u)$ is a homogeneous, P-linear basis of the web bimodule $\mathcal{W}[\mathtt{P}](u)$. (Similarly for any specialization of $\mathtt{P}$.) ■

Proof. Analogous to Lemma 2.25 and thus, omitted. See also [18, Lemma 4.14] for the proof with specialized parameters (which still works almost word-by-word). □

The following is now evident.

Corollary 2.39. All web bimodules are free P-modules of finite rank. Any specialization is of the same rank as for the 4-parameter version. □

Lemma 2.40. Let $u \in \mathrm{Hom}_{\mathfrak{F}[\mathtt{P}]}(\vec{k}, \vec{l})$ be a web. Then the left (bottom) action of $\mathfrak{W}[\mathtt{P}]_{\vec{k}}$ as well as the right (top) action of $\mathfrak{W}[\mathtt{P}]_{\vec{l}}$ on $\mathcal{W}[\mathtt{P}](u)$ are well-defined and commute. Hence, $\mathcal{W}[\mathtt{P}](u)$ is a $\mathfrak{W}[\mathtt{P}]_{\vec{k}}$-$\mathfrak{W}[\mathtt{P}]_{\vec{l}}$-bimodule and thus, a $\mathfrak{W}[\mathtt{P}]$-bimodule. (Similarly for any specialization of $\mathtt{P}$.) ■

Proof. Let $u \in \mathrm{Hom}_{\mathfrak{F}[\mathtt{P}]}(\vec{k}, \vec{l})$. The left (bottom) action of $\mathfrak{W}[\mathtt{P}]_{\vec{k}}$ and the right (top) action of $\mathfrak{W}[\mathtt{P}]_{\vec{l}}$ on $\mathcal{W}[\mathtt{P}](u)$ commute since they are "far apart". Hence, $\mathcal{W}[\mathtt{P}](u)$ is a $\mathfrak{W}[\mathtt{P}]_{\vec{k}}$-$\mathfrak{W}[\mathtt{P}]_{\vec{l}}$-bimodule (and thus, a $\mathfrak{W}[\mathtt{P}]$-bimodule). The same works word-by-word for any specialization of $\mathtt{P}$ which shows the statement. □

Proposition 2.41. All $\mathcal{W}[\mathtt{P}](u)$ are graded biprojective, $\mathfrak{W}[\mathtt{P}]$-bimodules which are free P-modules of finite rank. (Similarly for any specialization of $\mathtt{P}$.) ■

Proof. Clearly, they are graded. They are $\mathfrak{W}[\mathtt{P}]$-bimodules which are free P-modules of finite rank follows from Lemma 2.40 and Corollary 2.39. It remains to show that they are biprojective. This follows, since they are direct summands of some $\mathfrak{W}[\mathtt{P}]_{\vec{k}}$ (of $\mathfrak{W}[\mathtt{P}]_{\vec{l}}$) as left (right) modules and for suitable $\vec{k} \in \mathbb{bl}^{\diamond}$ (or $\vec{l} \in \mathbb{bl}^{\diamond}$). Again, the arguments are parameter independent which shows the statement. □

This motivates the definition of the following 2-category.

Definition 2.42. Let $\mathfrak{W}[\mathtt{P}]$-$\mathbf{biMod}^{p}_{\mathrm{gr}}$ be the following 2-category:

- Objects are the various $\vec{k} \in \mathbb{bl}^{\diamond}$.
- 1-morphisms are finite direct sums and tensor products (taken over the algebra $\mathfrak{W}[\mathtt{P}]$) of the $\mathfrak{W}[\mathtt{P}]$-bimodules $\mathcal{W}[\mathtt{P}](u)$.
- 2-morphisms are $\mathfrak{W}[\mathtt{P}]$-bimodule homomorphisms.
- The composition of web bimodules is the tensor product $\cdot \otimes_{\mathfrak{W}[\mathtt{P}]} \cdot$. The vertical composition of $\mathfrak{W}[\mathtt{P}]$-bimodule homomorphisms is the usual composition. The horizontal composition is given by tensoring (over $\mathfrak{W}[\mathtt{P}]$).

We consider $\mathfrak{W}[\mathtt{P}]$-$\mathbf{biMod}^{p}_{\mathrm{gr}}$ as a graded 2-category by turning the 2-hom-spaces into graded P-modules (in the sense of (1)) via Definition 2.19. ▲

As usual, we also consider specializations of $\mathfrak{W}[\mathtt{P}]$-$\mathbf{biMod}^{p}_{\mathrm{gr}}$, e.g. any specialization $\mathtt{p}(\boldsymbol{\alpha}) = 0$ yields a graded 2-category.

This 2-category provides a faithful 2-representation of the 2-category $\mathfrak{F}[\mathtt{P}]$ we are interested in as follows. Recall that the *additive closure* $\oplus(\mathfrak{C})$ of a 2-category $\mathfrak{C}$ has the same objects as $\mathfrak{C}$, but one allows finite formal direct sums of 1-morphisms from $\mathfrak{C}$ and matrices between these. The reader unfamiliar with this construction is referred to [3, Definition 3.2] for a thorough treatment.

Proposition 2.43. There is an embedding of graded, P-linear 2-categories

$$\Upsilon \colon \oplus(\mathfrak{F}[\mathtt{P}]) \hookrightarrow \mathfrak{W}[\mathtt{P}]\text{-}\mathbf{biMod}^{p}_{\mathrm{gr}},$$

which is bijective on objects and essential surjective on 1-morphisms. (Similarly for any specialization of $\mathtt{P}$.)

Remark 2.44. We will see later in Proposition 4.34 that Υ is an equivalence (by restricting to $\mathtt{Q}$). Note that this is non-trivial (and relies on the isomorphisms found in Section 4) since there could potentially be plenty of uncontrollable $\mathfrak{W}[\mathtt{P}]$-bimodule homomorphisms. ▲

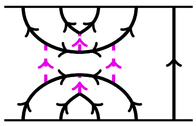
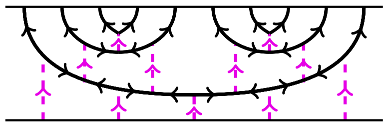
Proof. Define $\Upsilon \colon \oplus(\mathfrak{F}[\mathtt{P}]) \to \mathfrak{W}[\mathtt{P}]$-$\mathbf{biMod}^{p}_{\mathrm{gr}}$ via (and then extend additively):

- On objects $\vec{k}$ we set $\Upsilon(\vec{k}) = \vec{k}$.
- On 1-morphisms $u \in \mathrm{Hom}_{\mathfrak{F}[\mathtt{P}]}(\vec{k}, \vec{l})$ we set $\Upsilon(u) = \mathcal{W}[\mathtt{P}](u)$.
- On 2-morphisms $f \in 2\mathrm{Hom}_{\mathfrak{F}[\mathtt{P}]}(u, v)$ we set $\Upsilon(f) \colon \mathcal{W}[\mathtt{P}](u) \to \mathcal{W}[\mathtt{P}](v)$ given by stacking f on top of the elements of $\mathcal{W}[\mathtt{P}](u)$.

Note that $\Upsilon(f)$ is a $\mathfrak{W}[\mathtt{P}]$-bimodule homomorphism. This can be seen topologically: $\mathfrak{W}[\mathtt{P}]$ acts on elements of web bimodules "horizontally", while f is stacked "vertically" (the meticulous reader can copy the arguments from [24, Subsection 2.7]). Thus, Υ extends to a P-linear 2-functor. Since Υ is clearly bijective on objects, it remains to show that Υ is essential surjective on 1-morphisms and faithful.

►**Essential surjective on** 1**-morphisms.** Each 1-morphism of $\mathfrak{W}[\mathtt{P}]\text{-}\mathbf{biMod}^{p}_{\mathrm{gr}}$ is by definition of the form $\mathcal{W}[\mathtt{P}](u)$, a finite direct sum or a tensor product (over $\mathfrak{W}[\mathtt{P}]$) of these. Note that $\mathcal{W}[\mathtt{P}](u) \otimes_{\mathfrak{W}[\mathtt{P}]} \mathcal{W}[\mathtt{P}](v)$ is isomorphic to $\mathcal{W}[\mathtt{P}](uv)$. This follows as in [24, Theorem 1] (for the careful reader we note that Khovanov's arguments are parameter free). Thus, Υ is essential surjective on 1-morphisms. ▷

►**Faithful.** Let $\mathrm{clap}(u), \mathrm{clap}(v)$ be the webs obtained via "(left) clapping", e.g.:

$u =$  , $\mathrm{clap}(u) =$ 

Then $2\mathrm{Hom}_{\mathfrak{F}[\mathtt{P}]}(u,v) \cong 2\mathrm{Hom}_{\mathfrak{F}[\mathtt{P}]}(\mathrm{clap}(u), \mathrm{clap}(v))$ as graded, free P-modules. Next, as in [18, Lemma 2.26], we have

$$2\mathrm{Hom}_{\mathfrak{F}[\mathtt{P}]}(\mathrm{clap}(u), \mathrm{clap}(v)) \cong 2\mathrm{Hom}_{\mathfrak{F}[\mathtt{P}]}(\mathbf{1}_{2\omega_\ell}, \mathrm{clap}(u)\mathrm{clap}(v)^*)\{d(\vec{k})\}$$

as graded, free P-modules. Thus, we have a cup foam basis (as in Definition 2.24) for $2\mathrm{Hom}_{\mathfrak{F}[\mathtt{P}]}(u,v)$ using these identifications. By construction, these are sent via Υ to linearly independent $\mathfrak{W}[\mathtt{P}]$-homomorphisms. This shows faithfulness of Υ, since passing to the additive closure does not change the arguments from above. ▷

Clearly, Υ is degree preserving. Note also that the arguments from above are independent of the precise form of the parameters from $\mathtt{P}$. Hence, the same holds word-by-word for any specialization. The statement follows. □

Specializations 2.45. By Propositions 2.29 and 2.43 we get embeddings of graded, R-linear 2-categories (for R being either $\mathbb{Z}[\boldsymbol{\alpha}]$ in case one and four or $\mathbb{Z}[\boldsymbol{\alpha}, i]$ else)

$$\begin{gathered}
\oplus(\mathfrak{F}_{\mathbb{Z}[\boldsymbol{\alpha}]}[\mathbf{KBN}]) \hookrightarrow \mathfrak{W}_{\mathbb{Z}[\boldsymbol{\alpha}]}[\mathbf{KBN}]\text{-}\mathbf{biMod}^{p}_{\mathrm{gr}},\\
\oplus(\mathfrak{F}_{\mathbb{Z}[\boldsymbol{\alpha},i]}[\mathbf{Ca}]) \hookrightarrow \mathfrak{W}_{\mathbb{Z}[\boldsymbol{\alpha},i]}[\mathbf{Ca}]\text{-}\mathbf{biMod}^{p}_{\mathrm{gr}},\\
\oplus(\mathfrak{F}_{\mathbb{Z}[\boldsymbol{\alpha},i]}[\mathbf{CMW}]) \hookrightarrow \mathfrak{W}_{\mathbb{Z}[\boldsymbol{\alpha},i]}[\mathbf{CMW}]\text{-}\mathbf{biMod}^{p}_{\mathrm{gr}},\\
\oplus(\mathfrak{F}_{\mathbb{Z}[\boldsymbol{\alpha}]}[\mathbf{Bl}]) \hookrightarrow \mathfrak{W}_{\mathbb{Z}[\boldsymbol{\alpha}]}[\mathbf{Bl}]\text{-}\mathbf{biMod}^{p}_{\mathrm{gr}}.
\end{gathered}$$

We will see later in Specializations 4.35 that these are actually equivalences. ▲

3. A family of arc algebras

We are going to define a $\mathtt{Q}$-version of the arc algebra (recall that the parameters from $\mathtt{Q}$ are specializations of those from $\mathtt{P}$, see (3)). Combinatorially this will follow the framework in [18], but the multiplication will be more involved, incorporating $\mathtt{Q}$.

3.1. Combinatorics of arc diagrams.

In this subsection we summarize the combinatorics of arc diagrams. This part is still independent of the parameters and follows closely [18, Subsection 3.1].

Definition 3.1. A *(diagrammatical) weight* is a sequence $\lambda = (\lambda_i)_{i\in\mathbb{Z}}$ with entries $\lambda_i \in \{\circ, \times, \vee, \wedge\}$, such that $\lambda_i = \circ$ for $|i| \gg 0$. Two weights λ and μ are said to be equivalent if one can obtain μ from λ by permuting some symbols $\wedge$ and $\vee$. The equivalence classes of weights are called *blocks*, whose set will be denoted by $\mathtt{bl}$. ▲

If we display such weights or blocks, then the first entry which is not $\circ$ is assumed to be at $i = 0$ (if not stated otherwise).

As in [18, Definition 3.2] a block can be determined by giving a block sequence and demanding a certain number of symbols $\wedge$ and $\vee$ to appear in its weights.

Definition 3.2. Let $\Lambda \in \mathtt{bl}$ be a block. To Λ we associate its (well-defined) *block sequence* $\mathrm{seq}(\Lambda) = (\mathrm{seq}(\Lambda)_i)_{i\in\mathbb{Z}}$ by taking any $\lambda \in \Lambda$ and replacing the symbols $\wedge, \vee$ by $\star$. Moreover, we define $\mathrm{up}(\Lambda)$ respectively $\mathrm{down}(\Lambda)$ to be the total number of $\wedge$'s respectively $\vee$'s in Λ where we count $\times$ as both, $\wedge$ and $\vee$. ▲

The definitions presented in this section will only make use of *balanced* blocks, i.e. blocks Λ with $\mathrm{up}(\Lambda) = \mathrm{down}(\Lambda)$, see [18, Definition 3.3], and we denote by $\mathtt{bl}^\diamond \subset \mathtt{bl}$ the set of balanced blocks. As in [18, Subsection 3.1], the basic building block for the combinatorics of the arc algebras is the *cup diagram*. A *cup diagram* c is a finite collection of non-intersecting arcs inside $\mathbb{R} \times [-1, 0]$ such that each arc intersects the boundary exactly in its endpoints, and either connecting two distinct points $(i, 0)$ and $(j, 0)$ with $i, j \in \mathbb{Z}$ (called a *cup*), or connecting one point $(i, 0)$ with $i \in \mathbb{Z}$ with a point on the lower boundary of $\mathbb{R} \times [-1, 0]$ (called a *ray*). Furthermore, each point in the boundary is the endpoint of at most one arc. Two cup diagrams are equal if the arcs contained in them connect the same points. One can reflect a cup diagram c along the axis $\mathbb{R} \times \{0\}$, denote this operation by *, to obtain a *cap diagram* c^* (defined inside $\mathbb{R} \times [0, 1]$). Clearly, $(c^*)^* = c$.

A cup diagram c (and similarly a cap diagram d^*) is *compatible* with a block $\Lambda \in \mathtt{bl}$ if $\{(i, 0) \mid \mathrm{seq}(\Lambda)_i = \star\} = (\mathbb{R} \times \{0\}) \cap c$.

We will view a weight λ as labeling integral points, called *vertices*, of the horizontal line $\mathbb{R} \times \{0\} \subset \mathbb{R} \times [-1, 0]$ (or $\mathbb{R} \times \{0\} \subset \mathbb{R} \times [0, 1]$ for caps) by putting the symbol λ_i at position $(i, 0)$. Together with a cup diagram c this forms a new diagram $c\lambda$.

Definition 3.3. We say that $c\lambda$ is *oriented* if:

(I) An arc in c only contains vertices labeled $\wedge$ or $\vee$, and every vertex labeled $\wedge$ or $\vee$ is contained in an arc.

(II) The two vertices of a cup are labeled by exactly one $\wedge$ and one $\vee$.

(III) For $i < j$ with $\lambda_i = \vee$, $\lambda_j = \wedge$ at most one, λ_i or λ_j, is contained in a ray.▲

Similarly, a cap diagram d^* together with a weight λ forms a diagram λd^*, which is called *oriented* if $d\lambda$ is oriented. A cup respectively a cap in such diagrams is called *anticlockwise*, if its rightmost vertex is labeled $\wedge$ and *clockwise* otherwise.

Putting a cap diagram d^* on top of a cup diagram c such that they are connected to the line $\mathbb{R} \times \{0\}$ at the same points creates a *circle diagram*, denoted by cd^*. All connected component of this diagram that do not touch the boundary of $\mathbb{R} \times [-1, 1]$ are called *circles*, all others are called *lines*. Together with $\lambda \in \Lambda$ such that $c\lambda$ and λd^* are oriented it forms an *oriented circle diagram* $c\lambda d^*$.

Definition 3.4. We define the *degree* of an oriented cup diagram $c\lambda$, of an oriented cap diagram λd^* and of an oriented circle diagram $c\lambda d^*$ as follows.

$$\begin{aligned}
\deg(c\lambda) &= \text{number of clockwise cups in } c\lambda,\\
\deg(\lambda d^*) &= \text{number of clockwise caps in } \lambda d^*, \qquad (25)\\
\deg(c\lambda d^*) &= \deg(c\lambda) + \deg(\lambda d^*).
\end{aligned}$$ ▲

Example 3.5. The cup diagrams which we mostly use are all similar to the ones displayed in (46). In this case the block Λ has sequence $\star\ \star\ \star\ \star$. The weight λ given by $\vee\ \wedge\ \vee\ \wedge$ gives rise to an orientation for both diagrams. With these orientation the degree of the left cup diagram (46) would be 1 and of the right it would be 0. For more examples see [18, Example 3.6]. ▲

Finally, we associate to each $\lambda \in \Lambda$ a unique cup diagram, denoted by $\underline{\lambda}$, via:

(I) Connect neighboring pairs $\vee\wedge$ with a cup, ignoring symbols of the type $\circ$ and $\times$ as well as symbols already connected. Repeat this process until there are no more $\vee$'s to the left of any $\wedge$.
(II) Put a ray under any remaining symbols $\vee$ or $\wedge$.

It is an easy observation that $\underline{\lambda}$ always exists for a fixed λ. Furthermore, λ is the (unique) orientation of $\underline{\lambda}$, such that $\underline{\lambda}\lambda$ has minimal degree. Each cup diagram c is of the form $\underline{\lambda}$ for $\lambda \in \Lambda$, a block compatible with c.

Similarly we can define $\overline{\lambda} = \underline{\lambda}^*$, and, as before, in an oriented circle diagram $\underline{\lambda}\nu\overline{\mu}$ a circle C is said to be oriented *anticlockwise* if the rightmost vertex contained in the circle is $\wedge$ and *clockwise* otherwise.

Note that, by [18, Lemma 3.9], the contribution to the degree of the arcs contained in a given circle C inside an oriented circle diagram is equal to

$$\deg(C) = (\text{number of cups in } C) \pm 1, \tag{26}$$

with $+1$, if the circle C is oriented clockwise and -1 otherwise.

We also make use of the same statistics as defined in [18, Definition 3.10], namely *distance* and *saddle width*, which come up in the coefficients of the multiplication.

Definition 3.6. For $i \in \mathbb{Z}$ and a block Λ define the *position of* i as

$$\mathrm{p}_\Lambda(i) = \#\{j \mid j < i, \mathrm{seq}(\Lambda)_j = \bigstar\} + 2 \cdot \#\{j \mid j < i, \mathrm{seq}(\Lambda)_j = \times\}.$$

For a cup or cap γ in a diagram connecting vertices $(i,0)$ and $(j,0)$ we define its *distance* $\mathrm{d}_\Lambda(\gamma)$ and *saddle width* $s_\Lambda(\gamma)$ by

$$\mathrm{d}_\Lambda(\gamma) = |\mathrm{p}_\Lambda(i) - \mathrm{p}_\Lambda(j)| \qquad \text{respectively} \qquad s_\Lambda(\gamma) = \tfrac{1}{2}\left(\mathrm{d}_\Lambda(\gamma) + 1\right).$$

For a ray γ set $\mathrm{d}_\Lambda(\gamma) = 0$. For a collection $M = \{\gamma_1, \dots, \gamma_r\}$ of distinct arcs (e.g. a circle or sequence of arcs connecting two vertices) set

$$\mathrm{d}_\Lambda(M) = \sum_{1 \le k \le r} \mathrm{d}_\Lambda(\gamma_k). \qquad \blacktriangle$$

Example 3.7. Given $\Lambda \in \mathtt{bl}^\diamond$ with sequence $\bigstar\ \bigstar$ and the circle as in (36). Then $\mathrm{p}_\Lambda(0) = 0$ and $\mathrm{p}_\Lambda(1) = 1$. Moreover, if γ is either the cup or cap of the circle, then $\mathrm{d}_\Lambda(\gamma) = 1$, while the saddle (where the surgery is performed) in the multiplication has $s_\Lambda(\gamma) = 1$. Changing to $\bigstar\ \times\ \bigstar$ will leave the circle as it is diagrammatically. But now $\mathrm{p}_\Lambda(0) = 0$, $\mathrm{p}_\Lambda(1) = 1$, $\mathrm{p}_\Lambda(2) = 3$, $\mathrm{d}_\Lambda(\gamma) = 3$ and $s_\Lambda(\gamma) = 2$. $\blacktriangle$

3.2. **The linear structure of the arc algebras.** Fix a block $\Lambda \in \mathtt{bl}$, and consider the set $\mathbb{B}(\Lambda) = \{\underline{\lambda}\nu\overline{\mu} \mid \underline{\lambda}\nu\overline{\mu} \text{ is oriented and } \lambda, \mu, \nu \in \Lambda\}$. We call this set *basis set of oriented circle diagrams*. This set is subdivided into smaller sets of the form ${}_\lambda\mathbb{B}(\Lambda)_\mu$ which are those diagrams in $\mathbb{B}(\Lambda)$ which have $\underline{\lambda}$ as cup part and $\overline{\mu}$ as cap part.

From now on, we restrict to circle diagrams that only contain cups and caps. Formally this is done as follows: for a block $\Lambda \in \mathtt{bl}$ denote by Λ° the set of weights λ such that $\underline{\lambda}$ only contains cups. Note that $\Lambda^\circ \neq \emptyset$ iff Λ is balanced. Define

$$\mathbb{B}^\circ(\Lambda) = \{\underline{\lambda}\nu\overline{\mu} \mid \underline{\lambda}\nu\overline{\mu} \text{ is oriented and } \lambda, \mu \in \Lambda^\circ, \nu \in \Lambda\} = \bigcup_{\lambda,\mu \in \Lambda^\circ} {}_\lambda\mathbb{B}^\circ(\Lambda)_\mu. \tag{27}$$

We equip the elements of $\mathbb{B}(\Lambda)$ and of $\mathbb{B}^\circ(\Lambda)$ with the degree from Definition 3.4.

Example 3.8. Collapsing of the "middle" of each diagram in Subsection 3.3.4 gives typical elements from $\mathbb{B}^\circ(\Lambda)$. $\blacktriangle$

For any ring R let $\langle\cdot\rangle_R$ be the R-linear span. Now, similar to [18, Definition 3.11], we define graded, free Q-modules via

$$\mathfrak{A}[\mathtt{Q}]_\Lambda = \langle\mathbb{B}^\circ(\Lambda)\rangle_Q = \bigoplus_{(\underline{\lambda}\nu\overline{\mu}) \in \mathbb{B}^\circ(\Lambda)} Q(\underline{\lambda}\nu\overline{\mu}), \qquad \mathfrak{A}[\mathtt{Q}] = \bigoplus_{\Lambda \in \mathtt{bl}^\diamond} \mathfrak{A}[\mathtt{Q}]_\Lambda, \tag{28}$$

which we call *signed* 2*-parameter arc algebra for* $\Lambda \in \mathtt{bl}^{\diamond}$ respectively *(full) signed* 2*-parameter arc algebra.* As usual, we also have their specializations.

Denote by ${}_{\lambda}(\mathfrak{A}[\mathtt{Q}]_{\Lambda})_{\mu}$ the Q-linear span of the basis vectors inside ${}_{\lambda}\mathbb{B}^{\circ}(\Lambda)_{\mu}$.

Proposition 3.9. The map $\mathbf{mult}\colon \mathfrak{A}[\mathtt{Q}]_{\Lambda} \otimes \mathfrak{A}[\mathtt{Q}]_{\Lambda} \to \mathfrak{A}[\mathtt{Q}]_{\Lambda}$ given in Subsection 3.3 below endows $\mathfrak{A}[\mathtt{Q}]_{\Lambda}$ with the structure of a graded, unital algebra with pairwise orthogonal, primitive idempotents ${}_{\lambda}\mathbb{1}_{\lambda} = \underline{\lambda}\lambda\overline{\lambda}$ for $\lambda \in \Lambda$ and unit $\mathbb{1} = \sum_{\lambda \in \Lambda} {}_{\lambda}\mathbb{1}_{\lambda}$. (Similarly for the locally unital algebra $\mathfrak{A}[\mathtt{Q}]$ and any specialization of $\mathtt{Q}$.) ■

Proof. As in [18, Proposition 3.12] where we leave it to the reader to incorporate the parameters (which can be done without problems). □

Remark 3.10. Note that so far we do not know whether $\mathfrak{A}[\mathtt{Q}]_{\Lambda}$ is associative. It will follow from the identification of $\mathfrak{A}[\mathtt{Q}]_{\Lambda}$ with $\mathfrak{W}[\mathtt{Q}]_{\vec{k}}$ that **mult** is independent of the chosen order in which the surgeries are performed and that $\mathfrak{A}[\mathtt{Q}]_{\Lambda}$ is associative, see Corollary 4.9. (Similarly for any specialization of $\mathtt{Q}$.) ▲

3.3. **The algebra structure.** We define **mult** in two steps: we first recall the maps used in each step (without any parameters), compare to [18, Subsection 3.3], and afterward go into details about how we modify these maps incorporating $\mathtt{Q}$. The reader who wants to see examples may jump to Subsection 3.3.4.

For $\lambda, \mu, \mu', \eta \in \Lambda^{\circ}$ we define a map

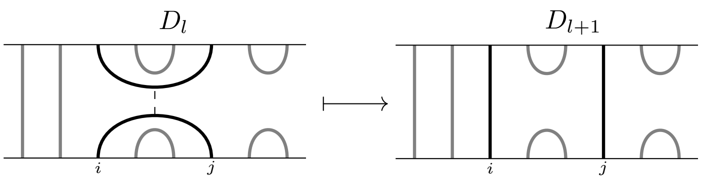
$$\mathbf{mult}_{\lambda,\mu}^{\mu',\eta}\colon {}_{\lambda}(\mathfrak{A}[\mathtt{Q}]_{\Lambda})_{\mu} \otimes {}_{\mu'}(\mathfrak{A}[\mathtt{Q}]_{\Lambda})_{\eta} \to {}_{\lambda}(\mathfrak{A}[\mathtt{Q}]_{\Lambda})_{\eta}$$

as follows. If $\mu \neq \mu'$ we declare the map to be identically zero. Thus, assume that $\mu = \mu'$, and stack the diagram, without orientations, $\underline{\mu}\overline{\eta}$ on top of the diagram $\underline{\lambda}\overline{\mu}$, creating a diagram $D_0 = \underline{\lambda}\overline{\mu}\underline{\mu}\overline{\eta}$. Given such a diagram D_l, starting with $l = 0$, we construct below a new diagram D_{l+1} by choosing a certain symmetric pair of a cup and a cap in the middle section. If r is the number of cups in $\underline{\mu}$, then this can be done a total number of r times. We call this procedure a *surgery* at the corresponding cup-cap pair. For each such step we define below a map $\mathbf{mult}_{D_l,D_{l+1}}$. Observing that the space of orientations of the final diagram D_r is equal to the space of orientations of the diagram $\underline{\lambda}\overline{\eta}$, we define

$$\mathbf{mult}_{\lambda,\mu}^{\mu',\eta} = \mathbf{mult}_{D_{r-1},D_r} \circ \ldots \circ \mathbf{mult}_{D_0,D_1}.$$

Then **mult** is defined as the direct sum of all of these. In order to make **mult** a priori well-defined, we always pick the *leftmost available cup-cap pair* (it will be a non-trivial fact that one could actually pick any pair).

3.3.1. *The surgery procedure.* To obtain D_{l+1} from $D_l = \underline{\lambda}c^{*}c\overline{\eta}$ (for some cup diagram c) choose the symmetric cup-cap pair with the leftmost endpoint in $c^{*}c$ that can be connected without crossing any arcs (this means that the cup and cap are not nested inside any other arcs). Cut open the cup and the cap and stitch the loose ends together to form a pair of vertical line segments, call this diagram D_{l+1}:



3.3.2. *The map without parameters.* The multiplication without parameters will closely resemble the one from [6]. One of the key differences is that we incorporate the parameter $\boldsymbol{\alpha}$ which changes some cases. The map $\mathbf{mult}_{D_{l+1},D_l}$, without any additional coefficients only depends on how the components change when going from D_l to D_{l+1}. The image of an orientation of D_l is constructed as follows in these cases (where we always leave the orientations on non-interacting arcs fixed).

Merge. If two circles, say C_i and C_j, are merged into a circle C proceed as follows.
$\succ$ *If C_i and C_j are oriented anticlockwise*, then orient C anticlockwise.
$\succ$ *If either C_i or C_j is oriented clockwise*, then orient C clockwise.
$\succ$ *If C_i and C_j are oriented clockwise*, then orient C anticlockwise.

Split. If one circle C splits into two circles, say C_i and C_j, proceed as follows.
$\succ$ *If C is oriented anticlockwise*, then take two copies of the diagram D_{l+1}. In one copy orient C_i clockwise and C_j anticlockwise, in the other vice versa.
$\succ$ *If C is oriented clockwise*, then take two copies of the diagram D_{l+1}. In one copy orient both C_i and C_j clockwise, in the other orient C_i and C_j anticlockwise.

3.3.3. *The map with parameters.* In general, the formulas below include signs (recall that $\boldsymbol{\varepsilon} \in \{\pm 1\}$) as well as coefficients coming from the parameters $\boldsymbol{\alpha}$ and $\boldsymbol{\omega}$.

The signs can be divided into the *dot moving signs*, the *topological sign* and the *saddle sign*. The latter two are topological in nature and quite involved. These signs are as follows (explained for each case in detail below).

$$\begin{gathered} \textit{Dot moving}\text{ signs: } \boldsymbol{\varepsilon}^{\mathrm{d}_\Lambda(\gamma_i^{\mathrm{dot}})} \quad\text{and}\quad \boldsymbol{\varepsilon}^{\mathrm{d}_\Lambda(\gamma_i^{\mathrm{ndot}})}. \\ \textit{Topological sign: } \boldsymbol{\varepsilon}^{\frac{1}{4}(\mathrm{d}_\Lambda(C_{\mathrm{in}})-2)}. \qquad \textit{Saddle sign: } \boldsymbol{\varepsilon}^{s_\Lambda(\gamma)}. \end{gathered} \tag{29}$$

The dot moving signs can appear in any situation, the topological sign will appear for nested merges and splits, and the saddle sign for nested merges and non-nested splits. Each case can pick up some extra factors $\boldsymbol{\alpha}$, $\boldsymbol{\varepsilon}$ or $\boldsymbol{\omega}$ as we are going to describe below. We note that one can always produce examples such that two of the three signs from (29) are trivial (that is, there exponents are 0 mod 2), but one is not (its exponent is 1 mod 2). Hence, all of them are needed for the general formula.

We distinguish whether the two circles, that are merged together or appear after a split, are nested inside each other or not. Fix for each circle

$$\mathrm{t}(C) = \text{(a choice of) a rightmost point in the circle } C. \tag{30}$$

Let γ denote the cup in the cup-cap pair we use to perform the surgery procedure in this step connecting vertices $i < j$.

Non-nested Merge. The non-nested circles C_i and C_j (containing vertices i respectively j) are merged into C. The cases from above are modified as follows.
$\succ$ *Both circles oriented anticlockwise.* As in Subsection 3.3.2 (no extra coefficients).
$\succ$ *One circle oriented clockwise, one oriented anticlockwise.* Let C_k (for $k = i$ or $k = j$) be the clockwise oriented circle and let γ_k^{dot} be a sequence of arcs in C connecting $\mathrm{t}(C_k)$ and $\mathrm{t}(C)$ (neither $\mathrm{t}(C_k), \mathrm{t}(C)$ nor γ_k^{dot} are unique, but possible choices differ in distance by 2, making the sign well-defined, see also [17, Lemma 5.7]). Proceed as in Subsection 3.3.2 and multiply by the dot moving sign

$$\boldsymbol{\varepsilon}^{\mathrm{d}_\Lambda(\gamma_k^{\mathrm{dot}})}. \tag{31}$$

$\succ$ *Both circles oriented clockwise.* Let γ_k^{dot} be a sequence of arcs in C connecting $\mathrm{t}(C_k)$ and $\mathrm{t}(C)$ (for both $k = i, j$). Proceed as in Subsection 3.3.2 and multiply by

$$\boldsymbol{\alpha} \cdot \boldsymbol{\varepsilon}^{\mathrm{d}_\Lambda(\gamma_i^{\mathrm{dot}})} \cdot \boldsymbol{\varepsilon}^{\mathrm{d}_\Lambda(\gamma_j^{\mathrm{dot}})}. \tag{32}$$

Nested Merge. The nested circles C_i and C_j (with notation as before) are merged into C. Denote by C_{in} the inner of the two original circles. Then:
$\succ$ *Both circles oriented anticlockwise.* Proceed as above, but multiply by

$$\boldsymbol{\varepsilon} \cdot \boldsymbol{\varepsilon}^{\frac{1}{4}(\mathrm{d}_\Lambda(C_{\text{in}})-2)} \cdot \boldsymbol{\varepsilon}^{s_\Lambda(\gamma)}, \tag{33}$$

where $s_\Lambda(\gamma)$ is the saddle width of the cup where the surgery is performed.
$\succ$ *One circle oriented clockwise, one oriented anticlockwise.* Again perform the surgery procedure as described in Subsection 3.3.2 and multiply by

$$\boldsymbol{\varepsilon} \cdot \boldsymbol{\varepsilon}^{\mathrm{d}_\Lambda(\gamma_k^{\text{dot}})} \cdot \boldsymbol{\varepsilon}^{\frac{1}{4}(\mathrm{d}_\Lambda(C_{\text{in}})-2)} \cdot \boldsymbol{\varepsilon}^{s_\Lambda(\gamma)},$$

where γ_k^{dot} (for $k = i$ or $k = j$) is defined as in (31), and $s_\Lambda(\gamma)$ as in (33).
$\succ$ *Both circles oriented clockwise.* Again perform the surgery procedure as described in Subsection 3.3.2 and multiply by

$$\boldsymbol{\alpha} \cdot \boldsymbol{\varepsilon} \cdot \boldsymbol{\varepsilon}^{\mathrm{d}_\Lambda(\gamma_i^{\text{dot}})} \cdot \boldsymbol{\varepsilon}^{\mathrm{d}_\Lambda(\gamma_j^{\text{dot}})} \cdot \boldsymbol{\varepsilon}^{\frac{1}{4}(\mathrm{d}_\Lambda(C_{\text{in}})-2)} \cdot \boldsymbol{\varepsilon}^{s_\Lambda(\gamma)},$$

where γ_i^{dot} and γ_j^{dot} are defined as in (32), and $s_\Lambda(\gamma)$ as in (33).

Non-nested Split. The circle C splits into the non-nested circles C_i and C_j (containing vertices i respectively j).
$\succ$ *C oriented anticlockwise.* Use the map as in Subsection 3.3.2, but the copy where C_i is oriented clockwise is multiplied with

$$\boldsymbol{\omega} \cdot \boldsymbol{\varepsilon}^{\mathrm{d}_\Lambda(\gamma_i^{\text{ndot}})} \cdot \boldsymbol{\varepsilon}^{s_\Lambda(\gamma)}, \tag{34}$$

while the one where C_j is oriented clockwise is multiplied with

$$\boldsymbol{\varepsilon} \cdot \boldsymbol{\omega} \cdot \boldsymbol{\varepsilon}^{\mathrm{d}_\Lambda(\gamma_j^{\text{ndot}})} \cdot \boldsymbol{\varepsilon}^{s_\Lambda(\gamma)}. \tag{35}$$

Here γ_i^{ndot} and γ_j^{ndot} are sequences of arcs connecting $(i, 0)$ and $\mathrm{t}(C_i)$ inside C_i respectively $(j, 0)$ and $\mathrm{t}(C_j)$ in C_j, and $s_\Lambda(\gamma)$ being again as in (33).
$\succ$ *C oriented clockwise.* Multiply the copy with both circles oriented clockwise by

$$\boldsymbol{\omega} \cdot \boldsymbol{\varepsilon}^{\mathrm{d}_\Lambda(\gamma_j^{\text{dot}})} \cdot \boldsymbol{\varepsilon}^{\mathrm{d}_\Lambda(\gamma_i^{\text{ndot}})} \cdot \boldsymbol{\varepsilon}^{s_\Lambda(\gamma)}$$

and the copy with both circles oriented anticlockwise by

$$\boldsymbol{\alpha} \cdot \boldsymbol{\varepsilon} \cdot \boldsymbol{\omega} \cdot \boldsymbol{\varepsilon}^{\mathrm{d}_\Lambda(\gamma_j^{\text{dot}})} \cdot \boldsymbol{\varepsilon}^{\mathrm{d}_\Lambda(\gamma_j^{\text{ndot}})} \cdot \boldsymbol{\varepsilon}^{s_\Lambda(\gamma)}.$$

Here γ_j^{dot} is a sequence of arcs connecting $\mathrm{t}(C)$ and $\mathrm{t}(C_j)$ in C and γ_i^{ndot} and γ_j^{ndot} are as before in (34) and (35). Moreover, $s_\Lambda(\gamma)$ is again as in (33).

Nested Split. We use here the same notations as in the non-nested split case, and we denote by C_{in} and C_{out} the inner and outer of the two circles C_i and C_j.
$\succ$ *C oriented anticlockwise.* We use the map as defined in Subsection 3.3.2, but the copy where C_{in} is oriented clockwise is multiplied with

$$\boldsymbol{\omega} \cdot \boldsymbol{\varepsilon}^{\frac{1}{4}(\mathrm{d}_\Lambda(C_{\text{in}})-2)},$$

while the copy where C_{out} is oriented clockwise is multiplied with

$$\boldsymbol{\varepsilon} \cdot \boldsymbol{\omega} \cdot \boldsymbol{\varepsilon}^{\frac{1}{4}(\mathrm{d}_\Lambda(C_{\text{in}})-2)}.$$

$\succ$ *C oriented clockwise.* Multiply the copy with both circles oriented clockwise by

$$\boldsymbol{\omega} \cdot \boldsymbol{\varepsilon}^{\frac{1}{4}(\mathrm{d}_\Lambda(C_{\text{in}})-2)},$$

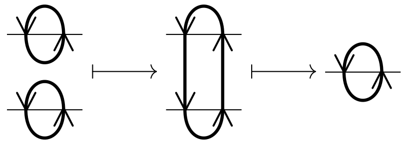
and the one with both circles oriented anticlockwise by

$$\boldsymbol{\alpha} \cdot \boldsymbol{\varepsilon} \cdot \boldsymbol{\omega} \cdot \boldsymbol{\varepsilon}^{\frac{1}{4}(\mathrm{d}_\Lambda(C_{\text{in}})-2)}.$$

3.3.4. *Examples for the surgery procedure.* We give below examples for some of the shapes that can occur during the surgery procedure and determine the coefficients. In all examples assume that outside of the shown strip all entries are $\circ$.

Example 3.11. In a simple, non-nested merge we have no coefficients at all:

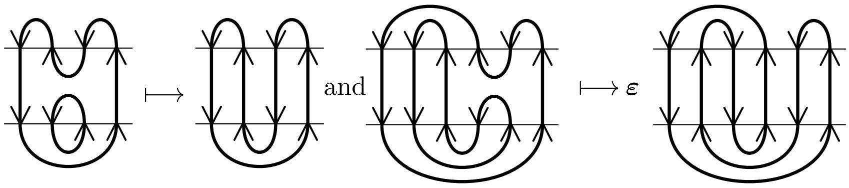
(36)



The rightmost step above, called *collapsing*, is always performed at the end of a multiplication procedure and is omitted in what follows.

Secondly, we consider a merge of two anticlockwise, nested circles:

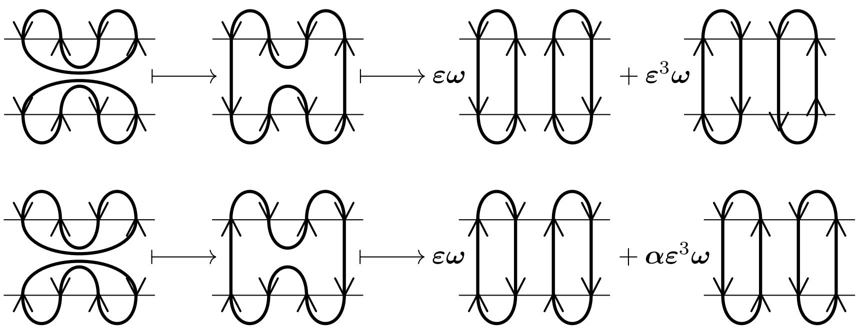
(37)



Here we have $s_\Lambda(\gamma) = 1$, but $\frac{1}{4}(\mathrm{d}_\Lambda(C_{\mathrm{in}}) - 2) = 0$ for the left multiplication step and $\frac{1}{4}(\mathrm{d}_\Lambda(C_{\mathrm{in}}) - 2) = 1$ for the right multiplication step. ▲

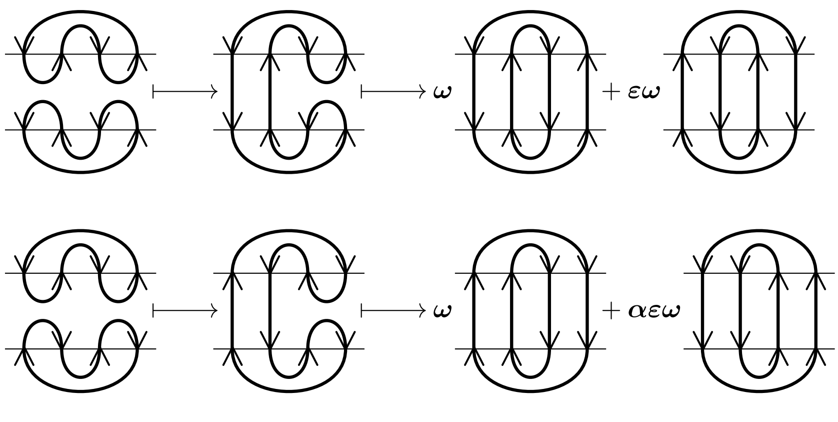
Example 3.12. In both examples given here a non-nested merge is performed, followed by a split into two non-nested respectively nested circles. First, the H-shape:

(38)



Here we have $s_\Lambda(\gamma) = 1$, but $\mathrm{d}_\Lambda(\gamma_i^{\mathrm{ndot}}) = 0$ and $\mathrm{d}_\Lambda(\gamma_j^{\mathrm{ndot}}) = 1$ for $i = 1$ and $j = 2$. Moreover, $\mathrm{d}_\Lambda(\gamma_j^{\mathrm{dot}}) = 0$ in the bottom case. Next, the C shape.

(39)

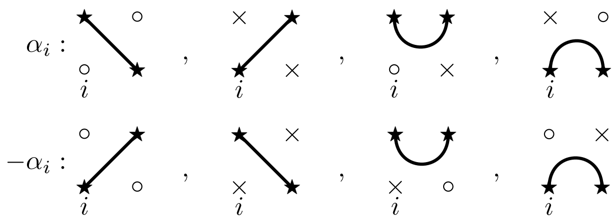


Here $\frac{1}{4}(\mathrm{d}_\Lambda(C_{\mathrm{in}}) - 2) = 0$, and again $\mathrm{d}_\Lambda(\gamma_j^{\mathrm{dot}}) = 0$ for the bottom. ▲

Remark 3.13. The Ↄ shape cannot appear as long as we impose the choice of the order of cup-cap pairs from left to right in the surgery procedure. ▲

Specializations 3.14. If we specialize $\mathtt{q}(\boldsymbol{\alpha}) = 0$, $\mathtt{q}(\boldsymbol{\varepsilon}) = 1$ and $\mathtt{q}(\boldsymbol{\omega}) = 1$, then we obtain the multiplication rules of the algebra from [6]. Specializing $\mathtt{q}(\boldsymbol{\alpha}) = 0$, $\mathtt{q}(\boldsymbol{\varepsilon}) = -1$ and $\mathtt{q}(\boldsymbol{\omega}) = 1$ gives the multiplication rule for the algebra from [18]. ▲

3.4. **Arc bimodules.** Very similar to [7, Section 3] and [18, Subsection 3.4], we define graded $\mathfrak{A}[\mathtt{Q}]$-bimodules by introducing additional diagrams moving from one block Λ to another block Γ. That is, fix two blocks $\Lambda, \Gamma \in \mathtt{bl}^{\diamond}$ such that $\mathrm{seq}(\Lambda)$ and $\mathrm{seq}(\Gamma)$ coincide except at positions i and $i+1$. Following [7], a (Λ, Γ)-*admissible matching (of type* $\pm\alpha_i$*)* is a diagram t inside $\mathbb{R} \times [0,1]$ consisting of vertical lines connecting $(k, 0)$ with $(k, 1)$ if we have that $\mathrm{seq}(\Lambda)_k = \mathrm{seq}(\Gamma)_k = \bigstar$ and, depending on the sign of α_i, an arc at positions i and $i+1$ of the form

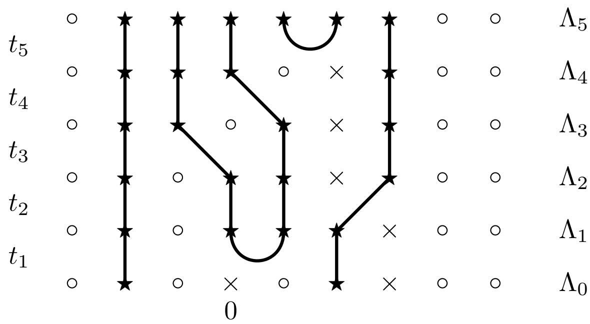
(40) 

where we view $\mathrm{seq}(\Lambda)$ as decorating the integral points of $\mathbb{R} \times \{0\}$ and $\mathrm{seq}(\Gamma)$ as decorating the integral points of $\mathbb{R} \times \{1\}$. Again, the first two moves in each row are called *rays*, the third ones *cups* and the last ones *caps*. Note that for the first arc in each row it holds $\mathrm{d}_{\Lambda}(\gamma) = 0$, while for the second it holds $\mathrm{d}_{\Lambda}(\gamma) = 2$.

For t a (Λ, Γ)-admissible matching, $\lambda \in \Lambda$, and $\mu \in \Gamma$ we say that $\lambda t \mu$ is *oriented* if its cups respectively caps connect one $\wedge$ and one $\vee$ in μ respectively λ, and rays connect the same symbols in λ and μ. For a sequence of blocks $\vec{\Lambda} = (\Lambda_0, \ldots, \Lambda_r)$ a $\vec{\Lambda}$-*admissible composite matching* is a sequence of diagrams $\vec{\mathrm{t}} = (t_1, \ldots, t_r)$ such that t_k is a $(\Lambda_{k-1}, \Lambda_k)$-admissible matching of some type. We view the sequence of matchings as being stacked on top of each other. A sequence of weights $\lambda_i \in \Lambda_i$ such that $\lambda_{k-1} t_k \lambda_k$ is oriented for all k is an *orientation* of the $\vec{\Lambda}$-admissible composite matching $\vec{\mathrm{t}}$. For short, we tend to drop the word admissible, since the only matchings we consider are admissible.

We stress that $\vec{\Lambda}$-composite matching can contain lines and “floating” circles.

Example 3.15. Below is a $(\Lambda_0, \Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4, \Lambda_5)$-composite matching (we assume that outside of the indicated areas all symbols are equal to $\circ$).



The types of the matchings are $-\alpha_0, \alpha_2, \alpha_{-1}, \alpha_0, \alpha_1$ (read from bottom to top). ▲

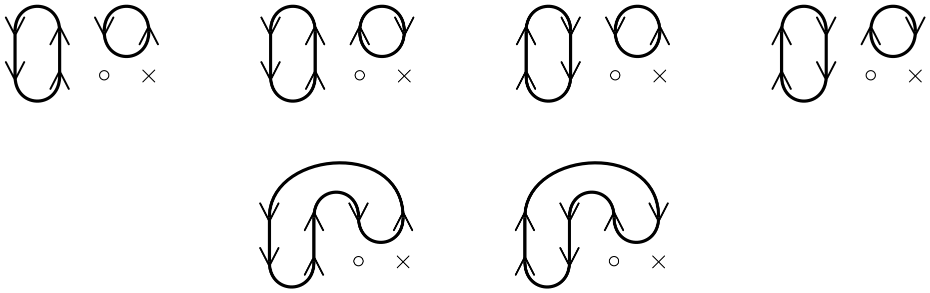
We now want to consider bimodules between arc algebras for different blocks, or said differently, bimodules for the algebra $\mathfrak{A}[\mathtt{Q}]$.

To a $\vec{\Lambda}$-composite matching $\vec{\mathrm{t}}$ we again associate a set of diagrams from which to create a graded, free Q-module (with degree as in Definition 3.4)

$$\mathbb{B}^{\circ}(\vec{\Lambda},\vec{\mathrm{t}}) = \left\{ \underline{\lambda}(\vec{\mathrm{t}},\vec{\nu})\overline{\mu} \;\middle|\; \begin{array}{l} \lambda \in \Lambda_0^{\circ},\, \mu \in \Lambda_r^{\circ},\, \vec{\nu} = (\nu_0,\dots,\nu_r) \text{ with } \nu_i \in \Lambda_i, \\ \underline{\lambda}\nu_0 \text{ oriented, } \nu_r\overline{\mu} \text{ oriented,} \\ \nu_{i-1}t_i\nu_i \text{ oriented for all } 1 \leq i \leq r. \end{array} \right\} \tag{41}$$

As before we obtain the set $\mathbb{B}^{\circ}(\vec{\Lambda},\vec{\mathrm{t}})$ by allowing $\lambda \in \Lambda_0$ and $\mu \in \Lambda_r$ in (41).

Example 3.16. Let Λ be the block with block sequence $\star\ \star\ \circ\ \times$, and Γ the block with sequence $\star\ \star\ \star\ \star$ (both with $\circ$ everywhere else). Assume both blocks are balanced. Then an example for a (Λ,Γ)-matching of type α_2 is the third diagram in the first row of (40) denoted here by t_1. Taking this as our composite matching we obtain a graded, free Q-module of rank 6 with basis consisting of



These are of degrees 0, 2, 2 and 4 (first row) respectively 2 and 4 (second row). ▲

Definition 3.17. Let $\vec{\mathrm{t}}$ be a $\vec{\Lambda}$-composite matching for $\vec{\Lambda} = (\Lambda_0,\dots,\Lambda_r)$. Set

$$\mathcal{A}[\mathtt{Q}](\vec{\Lambda},\vec{\mathrm{t}}) = \left\langle \mathbb{B}^{\circ}(\vec{\Lambda},\vec{\mathrm{t}}) \right\rangle_Q \{-(\mathrm{up}(\Lambda_k) + \mathrm{down}(\Lambda_k))\}, k \in \{0,\cdots,r\}$$

as a graded, free Q-module, using $\mathrm{up}(\Lambda_k)$ and $\mathrm{down}(\Lambda_k)$ from Definition 3.2 (neither $\mathrm{up}(\Lambda_k)$ nor $\mathrm{down}(\Lambda_k)$ depend on k). We call all such $\mathcal{A}[\mathtt{Q}](\vec{\Lambda},\vec{\mathrm{t}})$ *arc bimodules.*

The left (bottom) action of a basis element $\underline{\lambda}\nu\overline{\mu} \in \mathfrak{A}[\mathtt{Q}]_{\Lambda_0}$ on a basis element of the form $\underline{\mu'}(\vec{\mathrm{t}},\vec{\nu})\overline{\eta}$ is given similar as for the algebra itself. As before we obtain zero, if $\mu \neq \mu'$, and otherwise we perform the same surgeries as before. The only difference is that local moves from $\times\ \star$ to $\star\ \times$ and vice versa contribute length 2 to $\mathrm{d}_{\Lambda}(C)$ if they are contained in the circle C while those between $\times\ \circ$ and $\circ\ \times$ do not. The right (top) action is defined in complete analogy. ▲

It is not clear that the above actions are well-defined and commute and we need the translation between $\mathfrak{W}[\mathtt{Q}]$ and $\mathfrak{A}[\mathtt{Q}]$ from Subsection 4.1 to prove it.

Proposition 3.18. Let $\vec{\mathrm{t}}$ be a $\vec{\Lambda}$-composite matching with $\vec{\Lambda} = (\Lambda_0,\dots,\Lambda_r)$. Then the left action of $\mathfrak{A}[\mathtt{Q}]_{\Lambda_0}$ as well as the right action of $\mathfrak{A}[\mathtt{Q}]_{\Lambda_r}$ on $\mathcal{A}[\mathtt{Q}](\vec{\Lambda},\vec{\mathrm{t}})$ are well-defined and commute. Hence, $\mathcal{A}[\mathtt{Q}](\vec{\Lambda},\vec{\mathrm{t}})$ is a $\mathfrak{A}[\mathtt{Q}]_{\Lambda_0}$-$\mathfrak{A}[\mathtt{Q}]_{\Lambda_r}$-bimodule and thus, a $\mathfrak{A}[\mathtt{Q}]$-bimodule. (Similarly for any specialization of $\mathtt{Q}$.) ■

Proof. We identify $\mathfrak{A}[\mathtt{Q}]_{\Lambda}$ with $\mathfrak{W}[\mathtt{Q}]_{\vec{k}}$ via Theorem 4.7. Then we identify $\mathcal{A}[\mathtt{Q}](\vec{\Lambda},\vec{\mathrm{t}})$ with $\mathcal{W}[\mathtt{Q}](\mathrm{w}(\vec{\Lambda},\vec{\mathrm{t}}))$ via Lemma 4.5. The latter isomorphism intertwines the actions of the two algebras on the bimodules by construction and hence, proves the claim. □

Proposition 3.19. All $\mathcal{A}[\mathtt{Q}](\vec{\Lambda},\vec{\mathrm{t}})$ are graded biprojective, $\mathfrak{A}[\mathtt{Q}]$-bimodules which are free Q-modules of finite rank. (Similarly for any specialization of $\mathtt{Q}$.) ■

Proof. We show that they are projective as left $\mathfrak{A}[\mathtt{Q}]$-modules, the "right" case follows similarly. Denote by ${}_{\lambda\downarrow}\mathbb{1}_{\lambda\downarrow}$ the idempotent obtained from $\underline{\lambda}(\vec{\mathrm{t}},\vec{\nu})\overline{\mu}$ via downwards reduction (see [18, Subsection 3.4]). Then $\mathcal{A}[\mathtt{Q}](\vec{\Lambda},\vec{\mathrm{t}}) \cong \bigoplus_{\lambda\in\Lambda^{\circ}} \mathfrak{A}[\mathtt{Q}]\cdot{}_{\lambda\downarrow}\mathbb{1}_{\lambda\downarrow}$ and hence, it is projective. The other statements are clear and the claim follows. □

This proposition motivates the definition of the following 2-category.

Definition 3.20. Let $\mathfrak{A}[\mathtt{Q}]$-$\mathbf{biMod}^{p}_{\mathrm{gr}}$ be the following 2-category:

- Objects are the various $\Lambda \in \mathtt{bl}^{\diamond}$.
- 1-morphisms are finite direct sums and tensor products (taken over the algebra $\mathfrak{A}[\mathtt{Q}]$) of the $\mathfrak{A}[\mathtt{Q}]$-bimodules $\mathcal{A}[\mathtt{Q}](\vec{\Lambda},\vec{\mathrm{t}})$.
- 2-morphisms are $\mathfrak{A}[\mathtt{Q}]$-bimodule homomorphisms.
- The composition of arc bimodules is the tensor product $\cdot\otimes_{\mathfrak{A}[\mathtt{Q}]}\cdot$. The vertical composition of $\mathfrak{A}[\mathtt{Q}]$-bimodule homomorphisms is the usual composition. The horizontal composition is given by tensoring (over $\mathfrak{A}[\mathtt{Q}]$).

We consider $\mathfrak{A}[\mathtt{Q}]$-$\mathbf{biMod}^{p}_{\mathrm{gr}}$ as a graded 2-category by turning the 2-hom-spaces into graded Q-modules (in the sense of (1)) via Definition 3.4. ▲

As usual, we also consider specializations of $\mathfrak{A}[\mathtt{Q}]$-$\mathbf{biMod}^{p}_{\mathrm{gr}}$.

4. Isomorphisms, equivalences and their consequences

This section has two main goals. First, we will construct an isomorphism of graded algebras $\Phi\colon \mathfrak{W}[\mathtt{Q}]^{\circ} \xrightarrow{\cong} \mathfrak{A}[\mathtt{Q}]$ (where $\mathfrak{W}[\mathtt{Q}]^{\circ}$ is a certain subalgebra of $\mathfrak{W}[\mathtt{Q}]$ defined in (47)). This isomorphism works for any specialization of $\mathtt{Q}$ as well and provides an algebraic model of $\mathfrak{W}[\mathtt{Q}]$. Form this we obtain (with $\mathrm{w}(\cdot)$ as in (45)):

Theorem 4.1. There is an equivalence of graded, Q-linear 2-categories

$$\boldsymbol{\Phi}\colon \mathfrak{W}[\mathtt{Q}]\text{-}\mathbf{biMod}^{p}_{\mathrm{gr}} \xrightarrow{\cong} \mathfrak{A}[\mathtt{Q}]\text{-}\mathbf{biMod}^{p}_{\mathrm{gr}} \tag{42}$$

induced by Φ under which the web bimodules $\mathcal{W}[\mathtt{Q}](\mathrm{w}(\vec{\Lambda},\vec{\mathrm{t}}))$ and the arc bimodules $\mathcal{A}[\mathtt{Q}](\vec{\Lambda},\vec{\mathrm{t}})$ are identified. (Similarly for any specialization of $\mathtt{Q}$.) ■

Second, let $R[\boldsymbol{\alpha}]$ be a graded ring with $\deg_R(\alpha) = 4$ (and everything else in degree 0). Let $\mathtt{q}\colon Q \to R[\boldsymbol{\alpha}]$ be any ring homomorphism with $\mathtt{q}(\boldsymbol{\alpha}) = \boldsymbol{\alpha}$. Set

$$\mathfrak{A}[\boldsymbol{\alpha},\mathtt{q}(\boldsymbol{\varepsilon}),\mathtt{q}(\boldsymbol{\omega})] = \mathfrak{A}_{R[\boldsymbol{\alpha}]}[\boldsymbol{\alpha},\mathtt{q}(\boldsymbol{\varepsilon}),\mathtt{q}(\boldsymbol{\omega})]\otimes_{\mathbb{Z}} Q$$

(we need these scalar extensions for technical reasons, e.g. to make statements as "isomorphisms of Q-algebras", and we omit the ring in the subscript for these). We show (where we explicitly construct the isomorphism from (43) in Subsection 4.2):

Theorem 4.2. There is an isomorphism

$$\Psi\colon \mathfrak{A}[\mathtt{Q}] \xrightarrow{\cong} \mathfrak{A}[\boldsymbol{\alpha},\mathtt{q}(\boldsymbol{\varepsilon}),\mathtt{q}(\boldsymbol{\omega})] \tag{43}$$

of graded Q-algebras. (Similarly for any further simultaneous specialization of $\boldsymbol{\alpha}$.) ■

From this we obtain:

Theorem 4.3. Let $R[\boldsymbol{\alpha}]$, $\mathtt{q}$ and $\mathfrak{A}[\boldsymbol{\alpha},\mathtt{q}(\boldsymbol{\varepsilon}),\mathtt{q}(\boldsymbol{\omega})]$ be as above. There is an equivalence (which is, in fact, even an isomorphism) of graded, Q-linear 2-categories

$$\boldsymbol{\Psi}\colon \mathfrak{A}[\mathtt{Q}]\text{-}\mathbf{biMod}^{p}_{\mathrm{gr}} \xrightarrow{\cong} \mathfrak{A}[\boldsymbol{\alpha},\mathtt{q}(\boldsymbol{\varepsilon}),\mathtt{q}(\boldsymbol{\omega})]\text{-}\mathbf{biMod}^{p}_{\mathrm{gr}}$$

induced by Ψ under which $\mathcal{A}[\mathtt{Q}](\vec{\Lambda},\vec{\mathrm{t}})$ and $\mathcal{A}[\boldsymbol{\alpha},\mathtt{q}(\boldsymbol{\varepsilon}),\mathtt{q}(\boldsymbol{\omega})](\vec{\Lambda},\vec{\mathrm{t}})$ are identified. (Similarly for any further simultaneous specialization of $\boldsymbol{\alpha}$.) ■

Taking Proposition 2.29, the equivalences (42) and Theorem 4.3 together (and working over $\mathbb{Z}[\boldsymbol{\alpha},i]$), we obtain that $\mathfrak{F}_{\mathbb{Z}[\boldsymbol{\alpha},i]}[\mathbf{KBN}]$, $\mathfrak{F}_{\mathbb{Z}[\boldsymbol{\alpha},i]}[\mathbf{Ca}]$, $\mathfrak{F}_{\mathbb{Z}[\boldsymbol{\alpha},i]}[\mathbf{CMW}]$ and $\mathfrak{F}_{\mathbb{Z}[\boldsymbol{\alpha},i]}[\mathbf{Bl}]$ are all equivalent, see Corollary 4.37.

4.1. **Web and arc algebras.** We start by constructing a graded algebra isomorphism $\Phi\colon \mathfrak{W}[\mathtt{Q}]^{\circ} \to \mathfrak{A}[\mathtt{Q}]$. For this purpose, recall that there is a bijection

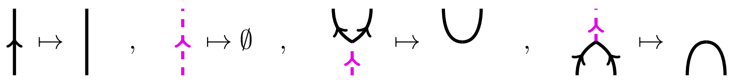
$$\mathbb{bl}^{\diamond} \to \mathtt{bl}^{\diamond}, \quad \vec{k} \mapsto \Lambda, \quad \text{given by } \; 0 \mapsto \circ, \quad 1 \mapsto \bigstar, \quad 2 \mapsto \times. \tag{44}$$

Here $\circ, \bigstar, \times$ are entries of $\mathrm{seq}(\Lambda)$ and Λ is determined demanding that Λ is balanced. We identify, using (44), such $\vec{k}$'s and Λ's in what follows. Moreover, recall that for $\Lambda \in \mathtt{bl}^{\diamond}$ and $\lambda \in \Lambda$, there is a unique web $\mathrm{w}(\lambda)$ associated to the cup diagram $\underline{\lambda}$, see [18, Lemma 4.8]. That is, there is a map

$$\mathrm{w}(\cdot)\colon \Lambda \to \mathrm{CUP}(\vec{k}), \quad \lambda \mapsto \mathrm{w}(\lambda) \tag{45}$$

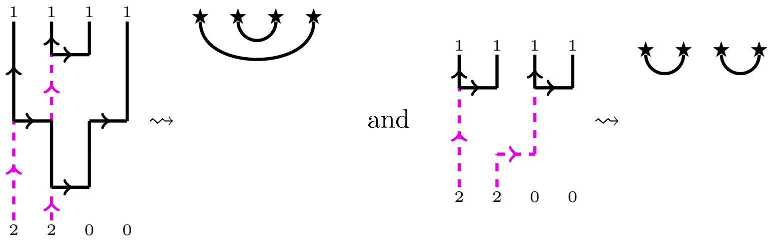
constructed from the cup diagram $\underline{\lambda}$. Similarly, for each $\vec{\Lambda}$-composite matching $\vec{\mathrm{t}}$ there is a unique associated web $\mathrm{w}(\vec{\Lambda}, \vec{\mathrm{t}})$ (given by an analog map). The images of these maps are called *basis webs*. All the reader needs to know about these basis webs is summarized in Example 4.4 below. Details can be found in [18, Subsection 4.1].

Example 4.4. Given a web u, then we can associate to it an arc diagram $\mathrm{a}(u)$ via



We do not consider any relations on the set of webs. Hence, isotopic webs are not equal and there are plenty of webs giving the same arc diagram, but there is a preferred choice of a preimage which defines a split of the map $u \mapsto \mathrm{a}(u)$ and gives the map $\mathrm{w}(\cdot)$. An example (where we use a "rectangular" presentation of webs) is

(46)



How this choice of preimage can be made precise is not important in what follows. That is, we only need the fact that there is a preferred choice. The only thing we additionally note is that this association is parameter independent. ▲

Moreover, as indicated in Example 4.4, the set of basis webs

$$\mathrm{Cup}(\vec{k}) = \{u \in \mathrm{CUP}(\vec{k}) \mid u = \mathrm{w}(\lambda) \text{ for some } \lambda \in \Lambda\}$$

is always a strict subset of $\mathrm{CUP}(\vec{k})$. Now, given $\lambda, \mu \in \Lambda$, let us denote

$$\mathfrak{W}[\mathtt{Q}]^{\circ}_{\vec{k}} = \bigoplus_{u,v \in \mathrm{Cup}(\vec{k})} {}_{u}(\mathfrak{W}[\mathtt{Q}]_{\vec{k}})_{v}, \qquad \mathfrak{W}[\mathtt{Q}]^{\circ} = \bigoplus_{\vec{k} \in \mathbb{bl}^{\diamond}} \mathfrak{W}[\mathtt{Q}]_{\vec{k}}. \tag{47}$$

Clearly, $\mathfrak{W}[\mathtt{Q}]^{\circ}$ is a graded subalgebra of $\mathfrak{W}[\mathtt{Q}]$.

Recalling the cup foam bases from Definitions 2.24 and 2.37 and the bases from (27) and (41), we have the following lemmas.

Lemma 4.5. Let $u, v \in \mathrm{Cup}(\vec{k})$ be webs such that $u = \mathrm{w}(\lambda)$ and $v = \mathrm{w}(\mu)$. There is an isomorphism of graded, free P-modules

$$\Phi^{\lambda\mu}_{uv}\colon {}_{u}(\mathfrak{W}[\mathtt{Q}]_{\vec{k}})_{v} \to {}_{\lambda}(\mathfrak{A}[\mathtt{Q}]_{\Lambda})_{\mu} \tag{48}$$

which sends ${}_{u}\mathbb{B}^{\circ}(\vec{k})_{v}$ to ${}_{\lambda}\mathbb{B}^{\circ}(\Lambda)_{\mu}$ by identifying the basis cup foams without dots with anticlockwise circles and the basis cup foams with dots with clockwise circles.

Let $u \in \mathrm{Hom}_{\mathfrak{F}[\mathtt{Q}]}(\vec{k}, \vec{l})$ be a web such that $u = \mathrm{w}(\vec{\Lambda}, \vec{\mathrm{t}})$. There is an isomorphism of graded, free Q-modules

$$\Phi_u^{(\vec{\Lambda},\vec{\mathrm{t}})} \colon \mathcal{W}[\mathtt{Q}](u) \to \mathcal{A}[\mathtt{Q}](\vec{\Lambda}, \vec{\mathrm{t}}) \tag{49}$$

which sends $\mathbb{B}^{\circ}(u)$ to $\mathbb{B}^{\circ}(\vec{\Lambda}, \vec{\mathrm{t}})$ by identifying the basis cup foams without dots with anticlockwise circles and the basis cup foams with dots with clockwise circles.

(For both statements: similarly for any specialization of $\mathtt{Q}$.) ■

Proof. We note that the arguments used in [18, Lemmas 4.15 and 4.16] as well as the construction of the two bases in question are parameter independent. Thus, we can adapt [18, Lemmas 4.15 and 4.16] without difficulties and the claim follows. □

Lemma 4.6. For any $\lambda, \mu \in \Lambda$ and $u = \mathrm{w}(\lambda), v = \mathrm{w}(\mu)$: the isomorphisms $\Phi_{uv}^{\lambda\mu}$ from (48) extend to isomorphisms of graded, free Q-modules

$$\Phi_{\vec{k}}^{\Lambda} \colon \mathfrak{W}[\mathtt{Q}]_{\vec{k}}^{\circ} \to \mathfrak{A}[\mathtt{Q}]_{\Lambda}, \qquad \Phi \colon \mathfrak{W}[\mathtt{Q}]^{\circ} \to \mathfrak{A}[\mathtt{Q}]. \tag{50}$$

(Similarly for any specialization of $\mathtt{Q}$.) ■

Proof. Clear by Lemma 4.5. □

Theorem 4.7. The maps from (50) are isomorphisms of graded algebras. (Similarly for any specialization of $\mathtt{Q}$.) ■

The non-trivial and lengthy proof of Theorem 4.7 is given in Section 6.

Remark 4.8. We use the specialization of the parameters $\mathtt{P}$ to $\mathtt{Q}$ from (3) to not having to worry about the difference between the “directions” in which we squeeze, migrate dots or perform ordinary-to-phantom neck cutting. Being more careful with the performed steps in the topological rewriting process leads to an analogue of Theorem 4.7 for $\mathtt{P}$ as well. Since this would require the introduction of some involved (but straightforward) notions for the diagram combinatorics keeping track of directions, we have decided, for brevity and clearness, to only do the $\mathtt{Q}$ case here (which includes our main list of examples anyway). ▲

Corollary 4.9. The multiplication rule from Subsection 3.3 is independent of the order in which the surgeries are performed. This turns $\mathfrak{A}[\mathtt{Q}]_{\Lambda}$ into a graded, associative, unital algebra. (Similarly for the locally unital algebra $\mathfrak{A}[\mathtt{Q}]$ and for any specialization of $\mathtt{Q}$.) ■

Proof. The claimed properties are clear for the web algebras $\mathfrak{W}[\mathtt{Q}]_{\vec{k}}^{\circ}$ and $\mathfrak{W}[\mathtt{Q}]^{\circ}$, see Lemma 2.32. Thus, using Theorem 4.7 provides the claim. □

We are now ready to prove our first main result, i.e. the equivalence from (42).

Proof of Theorem 4.1. The algebras $\mathfrak{W}[\mathtt{Q}]_{\vec{k}}$ and $\mathfrak{W}[\mathtt{Q}]_{\vec{k}}^{\circ}$ are graded Morita equivalent (this can be seen as in [18, Proof of Theorem 4.1]) and the statement follows from Theorem 4.7: the identification of the bimodules as graded, free Q-modules is clear by Lemma 4.5, while the actions agree by Theorem 4.7 and the construction of the actions. Everything in these arguments is independent of the parameters and thus, the theorem follows. □

4.2. **Arc algebras: isomorphisms.** In this subsection we show that the signed 2-parameter arc algebra $\mathfrak{A}[\mathtt{Q}]$ and the (scalar extended) **KBN** specialization

$$\mathfrak{A}[\mathbf{KBN}] = \mathfrak{A}_{\mathbb{Z}[\boldsymbol{\alpha}]}[\mathbf{KBN}] \otimes_{\mathbb{Z}} \mathbb{Z}[\boldsymbol{\omega}^{\pm 1}]$$

are isomorphic as graded algebras. Here, as usual, $\deg_{\mathbb{Z}[\mathtt{Q}]}(\alpha) = 4$ (and everything else in degree 0). As we explain, this gives rise to the isomorphisms from (43), which in turn enables us to prove Theorem 4.3.

Both, $\mathfrak{A}[\mathbf{KBN}]_\Lambda$ and $\mathfrak{A}[\mathtt{Q}]_\Lambda$, are isomorphic as graded, free Q-modules to $\langle \mathbb{B}^\circ(\Lambda)\rangle_Q$, with $\mathbb{B}^\circ(\Lambda)$ being as in (27). By definition, the multiplication differs only by the appearing coefficients in the result. Hence, we will give the isomorphism from $\mathfrak{A}[\mathbf{KBN}]_\Lambda$ to $\mathfrak{A}[\mathtt{Q}]_\Lambda$ by defining a coefficient for each of the diagrams appearing in the multiplication and show that the maps intertwine the two multiplication rules.

Definition 4.10. We call any diagram appearing in an intermediate step of the multiplication procedure from Subsection 3.3 a *stacked diagram*. We denote such diagrams throughout this subsection by D (possibly with decorations and indices), and choices of orientations of it by D^{or} (possibly with decorations and indices). ▲

Definition 4.11. We define sets of arcs inside a circle C in a fixed diagram D:

- $\cup_{\mathrm{ex}}(C)$ denotes all cups in C such that the exterior of C is above the cup.
- $\cap_{\mathrm{ex}}(C)$ denotes all caps in C such that the exterior of C is below the cap.
- $\cup_{\mathrm{in}}(C)$ denotes all cups in C such that the interior of C is above the cup.
- $\cap_{\mathrm{in}}(C)$ denotes all caps in C such that the interior of C is below the cap.

The exterior and interior is meant here with respect to the circle C only (ignoring all possible other components of D). ▲

Example 4.12. The outer circle C_{out} in the third diagram in (37) has $\cup_{\mathrm{ex}}(C_{\mathrm{out}})=1$, $\cap_{\mathrm{ex}}(C_{\mathrm{out}})=0$, $\cup_{\mathrm{in}}(C_{\mathrm{out}})=1$ and $\cap_{\mathrm{in}}(C_{\mathrm{out}})=2$. The inner circle C_{in} in the same stacked diagram has exactly the same numbers. The circle C in the rightmost diagram in (37) has $\cup_{\mathrm{ex}}(C)=1$, $\cap_{\mathrm{ex}}(C)=1$, $\cup_{\mathrm{in}}(C)=2$ and $\cap_{\mathrm{in}}(C)=2$. Moreover,

$$(\cup_{\mathrm{ex}}(C_{\mathrm{out}})\cup\cap_{\mathrm{ex}}(C_{\mathrm{out}})\cup\cup_{\mathrm{in}}(C_{\mathrm{in}})\cup\cap_{\mathrm{in}}(C_{\mathrm{in}}))\setminus\mathrm{surg}=\cup_{\mathrm{ex}}(C)\cup\cap_{\mathrm{ex}}(C).$$

Here "surg" means the set containing the cup-cap involved the in surgery. ▲

We denote by $\mathbb{B}(D)$ the set of all possible orientations of a given D.

Definition 4.13. For a fixed D, we define its Q-linear *coefficient map* via:

$$\begin{aligned}\mathrm{coeff}_D\colon\ \langle\mathbb{B}(D)\rangle_Q &\longrightarrow \langle\mathbb{B}(D)\rangle_Q\,,\\ D^{\mathrm{or}} &\longmapsto \left(\prod_{\mathrm{circles}}\mathrm{coeff}_{\boldsymbol{\varepsilon}}(C,D^{\mathrm{or}})\cdot\mathrm{coeff}_{\boldsymbol{\omega}}(C,D^{\mathrm{or}})\right)D^{\mathrm{or}}.\end{aligned}$$

Here the product runs over all circles in D, and the involved terms (i.e. for each such circle C) are defined as follows.

≻ If C is oriented anticlockwise (when looking at the orientation D^{or}), then set

$$\begin{aligned}\mathrm{coeff}_{\boldsymbol{\varepsilon}}(C,D^{\mathrm{or}}) &= \prod_{\gamma\in\cup_{\mathrm{ex}}(C)}\boldsymbol{\varepsilon}^{(s_\Lambda(\gamma)+1)\mathrm{p}_\Lambda(\gamma)}\cdot\prod_{\gamma\in\cap_{\mathrm{ex}}(C)}\boldsymbol{\varepsilon}^{s_\Lambda(\gamma)(\mathrm{p}_\Lambda(\gamma)+1)},\\ \mathrm{coeff}_{\boldsymbol{\omega}}(C,D^{\mathrm{or}}) &= \prod_{\gamma\in\cup_{\mathrm{ex}}(C)}\boldsymbol{\omega}^{-s_\Lambda(\gamma)}\cdot\prod_{\gamma\in\cap_{\mathrm{ex}}(C)}\boldsymbol{\omega}^{s_\Lambda(\gamma)-1},\end{aligned}$$

where, as usual, the γ's denote the corresponding cups and caps, $\mathrm{p}_\Lambda(\gamma)$ denotes the position of their leftmost points and $s_\Lambda(\gamma)$ is the saddle width as in Definition 3.6.

≻ If C is oriented clockwise (when looking at the orientation D^{or}), then we use the same coefficient and additionally multiply by $\boldsymbol{\varepsilon}^{\mathrm{t}(C)}$ (recalling $\mathrm{t}(C)$ from (30) – the reader might think of $\boldsymbol{\varepsilon}^{\mathrm{t}(C)}$ as keeping track of "dot moving" again). ▲

Since $\boldsymbol{\varepsilon}=\pm1$, its powers matter only mod 2.

Example 4.14. The circle C_{out} in the third diagram D_3 in (37) has only one cup γ "pushing inwards" with $s_\Lambda(\gamma)=1$ and $\mathrm{p}_\Lambda(\gamma)=3$. Thus, if D_3^{or} denotes the orientation from (37), then $\mathrm{coeff}(C_{\mathrm{out}},D_3^{\mathrm{or}})=\boldsymbol{\varepsilon}^6\boldsymbol{\omega}^{-1}=\boldsymbol{\omega}^{-1}$. Similarly one obtains

$\mathrm{coeff}(C_{\mathrm{in}}, D_3^{\mathrm{or}}) = \boldsymbol{\varepsilon}^4\boldsymbol{\omega}^{-1} = \boldsymbol{\omega}^{-1}$. Moreover, the circle C in the rightmost diagram D_4 in (37) has one cup γ and one cap γ' with $s_\Lambda(\gamma) = 2$, $\mathrm{p}_\Lambda(\gamma) = 1$, $s_\Lambda(\gamma') = 1$ and $\mathrm{p}_\Lambda(\gamma') = 1$ "pushing inwards". Thus, $\mathrm{coeff}(C, D_4^{\mathrm{or}}) = \boldsymbol{\varepsilon}^3\boldsymbol{\omega}^{-2} \cdot \boldsymbol{\varepsilon}^2 = \boldsymbol{\varepsilon}\boldsymbol{\omega}^{-2}$. ▲

We will usually write $\mathrm{coeff}(C^{\mathrm{anti}}) = \mathrm{coeff}(C, D^{\mathrm{or}})$ etc. to denote the coefficient for the circle C when the orientation is chosen such that C is oriented anticlockwise, and similarly $\mathrm{coeff}(C^{\mathrm{cl}}) = \mathrm{coeff}(C, D^{\mathrm{or}})$ when it is chosen such that C is oriented clockwise. For example, we have by definition

$$\mathrm{coeff}(C^{\mathrm{cl}}) = \mathrm{coeff}(C^{\mathrm{anti}}) \cdot \boldsymbol{\varepsilon}^{\mathrm{t}(C)}. \tag{51}$$

Definition 4.13 restricts to a homogeneous, Q-linear map

$$\mathrm{coeff}_{\lambda,\mu} \colon \langle {}_\lambda\mathbb{B}^\circ(\Lambda)_\mu \rangle_Q \to \langle {}_\lambda\mathbb{B}^\circ(\Lambda)_\mu \rangle_Q\,,$$

for $\lambda, \mu \in \Lambda$. By summing all of these up we obtain a homogeneous, Q-linear map

$$\mathrm{coeff}_\Lambda \colon \mathfrak{A}[\mathbf{KBN}]_\Lambda \to \mathfrak{A}[\mathtt{Q}]_\Lambda \tag{52}$$

by using $\mathfrak{A}[\mathbf{KBN}]_\Lambda \cong \langle \mathbb{B}^\circ(\Lambda) \rangle_Q \cong \mathfrak{A}[\mathtt{Q}]_\Lambda$ (as graded, free Q-modules).

In fact, the Q-linear map from (52) is actually an isomorphism of graded algebras:

Proposition 4.15. The maps from (52) are isomorphisms of graded, Q-algebras for all $\Lambda \in \mathtt{bl}^\diamond$. These can be extended to an isomorphism of graded Q-algebras

$$\mathrm{coeff} \colon \mathfrak{A}[\mathbf{KBN}] \xrightarrow{\cong} \mathfrak{A}[\mathtt{Q}]. \qquad \blacksquare$$

Again, the proof of Proposition 4.15 is rather lengthy and is given in Section 6. The main point hereby (as we explain in detail in the proof) is to show that

$$\mathrm{coeff}_{D_l}(D_l^{\mathrm{or}}) \cdot \mathrm{coeff}(\mathtt{Q}) = \mathrm{coeff}_{D_{l+1}}(\tilde{D}_{l+1}^{\mathrm{or}}). \tag{53}$$

Here $\mathrm{coeff}_{D_l}(D_l^{\mathrm{or}})$ and $\mathrm{coeff}_{D_{l+1}}(\tilde{D}_{l+1}^{\mathrm{or}})$ denotes the coefficients of the stacked diagrams before and after the l-th step in the multiplication procedure, and $\mathrm{coeff}(\mathtt{Q})$ denotes the coefficients (for $\mathfrak{A}[\mathtt{Q}]$) coming from this step. We give an example (which serves as a road map) illustrating the reasoning.

Example 4.16. In Example 4.14 we have already calculated $\mathrm{coeff}(C_{\mathrm{out}}) = \boldsymbol{\omega}^{-1}$, $\mathrm{coeff}(C_{\mathrm{in}}) = \boldsymbol{\omega}^{-1}$ and $\mathrm{coeff}(C) = \boldsymbol{\varepsilon}\boldsymbol{\omega}^{-2}$ for the three circles appearing in the diagram on the right-hand side of (37). Moreover, $\mathrm{coeff}(\mathtt{Q}) = \boldsymbol{\varepsilon}$. Thus, (53) holds. ▲

Given the setup as in the beginning of this section, we define the map Ψ from (43) as follows. Let $\mathrm{coeff}^{\boldsymbol{\alpha},\mathtt{q}(\boldsymbol{\varepsilon}),\mathtt{q}(\boldsymbol{\omega})} \colon \mathfrak{A}[\mathbf{KBN}] \to \mathfrak{A}[\boldsymbol{\alpha}, \mathtt{q}(\boldsymbol{\varepsilon}), \mathtt{q}(\boldsymbol{\omega})]$ be the homogeneous, Q-linear map obtained in the same way as $\mathrm{coeff} \colon \mathfrak{A}[\mathbf{KBN}] \to \mathfrak{A}[\mathtt{Q}]$, but using the specialized parameters $\mathtt{q}(\boldsymbol{\varepsilon})$ and $\mathtt{q}(\boldsymbol{\omega})$. Then, by Proposition 4.15, set

$$\Psi \colon \mathfrak{A}[\mathtt{Q}] \to \mathfrak{A}[\boldsymbol{\alpha}, \mathtt{q}(\boldsymbol{\varepsilon}), \mathtt{q}(\boldsymbol{\omega})],\ \Psi = \mathrm{coeff}^{\boldsymbol{\alpha},\mathtt{q}(\boldsymbol{\varepsilon}),\mathtt{q}(\boldsymbol{\omega})} \circ (\mathrm{coeff})^{-1}. \tag{54}$$

We are now ready to prove Theorem 4.2 (assuming Proposition 4.15).

Proof of Theorem 4.2. The proof of Proposition 4.15 only uses that $\boldsymbol{\varepsilon} = \pm 1$ and that $\boldsymbol{\omega}$ is invertible. Thus, the same arguments work for any $\mathtt{q}(\boldsymbol{\varepsilon})$ and $\mathtt{q}(\boldsymbol{\omega})$ providing a homogeneous isomorphism $\mathrm{coeff}^{\boldsymbol{\alpha},\mathtt{q}(\boldsymbol{\varepsilon}),\mathtt{q}(\boldsymbol{\omega})}$ between $\mathfrak{A}[\mathbf{KBN}]$ and $\mathfrak{A}[\boldsymbol{\alpha}, \mathtt{q}(\boldsymbol{\varepsilon}), \mathtt{q}(\boldsymbol{\omega})]$. (Similarly for any further simultaneous specialization of $\boldsymbol{\alpha}$.) □

4.3. **Arc bimodules: bimodule homomorphisms.** In the last subsection we have identified $\mathfrak{A}[\mathbf{KBN}]$ with $\mathfrak{A}[\mathtt{Q}]$ using the coefficient map. Thus, there is also an identification of their bimodules. The aim of this subsection is to make this explicit. For the identification of the bimodules $\mathcal{A}[\mathbf{KBN}](\vec{\Lambda}, \vec{\mathrm{t}})$ and $\mathcal{A}[\mathtt{Q}](\vec{\Lambda}, \vec{\mathrm{t}})$ for a fixed admissible matching $(\vec{\Lambda}, \vec{\mathrm{t}})$ we need to introduce some additional notations and slightly modify the coefficient map. But otherwise the identification works as for the algebras.

Definition 4.17. As in Definition 4.10, we call any diagram appearing in the intermediate step of the multiplication procedure from Subsection 3.4 a *stacked diagram* (using similar notations). Furthermore, fixing a circle C in such a stacked diagram, we define subsets of arcs containing the arcs in the basic moves of the second type for α_i and $-\alpha_i$, i.e. local moves from $\star\ \times$ to $\times\ \star$, or from $\times\ \star$ to $\star\ \times$, see (40). We divide these depending on the exterior or interior of C:

- ${}_{\mathrm{ex}}\!\diagdown^{\!\times}(C)$ denotes the arcs in local moves from $\times\ \star$ to $\star\ \times$, where the exterior of the circle is to the lower left of the arc.
- ${}^{\times}\!\diagup_{\!\mathrm{ex}}(C)$ denotes the arcs in local moves from $\star\ \times$ to $\times\ \star$, where the exterior of the circle is to the lower right of the arc.
- ${}_{\mathrm{in}}\!\diagdown^{\!\times}(C)$ denotes the arcs in local moves from $\times\ \star$ to $\star\ \times$, where the interior of the circle is to the lower left of the arc.
- ${}^{\times}\!\diagup_{\!\mathrm{in}}(C)$ denotes the arcs in local moves from $\star\ \times$ to $\times\ \star$, where the interior of the circle is to the lower right of the arc.

Again, the exterior and interior is meant here with respect to the circle C only. ▲

Definition 4.18. For a fixed D, we define its Q-linear *coefficient map* via:

$$\begin{aligned}\mathrm{coeff}_D\colon \langle\mathbb{B}(D)\rangle_Q &\longrightarrow \langle\mathbb{B}(D)\rangle_Q,\\ D^{\mathrm{or}} &\longmapsto \Big(\prod_{\text{circles}} \mathrm{coeff}_{\boldsymbol{\varepsilon}}(C,D^{\mathrm{or}})\cdot \mathrm{coeff}_{\boldsymbol{\omega}}(C,D^{\mathrm{or}})\Big)D^{\mathrm{or}}.\end{aligned}$$

Here the product runs over all circles in D, and the involved terms (i.e. for each such circle C) are defined as follows.

$\succ$ If C is oriented anticlockwise (when looking at the orientation D^{or}), then set

$$\begin{aligned}\mathrm{coeff}_{\boldsymbol{\varepsilon}}(C,D^{\mathrm{or}}) &= \prod_{\gamma\in\cup^{\mathrm{ex}}(C)} \boldsymbol{\varepsilon}^{(s_\Lambda(\gamma)+1)\mathrm{p}_\Lambda(\gamma)}\cdot \prod_{\gamma\in\cap_{\mathrm{ex}}(C)} \boldsymbol{\varepsilon}^{s_\Lambda(\gamma)(\mathrm{p}_\Lambda(\gamma)+1)}\\ &\quad\cdot \prod_{\gamma\in{}_{\mathrm{ex}}\diagdown^{\times}(C)} \boldsymbol{\varepsilon}^{\mathrm{p}_\Lambda(\gamma)}\cdot \prod_{\gamma\in{}^{\times}\diagup_{\mathrm{ex}}(C)} \boldsymbol{\varepsilon}^{\mathrm{p}_\Lambda(\gamma)+1},\\ \mathrm{coeff}_{\boldsymbol{\omega}}(C,D^{\mathrm{or}}) &= \prod_{\gamma\in\cup^{\mathrm{ex}}(C)} \boldsymbol{\omega}^{-s_\Lambda(\gamma)} \prod_{\gamma\in\cap_{\mathrm{ex}}(C)} \boldsymbol{\omega}^{s_\Lambda(\gamma)-1}\cdot \boldsymbol{\omega}^{\#\left({}_{\mathrm{ex}}\diagdown^{\times}(C)\,\cup\,{}^{\times}\diagup_{\mathrm{ex}}(C)\right)},\end{aligned}\tag{55}$$

where we use the same notations as in Definition 4.13.

$\succ$ If C is oriented clockwise (when looking at the orientation D^{or}), then we use the same coefficient and additionally multiply by $\boldsymbol{\varepsilon}^{\mathrm{t}(C)}$. ▲

Similar to (52), we use these maps to define a homogeneous, Q-linear map

$$\mathrm{coeff}_{\vec{\Lambda},\vec{\mathrm{t}}}\colon \mathcal{A}[\mathbf{KBN}](\vec{\Lambda},\vec{\mathrm{t}})\to\mathcal{A}[\mathtt{Q}](\vec{\Lambda},\vec{\mathrm{t}}).\tag{56}$$

Proposition 4.19. The map

$$\mathrm{coeff}_{\vec{\Lambda},\vec{\mathrm{t}}}\colon \mathcal{A}[\mathbf{KBN}](\vec{\Lambda},\vec{\mathrm{t}})\to\mathcal{A}[\mathtt{Q}](\vec{\Lambda},\vec{\mathrm{t}})$$

is an isomorphism of graded, free Q-modules that intertwines the actions of $\mathfrak{A}[\mathbf{KBN}]$ and $\mathfrak{A}[\mathtt{Q}]$, i.e. for any $x\in\mathfrak{A}[\mathbf{KBN}]$ and any $m\in\mathcal{A}[\mathbf{KBN}](\vec{\Lambda},\vec{\mathrm{t}})$ it holds (similarly for the right action)

$$\mathrm{coeff}_{\vec{\Lambda},\vec{\mathrm{t}}}(x\cdot m)=\mathrm{coeff}(x)\cdot\mathrm{coeff}_{\vec{\Lambda},\vec{\mathrm{t}}}(m).\qquad\blacksquare$$

Again, the proof of this proposition appears in Section 6

We assume the setup from the beginning of this section. As before in (54), we use Proposition 4.19 to define $\mathrm{coeff}^{\boldsymbol{\alpha},\mathtt{q}(\boldsymbol{\varepsilon}),\mathtt{q}(\boldsymbol{\omega})}_{\vec{\Lambda},\vec{\mathrm{t}}}\colon \mathcal{A}[\mathtt{Q}](\vec{\Lambda},\vec{\mathrm{t}})\to\mathcal{A}[\boldsymbol{\alpha},\mathtt{q}(\boldsymbol{\varepsilon}),\mathtt{q}(\boldsymbol{\omega})](\vec{\Lambda},\vec{\mathrm{t}})$ to be the homogeneous, Q-linear map obtained in the same way as $\mathrm{coeff}_{\vec{\Lambda},\vec{\mathtt{t}}}$ from (56), but using the specialized parameters $\mathtt{q}(\boldsymbol{\varepsilon})$ and $\mathtt{q}(\boldsymbol{\omega})$ instead of $\boldsymbol{\varepsilon}$ and $\boldsymbol{\omega}$. Then set

$$\begin{aligned}&\mathrm{coeff}_{\Psi}\colon \mathcal{A}[\mathtt{Q}](\vec{\Lambda},\vec{\mathtt{t}})\to\mathcal{A}[\boldsymbol{\alpha},\mathtt{q}(\boldsymbol{\varepsilon}),\mathtt{q}(\boldsymbol{\omega})](\vec{\Lambda},\vec{\mathtt{t}}),\\ &\mathrm{coeff}_{\Psi}=\mathrm{coeff}_{\vec{\Lambda},\vec{\mathtt{t}}}^{\boldsymbol{\alpha},\mathtt{q}(\boldsymbol{\varepsilon}),\mathtt{q}(\boldsymbol{\omega})}\circ(\mathrm{coeff}_{\vec{\Lambda},\vec{\mathtt{t}}})^{-1}.\end{aligned}\tag{57}$$

Corollary 4.20. The map coeff_{Ψ} is an isomorphism of graded, free Q-modules that intertwines the actions of $\mathfrak{A}[\mathtt{Q}]$ and $\mathfrak{A}[\boldsymbol{\alpha},\mathtt{q}(\boldsymbol{\varepsilon}),\mathtt{q}(\boldsymbol{\omega})]$. ■

Proof. As in the proof Theorem 4.2, but using Proposition 4.19. □

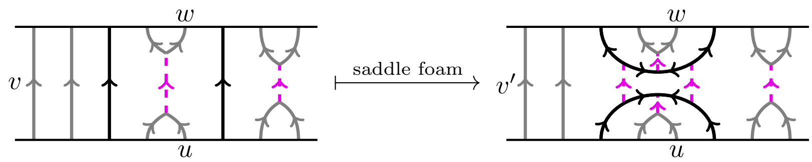
We obtain Theorem 4.3:

Proof of Theorem 4.3. This follows from Theorem 4.2 and Corollary 4.20. □

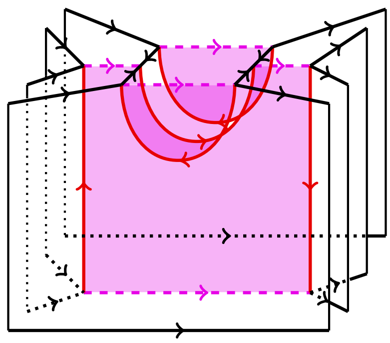
4.4. **Arc bimodules: co-structure.** The aim of this subsection is to describe a *co-structure* topologically on web bimodules $\mathcal{W}[\mathtt{Q}](v)$ and algebraically on arc bimodules $\mathcal{A}[\mathtt{Q}](\vec{\Lambda},\vec{\mathtt{t}})$. Then we match theses structures (which again comes with sophisticated scalars) for different specializations of $\mathtt{Q}$ using an isomorphism similar, but not equal, to the coefficient map from (56).

We start on the side of $\mathfrak{W}[\mathtt{Q}]$ (and we note that the whole definition works of course more general for $\mathtt{P}$). We, as usually, only consider balanced $\vec{k},\vec{l}\in\mathbb{bl}^{\diamond}$.

Definition 4.21. Let $v\in\mathrm{Hom}_{\mathfrak{F}[\mathtt{Q}]}(\vec{k},\vec{l})$. Recalling that we consider in $\mathfrak{F}[\mathtt{Q}]$ webs without relations, we can *pick* any pair of neighboring vertical usual edges (ignoring possible phantom edges) as below and perform a "reverse surgery" on $\mathcal{W}[\mathtt{Q}](v)$:



Here the saddle foam is locally of the form as in (24), but "read from top to bottom":



(and the identity elsewhere). One ends up with a new web $v'\in\mathrm{Hom}_{\mathfrak{F}[\mathtt{Q}]}(\vec{k},\vec{l})$. This should be read as follows: start with $f\in 2\mathrm{Hom}_{\mathfrak{F}[\mathtt{P}]}(\mathbf{1}_{2\omega_{\ell}},uv^{*}vw)$ and stack on top of it a foam which is the identity at the bottom (u part) and top (w part) of the web, and the saddle in between. Repeat this for all $u\in\mathrm{CUP}(\vec{k}),w\in\mathrm{CUP}(\vec{l})$. ▲

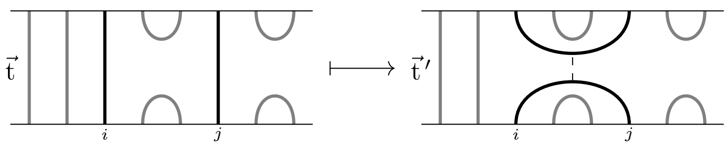
Note that we make a certain choice where to perform the reverse surgery. But fixing v' determines this choice. Thus, we can write $\mathbf{rMult}_{v}^{v'}$ etc. without ambiguity.

Lemma 4.22. The procedure from Definition 4.21 defines a $\mathfrak{W}[\mathtt{Q}]$-bimodule homomorphism $\mathbf{rMult}_{v}^{v'}\colon\mathcal{W}[\mathtt{Q}](v)\to\mathcal{W}[\mathtt{Q}](v')$. ■

Proof. Clear by construction. □

We define the same on the side of $\mathfrak{A}[\mathtt{Q}]$. As usual, all blocks are balanced.

Definition 4.23. Let $\vec{\mathrm{t}}$ be a $\vec{\Lambda}$-composite matching. Recalling that we construct these using the basic moves from (40), we can pick any pair of neighboring vertical arcs (ignoring possible symbols $\circ$ or $\bigstar$ in between) as below and perform a "reverse surgery" on $\mathcal{A}[\mathtt{Q}](\vec{\Lambda},\vec{\mathrm{t}})$ giving us a new composite matching $\vec{\mathrm{t}}'$ for $\vec{\Lambda}'$:



Define a Q-linear map $\mathbf{rmult}_{\vec{\mathrm{t}}}^{\vec{\mathrm{t}}'}\colon \mathcal{A}[\mathtt{Q}](\vec{\Lambda},\vec{\mathrm{t}}) \to \mathcal{A}[\mathtt{Q}](\vec{\Lambda}',\vec{\mathrm{t}}')$ on basis elements of $\mathcal{A}[\mathtt{Q}](\vec{\Lambda},\vec{\mathrm{t}})$ precisely as in Subsection 3.3 to be the identity on circles not involved in the reversed surgery, and with the following differences for involved circles:

Non-nested Merge. The non-nested circles C_i and C_j are merged into C. We use the same conventions and spread the same scalars as in Subsection 3.3.3.

Nested Merge. The nested circles C_i and C_j are merged into C. We use the same conventions and scalars as in Subsection 3.3.3, but additionally multiply with $\boldsymbol{\varepsilon}$.

Non-nested Split. The circle C splits into the non-nested circles C_{bot} and C_{top} (being at the bottom or top of the picture). We use the same conventions and spread almost the same scalars as in Subsection 3.3.3, but in case C is oriented anticlockwise, we use (for bottom respectively top circle oriented clockwise)

$$\boldsymbol{\omega}\cdot\boldsymbol{\varepsilon}^{\mathrm{d}_\Lambda(\gamma_{\mathrm{bot}}^{\mathrm{ndot}})}\cdot\boldsymbol{\varepsilon}^{s_\Lambda(\gamma)} \quad \text{respectively} \quad \boldsymbol{\varepsilon}\cdot\boldsymbol{\omega}\cdot\boldsymbol{\varepsilon}^{\mathrm{d}_\Lambda(\gamma_{\mathrm{top}}^{\mathrm{ndot}})}\cdot\boldsymbol{\varepsilon}^{s_\Lambda(\gamma)}.$$

Here $\gamma_{\mathrm{bot}}^{\mathrm{ndot}}$ respectively $\gamma_{\mathrm{top}}^{\mathrm{ndot}}$ are to be understood similar to (34) and (35). In case C is oriented clockwise, we use

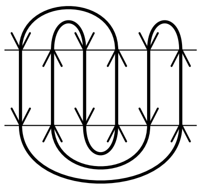
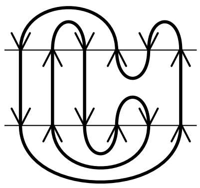
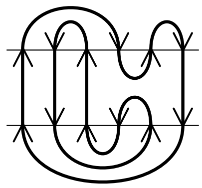
$$\boldsymbol{\omega}\cdot\boldsymbol{\varepsilon}^{\mathrm{d}_\Lambda(\gamma_{\mathrm{top}}^{\mathrm{dot}})}\cdot\boldsymbol{\varepsilon}^{\mathrm{d}_\Lambda(\gamma_{\mathrm{bot}}^{\mathrm{ndot}})}\cdot\boldsymbol{\varepsilon}^{s_\Lambda(\gamma)} \text{ respectively } \boldsymbol{\alpha}\cdot\boldsymbol{\varepsilon}\cdot\boldsymbol{\omega}\cdot\boldsymbol{\varepsilon}^{\mathrm{d}_\Lambda(\gamma_{\mathrm{top}}^{\mathrm{dot}})}\cdot\boldsymbol{\varepsilon}^{\mathrm{d}_\Lambda(\gamma_{\mathrm{top}}^{\mathrm{ndot}})}\cdot\boldsymbol{\varepsilon}^{s_\Lambda(\gamma)}$$

for both circles oriented clockwise respectively anticlockwise.

Nested Split. The circle C splits into the non-nested circles C_{in} and C_{out}. We use the same conventions and spread the same scalars as in Subsection 3.3.3. ▲

Remark 4.24. We note that the web algebra is in fact a (symmetric) Frobenius algebra (this can be seen by copying [24, Proposition 30] or [34, Theorem 3.9]). The same holds for the arc algebra (this can be seen by copying [6, Theorem 6.3]). Thus, both come with co-multiplications. The reverse surgeries from above can be used to give rise to these co-multiplications. We skip the details, since we do not use this co-multiplication in this paper. We only point out that our results of this subsection match the various co-multiplications on web or arc algebras for different parameters (similar, but "co", as in Subsection 4.2), but not the Frobenius structures since the isomorphisms in the present subsection are different from the ones in Subsection 4.2. ▲

Example 4.25. An illustration of the reverse multiplication is

 $\longmapsto \omega\varepsilon^1$  $+\ \varepsilon\omega\varepsilon^1$ 

▲

Lemma 4.26. The procedure from Definition 4.23 defines an $\mathfrak{A}[\mathtt{Q}]$-bimodule homomorphism $\mathbf{rmult}\colon \mathcal{A}[\mathtt{Q}](\vec{\Lambda},\vec{\mathrm{t}}) \to \mathcal{A}[\mathtt{Q}](\vec{\Lambda}',\vec{\mathrm{t}}')$. ■

Proof. As usual, this is not clear and needs the translation to the side of the web algebras from Proposition 4.27 below. □

As before, we have the following (recalling the equivalence $\mathbf{\Phi}$ from Theorem 4.1 induced by the isomorphism Φ from (50) under which the web bimodules $\mathcal{W}[\mathtt{Q}](\mathrm{w}(\vec{\Lambda},\vec{\mathrm{t}}))$ and the arc bimodules $\mathfrak{A}[\mathtt{Q}](\vec{\Lambda},\vec{\mathrm{t}})$ are identified).

Proposition 4.27. Fixing the Q-linear isomorphisms $\Phi^{\cdot}_{\cdot}$ from (49):

$$\Phi^{(\vec{\Lambda}',\vec{\mathrm{t}}')}_{\mathrm{w}(\vec{\Lambda}',\vec{\mathrm{t}}')} \circ \mathbf{rMult}^{\mathrm{w}(\vec{\Lambda}',\vec{\mathrm{t}}')}_{\mathrm{w}(\vec{\Lambda},\vec{\mathrm{t}})} = \mathbf{rmult}^{\vec{\mathrm{t}}'}_{\vec{\mathrm{t}}} \circ \Phi^{(\vec{\Lambda},\vec{\mathrm{t}})}_{\mathrm{w}(\vec{\Lambda},\vec{\mathrm{t}})}.$$

(Similarly for any specialization of $\mathtt{Q}$.) ■

Proof. Very similar to the proof of Theorem 4.7. Indeed, we can use the same argumentation as given there (noting that the shifting basic moves as in the first two columns of (40) can be incorporated without difficulties), but we turn the corresponding pictures by $\frac{\pi}{2}$ (which gives the slight changes for the scalars in the algebraic setting). We skip the calculations for brevity. □

We now aim to match the bimodule maps for different specializations of $\mathtt{Q}$ as in Subsections 4.2 and 4.3. For this purpose, we define a coefficient map which is again slightly modified. In particular, we use the same notations as in Definition 4.17.

Definition 4.28. For fixed D, we define its *reverse coefficient map* $\overline{\mathrm{coeff}}$ as

$$\begin{aligned}
\overline{\mathrm{coeff}}_{\boldsymbol{\varepsilon}}(C,D^{\mathrm{or}}) \;=\; & \prod_{\gamma\in \mathrm{in}_{\cup}(C)} \boldsymbol{\varepsilon}^{s_\Lambda(\gamma)(\mathrm{p}_\Lambda(\gamma)+1)} \cdot \prod_{\gamma\in \mathrm{in}_{\cap}(C)} \boldsymbol{\varepsilon}^{(s_\Lambda(\gamma)+1)\mathrm{p}_\Lambda(\gamma)} \\
& \cdot \prod_{\gamma\in \mathrm{in}_{\times\searrow}(C)} \boldsymbol{\varepsilon}^{\mathrm{p}_\Lambda(\gamma)+1} \cdot \prod_{\gamma\in \mathrm{in}_{\times\nearrow}(C)} \boldsymbol{\varepsilon}^{\mathrm{p}_\Lambda(\gamma)}, \\
\overline{\mathrm{coeff}}_{\boldsymbol{\omega}}(C,D^{\mathrm{or}}) \;=\; & \prod_{\gamma\in \mathrm{in}_{\cup}(C)} \boldsymbol{\omega}^{-s_\Lambda(\gamma)+1} \cdot \prod_{\gamma\in \mathrm{in}_{\cap}(C)} \boldsymbol{\omega}^{s_\Lambda(\gamma)} \cdot \boldsymbol{\omega}^{\#\left(\mathrm{in}_{\times\searrow}(C)\,\cup\,\mathrm{in}_{\times\nearrow}(C)\right)}
\end{aligned}$$

instead of (55), and a further factor of $\boldsymbol{\varepsilon}^{\mathrm{t}(C)}$ for the clockwise circle. ▲

For the following proposition we use the evident notation to distinguish the reverse multiplication maps from Definition 4.21 for different choices of specializations.

Proposition 4.29. The homogeneous, Q-linear map (defined as in (56), but using $\overline{\mathrm{coeff}}$ instead of coeff)

$$\overline{\mathrm{coeff}}_{\vec{\Lambda},\vec{\mathrm{t}}}\colon \mathcal{A}[\mathbf{KBN}](\vec{\Lambda},\vec{\mathrm{t}}) \to \mathcal{A}[\mathtt{Q}](\vec{\Lambda},\vec{\mathrm{t}})$$

is an isomorphism of graded, free Q-modules such that the following commutes:

$$\begin{tikzcd}
\mathcal{A}[\mathbf{KBN}](\vec{\Lambda},\vec{\mathtt{t}}) \arrow[r, "{\mathbf{rmult}[\mathbf{KBN}]_{\vec{\mathtt{t}}}^{\vec{\mathtt{t}}'}}"] \arrow[d, "\overline{\mathrm{coeff}}_{\vec{\Lambda},\vec{\mathtt{t}}}"'] & \mathcal{A}[\mathbf{KBN}](\vec{\Lambda}',\vec{\mathtt{t}}') \arrow[d, "\overline{\mathrm{coeff}}_{\vec{\Lambda}',\vec{\mathtt{t}}'}"] \\
\mathcal{A}[\mathtt{Q}](\vec{\Lambda},\vec{\mathtt{t}}) \arrow[r, "{\mathbf{rmult}[\mathtt{Q}]_{\vec{\Lambda}}^{\vec{\mathtt{t}}'}}"'] & \mathcal{A}[\mathtt{Q}](\vec{\Lambda}',\vec{\mathtt{t}}').
\end{tikzcd} \tag{58}$$
■

Proof. We omit the details of this proof. It can be proven similar (but "co") to the proof that $\mathrm{coeff}_{\vec{\Lambda},\vec{\mathtt{t}}}\colon \mathcal{A}[\mathbf{KBN}](\vec{\Lambda},\vec{\mathtt{t}}) \to \mathcal{A}[\mathtt{Q}](\vec{\Lambda},\vec{\mathtt{t}})$ from Proposition 4.19 intertwines the actions of $\mathfrak{A}[\mathbf{KBN}]$ and $\mathfrak{A}[\mathtt{Q}]$ (again checking the cases (i)-(iv) as in the proof of Theorem 4.7) with the following differences: the non-nested cases work analogously, while in the nested cases one needs to successively apply Lemma 6.6 as in the presented nested merge case in the proof of Proposition 4.19. □

Similar to (57), but using Proposition 4.29 and the corresponding maps, we define

$$\overline{\mathrm{coeff}}_{\Psi} : \mathcal{A}[\mathtt{Q}](\vec{\Lambda},\vec{\mathtt{t}}) \to \mathcal{A}[\mathbf{KBN}](\vec{\Lambda},\vec{\mathtt{t}}) \to \mathcal{A}[\boldsymbol{\alpha},\mathtt{q}(\boldsymbol{\varepsilon}),\mathtt{q}(\boldsymbol{\omega})](\vec{\Lambda},\vec{\mathtt{t}}).$$

The following is now clear because the proof of Proposition 4.29 does not use the specific form of the parameters in question.

Corollary 4.30. The map $\overline{\mathrm{coeff}}_{\Psi}$ is an isomorphism of graded, free Q-modules such that the corresponding diagram in (58) commutes. (Similarly for any further simultaneous specialization of $\boldsymbol{\alpha}$.) □

Example 4.31. Denote the diagrams in Example 4.25 from left to right by D_1, D_2 and D_3. Then

$$\overline{\mathrm{coeff}}_{D_1}(D_1^{\mathrm{or}}) = \boldsymbol{\varepsilon}\cdot\boldsymbol{\omega}\cdot D_1^{\mathrm{or}}, \qquad \overline{\mathrm{coeff}}_{D_2}(D_2^{\mathrm{or}}) = 1\cdot D_2^{\mathrm{or}}, \qquad \overline{\mathrm{coeff}}_{D_3}(D_3^{\mathrm{or}}) = \boldsymbol{\varepsilon}\cdot D_3^{\mathrm{or}}.$$

Thus, we see that (58) commutes in this example. ▲

Remark 4.32. By Lemma 6.7 (which we state later), $\overline{\mathrm{coeff}}_D(D^{\mathrm{or}})$ can be expressed in terms of $\mathrm{coeff}_D(D^{\mathrm{or}})$ times a constant that can either be determined by counting cups or by counting caps as well as shifts. Hence, it is evident that for $x \in \mathfrak{A}[\mathbf{KBN}]$ and any $m \in \mathcal{A}[\mathbf{KBN}](\vec{\Lambda},\vec{\mathtt{t}})$ it holds that (similarly for the right action)

$$\overline{\mathrm{coeff}}_{\vec{\Lambda},\vec{\mathtt{t}}}(x\cdot m) = \mathrm{coeff}(x)\cdot\overline{\mathrm{coeff}}_{\vec{\Lambda},\vec{\mathtt{t}}}(m).$$

Thus, $\overline{\mathrm{coeff}}$ is a graded, free Q-modules isomorphism intertwining the two actions. ▲

Lemma 4.33. The compositions

$$\overline{\mathrm{coeff}}_{\Psi}^{-1}\circ\mathrm{coeff}_{\Psi},\mathrm{coeff}_{\Psi}^{-1}\circ\overline{\mathrm{coeff}}_{\Psi} : \mathcal{A}[\mathtt{Q}](\vec{\Lambda},\vec{\mathtt{t}}) \to \mathcal{A}[\mathtt{Q}](\vec{\Lambda},\vec{\mathtt{t}})$$

are $\mathfrak{A}[\mathtt{Q}]$-bimodule maps. (Similarly for any specialization of $\mathtt{Q}$.) ■

Proof. This follows from Remark 4.32 and Proposition 4.19 (and, as before, that our arguments do not use the specific form of the parameters in question). □

4.5. **Consequences.** Using our identifications from Theorems 4.7, 4.2 and 4.3, we have the following (recalling Υ from Proposition 2.43).

Proposition 4.34. There is an equivalence of graded, Q-linear 2-categories

$$\Upsilon\colon \mathfrak{F}[\mathtt{Q}] \xrightarrow{\cong} \mathfrak{W}[\mathtt{Q}]\text{-}\mathbf{biMod}_{\mathrm{gr}}^{p},$$

which is bijective on objects. Similarly for $\mathtt{q}\colon Q \to R$ such that either:

(a) $\mathtt{q}(\boldsymbol{\alpha}) = \boldsymbol{\alpha}$ is generic or $\mathtt{q}(\boldsymbol{\alpha}) = 0$.

(b) $\mathtt{q}(\boldsymbol{\alpha})$ is invertible, $\sqrt{\mathtt{q}(\boldsymbol{\alpha})} \in R$ and $\frac{1}{2} \in R$. ■

The proof is given in Section 6 (the main step is to calculate the rank of hom-spaces between bimodules). We only note here that the algebras in question are semisimple under the circumstances of (b).

Specializations 4.35. The embeddings from Specializations 2.45 are, by Proposition 2.43, actually equivalences. ▲

Theorem 4.36. Let $R[\boldsymbol{\alpha}]$ and $\mathtt{q}$ be as in Theorem 4.3. Then there are equivalences of graded, Q-linear 2-categories

$$\mathfrak{F}[\mathtt{Q}] \cong \mathfrak{F}[\boldsymbol{\alpha}, \mathtt{q}(\boldsymbol{\varepsilon}), \mathtt{q}(\boldsymbol{\omega})].$$

(Similarly for any simultaneous specialization of $\boldsymbol{\alpha}$ satisfying the conditions (a) or (b) from Proposition 4.34.) ■

Proof. This is just assembling all the pieces. First we use Proposition 4.34 to see that both sides are equivalent to the corresponding module categories of (specialized) web algebras. Then we use Theorem 4.1 to translate it to the corresponding arc algebras. Finally, using Theorem 4.3 provides the statement. □

If one works over $\mathbb{Z}[\boldsymbol{\alpha}]$, then Theorem 4.36 shows that the 2-categories coming from the **KBN** and **Bl** setups are equivalent. Having a square root of -1 gives a stronger result, i.e. the following is a direct consequence of Proposition 2.29 and Theorem 4.36.

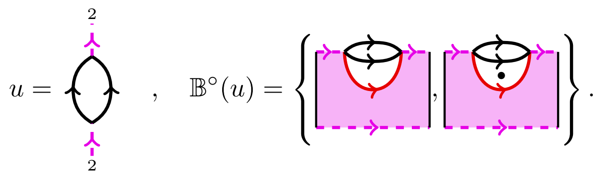
Corollary 4.37. There are equivalences of graded, $\mathbb{Z}[\boldsymbol{\alpha}, i]$-linear 2-categories

$$\mathfrak{F}_{\mathbb{Z}[\boldsymbol{\alpha},i]}[\mathbf{KBN}] \cong \mathfrak{F}_{\mathbb{Z}[\boldsymbol{\alpha},i]}[\mathbf{Ca}] \cong \mathfrak{F}_{\mathbb{Z}[\boldsymbol{\alpha},i]}[\mathbf{CMW}] \cong \mathfrak{F}_{\mathbb{Z}[\boldsymbol{\alpha},i]}[\mathbf{Bl}]. \tag{59}$$

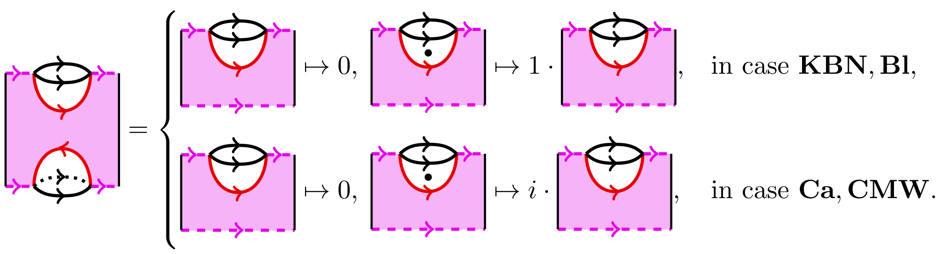
(Similarly, by using $\mathbb{Z}[\sqrt{\mathtt{q}(\boldsymbol{\alpha})}^{\pm 1}, \frac{1}{2}, i]$, for any simultaneous specialization of $\boldsymbol{\alpha}$ satisfying the condition (b) from Proposition 4.34.) □

Example 4.38. The equivalences from Corollary 4.37 are obtained by using the translation from web to arc algebras. In particular, these equivalences are given by evaluating foams on the cup foam basis.

Let us compare for instance an endomorphism f of the web u (both given below) for the 2-categories from (59). In this case the cup foam basis is, for all of them,



In general we need to match the two bases via the coefficient map from Definition 4.18. But in this case we have the following identification (given on the cup foam basis):



Thus, the equivalence from $\mathfrak{F}_{\mathbb{Z}[\boldsymbol{\alpha},i]}[\mathbf{KBN}]$ to $\mathfrak{F}_{\mathbb{Z}[\boldsymbol{\alpha},i]}[\mathbf{Ca}]$, $\mathfrak{F}_{\mathbb{Z}[\boldsymbol{\alpha},i]}[\mathbf{CMW}]$ respectively $\mathfrak{F}_{\mathbb{Z}[\boldsymbol{\alpha},i]}[\mathbf{Bl}]$ rescales $f \mapsto i \cdot f$ respectively $f \mapsto f$. Similarly for more complicated situations (where also the cup foam basis might be already changed). ▲

5. Applications

Now we discuss some applications of our isomorphisms and equivalences.

5.1. **Connection to category $\mathcal{O}$.** To obtain a connection to parabolic category $\mathcal{O}^{\mathfrak{p}}$ for some maximal parabolic of the complex general linear Lie algebra, we first need to define the *generalized signed* 1*-parameter arc algebra* $\mathfrak{C}[0,\boldsymbol{\varepsilon},\boldsymbol{\omega}]_\Lambda$. This algebra might be seen as the quasi-hereditary cover of the signed 1-parameter arc algebra (as it was shown in the **KBN** case in [6, Corollary 5.4]).

Let us give details. To this end, we work over $\mathbb{Z}[\boldsymbol{\omega}^{\pm 1}]$ or $\mathbb{C}$ and specialize $\boldsymbol{\alpha} = 0$ throughout this subsection. Denote by $\mathbf{KBN}_0$ the further specialization of the **KBN** setup with $\mathtt{q}(\boldsymbol{\alpha}) = 0$ (similarly for the $\mathbf{Ca}, \mathbf{CMW}$ and $\mathbf{Bl}$ specializations). The corresponding algebras are called *signed* 1*-parameter (web or arc) algebras*.

Our construction here follows closely [6, Section 4] and [17, Section 6]. In particular, we fix a not necessarily balanced block $\Lambda \in \mathtt{bl}$. Further, fix $n \in \mathbb{Z}_{\geq 2}$ and take two integers $p, q \in \mathbb{Z}_{\geq 0}$ such that $p + q = n$. Consider $\mathfrak{gl}_n$ with fixed Cartan and Borel subalgebras $\mathfrak{h} \subset \mathfrak{b}$, and fix the standard parabolic subalgebra $\mathfrak{p} = \mathfrak{p}_p^q$ with respect to $\mathfrak{b}$ such that its Levi factor is isomorphic to $\mathfrak{gl}_p \times \mathfrak{gl}_q$.

Denote by $\mathcal{O}^{p,q}$ the corresponding *parabolic category* $\mathcal{O}$, i.e. the full subcategory of the category $\mathcal{O}$ for $\mathfrak{gl}_n$ consisting of modules which are locally finite for the action of $\mathfrak{p}$ (the reader unfamiliar with this construction is referred to e.g. [19, Chapter 9]). We can associate to Λ with $\mathrm{up}(\Lambda) = p$ and $\mathrm{down}(\Lambda) = q$ a block of $\mathcal{O}^{p,q}$ which we denote by $\mathcal{O}^{p,q}_\Lambda$ (this works as in [8, (1.3) and (1.4)]). Our aim is to match our signed 1-parameter arc algebra with the projective-injective modules in $\mathcal{O}^{p,q}_\Lambda$, see Remark 5.10, and furthermore construct a generalized signed 1-parameter arc algebra describing these parabolic blocks, see Theorem 5.8.

The idea for the construction is to embed the set of elements of the basis $\mathbb{B}(\Lambda)$ into a set $\mathbb{B}^\circ(\mathrm{hl}_m(\Lambda))$ for some balanced block $\mathrm{hl}_m(\Lambda) \in \mathtt{bl}^\diamond$ called the *m-hull of* Λ. This will enable us to define the generalized signed 1-parameter arc algebra $\mathfrak{C}[0,\boldsymbol{\varepsilon},\boldsymbol{\omega}]_\Lambda$ as a subquotient of the signed 1-parameter arc algebra $\mathfrak{A}[0,\boldsymbol{\varepsilon},\boldsymbol{\omega}]_{\mathrm{hl}_m(\Lambda)}$.

We start by introducing the m-closure of a weight and corresponding to this the m-hull of a block. Morally one puts "enough symbols $\vee$ respectively $\wedge$ to the left respectively right of Λ such that one can close any diagram bounding Λ".

Definition 5.1. Fix $\lambda \in \Lambda$. Let $m \gg 0$ such that $\lambda_i = \mathrm{seq}(\Lambda)_i = \circ$ for $|i| > m$. The *m-closure* $\mathrm{cl}_m(\lambda)$ of λ is defined as the sequence

$$\mathrm{cl}_m(\lambda)_i = \begin{cases} \lambda_i, & \text{for } |i| \leq m, \\ \vee, & \text{for } -m - \mathrm{up}(\Lambda) \leq i < -m, \\ \wedge, & \text{for } m < i \leq m + \mathrm{down}(\Lambda), \\ \circ, & \text{otherwise.} \end{cases}$$

The *m-hull* $\mathrm{hl}_m(\Lambda)$ of Λ is the equivalence class (modulo permutations of $\vee$ and $\wedge$) of $\mathrm{cl}_m(\lambda)$. In addition, we have the subset $\mathrm{cl}_m(\Lambda) = \{\mathrm{cl}_m(\lambda) | \lambda \in \Lambda\}$ inside $\mathrm{hl}_m(\Lambda)$. ▲

Recalling that ${}_\nu\mathbb{1}_\nu = \underline{\nu}\nu\overline{\nu}$, we fix the idempotent

$$\mathbb{1}_{\mathrm{cl}_m(\Lambda)} = \sum_{\nu \in \mathrm{cl}_m(\Lambda)} {}_\nu\mathbb{1}_\nu \in \mathfrak{A}[0,\boldsymbol{\varepsilon},\boldsymbol{\omega}]_{\mathrm{hl}_m(\Lambda)}.$$

Lemma 5.2. Fix an m-hull $\mathrm{hl}_m(\Lambda)$. Then the graded, free $\mathbb{Z}[\boldsymbol{\omega}^{\pm 1}]$-module

$$\mathbf{I}(\Lambda, m) = \big\langle \underline{\lambda'}\nu'\overline{\mu'} \mid \lambda', \mu' \in \mathrm{cl}_m(\Lambda), \nu' \in \mathrm{hl}_m(\Lambda) \setminus \mathrm{cl}_m(\Lambda) \big\rangle_{\mathbb{Z}[\boldsymbol{\omega}^{\pm 1}]}$$

is an ideal in $\mathfrak{A}[0,\boldsymbol{\varepsilon},\boldsymbol{\omega}]_{\mathrm{cl}_m(\Lambda)} = \mathbb{1}_{\mathrm{cl}_m(\Lambda)} \cdot \mathfrak{A}[0,\boldsymbol{\varepsilon},\boldsymbol{\omega}]_{\mathrm{hl}_m(\Lambda)} \cdot \mathbb{1}_{\mathrm{cl}_m(\Lambda)}$. ■

Proof. The proof follows the same arguments as in [17, Subsection 5.3] (in fact, it is easier since one does not need to consider the dotted cups from [17]), since the exact coefficient do not matter in the argument given therein. □

Remark 5.3. The set $\mathbf{I}(\Lambda, m)$ is not an ideal if $\boldsymbol{\alpha}$ is not specialized to 0. This is evident from the arguments in [17, Subsection 5.3]. ▲

Remark 5.4. Using our isomorphism from Theorem 4.7: in terms of the web algebra the ideal above is given by cup foam basis elements which have a dot on a component touching the boundary in a point with $|i| > m$. ▲

Note that the generalized arc algebras for different hulls of Λ are isomorphic.

Lemma 5.5. Fix $m' \geq m \gg 0$ such that $\mathrm{seq}(\Lambda)_i = \circ$ for $|i| > m$. Then there is an isomorphism of graded $\mathbb{Z}[\boldsymbol{\omega}^{\pm 1}]$-algebras

$$\mathfrak{A}[0, \boldsymbol{\varepsilon}, \boldsymbol{\omega}]_{\mathrm{hl}_m(\Lambda)} \cong \mathfrak{A}[0, \boldsymbol{\varepsilon}, \boldsymbol{\omega}]_{\mathrm{hl}_{m'}(\Lambda)}.$$

This isomorphism identifies the subalgebras $\mathfrak{A}[0, \boldsymbol{\varepsilon}, \boldsymbol{\omega}]_{\mathrm{cl}_m(\Lambda)}$ and $\mathfrak{A}[0, \boldsymbol{\varepsilon}, \boldsymbol{\omega}]_{\mathrm{cl}_{m'}(\Lambda)}$ as well as the ideals $\mathbf{I}(\Lambda, m)$ and $\mathbf{I}(\Lambda, m')$. ■

Proof. The claim follows immediately since the only difference between the two algebras is the number of symbols $\circ$ between the symbols from the block and the newly added ones, and these do not interfere at all with the multiplication rules. □

We then define the generalized version via the indicated quotient construction.

Definition 5.6. The *generalized* signed 1-parameter arc algebra $\mathfrak{C}[0, \boldsymbol{\varepsilon}, \boldsymbol{\omega}]_\Lambda$ is defined as

$$\mathfrak{C}[0, \boldsymbol{\varepsilon}, \boldsymbol{\omega}]_\Lambda = \mathfrak{A}[0, \boldsymbol{\varepsilon}, \boldsymbol{\omega}]_{\mathrm{cl}_m(\Lambda)} / \mathbf{I}(\Lambda, m).$$ ▲

Up to isomorphisms (induced from Lemma 5.5), this is independent of $m \gg 0$. Moreover, everything above works for specializations of $\boldsymbol{\varepsilon}$ and $\boldsymbol{\omega}$ as well.

Remark 5.7. By Theorem 4.7 we have indeed no problems to define the same notions as in Definition 5.6 on the side of the web algebras. The result for the $\mathbf{KBN}_0$ specialization of this will be exactly as in [14], see also Remark 5.4. ▲

We are now ready to give the representation theoretical meaning of $\mathfrak{C}_{\mathbb{C}}[0, \boldsymbol{\varepsilon}, \boldsymbol{\omega}]_\Lambda$ in case the ground ring is $\mathbb{C}$. By [8, Theorem 1.1] there is an equivalence of categories

$$\mathcal{O}_\Lambda^{p,q} \cong \mathfrak{C}_{\mathbb{C}}[\mathbf{KBN}_0]_\Lambda\text{-}\mathbf{Mod}^{\mathrm{fd}}, \tag{60}$$

sending a minimal projective generator to $\mathfrak{C}_{\mathbb{C}}[\mathbf{KBN}_0]_\Lambda$. Here $\cdot\text{-}\mathbf{Mod}^{\mathrm{fd}}$ denotes categories of finite-dimensional modules. Since $\mathfrak{C}_{\mathbb{C}}[\mathbf{KBN}_0]_\Lambda$ is clearly graded, this allows to define the block $\mathcal{O}_\Lambda^{p,q}$ of graded category $\mathcal{O}$ as the category of graded, finite-dimensional modules of $\mathfrak{C}_{\mathbb{C}}[\mathbf{KBN}_0]_\Lambda$.

Thus, using our results from Section 4, we obtain an alternative algebraic description as well as a "singular TQFT model" of category $\mathcal{O}$:

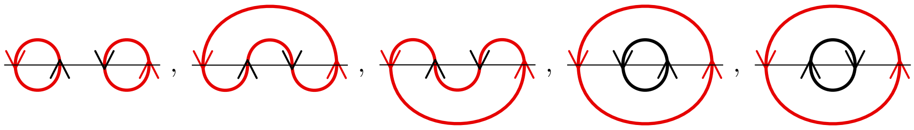
Theorem 5.8. For any specialization $\mathtt{q}\colon \mathtt{Q} \to \mathbb{C}$ with $\mathtt{q}(\boldsymbol{\alpha}) = 0$ it holds that

$$\begin{aligned} \mathcal{O}_\Lambda^{p,q} &\cong \mathfrak{C}_{\mathbb{C}}[0, \mathtt{q}(\boldsymbol{\varepsilon}), \mathtt{q}(\boldsymbol{\omega})]\text{-}\mathbf{Mod}_\Lambda^{\mathrm{fd}} \\ &\cong \mathfrak{C}_{\mathbb{C}}[\mathbf{KBN}_0]_\Lambda\text{-}\mathbf{Mod}^{\mathrm{fd}} \quad \cong \mathfrak{C}_{\mathbb{C}}[\mathbf{Ca}_0]_\Lambda\text{-}\mathbf{Mod}^{\mathrm{fd}} \\ &\cong \mathfrak{C}_{\mathbb{C}}[\mathbf{CMW}_0]_\Lambda\text{-}\mathbf{Mod}^{\mathrm{fd}} \quad \cong \mathfrak{C}_{\mathbb{C}}[\mathbf{Bl}_0]_\Lambda\text{-}\mathbf{Mod}^{\mathrm{fd}}. \end{aligned}$$

(Similarly for the corresponding web algebras.) ■

Proof. This follows directly from (60) and Proposition 4.15. The claim about the web algebras follows then from Theorem 4.1. □

Example 5.9. Take Λ balanced with block sequence $\star\ \star$. For $m = 2$ we obtain for $\lambda \in \Lambda$ with sequence $\wedge\ \vee$ the 2-closure with sequence $\vee\ \wedge\ \vee\ \wedge$, while for $\mu \in \Lambda$ with sequence $\vee\ \wedge$ we have the 2-closure with sequence $\vee\ \vee\ \wedge\ \wedge$. Thus, since $\mathbf{I}(\Lambda, 2)$ "consists of cup foam basis elements without a dot touching the two outer points”, we have (where we marked the components touching the outer points)



as a $\mathbb{Z}[\boldsymbol{\omega}^{\pm 1}]$-basis of $\mathfrak{C}[0, \mathsf{q}(\boldsymbol{\varepsilon}), \mathsf{q}(\boldsymbol{\omega})]$ (denoted from left to right by $\mathbb{1}_\lambda, x^\mu_\lambda, x^\lambda_\mu, \mathbb{1}_\mu, x^\mu_\mu$). This is the path algebra of the quiver (with $x^\lambda_\mu \circ x^\mu_\lambda = 0$ and $x^\mu_\lambda \circ x^\lambda_\mu = \mathsf{q}(\boldsymbol{\omega}) x^\mu_\mu$)

$$\begin{tikzcd} \mathbb{1}_\lambda \arrow[r, shift right, "x^\mu_\lambda"'] & \mathbb{1}_\mu \arrow[l, shift right, "x^\lambda_\mu"'] \arrow[loop right, "x^\mu_\mu"] \end{tikzcd}$$

Working over $\mathbb{C}$, this is the quiver of the principal block of category $\mathcal{O}$ for $\mathfrak{gl}_2$, see for example [43, Subsection 5.1.1] for an explicit calculation of this quiver. ▲

Remark 5.10. Let $\mathrm{pi}(\mathcal{O}^{p,q}_\Lambda)$ be the category of finite-dimensional modules for the endomorphism algebra of all indecomposable projective-injectives in $\mathcal{O}^{p,q}_\Lambda$. We have

$$\mathrm{pi}(\mathcal{O}^{p,q}_\Lambda) \cong \mathfrak{A}_\mathbb{C}[0, \mathsf{q}(\boldsymbol{\varepsilon}), \mathsf{q}(\boldsymbol{\omega})]\text{-}\mathbf{Mod}^{\mathrm{fd}}.$$

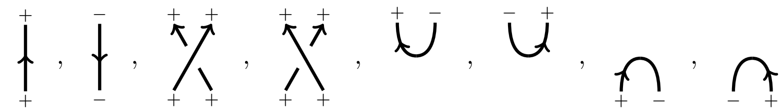
This can be seen as in the proof of Theorem 5.8. That is, one first uses the known equivalence to the $\mathbf{KBN}_0$ setup, see [8, Lemma 4.3], and then the equivalence to any other specialization in $\mathbb{C}$ with $\mathsf{q}(\boldsymbol{\alpha}) = 0$ from Theorem 4.2. ▲

5.2. **Connection with link and tangle invariants.** Given a tangle diagram $\overline{T}$, we will now construct a chain complex $[\![\cdot]\!]^{\mathsf{P}} \colon \overline{T} \mapsto [\![\overline{T}]\!]^{\mathsf{P}}$ with values in $\mathfrak{W}[\mathsf{P}]$-$\mathbf{biMod}^p_{\mathrm{gr}}$. We show in Proposition 5.23 that its homotopy type is an invariant. Hence, $[\![\cdot]\!]^{\mathsf{P}}$ extends to a complex for tangles (and thus, for links) called the 4-*parameter complex*.

The 4-parameter complex specializes to the original **KBN** complex $[\![\cdot]\!]^{\mathbf{KBN}}_{\mathbb{Z}[\boldsymbol{\alpha}]}$, as well as to the versions $[\![\cdot]\!]^{\mathbf{Ca}}_{\mathbb{Z}[\boldsymbol{\alpha},i]}$, $[\![\cdot]\!]^{\mathbf{CMW}}_{\mathbb{Z}[\boldsymbol{\alpha},i]}$ and $[\![\cdot]\!]^{\mathbf{Bl}}_{\mathbb{Z}[\boldsymbol{\alpha}]}$. Using our various isomorphisms, we can show in Theorem 5.25 that all of these give the same tangle invariant.

Tangles and tangled webs. Akin to Definition 2.1 we define *tangles* (algebraically).

Definition 5.11. An (oriented) *tangle diagram* is an oriented, four-valent graph, whose vertices are labeled by crossings, which can be obtained by gluing (whenever this makes sense) or juxtaposition of finitely many of the following pieces:

(61) 

The third generator below is called a *positive crossing* and the fourth a *negative crossing*. We assume that these are embedded in $\mathbb{R} \times [-1, 1]$ in the same ways as webs. In particular, we can associate to each such tangle diagram a (bottom) source $\vec{s}$ and a (top) target $\vec{t}$ (both being elements of $\{0, +, -\}^{\mathbb{Z}}$) in the evident way using the conventions indicated in (61). We only consider tangle diagrams with an even number of bottom and top boundary points (called even tangle diagrams). ▲

Remark 5.12. The restriction to even tangle diagrams comes from the fact that we work with $\mathfrak{W}[\cdot]$ and $\mathfrak{A}[\cdot]$. One could treat arbitrary tangles by using the generalized algebras from Subsection 5.1 (restricting the parameters as therein): it is clear by our results that one can follow [14, Section 5] or [44, Section 5] to define parameter dependent complexes of bimodules for these generalized algebras giving rise to an invariant of arbitrary tangles. We have decided for brevity to only do the $\mathfrak{W}[\cdot]$ and $\mathfrak{A}[\cdot]$ versions here (since we treat these in detail in this paper). ▲

We study the following category, see Kassel [22, Theorem XII.2.2] (apart from the fact that he uses downwards pointing crossing generators).

Definition 5.13. The *category of tangles* 1-**Tan** consists of:

- Objects are sequences $\vec{s}, \vec{t} \in \{0,+,-\}^{\mathbb{Z}}$ with only a finite and even number of non-zero entries (which includes $\emptyset = (,\dots,0,0,0,\dots,))$.
- 1-Morphisms from $\vec{s}$ to $\vec{t}$ are all tangle diagrams with source $\vec{s}$ and target $\vec{t}$.
- The relations are the usual *tangle Reidemeister moves*, which can be found in [22, Section XII.2, Figures 2.2 to 2.9]).
- Composition of tangles is given via the evident gluing.

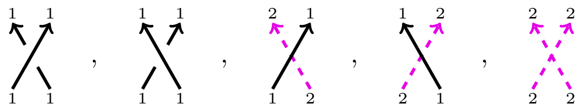
The elements from $\mathrm{Hom}_{1\text{-}\mathbf{Tan}}(\emptyset,\emptyset)$ are called *links*. ▲

Remark 5.14. The tangle Reidemeister moves can be roughly described as:

(I) "Isotopies", i.e. zig-zag moves and relations to twist crossings.
(II) The usual Reidemeister moves **R1**, **R2**, and **R3** (with the latter two seen as braid moves, i.e. pointing only upwards).
(III) Some **R2** moves with an upwards and a downwards pointing strand. ▲

Note that 1-**Tan** gives a generators and relations description of the topological category of tangles (as explained e.g. in [22, Section XII.2]). We denote by ${}_{\vec{s}}T_{\vec{t}}$ a 1-morphisms in $\mathrm{Hom}_{1\text{-}\mathbf{Tan}}(\vec{s},\vec{t})$ and by ${}_{\vec{s}}\overline{T}_{\vec{t}}$ a choice of a diagram representing ${}_{\vec{s}}T_{\vec{t}}$.

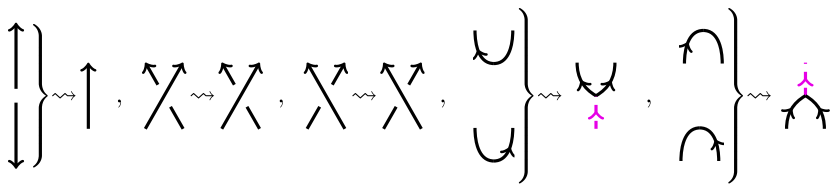
Definition 5.15. Define *tangled webs*, i.e. webs as in Definition 2.1, but additionally allow local generators of the form

(62) 

called *positive crossing*, *negative crossing* and *phantom crossings*. Clearly, tangle webs have the same boundary sequences as webs, e.g. $\vec{k},\vec{l} \in \mathbb{bl}^{\diamond}$. We write $\mathrm{Hom}_{\mathrm{tw}}(\vec{k},\vec{l})$ for the set of tangle webs with bottom sequences $\vec{k}$ and top sequence $\vec{l}$ (as usual, without taking any relations into account). ▲

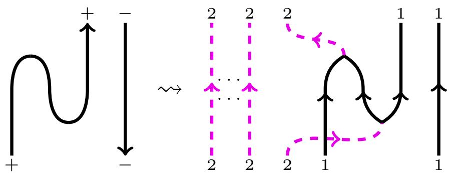
We now give a (straightforward) translation of a tangle diagram ${}_{\vec{s}}\overline{T}_{\vec{t}}$ into a tangled web. We assume for simplicity that we fix an $\ell \in \mathbb{Z}_{\geq 0}$ which is $\ell \gg 0$ ("big enough"). Moreover, we assume that $\vec{s}$ has $s \in 2\mathbb{Z}_{\geq 0}$ non-zero elements etc. in what follows.

Definition 5.16. Given a tangle diagram ${}_{\vec{s}}\overline{T}_{\vec{t}}$, we define a map $\mathrm{w}(\cdot)\colon {}_{\vec{s}}\overline{T}_{\vec{t}} \mapsto \mathrm{w}({}_{\vec{s}}\overline{T}_{\vec{t}})$ to tangled webs in $\mathrm{Hom}_{\mathrm{tw}}(\vec{k},\vec{l})$ with $\vec{k} = \omega_{\ell+s} + \omega_s$ and $\vec{l} = \omega_{\ell+t} + \omega_t$ locally as

(63) 

(By using the phantom crossings from (62), one can always rearrange everything such that one can start in $\omega_{\ell+s} + \omega_s$ and end in $\omega_{\ell+t} + \omega_t$. This association is far from being unique, but what we are going to do will not depend on the choice of the map $\mathrm{w}(\cdot)$ and the concerned reader can pick any such choice). ▲

Example 5.17. One example of this association is the following.

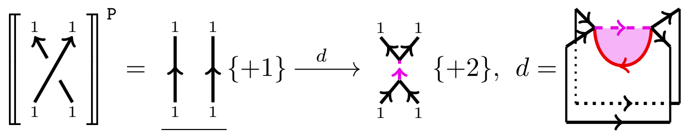


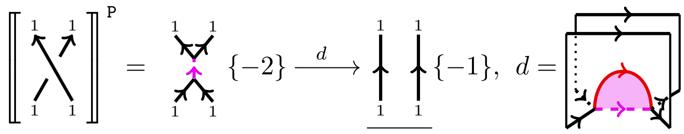
Here we have to pull the phantom edge to the right, because we demand that we start and end in a sequences all of whose entries equal to 2 are placed on the left. ▲

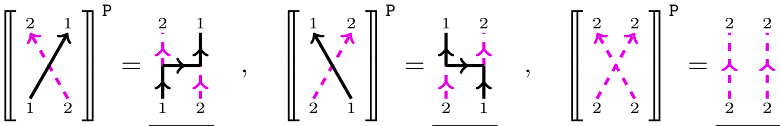
The 4-parameter complex: definition and invariance. Fix an additive 2-category $\mathfrak{X}$. A *chain complex* $(C_i, d_i)_{i\in\mathbb{Z}}$ *with values in* $\mathfrak{X}$ is a chain complex whose chain groups C_i are the 1-morphisms from $\mathfrak{X}$ and whose differentials d_i are the 2-morphisms of $\mathfrak{X}$ such that $d_{i+1} \circ d_i = 0$ for all $i \in \mathbb{Z}$. Such a complex $(C_i, d_i)_{i\in\mathbb{Z}}$ is called *bounded*, if $C_i = 0$ for $|i| \gg 0$. Denote by 1-$\mathbf{CC}(\mathfrak{X})$ *the category of bounded complexes with values in* $\mathfrak{X}$. These can be related via 2-morphism, i.e. chain maps with entries from $\mathfrak{X}$. We consider these in the graded setup (allowing only 2-morphisms of degree 0).

To construct $[\![{}_{\vec{s}}T_{\vec{t}}]\!]^{\mathtt{P}}$ for an oriented tangle diagram ${}_{\vec{s}}T_{\vec{t}}$, we first define $[\![u_t]\!]^{\mathtt{P}}$ for a tangle web $u_t \in \mathrm{Hom}_{\mathrm{tw}}(\vec{k}, \vec{l})$. To this end, we define the following basic complexes (where we denote by $\{\cdot\}$ the shift of the chain groups in their internal degree).

Definition 5.18. The *basic complexes* are

(64) 

(65) 

(66) 

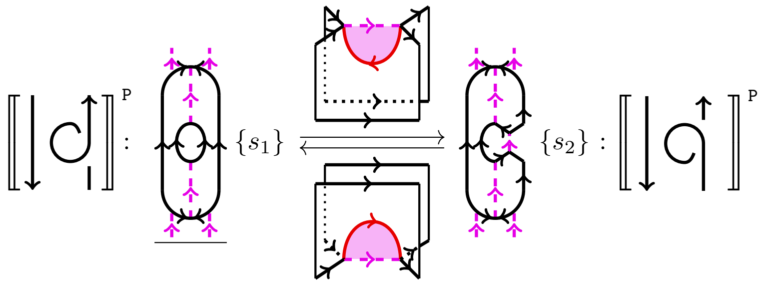
(with the underlined terms sitting in homological degree zero). We see these as objects in 1-$\mathbf{CC}(\mathfrak{W}[\mathtt{P}]\text{-}\mathbf{biMod}^{p}_{\mathrm{gr}})$, i.e. the chain groups are (shifted) web bimodules and the differentials are $\mathfrak{W}[\mathtt{P}]$-bimodule homomorphisms.

Using these, we can associate to any tangled web $u_t \in \mathrm{Hom}_{\mathrm{tw}}(\vec{k}, \vec{l})$ an object $[\![u_t]\!]^{\mathtt{P}}$ via the "tensor products" of the basic complexes. (Similarly for any specialization of $\mathtt{P}$.) We skip the details of this "tensor product procedure" and refer the reader to [24, Subsection 3.4] (which we can copy, incorporating $\mathtt{P}$, without problems). ▲

Remark 5.19. The whole definition of $[\![u_t]\!]^{\mathtt{P}}$ can be also made using Khovanov's famous cube construction (as in [23, Section 3] or [3, Subsection 2.3]). Here the basic complexes from (64) and (65) give rise to 0/1-resolutions, while the basic complexes from (66) have only one resolution (the one given in (66)). ▲

Definition 5.20. Let ${}_{\vec{s}}T_{\vec{t}}$ be a tangle diagram. Set $[\![{}_{\vec{s}}T_{\vec{t}}]\!]^{\mathtt{P}} = [\![\mathrm{w}({}_{\vec{s}}T_{\vec{t}})]\!]^{\mathtt{P}}$, which is an object of 1-$\mathbf{CC}(\mathfrak{W}[\mathtt{P}]\text{-}\mathbf{biMod}^{p}_{\mathrm{gr}})$. (Similarly for any specialization of $\mathtt{P}$.) ▲

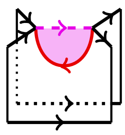
Example 5.21. Recalling that we have to close the associated tangled webs in all possible ways (in this case there is only one), we obtain for example



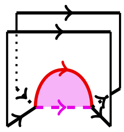
Reading from left to right is the complex for a positive crossing, while reading from right to left is the complex for a negative crossing. In particular, the shifts are $s_1 = +1$ and $s_2 = +2$ for the positive crossing, and $s_1 = -2$ and $s_2 = -1$ for the negative crossing. Seen as bimodules over $\mathfrak{W}[\mathtt{P}]$, the two chain groups have a cup foam basis consisting of four respectively two elements for the left respectively right web bimodule. If we order these

$$(1 \otimes 1, 1 \otimes \mathrm{dot}_{\mathrm{in}}, \mathrm{dot}_{\mathrm{out}} \otimes 1, \mathrm{dot}_{\mathrm{out}} \otimes \mathrm{dot}_{\mathrm{in}}) \quad , \quad (1, \mathrm{dot})$$

(using the evident notation), then the two differentials above are



$$\rightsquigarrow \begin{pmatrix} \boldsymbol{\tau}\boldsymbol{\omega}_+^{-2} & 0 & 0 & 0 \\ 0 & \boldsymbol{\tau}\boldsymbol{\omega}_+^{-1}\boldsymbol{\omega}_-^{-1} & \boldsymbol{\tau}\boldsymbol{\omega}_+^{-2} & \boldsymbol{\alpha}\boldsymbol{\tau}\boldsymbol{\omega}_+^{-1}\boldsymbol{\omega}_-^{-1} \end{pmatrix} \xmapsto{\mathtt{P}\to\mathtt{Q}} \begin{pmatrix} \boldsymbol{\varepsilon} & 0 & 0 & \boldsymbol{\alpha} \\ 0 & 1 & \boldsymbol{\varepsilon} & 0 \end{pmatrix},$$



$$\rightsquigarrow \begin{pmatrix} 0 & \boldsymbol{\alpha}\boldsymbol{\tau}\boldsymbol{\omega}_+\boldsymbol{\omega}_-^{-2} \\ \boldsymbol{\tau}\boldsymbol{\omega}_-^{-1} & 0 \\ \boldsymbol{\tau}\boldsymbol{\omega}_+\boldsymbol{\omega}_-^{-2} & 0 \\ 0 & \boldsymbol{\tau}\boldsymbol{\omega}_-^{-1} \end{pmatrix} \xmapsto{\mathtt{P}\to\mathtt{Q}} \begin{pmatrix} 0 & \boldsymbol{\alpha}\boldsymbol{\varepsilon}\boldsymbol{\omega} \\ \boldsymbol{\omega} & 0 \\ \boldsymbol{\varepsilon}\boldsymbol{\omega} & 0 \\ 0 & \boldsymbol{\omega} \end{pmatrix}.$$

Note that all the chain groups are isomorphic to the one in the **KBN** setup from [24], but the differential are crucially different from the **KBN** differentials (and e.g. not just a scalar times the **KBN** differentials). ▲

Specializations 5.22. We have

$$[\![{}_{\vec{s}}T_{\vec{t}}]\!]^{\boldsymbol{\alpha},1,1,1}_{\mathbb{Z}[\boldsymbol{\alpha}]} \cong [\![{}_{\vec{s}}T_{\vec{t}}]\!]^{\mathbf{KBN}}_{\mathbb{Z}[\boldsymbol{\alpha}]}, \qquad [\![{}_{\vec{s}}T_{\vec{t}}]\!]^{\boldsymbol{\alpha},1,i,-i}_{\mathbb{Z}[\boldsymbol{\alpha},i]} \cong [\![{}_{\vec{s}}T_{\vec{t}}]\!]^{\mathbf{Ca}}_{\mathbb{Z}[\boldsymbol{\alpha},i]},$$
$$[\![{}_{\vec{s}}T_{\vec{t}}]\!]^{\boldsymbol{\alpha},1,i,-i}_{\mathbb{Z}[\boldsymbol{\alpha},i]} \cong [\![{}_{\vec{s}}T_{\vec{t}}]\!]^{\mathbf{CMW}}_{\mathbb{Z}[\boldsymbol{\alpha},i]}, \qquad [\![{}_{\vec{s}}T_{\vec{t}}]\!]^{\boldsymbol{\alpha},-1,1,-1}_{\mathbb{Z}[\boldsymbol{\alpha}]} \cong [\![{}_{\vec{s}}T_{\vec{t}}]\!]^{\mathbf{Bl}}_{\mathbb{Z}[\boldsymbol{\alpha}]},$$

(as chain complexes) with specializations denoted in our usual 4-term notation. ▲

Denote by 1-$\mathbf{HCC}(\mathfrak{W}[\mathtt{P}]\text{-}\mathbf{biMod}^p_{\mathrm{gr}})$ the same category as the one defined before Definition 5.18, but modulo chain homotopy (where we only use $\mathfrak{W}[\mathtt{P}]$-bimodule homomorphisms for all maps in question).

Proposition 5.23. The association $[\![\cdot]\!]^{\mathtt{P}}$ from Definition 5.20 extends to a functor

$$[\![\cdot]\!]^{\mathtt{P}} : 1\text{-}\mathbf{Tan} \to 1\text{-}\mathbf{HCC}(\mathfrak{W}[\mathtt{P}]\text{-}\mathbf{biMod}^p_{\mathrm{gr}}).$$

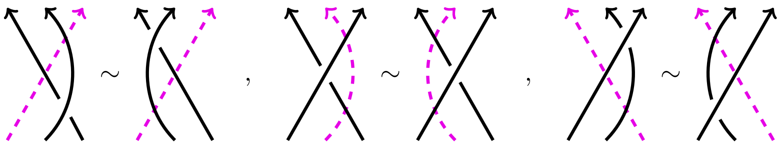
Thus, $[\![\cdot]\!]^{\mathtt{P}}$ is an invariant of tangles. (Similarly for any specialization of $\mathtt{P}$.) ■

Proof. Note that composition of tangles corresponds, by construction, to tensoring of bimodule complexes (the careful reader might want to copy [24, Proposition 13] to see this). Thus, there are two things to show: we have to show that $[\![\cdot]\!]^{\mathtt{P}}$ does not depend on our choice of the map $\mathrm{w}(\cdot)$ from Definition 5.16, and we have to show invariance under the tangle Reidemeister moves from Definition 5.13.

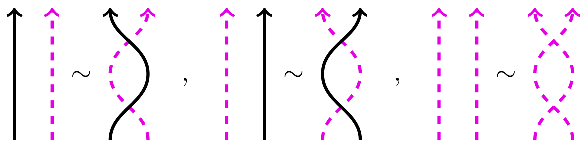
▶**Independence of choice**. Given some ${}_{\vec{s}}\overline{T}_{\vec{t}}$. Assume that we have two different choices $\mathrm{w}_1({}_{\vec{s}}\overline{T}_{\vec{t}})$ and $\mathrm{w}_2({}_{\vec{s}}\overline{T}_{\vec{t}})$. To show independence we have show that

$$[\![\mathrm{w}_1({}_{\vec{s}}\overline{T}_{\vec{t}})]\!]^{\mathtt{P}} = [\![\mathrm{w}_2({}_{\vec{s}}\overline{T}_{\vec{t}})]\!]^{\mathtt{P}} \text{ as complexes in } 1\text{-}\mathbf{HCC}(\mathfrak{W}[\mathtt{P}]\text{-}\mathbf{biMod}^p_{\mathrm{gr}}),$$

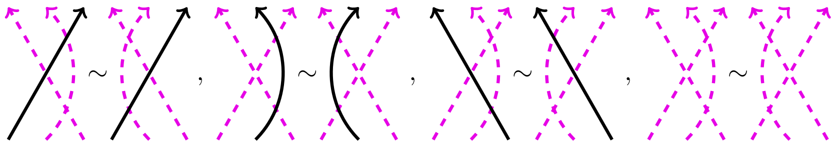
Note that two different choices $\mathrm{w}_1({}_{\vec{s}}\overline{T}_{\vec{t}})$ and $\mathrm{w}_2({}_{\vec{s}}\overline{T}_{\vec{t}})$ can only differ by the following local moves. First, the ordinary-ordinary-phantom **R3** moves (similarly for a negative crossing)

(67) 

Second, the ordinary-phantom **R2** moves and the pure-phantom **R2** move

(68) 

Third, the ordinary-phantom-phantom **R3** moves and the pure-phantom **R3** move

(69) 

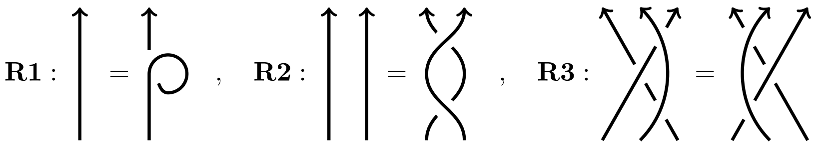
Thus, it suffices to show that the chain complex stay the same (up to chain homotopy) if the two choices $\mathrm{w}_1({}_{\vec{s}}\overline{T}_{\vec{t}})$ and $\mathrm{w}_2({}_{\vec{s}}\overline{T}_{\vec{t}})$ differ by one of these moves. For this purpose, we have by [18, Lemmas 4.4 and 4.5] (which work for $\mathtt{P}$ as well) that the chain groups (i.e. the web bimodules) are isomorphic ("isotopic webs give isomorphic web bimodules"). These isomorphisms are given by the evident foams (which are clearly $\mathfrak{W}[\mathtt{P}]$-bimodule homomorphisms). Thus, it remains to show that these commute with the differentials in the associated complexes. This is clear for a difference as in (68) or (69). For a difference as in (67) one easily checks that these isomorphisms commute with the differentials. Indeed, instead of checking that these work out locally, we could also perform the necessary arrangements globally, i.e. outside of the illustrated picture (recalling that these always close in some way within the web bimodules). We thus, avoid the crossings and the saddle foams will commute with these, since all non-trivial changes are "far apart". ▷

▶**Tangle Reidemeister moves**. Denote by ${}_{\vec{s}}\overline{T}_{\vec{t}}$ and ${}_{\vec{s}}\overline{\mathcal{T}}_{\vec{t}}$ two tangle diagrams that differ by one of the moves (I)-(III) from Remark 5.14. Again, if we show that

$$[\![{}_{\vec{s}}\overline{T}_{\vec{t}}]\!]^{\mathtt{P}} = [\![{}_{\vec{s}}\overline{\mathcal{T}}_{\vec{t}}]\!]^{\mathtt{P}} \text{ as complexes in } 1\text{-}\mathbf{HCC}(\mathfrak{W}[\mathtt{P}]\text{-}\mathbf{biMod}^p_{\mathrm{gr}}),$$

then we are done. The main point is invariance under a move from (II). Indeed, invariance under a move from (I) follows as above, because e.g. we can by the above assume that a zig-zag move looks locally as in Example 5.17 and then use the same arguments as before ("isotopic webs give isomorphic web bimodules"). Invariance under a move from (III) follows also as above, if we already know invariance under the **R2** moves from (II). Hence, it remains to show invariance under the moves from

(II), i.e. we have to check the following (together with variations of these):



(i) **R1** *move, right curl with a positive or negative crossing.* For the positive crossing we take the same cobordisms as in [4, Subsection 3.3] (adding phantom edges/facets similar to (63)), but set $f_{\mathtt{P}} = \boldsymbol{\tau}\boldsymbol{\omega}_+^{-2} f_{\mathbf{Bl}}$, $g_{\mathtt{P}}^{\mathrm{l}} = -\boldsymbol{\omega}_- g_{\mathbf{Bl}}^{\mathrm{l}}$, $g_{\mathtt{P}}^{\mathrm{r}} = \boldsymbol{\omega}_+ g_{\mathbf{Bl}}^{\mathrm{r}}$ and $D_{\mathtt{P}} = \boldsymbol{\tau}\boldsymbol{\omega}_+^{-2} D_{\mathbf{Bl}}$ (we use Blanchet's notation, where $g_{\mathbf{Bl}}^{\mathrm{l}}$ and $g_{\mathbf{Bl}}^{\mathrm{r}}$ mean the left and right summand of Blanchet's g from [4, Fig. 13] with the dot in the back for $g_{\mathbf{Bl}}^{\mathrm{r}}$). Similarly for the negative crossing with exchanged roles of f and g (beware the slight typos in [4, Subsection 3.3]).

(ii) **R1** *move, left curl with a positive or negative crossing.* Similarly as for the right curl, but exchanging the roles of $\boldsymbol{\omega}_+$ and $\boldsymbol{\omega}_-$.

(iii) **R2** *move, both versions.* We use the same cobordisms (plus phantom edges and facets) and coefficients as in [4, Subsections 3.4 and 3.5].

(iv) **R3** *move, both versions.* This can be showed using the usual (abstract) Gauss elimination argument (as pioneered in [2]). To be precise, one uses the $\mathtt{P}$-analog of [18, Lemma 4.3] (the "circle removal") and then twice the Gauss elimination from [2, Lemma 3.2]. One obtains that the two complexes for both sides of the **R3** move have isomorphic chain groups. These can then be matched directly. We leave the details to the reader, where we note that all appearing coefficients are trivial (because the "complicated" maps in the Gauss elimination are at extremal parts of the complexes). $\triangleright$

Everything above works for any specialization of $\mathtt{P}$ and the claim follows. $\square$

Thus, we can write $[\![{}_{\vec{s}}T_{\vec{t}}]\!]^{\mathtt{P}}$ etc. without ambiguity.

The signed 2-parameter complex: comparison. Our results of Section 4 almost immediately imply that the link and tangle homologies from above "are the same". Let us make this precise (using our results from Section 4).

Denote by $[\![\cdot]\!]^{\mathbf{KBN}}$ the functor obtained via specializing $\mathtt{q}(\boldsymbol{\alpha}) = \boldsymbol{\alpha}$, $\mathtt{q}(\boldsymbol{\varepsilon}) = 1$ and $\mathtt{q}(\boldsymbol{\omega}) = 1$ (and scalar extension). Moreover, by abusing notation, we denote by

$$\boldsymbol{\Psi}\colon \mathfrak{W}[\mathbf{KBN}]\text{-}\mathbf{biMod}^p_{\mathrm{gr}} \xrightarrow{\cong} \mathfrak{W}[\mathtt{Q}]\text{-}\mathbf{biMod}^p_{\mathrm{gr}}$$

the equivalence obtained by combining Theorems 4.1 and 4.3.

Proposition 5.24. The following diagram commutes.

$$\begin{tikzcd}
 & 1\text{-}\mathbf{HCC}(\mathfrak{W}[\mathbf{KBN}]\text{-}\mathbf{biMod}^p_{\mathrm{gr}}) \arrow[dd, "\cong"', "\boldsymbol{\Psi}"] \\
1\text{-}\mathbf{Tan} \arrow[ur, "{[\![\cdot]\!]^{\mathbf{KBN}}}"] \arrow[dr, "{[\![\cdot]\!]^{\mathtt{Q}}}"'] & \\
 & 1\text{-}\mathbf{HCC}(\mathfrak{W}[\mathtt{Q}]\text{-}\mathbf{biMod}^p_{\mathrm{gr}}).
\end{tikzcd}$$

(Similarly for any further simultaneous specialization of $\boldsymbol{\alpha}$.) $\blacksquare$

Proof. Let ${}_{\vec{s}}T_{\vec{t}}$ be any tangle. The Khovanov cubes associated to $\boldsymbol{\Psi}([\![{}_{\vec{s}}T_{\vec{t}}]\!]^{\mathbf{KBN}})$ and $[\![{}_{\vec{s}}T_{\vec{t}}]\!]^{\mathtt{Q}}$ (mentioned in Remark 5.19) are the same combinatorially, i.e. all vertices and all edges are at the same positions. Moreover, by Theorems 4.1 and 4.3, the web bimodules associated to vertices are isomorphic. Recall that edges of a Khovanov cube have an associated "multiplication foam" $f\colon \mathcal{W}(u) \to \mathcal{W}(v)$ (as in (65)) or a "reversed multiplication foam" $g\colon \mathcal{W}(v) \to \mathcal{W}(u)$ (as in (64)). Clearly, the "type" of such a foam associated to an edges is the same for the two complexes under consideration. To be precise, for each edge in the Khovanov cubes we have

$$f\colon \mathcal{W}[\mathbf{KBN}](u) \to \mathcal{W}[\mathbf{KBN}](v) \xmapsto{\mathbf{\Psi}} \mathrm{coeff}_v \circ f \circ \mathrm{coeff}_u^{-1}\colon \mathcal{W}[\mathtt{Q}](u) \to \mathcal{W}[\mathtt{Q}](v),$$

$$g\colon \mathcal{W}[\mathbf{KBN}](v) \to \mathcal{W}[\mathbf{KBN}](u) \xmapsto{\mathbf{\Psi}} \mathrm{coeff}_u \circ g \circ \mathrm{coeff}_v^{-1}\colon \mathcal{W}[\mathtt{Q}](v) \to \mathcal{W}[\mathtt{Q}](u)$$

where we use the evident notation from Section 4, but for the web algebras side instead of the arc algebra used therein. It remains to analyze these differentials, i.e. we have to compare $\mathbf{\Psi}(f)$ to $f_{\mathtt{Q}}$ and $\mathbf{\Psi}(g)$ to $g_{\mathtt{Q}}$ (with the latter being the differentials for $[\![{}_{\vec{s}}T_{\vec{t}}]\!]^{\mathtt{Q}}$). With the work already done this is not hard. Indeed, it follows from Proposition 4.19 that $\mathbf{\Psi}(f) = f_{\mathtt{Q}}$, while it follows from Proposition 4.29 that

$$g_{\mathtt{Q}} = \overline{\mathrm{coeff}}_v \circ \mathbf{\Psi}(g) \circ \overline{\mathrm{coeff}}_u^{-1}.$$

Thus, the differentials of two complexes $\mathbf{\Psi}([\![{}_{\vec{s}}T_{\vec{t}}]\!]^{\mathbf{KBN}})$ and $[\![{}_{\vec{s}}T_{\vec{t}}]\!]^{\mathtt{Q}}$ differ only by "units", and we can use the usual unit sprinkling (see [15, Lemma 4.5]) to get a chain isomorphism between them. Hence, it remains to verify that the maps used in this chain isomorphism are actually entrywise $\mathfrak{W}[\mathtt{Q}]$-bimodule homomorphisms. This is true by the above and Lemma 4.33. The statement follows. $\square$

Let $R[\boldsymbol{\alpha}]$ and $\mathtt{q}$ be as at the beginning of Section 4. Moreover, denote by $[\![\cdot]\!]^{\mathtt{q}}$ the functor obtained from $[\![\cdot]\!]^{\mathtt{Q}}$ via the specialization $\mathtt{q}\colon Q \to R[\boldsymbol{\alpha}]$ (and scalar extension). Abusing notation, we keep on writing $\mathbf{\Psi}$ for the equivalence used below.

Theorem 5.25. The following diagram commutes.

$$\begin{tikzcd}
 & \text{1-}\mathbf{HCC}(\mathfrak{W}[\mathtt{Q}]\text{-}\mathbf{biMod}^p_{\mathrm{gr}}) \arrow[dd, "\cong"', "\mathbf{\Psi}"] \\
\text{1-}\mathbf{Tan} \arrow[ur, "{[\![\cdot]\!]^{\mathtt{Q}}}"] \arrow[dr, "{[\![\cdot]\!]^{\mathtt{q}}}"'] & \\
 & \text{1-}\mathbf{HCC}(\mathfrak{W}[\boldsymbol{\alpha}]\text{-}\mathbf{biMod}^p_{\mathrm{gr}}).
\end{tikzcd} \tag{70}$$

(Similarly for any further simultaneous specialization of $\boldsymbol{\alpha}$.) ■

Proof. Exactly as in the proof of Proposition 5.24, since we have not used the specific form of the parameters in question. $\square$

These result are stronger than just saying that the corresponding chain complexes are homotopy equivalent since we match the bimodules structures as well.

Let us write $\approx$ for short if two homologies obtained via specialization of $[\![\cdot]\!]^{\mathtt{Q}}$ can be matched as in (70). In this case, we say that they *give the same invariant.*

Specializations 5.26. Set $R = \mathbb{Z}[\boldsymbol{\alpha}]$ and specialize $\mathtt{q}(\boldsymbol{\alpha}) = \boldsymbol{\alpha}$, $\mathtt{q}(\boldsymbol{\varepsilon}) = 1$ and $\mathtt{q}(\boldsymbol{\omega}) = 1$ respectively $\mathtt{q}(\boldsymbol{\alpha}) = \boldsymbol{\alpha}$, $\mathtt{q}(\boldsymbol{\varepsilon}) = -1$ and $\mathtt{q}(\boldsymbol{\omega}) = 1$. Then

$$[\![\cdot]\!]^{\mathbf{KBN}}_{\mathbb{Z}[\boldsymbol{\alpha}]} \approx [\![\cdot]\!]^{\mathbf{Bl}}_{\mathbb{Z}[\boldsymbol{\alpha}]}.$$

(Similarly for e.g. $\mathtt{q}(\boldsymbol{\alpha}) = 0$.) This shows that Khovanov's original link homology and Blanchet's version of it give the same invariant (even for tangles). ▲

Remark 5.27. The result of Subsection 5.1 give a way to relate our link and tangle invariants constructed here to the link and tangle invariants $[\![\cdot]\!]^{\mathcal{O}}_{\mathbb{C}}$ constructed from category $\mathcal{O}$. We refer the reader to [44, Subsection 5.10] for details. ▲

Working over $R=\mathbb{Z}[\boldsymbol{\alpha},i]$ or $R=\mathbb{C}$ gives a stronger result whose proof is now evident (by using Theorem 5.25 and Remark 5.27).

Corollary 5.28. We have (with the last $\approx$ only for $R=\mathbb{C}$ and $\mathtt{q}(\boldsymbol{\alpha})=0$)

$$[\![\cdot]\!]_R^{\mathbf{KBN}} \approx [\![\cdot]\!]_R^{\mathbf{Ca}} \approx [\![\cdot]\!]_R^{\mathbf{CMW}} \approx [\![\cdot]\!]_R^{\mathbf{Bl}} \underset{\boldsymbol{\alpha}=0}{\overset{R=\mathbb{C}}{\approx}} [\![\cdot]\!]_{\mathbb{C}}^{\mathcal{O}}.$$

(Similarly for any further simultaneous specialization of $\boldsymbol{\alpha}$.) □

This was known for links, but, to the best of our knowledge, not for tangles.

The signed 2*-parameter complex: functoriality.* In fact, our results are even stronger: it follows from Section 4 that all of them can be arranged such that they give functorial invariants of links. Moreover, using the arc algebra setup, calculations of these functorial invariants can be made explicit.

In order to give some more detail, let us denote by 2-**Tan** the 2*-category of tangles*. This is the 2-category whose underlying 1-category is 1-**Tan** and whose 2-morphisms are certain cobordisms called 2*-tangles*. There is a generators and relations description of 2-**Tan** in the spirit of the one for 1-**Tan** as well (with relations given by the so-called *movie moves*). We do not recall the details here and refer the reader to [3, Section 8] or [12, Chapter 1].

Hence, in the light of Proposition 5.23, it makes sense to ask, if there is a 2-functor

$$[\![\cdot]\!]^{\mathtt{P}}\colon 2\text{-}\mathbf{Tan}\to 2\text{-}\mathbf{HCC}(\mathfrak{W}[\mathtt{P}]\text{-}\mathbf{biMod}^p_{\mathrm{gr}})?$$

Here $2\text{-}\mathbf{HCC}(\mathfrak{W}[\mathtt{P}]\text{-}\mathbf{biMod}^p_{\mathrm{gr}})$ means that we identify homotopic 2-morphisms.

Again, we specialize to $\mathtt{Q}$. Let $\overline{2\text{-}\mathbf{Tan}_{\emptyset}}$ be the 2-subcategory consisting of only links (which we do not consider up to isotopy, see [4, Remark 5.2] for the reason for this). In this case we have $2\text{-}\mathbf{HCC}(Q\text{-}\mathbf{Mod}_{\mathrm{free}})$ as a target 2-category. Now, Caprau, Clark-Morrison-Walker and Blanchet showed that their construction of Khovanov homology extends to 2-functors (for $R=\mathbb{Z}[i]$ or $R=\mathbb{Z}[\frac{1}{2}]$ and $\mathtt{q}(\boldsymbol{\alpha})\in\{0,1\}$)

$$[\![\cdot]\!]^{\mathbf{Ca}}_{\mathbb{Z}[i]}, [\![\cdot]\!]^{\mathbf{CMW}}_{\mathbb{Z}[i]}, [\![\cdot]\!]^{\mathbf{Bl}}_{\mathbb{Z}[\frac{1}{2}]}\colon \overline{2\text{-}\mathbf{Tan}_{\emptyset}}\to 2\text{-}\mathbf{HCC}(R\text{-}\mathbf{Mod}_{\mathrm{free}}), \tag{71}$$

see [11, Theorem 3.5], [15, Theorem 1.1] and [4, Theorem 5.1].

Thus, the above gives a way to fix functoriality of Khovanov homology without changing the easy framework of **KBN**. Namely, use any of the functorial invariants from (71) and "pull it over". To be more precise, one uses the coefficient maps (from the **KBN** setup to any of **Ca**, **CMW** or **Bl**) from Definition 4.18 on the chain groups (bimodules) to get a different, scalar adjusted, cup foam basis. Then one can rearrange the differentials (web bimodule homomorphisms) as in Example 4.38. The resulting complex is functorial.

6. Main proofs

In this final section we give the more technical proofs of our main statements (together with some technical lemmas needed to proof these statements).

Proof of Theorem 4.7. We will use the notation from Subsection 4.1.

Proof. Our proof here follows [18, proof of Theorem 4.18]. That is, we show that each step in the multiplication procedure from Definition 2.31 locally agrees with the one from Subsection 3.3. Here we use Lemma 4.6, i.e. it suffices to show that they agree on the cup foam basis on the side of $\mathfrak{W}[\mathtt{Q}]^{\circ}$. Note that the setup of $\mathfrak{W}[\mathtt{Q}]^{\circ}$ is more flexible and thus, harder to work with. That is, throughout the whole proof, we first check the multiplication steps for $\mathfrak{W}[\mathtt{Q}]^{\circ}$ where some rewriting has to be done, and then for $\mathfrak{A}[\mathtt{Q}]$ where we can read-off the multiplication directly using the rules from Subsection 3.3.

Now, there are four topologically different situations to check:

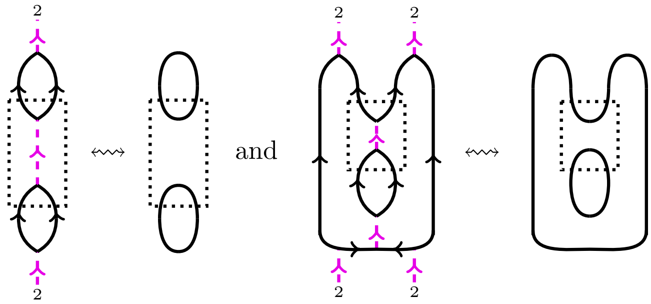
(i) **Non-nested merge:** two non-nested circles are replaced by one circle.
(ii) **Nested merge:** two nested circles are replaced by one circle.
(iii) **Non-nested split:** one circle is replaced by two non-nested circles.
(iv) **Nested split:** one circle is replaced by two nested circles.

As in [18, proof of Theorem 4.18], we will go through the following cases:

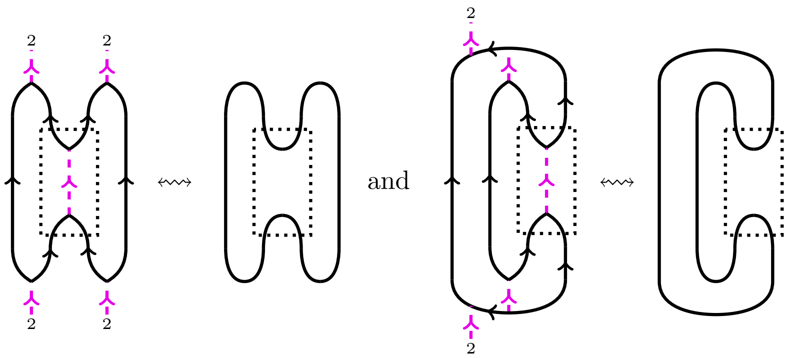
(A) **Basic shape:** the involved components are as small as possible with the minimal number of phantom edges.
(B) **Minimal saddle:** while the components themselves are allowed to be of any shape, the involved saddle has only a single phantom facet.
(C) **General case:** both, the shape as well as the saddle, are arbitrary.

Our proof here is in principle the same as [18, proof of Theorem 4.18], but harder and more delicate, because the appearing factors are more involved. Thus, for brevity, we only do here the basic shapes in detail and sketch the remaining ones.

We start with (A). In cases (i) and (ii) we have

(72)


(as well as a horizontal flip of the second situation). In cases (iii) and (iv) we have

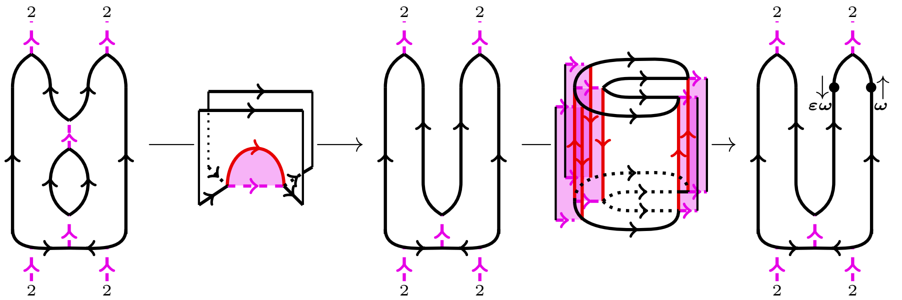
(73)


i.e. the H-shape and C-shape (recall that, by our convention, the Ɔ-shape does not occur). Here we have displayed both, the web and its corresponding arc diagram.

To understand the calculation below recall that we use the specialization from (3). What is of paramount importance about this specialization is that we do not have to worry about the "direction" in which we apply squeezing (16), dot migrations (17) or ordinary-to-phantom neck cutting (18) (since all of them will just contribute an $\boldsymbol{\varepsilon}$).

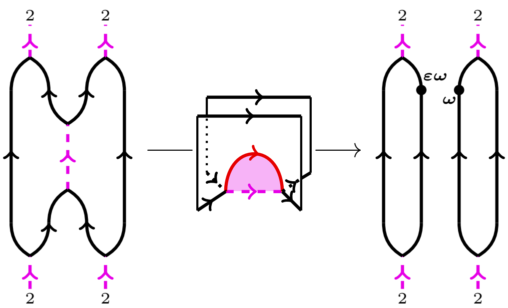
►**Non-nested merge - basic shape.** This case works almost exactly in the same way as in [18, proof of Theorem 4.18]. That is, multiplication of basis cup foams yields topologically basis cup foams again, except in case where we start with two dotted basis cup foams. But in this case we can use (10) to create a basis cup foam without dots and a factor $\boldsymbol{\alpha}$. The same happens for $\mathfrak{A}[\mathtt{Q}]$, see (36). ▷

▸**Nested merge - basic shape.** In this case something has to be done on the side of $\mathfrak{W}[\mathtt{P}]^{\circ}$. In fact, this is the most complicated case and we go through all details and will be shorter in the other cases afterward. The multiplication step is given by



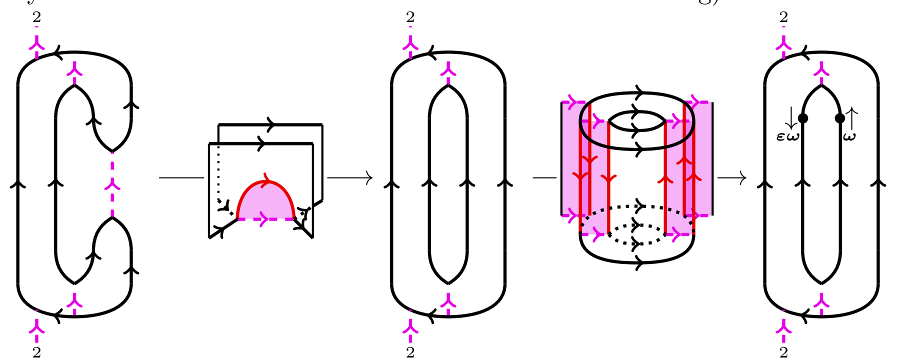
The right foam above is shown to illustrate the cylinder we cut together with the dots, their positions (with $\downarrow$ and $\uparrow$ meaning the dot sits on a facet touching the corresponding edge under or over the part where we cut the cylinder) and factors created in this cutting procedure. Now, if a basis cup foam is sitting underneath the leftmost picture, then the multiplication result is topologically not a basis cup foam. Thus, we need to turn it into a basis cup foam. In order to do so, we apply (15) to the cylinder illustrated above. Here we have to use (16) first, which gives an overall factor ε (we squeeze the left part of the cylinder). Cutting the cylinder gives a sum of two foams, one with a dot on the top and one with a dot on the bottom. The one with a dot on the bottom will be of importance and it comes with a factor $\varepsilon\omega$, as illustrated above. After neck cutting the cylinder we create a "bubble" (recalling that a basis cup foam is sitting underneath) with two internal phantom facets in the bottom part of the picture. By (18), we can remove the phantom facets (we remove the left phantom facet), we pick up a factor ε and create an "honest" bubble instead. Thus, by (14), only the term in (15) with the dot on the bottom survives. By (13) the remaining bubble evaluates to $\varepsilon\omega^{-1}$. Hence, we get in total a basis cup foam without dots and a factor $\varepsilon\cdot\varepsilon\omega\cdot\varepsilon\cdot\varepsilon\omega^{-1}=\varepsilon^4=1$. This is the same as for $\mathfrak{A}[\mathtt{Q}]$ which was computed in (37). If we start with a dotted basis cup foam, then we can move the dot topologically aside and proceed as above (in particular, we pick up the same coefficients). After the topological rearrangement, we have to move the dot to the rightmost facets to produce a basis cup foam again. Thus, using dot migration (17), we get the same result as in (37), because the dot moving sign from (29) reflects the dot migration. More precisely, the dot migration gives a factor ε which is as in (29), since $\mathrm{d}_\Lambda(\gamma_i^{\mathrm{dot}})=1$ (we have to move across one cap of length 1). The same works word-by-word in the horizontally flipped cases as well, which proves this case.▹

▸**Non-nested split - basic shape.** The multiplication step is



(we have again illustrated the dots which are created while topologically rearranging the resulting foam). Assuming that a basis cup foam without dots is sitting underneath the leftmost picture, we see that we almost get a basis cup foam after stacking the saddle on top: we get two cup foams sitting underneath the left and right circle which touch each other in the middle in a closed singular seam (and a corresponding phantom facet). Thus, by using the singular seam removal (19) (creating dots as illustrated above) and dot migration (17), we get two basis cup foams, one with a dot on the rightmost facets of the left circle and one with a dot on the rightmost facet of the right circle. The singular seam removal gives a factor $\varepsilon\omega$ for the first and a factor ω for the second. Additionally, the second gets a factor ε coming from the dot migration. Recalling $\varepsilon = \pm 1$, this matches the side of $\mathfrak{A}[\mathtt{Q}]$ which was computed in (38). On the other hand, if a basis cup foam with a dot (on the rightmost facet) is sitting underneath the leftmost picture, we can move the dot topologically aside, proceed as above and create, after using the singular seam removal (19) and dot migration (17), two basis cup foams. Remembering that we started with a dot, we see that these two are now a basis cup foam with one dot on the rightmost facets of the two circles and a foam that is topological a basis cup foam, but with two dots on the rightmost facet of the right circles. Thus, using (10), we get the same result as for $\mathfrak{A}[\mathtt{Q}]$, see (38). ▷

▶**Nested split - basic shape.** The multiplication foam is now (indicating again the cylinder we want to cut and the dots we created via cutting)



Again we can apply neck cutting. This time to the internal cylinder in the second foam between the middle web and the rightmost web connecting the two nested circles that we can cut using (15). First assume that the original basis cup foam sitting underneath has no dots. After neck cutting we get a sum of two basis cup foams (so nothing needs to be done topologically). One has a factor ω and a dot sitting on the rightmost facet of the nested circle, the other has a factor $\varepsilon\omega$ and a dot sitting on the next to leftmost facets of the outer circle (as illustrated above). Moving this dot across two phantom facets to the rightmost facets picks up, by dot migration (17), a factor $\varepsilon^2 = 1$ (recalling that the dot is sitting underneath the place where we applied neck cutting and hence, is on a foam with a generic slice as in the leftmost picture above). Thus, we end with the same as for $\mathfrak{A}[\mathtt{Q}]$, see (39). Similarly, starting with a basis cup foam sitting underneath having a dot on the rightmost facet, we can move the dot topologically aside and proceed as before. As above in the non-nested split case, we get a sum of two basis cup foams, one with one dot on each rightmost facet, and one with two dots on the rightmost facet of the outer circle. Hence, using (10) again, we get the same result as in (39). ▷

The remaining cases (B) and (C) from above can be proven by copying the arguments from [18, Proof of Theorem 4.18]. In particular, non-interfering foam parts can be topologically moved away and do not matter in the rewriting process. The only thing that changes is that the dot moving signs, the topological sign and the saddle sign from (29) are now powers of ε instead of powers of -1.

▶**General shape, but minimal saddle.** The dot moving signs are precisely the same on both sides (recalling that moving across phantom facets always gives ε). Furthermore, we can always move existing dots topologically aside and we dot not have to worry about them until the very end where we possible apply (9) to remove two of them. In particular, if we understand the undotted case, then the dotted follows. So let us consider only basis cup foams without dots. In case of the non-nested merge, the resulting foams are topologically basis cup foams and we are done. In case of the nested merge we have to topologically manipulate the result until it is a basis cup foam again. This can be done as in [18, Proof of Theorem 4.18] with the difference that the formula [18, (43)] gives

$$\varepsilon^{\frac{1}{4}(\mathrm{d}_\Lambda(C_{\mathrm{in}})-2)} \cdot \varepsilon^1 \quad \text{instead of} \quad (-1)^{\frac{1}{4}(\mathrm{d}_\Lambda(C_{\mathrm{in}})-2)} \cdot (-1)^1.$$

This matches the side of $\mathfrak{A}[\mathtt{Q}]$. For the case of the non-nested split we can proceed as above and we get the same factors which matches the case of $\mathfrak{A}[\mathtt{Q}]$. Last, for the nested split we copy the argument in [18, Proof of Theorem 4.18], but picking up

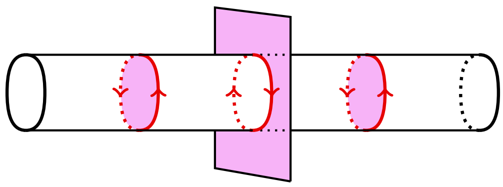
$$\varepsilon^{\frac{1}{4}(\mathrm{d}_\Lambda(C_{\mathrm{in}})-2)} \quad \text{instead of} \quad (-1)^{\frac{1}{4}(\mathrm{d}_\Lambda(C_{\mathrm{in}})-2)}.$$

Again, this is as in case $\mathfrak{A}[\mathtt{Q}]$. ▷

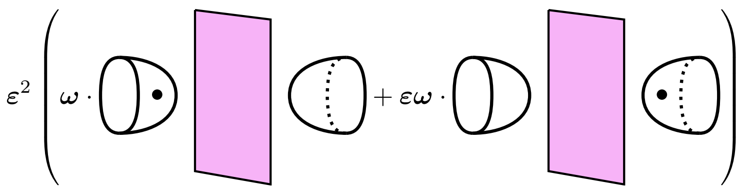
▶**General shape.** The non-nested merge works as above, i.e. this case does not depend on the "size" of the saddle. Incorporating a general saddle in the cases of a nested merge is as in [18, Proof of Theorem 4.18] but with

$$\varepsilon^{s_\Lambda(\gamma)} \quad \text{instead of} \quad (-1)^{s_\Lambda(\gamma)}.$$

The non-nested split case can be done as above for the basic shape, but the two cups foams touch each other now locally as (we have illustrated the case $s_\Lambda(\gamma) = 2$)



By using (19) once followed by $s_\Lambda(\gamma) - 1$ applications of (18) (as well as $2(s_\Lambda(\gamma) - 1)$ applications of (17) which do not contribute because $\varepsilon = \pm 1$) reduces the above locally to (here again $s_\Lambda(\gamma) = 2$ - the left ε has an exponent $s_\Lambda(\gamma)$ in general)



The case of a nested split does not depend on the saddle and can be done as above in case of the minimal saddle. In all cases, we get the same on the side of $\mathfrak{A}[\mathtt{Q}]$ which finishes the arguments for the general cases. ▷

The case of specialized $\mathtt{Q}$ works analogously. Thus, the claim follows. □

Proof of Proposition 4.15. We will use the Notation from Subsection 4.2 in the proof of Proposition 4.15 given below.

Proof. First note that the maps from (52) are clearly homogeneous and Q-linear. Moreover, it suffices to show the isomorphism for some fixed, but arbitrary, $\Lambda \in \mathtt{bl}^{\diamond}$. Thus, we fix $\Lambda \in \mathtt{bl}^{\diamond}$ in what follows.

The main idea of the proof is to show that the maps coeff_D successively intertwine the two multiplication rules for $\mathfrak{A}[\mathbf{KBN}]_\Lambda$ and $\mathfrak{A}[\mathtt{Q}]_\Lambda$. Consequently, we compare two intermediate multiplications steps in the following fashion:

$$\begin{tikzcd} D_l \arrow[r, "\mathbf{mult}^{\mathbf{KBN}}_{D_l,D_{l+1}}"] \arrow[d, "\mathrm{coeff}_{D_l}"'] & D_{l+1} \arrow[d, "\mathrm{coeff}_{D_{l+1}}"] \\ D_l \arrow[r, "\mathbf{mult}^{\mathtt{Q}}_{D_l,D_{l+1}}"'] & D_{l+1} \end{tikzcd}$$

(with the notation as in Subsection 3.3). The goal is to show that each such diagram, i.e. for each D_l and D_{l+1}, commutes. Since the multiplication in $\mathfrak{A}[\mathbf{KBN}]_\Lambda$ has always trivial coefficients (up to a factor $\boldsymbol{\alpha}$), and

$$\mathbf{mult}^{\mathtt{Q}}_{D_l,D_{l+1}}(D_l^{\mathrm{or}}) = \mathrm{coeff}(\mathtt{Q}) \cdot \tilde{D}^{\mathrm{or}}_{l+1} + \mathrm{REST}$$

(where $\mathrm{coeff}(\mathtt{Q})$ is the coefficient coming from $\mathfrak{A}[\mathtt{Q}]_\Lambda$), this amounts to prove

$$\mathrm{coeff}_{D_l}(D_l^{\mathrm{or}}) \cdot \mathrm{coeff}(\mathtt{Q}) = \mathrm{coeff}_{D_{l+1}}(\tilde{D}^{\mathrm{or}}_{l+1}) \tag{74}$$

(up to a factor $\boldsymbol{\alpha}$ which always appears on neither side or on both sides of (74)).

To this end, we need to check the same cases (i)-(iv) as in the proof of Theorem 4.7.

In contrast to the situation of Theorem 4.7, we additionally need to distinguish the cases with different orientations of the circles in question (all circles not involved in the surgery from D_l to D_{l+1} remain unchanged, and we ignore them in the following).

►**Non-nested merge.** Assume that circles C_i and C_j are merged into a circle C. In this case we have (as one easily sees)

$$\overset{\mathrm{ex}}{\smile}(C_i) \cup \underset{\mathrm{ex}}{\frown}(C_i) \cup \overset{\mathrm{ex}}{\smile}(C_j) \cup \underset{\mathrm{ex}}{\frown}(C_j) = \overset{\mathrm{ex}}{\smile}(C) \cup \underset{\mathrm{ex}}{\frown}(C). \tag{75}$$

For an example see (36). Now let us look at possible orientations.

Both, C_i and C_j, are oriented anticlockwise. By (75), we directly obtain

$$\mathrm{coeff}(C_i^{\mathrm{anti}}) \cdot \mathrm{coeff}(C_j^{\mathrm{anti}}) = \mathrm{coeff}(C^{\mathrm{anti}}). \tag{76}$$

Since $\mathrm{coeff}(\mathtt{Q}) = 1$ in this case, we see that (74) holds.

One circle is oriented anticlockwise, the other clockwise. If C_i is oriented clockwise, then the left-hand side of (76) picks up the coefficient $\boldsymbol{\varepsilon}^{\mathrm{d}_\Lambda(\gamma_i^{\mathrm{dot}})} = \boldsymbol{\varepsilon}^{\mathrm{t}(C)-\mathrm{t}(C_i)}$ from the multiplication rule for $\mathbf{mult}^{\mathtt{Q}}_{D_l,D_{l+1}}$. We again obtain (74):

$$\begin{aligned} & \mathrm{coeff}(C_i^{\mathrm{cl}}) \cdot \mathrm{coeff}(C_j^{\mathrm{anti}}) \cdot \boldsymbol{\varepsilon}^{\mathrm{t}(C)-\mathrm{t}(C_i)} \\ \overset{(51)}{=} \; & \mathrm{coeff}(C_i^{\mathrm{anti}}) \cdot \boldsymbol{\varepsilon}^{\mathrm{t}(C_i)} \cdot \mathrm{coeff}(C_j^{\mathrm{anti}}) \cdot \boldsymbol{\varepsilon}^{\mathrm{t}(C)-\mathrm{t}(C_i)} \\ \overset{(76)}{=} \; & \mathrm{coeff}(C^{\mathrm{anti}}) \cdot \boldsymbol{\varepsilon}^{\mathrm{t}(C)} \\ \overset{(51)}{=} \; & \mathrm{coeff}(C^{\mathrm{cl}}). \end{aligned}$$

The case of C_j being clockwise and C_i being anticlockwise instead is similar.

Both, C_i and C_j, are oriented clockwise. In this case we have

$$\begin{aligned}
&\mathrm{coeff}(C_i^{\mathrm{cl}})\cdot\mathrm{coeff}(C_j^{\mathrm{cl}})\cdot\boldsymbol{\alpha}\cdot\boldsymbol{\varepsilon}^{\mathrm{t}(C)-\mathrm{t}(C_i)}\cdot\boldsymbol{\varepsilon}^{\mathrm{t}(C)-\mathrm{t}(C_j)}\\
\overset{(51)}{=}\;&\mathrm{coeff}(C_i^{\mathrm{anti}})\cdot\mathrm{coeff}(C_j^{\mathrm{anti}})\cdot\boldsymbol{\alpha}\\
\overset{(76)}{=}\;&\mathrm{coeff}(C^{\mathrm{anti}})\cdot\boldsymbol{\alpha},
\end{aligned}$$

which again give (74), because the multiplication rule for $\mathbf{mult}^{\mathtt{Q}}_{D_l,D_{l+1}}$ picks up the coefficient $\mathrm{coeff}(\mathtt{Q})=\boldsymbol{\alpha}\boldsymbol{\varepsilon}^{\mathrm{d}_\Lambda(\gamma_i^{\mathrm{dot}})}\boldsymbol{\varepsilon}^{\mathrm{d}_\Lambda(\gamma_j^{\mathrm{dot}})}=\boldsymbol{\alpha}\boldsymbol{\varepsilon}^{\mathrm{t}(C)-\mathrm{t}(C_i)}\boldsymbol{\varepsilon}^{\mathrm{t}(C)-\mathrm{t}(C_j)}$. ▷

►**Nested merge.** In this case two nested circles, C_{out} and C_{in}, are merged into one circle C. In the nested situation (also for the nested split below) the notion of exterior and interior swaps for the nested circle C_{in}. Moreover, in case of the nested merge, the cup-cap pair involved in the surgery is of the form $\cup_{\mathrm{ex}}$-$\cap_{\mathrm{in}}$ or of the form $\cup_{\mathrm{in}}$-$\cap_{\mathrm{ex}}$ (and hence, is "lost" after the surgery). That is, we have altogether

$$(77)\qquad (\cup_{\mathrm{ex}}(C_{\mathrm{out}})\cup\cap_{\mathrm{ex}}(C_{\mathrm{out}})\cup\cup_{\mathrm{in}}(C_{\mathrm{in}})\cup\cap_{\mathrm{in}}(C_{\mathrm{in}}))\setminus\mathrm{surg}=\cup_{\mathrm{ex}}(C)\cup\cap_{\mathrm{ex}}(C).$$

Here "surg" is the set containing the cup-cap of the surgery. For an example see (37).

Both, C_{out} and C_{in}, are oriented anticlockwise. First note that we get the coefficient $\mathrm{coeff}(\mathtt{Q})=\boldsymbol{\varepsilon}\cdot\boldsymbol{\varepsilon}^{\frac{1}{4}(\mathrm{d}_\Lambda(C_{\mathrm{in}})-2)}\cdot\boldsymbol{\varepsilon}^{s_\Lambda(\gamma)}$ from $\mathbf{mult}^{\mathtt{Q}}_{D_l,D_{l+1}}$. We get (74):

$$(78)\qquad\begin{aligned}
&\mathrm{coeff}(C_{\mathrm{out}}^{\mathrm{anti}})\cdot\mathrm{coeff}(C_{\mathrm{in}}^{\mathrm{anti}})\cdot\boldsymbol{\varepsilon}\cdot\boldsymbol{\varepsilon}^{\frac{1}{4}(\mathrm{d}_\Lambda(C_{\mathrm{in}})-2)}\cdot\boldsymbol{\varepsilon}^{s_\Lambda(\gamma)}\\
=\;&\mathrm{coeff}(C_{\mathrm{out}}^{\mathrm{anti}})\cdot\prod_{\gamma'\in\cup_{\mathrm{ex}}(C_{\mathrm{in}})}\boldsymbol{\varepsilon}^{(s_\Lambda(\gamma')+1)\mathrm{p}_\Lambda(\gamma')}\cdot\prod_{\gamma'\in\cap_{\mathrm{ex}}(C_{\mathrm{in}})}\boldsymbol{\varepsilon}^{s_\Lambda(\gamma')(\mathrm{p}_\Lambda(\gamma')+1)}\\
&\cdot\prod_{\gamma'\in\cup_{\mathrm{ex}}(C_{\mathrm{in}})}\boldsymbol{\omega}^{-s_\Lambda(\gamma')}\cdot\prod_{\gamma'\in\cap_{\mathrm{ex}}(C_{\mathrm{in}})}\boldsymbol{\omega}^{s_\Lambda(\gamma')-1}\cdot\boldsymbol{\varepsilon}\cdot\boldsymbol{\varepsilon}^{\frac{1}{4}(\mathrm{d}_\Lambda(C_{\mathrm{in}})-2)}\cdot\boldsymbol{\varepsilon}^{s_\Lambda(\gamma)}\\
\overset{(\mathrm{I})}{=}\;&\mathrm{coeff}_{\boldsymbol{\varepsilon}}(C_{\mathrm{out}}^{\mathrm{anti}})\cdot\prod_{\gamma'\in\cup_{\mathrm{in}}(C_{\mathrm{in}})}\boldsymbol{\varepsilon}^{(s_\Lambda(\gamma')+1)\mathrm{p}_\Lambda(\gamma')}\cdot\prod_{\gamma'\in\cap_{\mathrm{in}}(C_{\mathrm{in}})}\boldsymbol{\varepsilon}^{s(\gamma')(\mathrm{p}_\Lambda(\gamma')+1)}\\
&\cdot\mathrm{coeff}_{\boldsymbol{\omega}}(C_{\mathrm{out}}^{\mathrm{anti}})\cdot\prod_{\gamma'\in\cup_{\mathrm{ex}}(C_{\mathrm{in}})}\boldsymbol{\omega}^{-s_\Lambda(\gamma')}\cdot\prod_{\gamma'\in\cap_{\mathrm{ex}}(C_{\mathrm{in}})}\boldsymbol{\omega}^{s_\Lambda(\gamma')-1}\cdot\boldsymbol{\varepsilon}^{\mathrm{p}_\Lambda(\gamma)+s_\Lambda(\gamma)}\\
\overset{(\mathrm{II})}{=}\;&\mathrm{coeff}_{\boldsymbol{\varepsilon}}(C_{\mathrm{out}}^{\mathrm{anti}})\cdot\prod_{\gamma'\in\cup_{\mathrm{in}}(C_{\mathrm{in}})}\boldsymbol{\varepsilon}^{(s_\Lambda(\gamma')+1)\mathrm{p}_\Lambda(\gamma')}\cdot\prod_{\gamma'\in\cap_{\mathrm{in}}(C_{\mathrm{in}})}\boldsymbol{\varepsilon}^{s_\Lambda(\gamma')(\mathrm{p}_\Lambda(\gamma')+1)}\\
&\cdot\mathrm{coeff}_{\boldsymbol{\omega}}(C_{\mathrm{out}}^{\mathrm{anti}})\cdot\prod_{\gamma'\in\cup_{\mathrm{in}}(C_{\mathrm{in}})}\boldsymbol{\omega}^{-s_\Lambda(\gamma')}\cdot\prod_{\gamma'\in\cap_{\mathrm{in}}(C_{\mathrm{in}})}\boldsymbol{\omega}^{s_\Lambda(\gamma')-1}\cdot\boldsymbol{\varepsilon}^{\mathrm{p}_\Lambda(\gamma)+s_\Lambda(\gamma)}\cdot\boldsymbol{\omega}\\
\overset{(77)}{=}\;&\mathrm{coeff}_{\boldsymbol{\varepsilon}}(C^{\mathrm{anti}})\cdot\mathrm{coeff}_{\boldsymbol{\omega}}(C^{\mathrm{anti}})=\mathrm{coeff}(C^{\mathrm{anti}}).
\end{aligned}$$

Here (I) follows from Lemmas 6.2 and 6.4 (since $\boldsymbol{\varepsilon}=\pm1$), and (II) from Lemma 6.3. Moreover, note that $\boldsymbol{\varepsilon}^{\mathrm{p}_\Lambda(\gamma)+s_\Lambda(\gamma)}\boldsymbol{\omega}$ is the inverse of the coefficient coming from the cup-cap pair in the surgery (counting them both).

C_{out} is oriented clockwise and C_{in} anticlockwise. In this case both sides are just multiplied with $\boldsymbol{\varepsilon}^{\mathrm{t}(C)}=\boldsymbol{\varepsilon}^{\mathrm{t}(C_{\mathrm{out}})}$. Hence, the calculation from (78) gives

$$\mathrm{coeff}(C_{\mathrm{out}}^{\mathrm{cl}})\cdot\mathrm{coeff}(C_{\mathrm{in}}^{\mathrm{anti}})\cdot\boldsymbol{\varepsilon}\cdot\boldsymbol{\varepsilon}^{\frac{1}{4}(\mathrm{d}_\Lambda(C_{\mathrm{in}})-2)}\cdot\boldsymbol{\varepsilon}^{s_\Lambda(\gamma)}=\mathrm{coeff}(C^{\mathrm{cl}}).$$

Thus, we again obtain (74), since $\mathbf{mult}^{\mathtt{Q}}_{D_l,D_{l+1}}$ does not give extra factors additionally to the ones from above.

C_{in} is oriented clockwise and C_{out} anticlockwise. In this case the coefficient of C is multiplied with $\boldsymbol{\varepsilon}^{\mathrm{t}(C)}$, while the one for C_{in} is multiplied with $\boldsymbol{\varepsilon}^{\mathrm{t}(C_{\mathrm{in}})}$. But in

addition the multiplication also introduces a dot moving. Hence, by (78),

$$\mathrm{coeff}(C_{\mathrm{out}}^{\mathrm{anti}}) \cdot \mathrm{coeff}(C_{\mathrm{in}}^{\mathrm{cl}}) \cdot \boldsymbol{\varepsilon} \cdot \boldsymbol{\varepsilon}^{\mathrm{t}(C)-\mathrm{t}(C_{\mathrm{in}})} \cdot \boldsymbol{\varepsilon}^{\frac{1}{4}(\mathrm{d}_\Lambda(C_{\mathrm{in}})-2)} \cdot \boldsymbol{\varepsilon}^{s_\Lambda(\gamma)} = \mathrm{coeff}(C^{\mathrm{cl}}), \tag{79}$$

which again gives (74), since $\mathbf{mult}^{\mathtt{Q}}_{D_l,D_{l+1}}$ gives, additionally to the factors from above, the extra coefficient $\boldsymbol{\varepsilon}^{\mathrm{d}_\Lambda(\gamma_{\mathrm{in}}^{\mathrm{dot}})} = \boldsymbol{\varepsilon}^{\mathrm{t}(C)-\mathrm{t}(C_{\mathrm{in}})}$.

Both, C_{in} and C_{out}, are oriented clockwise. In this case we obtain two dot moving signs, but the one for C_{out} is, as before, equal to 1. Thus, we obtain the same as in (79), but multiplied on both sides with $\boldsymbol{\alpha} \cdot \boldsymbol{\varepsilon}^{\mathrm{t}(C)}$ which shows (74). ▷

▶**Non-nested split.** In this case a circle C is split into two non-nested circles C_i and C_j (containing the vertexes at positions i or j). We clearly have

$$\cup^{\mathrm{ex}}(C) \cup \cap^{\mathrm{ex}}(C) = \cup^{\mathrm{ex}}(C_i) \cup \cap^{\mathrm{ex}}(C_i) \cup \cup^{\mathrm{ex}}(C_j) \cup \cap^{\mathrm{ex}}(C_j) \cup \mathrm{surg}, \tag{80}$$

where "surg" is as above. For an example see (38).

C is oriented anticlockwise. By (80), we get

$$\mathrm{coeff}(C^{\mathrm{anti}}) = \mathrm{coeff}(C_i^{\mathrm{anti}}) \cdot \mathrm{coeff}(C_j^{\mathrm{anti}}) \cdot \boldsymbol{\varepsilon}^{\mathrm{p}_\Lambda(\gamma)+s_\Lambda(\gamma)} \cdot \boldsymbol{\omega}^{-1}, \tag{81}$$

since, as above, $\boldsymbol{\varepsilon}^{\mathrm{p}_\Lambda(\gamma)+s_\Lambda(\gamma)}\boldsymbol{\omega}^{-1}$ is the coefficient coming from the cup-cap pair in the surgery (recalling that $\boldsymbol{\varepsilon} = \pm 1$). Now, we have $\mathrm{coeff}(\mathtt{Q}) = \boldsymbol{\omega}\boldsymbol{\varepsilon}^{\mathrm{d}_\Lambda(\gamma_i^{\mathrm{ndot}})}\boldsymbol{\varepsilon}^{s_\Lambda(\gamma)}$. By Lemma 6.2 and $\boldsymbol{\varepsilon} = \pm 1$ we have $\boldsymbol{\varepsilon}^{\mathrm{d}_\Lambda(\gamma_i^{\mathrm{ndot}})} = \boldsymbol{\varepsilon}^{\mathrm{t}(C_i)-\mathrm{p}_\Lambda(\gamma)}$. This in turn gives

$$\begin{aligned}
&\mathrm{coeff}(C^{\mathrm{anti}}) \cdot \boldsymbol{\omega} \cdot \boldsymbol{\varepsilon}^{\mathrm{d}_\Lambda(\gamma_i^{\mathrm{ndot}})} \cdot \boldsymbol{\varepsilon}^{s_\Lambda(\gamma)} \\
\overset{(81)}{=}\ &\mathrm{coeff}(C_i^{\mathrm{anti}}) \cdot \mathrm{coeff}(C_j^{\mathrm{anti}}) \cdot \boldsymbol{\omega} \cdot \boldsymbol{\varepsilon}^{\mathrm{t}(C_i)-\mathrm{p}_\Lambda(\gamma)} \cdot \boldsymbol{\varepsilon}^{s_\Lambda(\gamma)} \cdot \boldsymbol{\varepsilon}^{\mathrm{p}_\Lambda(\gamma)+s_\Lambda(\gamma)} \cdot \boldsymbol{\omega}^{-1} \\
=\ &\mathrm{coeff}(C_i^{\mathrm{anti}}) \cdot \boldsymbol{\varepsilon}^{\mathrm{t}(C_i)} \cdot \mathrm{coeff}(C_j^{\mathrm{anti}}) \overset{(51)}{=} \mathrm{coeff}(C_i^{\mathrm{cl}}) \cdot \mathrm{coeff}(C_j^{\mathrm{anti}}).
\end{aligned}$$

The term with C_j is oriented clockwise instead is dealt with completely analogously using the fact that $\boldsymbol{\varepsilon}^{\mathrm{p}_\Lambda(j)} = \boldsymbol{\varepsilon}^{\mathrm{p}_\Lambda(\gamma)+1}$ (by definition). We obtain (74).

C is oriented clockwise. We first compare the coefficients for the term where both, C_i and C_j, are oriented clockwise (thus, $\mathrm{coeff}(\mathtt{Q}) = \boldsymbol{\omega}\boldsymbol{\varepsilon}^{\mathrm{d}_\Lambda(\gamma_j^{\mathrm{dot}})}\boldsymbol{\varepsilon}^{\mathrm{d}_\Lambda(\gamma_i^{\mathrm{ndot}})}\boldsymbol{\varepsilon}^{s_\Lambda(\gamma)}$) and obtain by rewriting the dot moving signs similar as above (using $\boldsymbol{\varepsilon} = \pm 1$)

$$\begin{aligned}
&\mathrm{coeff}(C^{\mathrm{cl}}) \cdot \boldsymbol{\omega} \cdot \boldsymbol{\varepsilon}^{\mathrm{d}_\Lambda(\gamma_j^{\mathrm{dot}})} \cdot \boldsymbol{\varepsilon}^{\mathrm{d}_\Lambda(\gamma_i^{\mathrm{ndot}})} \cdot \boldsymbol{\varepsilon}^{s_\Lambda(\gamma)} \\
=\ &\mathrm{coeff}(C^{\mathrm{cl}}) \cdot \boldsymbol{\omega} \cdot \boldsymbol{\varepsilon}^{\mathrm{t}(C)-\mathrm{t}(C_j)} \cdot \boldsymbol{\varepsilon}^{\mathrm{t}(C_i)-\mathrm{p}_\Lambda(\gamma)} \cdot \boldsymbol{\varepsilon}^{s_\Lambda(\gamma)} \\
\overset{(51)}{=}\ &\mathrm{coeff}(C^{\mathrm{anti}}) \cdot \boldsymbol{\omega} \cdot \boldsymbol{\varepsilon}^{\mathrm{t}(C_j)} \cdot \boldsymbol{\varepsilon}^{\mathrm{t}(C_i)-\mathrm{p}_\Lambda(\gamma)} \cdot \boldsymbol{\varepsilon}^{s_\Lambda(\gamma)} \\
\overset{(81)}{=}\ &\mathrm{coeff}(C_i^{\mathrm{anti}}) \cdot \boldsymbol{\varepsilon}^{\mathrm{t}(C_i)} \cdot \mathrm{coeff}(C_j^{\mathrm{anti}}) \cdot \boldsymbol{\varepsilon}^{\mathrm{t}(C_j)} \\
\overset{(51)}{=}\ &\mathrm{coeff}(C_i^{\mathrm{cl}}) \cdot \mathrm{coeff}(C_j^{\mathrm{cl}}).
\end{aligned}$$

Hence, we have (74). For the term where both C_i and C_j are oriented anticlockwise (where we have $\mathrm{coeff}(\mathtt{Q}) = \boldsymbol{\alpha}\boldsymbol{\varepsilon}\boldsymbol{\omega}\boldsymbol{\varepsilon}^{\mathrm{d}_\Lambda(\gamma_j^{\mathrm{dot}})}\boldsymbol{\varepsilon}^{\mathrm{d}_\Lambda(\gamma_j^{\mathrm{ndot}})}\boldsymbol{\varepsilon}^{s_\Lambda(\gamma)}$) we obtain

$$\begin{aligned}
&\mathrm{coeff}(C^{\mathrm{cl}}) \cdot \boldsymbol{\alpha} \cdot \boldsymbol{\varepsilon} \cdot \boldsymbol{\omega} \cdot \boldsymbol{\varepsilon}^{\mathrm{d}_\Lambda(\gamma_j^{\mathrm{dot}})} \cdot \boldsymbol{\varepsilon}^{\mathrm{d}_\Lambda(\gamma_j^{\mathrm{ndot}})} \cdot \boldsymbol{\varepsilon}^{s_\Lambda(\gamma)} \\
=\ &\mathrm{coeff}(C^{\mathrm{cl}}) \cdot \boldsymbol{\alpha} \cdot \boldsymbol{\varepsilon} \cdot \boldsymbol{\omega} \cdot \boldsymbol{\varepsilon}^{\mathrm{t}(C_j)-\mathrm{p}_\Lambda(j)} \cdot \boldsymbol{\varepsilon}^{\mathrm{t}(C)-\mathrm{t}(C_j)} \cdot \boldsymbol{\varepsilon}^{s_\Lambda(\gamma)} \\
\overset{(51)}{=}\ &\mathrm{coeff}(C^{\mathrm{anti}}) \cdot \boldsymbol{\alpha} \cdot \boldsymbol{\varepsilon} \cdot \boldsymbol{\omega} \cdot \boldsymbol{\varepsilon}^{\mathrm{p}_\Lambda(j)} \cdot \boldsymbol{\varepsilon}^{s(\gamma)} \\
=\ &\mathrm{coeff}(C^{\mathrm{anti}}) \cdot \boldsymbol{\alpha} \cdot \boldsymbol{\varepsilon}^{\mathrm{p}_\Lambda(\gamma)+s_\Lambda(\gamma)} \cdot \boldsymbol{\omega} \\
\overset{(81)}{=}\ &\mathrm{coeff}(C_i^{\mathrm{anti}}) \cdot \mathrm{coeff}(C_j^{\mathrm{anti}}) \cdot \boldsymbol{\alpha},
\end{aligned}$$

where we again use the crucial fact that $\boldsymbol{\varepsilon} = \pm 1$. Thus, we obtain (74). ▷

►**Nested split.** In this case one circle C is split into two nested circles C_{out} and C_{in}. The steps are very similar to the case of the nested merge before, with the main difference that, instead of (77), we have

$$\text{cup}_{\mathrm{ex}}(C)\cup\text{cap}_{\mathrm{ex}}(C)=\text{cup}_{\mathrm{ex}}(C_{\mathrm{out}})\cup\text{cap}_{\mathrm{ex}}(C_{\mathrm{out}})\cup\text{cup}_{\mathrm{in}}(C_{\mathrm{in}})\cup\text{cap}_{\mathrm{in}}(C_{\mathrm{in}}). \tag{82}$$

For an example see (39). By (82), we obtain (with (III) similar as in (78))

$$\begin{aligned}
&\mathrm{coeff}(C^{\mathrm{anti}})\\
={}&\mathrm{coeff}_{\boldsymbol{\varepsilon}}(C^{\mathrm{anti}}_{\mathrm{out}})\cdot\prod_{\gamma'\in\text{cap}_{\mathrm{in}}(C_{\mathrm{in}})}\boldsymbol{\varepsilon}^{s_\Lambda(\gamma')(\mathrm{p}_\Lambda(\gamma')+1)}\cdot\prod_{\gamma'\in\text{cup}_{\mathrm{in}}(C_{\mathrm{in}})}\boldsymbol{\varepsilon}^{(s_\Lambda(\gamma')+1)\mathrm{p}_\Lambda(\gamma')}\\
&\cdot\ \mathrm{coeff}_{\boldsymbol{\omega}}(C^{\mathrm{anti}}_{\mathrm{out}})\cdot\prod_{\gamma'\in\text{cap}_{\mathrm{in}}(C_{\mathrm{in}})}\boldsymbol{\omega}^{s_\Lambda(\gamma')-1}\cdot\prod_{\gamma'\in\text{cup}_{\mathrm{in}}(C_{\mathrm{in}})}\boldsymbol{\omega}^{-s_\Lambda(\gamma')}\\
\overset{(\mathrm{III})}{=}{}&\mathrm{coeff}(C^{\mathrm{anti}}_{\mathrm{out}})\cdot\mathrm{coeff}(C^{\mathrm{anti}}_{\mathrm{in}})\cdot\boldsymbol{\omega}^{-1}\cdot\boldsymbol{\varepsilon}^{\frac{1}{4}(\mathrm{d}_\Lambda(C_{\mathrm{in}})-2)}\cdot\boldsymbol{\varepsilon}^{\mathrm{t}(C_{\mathrm{in}})}.
\end{aligned}$$

C is oriented anticlockwise. We have to multiply the coefficient $\mathrm{coeff}(C^{\mathrm{anti}})$ by

$$\mathrm{coeff}(\mathtt{Q})=\boldsymbol{\omega\varepsilon}^{\frac{1}{4}(\mathrm{d}_\Lambda(C_{\mathrm{in}})-2)}\quad\text{respectively}\quad\mathrm{coeff}(\mathtt{Q})=\boldsymbol{\varepsilon\omega\varepsilon}^{\frac{1}{4}(\mathrm{d}_\Lambda(C_{\mathrm{in}})-2)}$$

and compare it to the coefficient of $\mathrm{coeff}(C^{\mathrm{anti}}_{\mathrm{out}})\mathrm{coeff}(C^{\mathrm{cl}}_{\mathrm{in}})$ respectively to the coefficient of $\mathrm{coeff}(C^{\mathrm{cl}}_{\mathrm{out}})\mathrm{coeff}(C^{\mathrm{anti}}_{\mathrm{in}})$. In both cases (74) follows then by Lemma 6.4.

C is oriented clockwise. This is done in an analogous way. Since we have $\mathrm{coeff}(C^{\mathrm{cl}})=\mathrm{coeff}(C^{\mathrm{anti}})\boldsymbol{\varepsilon}^{\mathrm{t}(C_{\mathrm{out}})}$, this fits for both appearing terms. ▷

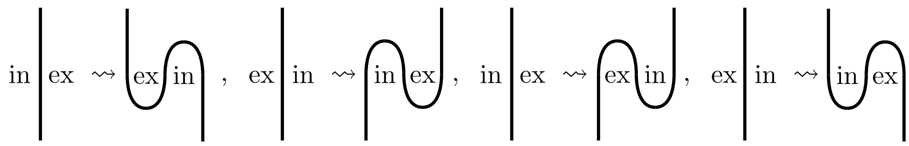
Taking everything together proves the theorem. □

Some rather dull lemmas needed for the proof of Proposition 4.15. We fix $\Lambda\in\mathtt{bl}^{\diamond}$.

Lemma 6.1. For a circle $C\in D$ it holds that

$$\#\left(\text{cup}_{\mathrm{ex}}(C)\right)+1=\#\left(\text{cap}_{\mathrm{in}}(C)\right)\quad\text{and}\quad\#\left(\text{cap}_{\mathrm{ex}}(C)\right)+1=\#\left(\text{cup}_{\mathrm{in}}(C)\right). \tag{83}$$ ■

Proof. This is clear for a circle containing only a single cup and cap. Any other circle can be constructed from such a small circle by successively adding "zig-zags":

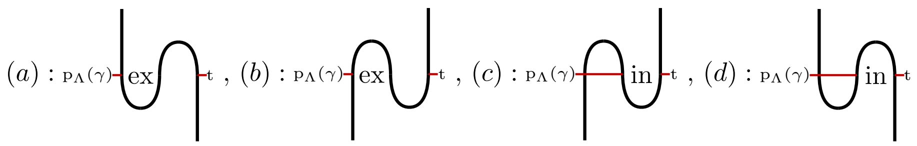


This increases both sides of the equalities from (83) by 0 or 1. The claim follows. □

Lemma 6.2. Let C be any circle in a stacked diagram D.

(a) If $\gamma\in\text{cup}_{\mathrm{ex}}(C)$, then $\mathrm{p}_\Lambda(\gamma)\equiv\mathrm{t}(C)\bmod 2$.
(b) If $\gamma\in\text{cap}_{\mathrm{ex}}(C)$, then $\mathrm{p}_\Lambda(\gamma)\equiv\mathrm{t}(C)\bmod 2$.
(c) If $\gamma\in\text{cup}_{\mathrm{in}}(C)$, then $\mathrm{p}_\Lambda(\gamma)\equiv\mathrm{t}(C)+1\bmod 2$.
(d) If $\gamma\in\text{cap}_{\mathrm{in}}(C)$, then $\mathrm{p}_\Lambda(\gamma)\equiv\mathrm{t}(C)+1\bmod 2$. ■

Proof. All four statements are clear for a circle C' with a single cup and cap. The circle C is obtained by adding successively "zig-zags" to C'. Adding such a zig-zag somewhere gives the following (we have illustrated where to read off $\mathrm{p}_\Lambda(\gamma)$ and t)



Observe that t might not be the rightmost point $\mathrm{t}(C)$ on the circle C. But since clearly $\mathrm{t}\equiv\mathrm{t}(C)$ mod 2, these do not change the congruences and we are done. □

Lemma 6.3. Let C be any circle in a stacked diagram D. Then

$$-\sum_{\gamma\in\cup^{\mathrm{ex}}(C)} s_\Lambda(\gamma)+\sum_{\gamma\in\cap^{\mathrm{ex}}(C)}(s_\Lambda(\gamma)-1)=1-\sum_{\gamma\in\cup^{\mathrm{in}}(C)} s_\Lambda(\gamma)+\sum_{\gamma\in\cap^{\mathrm{in}}(C)}(s_\Lambda(\gamma)-1). \qquad \blacksquare$$

Proof. We have (which follows by comparison of definitions)

$$\sum_{\gamma\in\cup^{\mathrm{ex}}(C)} s_\Lambda(\gamma)+\sum_{\gamma\in\cap^{\mathrm{in}}(C)}(s_\Lambda(\gamma)-1)=\tfrac{1}{4}\left(\mathrm{d}_\Lambda(C)-2\right),$$
$$\sum_{\gamma\in\cap^{\mathrm{ex}}(C)} s_\Lambda(\gamma)+\sum_{\gamma\in\cup^{\mathrm{in}}(C)}(s_\Lambda(\gamma)-1)=\tfrac{1}{4}\left(\mathrm{d}_\Lambda(C)-2\right).$$

Now we apply Lemma 6.1. □

Lemma 6.4. Let $C\in D$. Then

$$\begin{aligned}&\sum_{\gamma\in\cup^{\mathrm{ex}}(C_{\mathrm{in}})}(s_\Lambda(\gamma)+1)\mathrm{p}_\Lambda(\gamma)+\sum_{\gamma\in\cap^{\mathrm{ex}}(C_{\mathrm{in}})} s_\Lambda(\gamma)(\mathrm{p}_\Lambda(\gamma)+1)+\mathrm{t}(C)\\ \equiv\ &\sum_{\gamma\in\cup^{\mathrm{in}}(C_{\mathrm{in}})}(s_\Lambda(\gamma)+1)\mathrm{p}_\Lambda(\gamma)+\sum_{\gamma\in\cap^{\mathrm{in}}(C_{\mathrm{in}})} s_\Lambda(\gamma)(\mathrm{p}_\Lambda(\gamma)+1)+\tfrac{1}{4}\left(\mathrm{d}_\Lambda(C)-2\right)\ \mathrm{mod}\ 2. \quad \blacksquare\end{aligned}$$

Proof. Via a direct calculation: one starts with the first line and rewrites all $\mathrm{p}_\Lambda(\gamma)$ in terms of $\mathrm{t}(C)$ using Lemma 6.2. Then we use the same equalities as in the proof of Lemma 6.3. Finally one has to use Lemma 6.1 to arrive at the second line. □

Proof of Proposition 4.19. We use the notation from Subsection 4.3 in the proof of Proposition 4.19 given below.

Proof. The proof is done in complete analogy to the proof of Proposition 4.15. We show that in each step the coefficient maps defined above for stacked diagrams intertwine the multiplication steps, i.e. in each step Equality (74) holds true. Since the coefficient map is only modified slightly, it is clear that all arguments for the non-nested merge and non-nested split are valid in the exact same way as before. For the nested cases the swap of exterior and interior of the inner circle C_{in} is more involved. We illustrate this by giving the proof for the nested merge situation.

►**Nested merge.** Two nested circles C_{out} and C_{in} are merged into one circle C. As in the proof of Proposition 4.15, the notion of exterior and interior swaps for the nested circle C_{in}. Overall the situation is similar in the sense that the cup-cap pair involved in the surgery is of the form $\cup^{\mathrm{ex}}$ - $\cap^{\mathrm{in}}$ or of the form $\cup^{\mathrm{in}}$ - $\cap^{\mathrm{ex}}$. That is, we have

$$(\cup^{\mathrm{ex}}(C_{\mathrm{out}})\cup\cap^{\mathrm{ex}}(C_{\mathrm{out}})\cup\cup^{\mathrm{in}}(C_{\mathrm{in}})\cup\cap^{\mathrm{in}}(C_{\mathrm{in}}))\setminus\mathrm{surg}=\cup^{\mathrm{ex}}(C)\cup\cap^{\mathrm{ex}}(C), \tag{84}$$

where "surg" is the set containing the cup and cap of the surgery, and we have

$$\times_{\mathrm{ex}}^{\searrow}(C_{\mathrm{out}})\cup\times_{\mathrm{ex}}^{\nearrow}(C_{\mathrm{out}})\cup\times_{\mathrm{in}}^{\searrow}(C_{\mathrm{in}})\cup\times_{\mathrm{in}}^{\nearrow}(C_{\mathrm{in}})=\times_{\mathrm{ex}}^{\searrow}(C)\cup\times_{\mathrm{ex}}^{\nearrow}(C). \tag{85}$$

Both, C_{out} and C_{in}, are oriented anticlockwise. Similarly as before, we get (74):

$$
\begin{aligned}
& \mathrm{coeff}_{\boldsymbol{\varepsilon}}(C^{\mathrm{anti}}_{\mathrm{out}}) \cdot \mathrm{coeff}_{\boldsymbol{\varepsilon}}(C^{\mathrm{anti}}_{\mathrm{in}}) \cdot \boldsymbol{\varepsilon} \cdot \boldsymbol{\varepsilon}^{\frac{1}{4}(\mathrm{d}_\Lambda(C_{\mathrm{in}})-2)} \cdot \boldsymbol{\varepsilon}^{s_\Lambda(\gamma)} \\
= \; & \mathrm{coeff}_{\boldsymbol{\varepsilon}}(C^{\mathrm{anti}}_{\mathrm{out}}) \cdot \prod_{\gamma' \in \cup_{\mathrm{ex}}(C_{\mathrm{in}})} \boldsymbol{\varepsilon}^{(s_\Lambda(\gamma')+1)\mathrm{p}_\Lambda(\gamma')} \cdot \prod_{\gamma' \in \cap_{\mathrm{ex}}(C_{\mathrm{in}})} \boldsymbol{\varepsilon}^{s_\Lambda(\gamma')(\mathrm{p}_\Lambda(\gamma')+1)} \\
& \cdot \prod_{\gamma' \in {}_{\mathrm{ex}}\diagdown^{\times}(C_{\mathrm{in}})} \boldsymbol{\varepsilon}^{\mathrm{p}_\Lambda(\gamma')} \cdot \prod_{\gamma' \in {}^{\times}\diagup_{\mathrm{ex}}(C_{\mathrm{in}})} \boldsymbol{\varepsilon}^{\mathrm{p}_\Lambda(\gamma')+1} \cdot \boldsymbol{\varepsilon} \cdot \boldsymbol{\varepsilon}^{\frac{1}{4}(\mathrm{d}_\Lambda(C_{\mathrm{in}})-2)} \cdot \boldsymbol{\varepsilon}^{s_\Lambda(\gamma)} \\
\overset{(\mathrm{I})}{=} \; & \mathrm{coeff}_{\boldsymbol{\varepsilon}}(C^{\mathrm{anti}}_{\mathrm{out}}) \cdot \prod_{\gamma' \in \cup_{\mathrm{in}}(C_{\mathrm{in}})} \boldsymbol{\varepsilon}^{(s_\Lambda(\gamma')+1)\mathrm{p}_\Lambda(\gamma')} \cdot \prod_{\gamma' \in \cap_{\mathrm{in}}(C_{\mathrm{in}})} \boldsymbol{\varepsilon}^{s_\Lambda(\gamma')(\mathrm{p}_\Lambda(\gamma')+1)} \\
& \cdot \prod_{\gamma' \in {}_{\mathrm{in}}\diagdown^{\times}(C_{\mathrm{in}})} \boldsymbol{\varepsilon}^{\mathrm{p}_\Lambda(\gamma')} \cdot \prod_{\gamma' \in {}^{\times}\diagup_{\mathrm{in}}(C_{\mathrm{in}})} \boldsymbol{\varepsilon}^{\mathrm{p}_\Lambda(\gamma')+1} \cdot \boldsymbol{\varepsilon}^{\mathrm{p}_\Lambda(\gamma)+s_\Lambda(\gamma)} \\
\overset{(84)+(85)}{=} & \mathrm{coeff}_{\boldsymbol{\varepsilon}}(C^{\mathrm{anti}}),
\end{aligned}
$$

$$
\begin{aligned}
& \mathrm{coeff}_{\boldsymbol{\omega}}(C^{\mathrm{anti}}_{\mathrm{out}}) \cdot \mathrm{coeff}_{\boldsymbol{\omega}}(C^{\mathrm{anti}}_{\mathrm{in}}) \\
= \; & \mathrm{coeff}_{\boldsymbol{\omega}}(C^{\mathrm{anti}}_{\mathrm{out}}) \cdot \prod_{\gamma' \in \cup_{\mathrm{ex}}(C_{\mathrm{in}})} \boldsymbol{\omega}^{-s_\Lambda(\gamma')} \cdot \prod_{\gamma' \in \cap_{\mathrm{ex}}(C_{\mathrm{in}})} \boldsymbol{\omega}^{s_\Lambda(\gamma')-1} \cdot \boldsymbol{\omega}^{\#\left({}_{\mathrm{ex}}\diagdown^{\times}(C) \cup {}^{\times}\diagup_{\mathrm{ex}}(C)\right)} \\
\overset{(\mathrm{II})}{=} \; & \mathrm{coeff}_{\boldsymbol{\omega}}(C^{\mathrm{anti}}_{\mathrm{out}}) \cdot \prod_{\gamma' \in \cup_{\mathrm{in}}(C_{\mathrm{in}})} \boldsymbol{\omega}^{-s_\Lambda(\gamma')} \cdot \prod_{\gamma' \in \cap_{\mathrm{in}}(C_{\mathrm{in}})} \boldsymbol{\omega}^{s_\Lambda(\gamma')-1} \cdot \boldsymbol{\omega}^{\#\left({}_{\mathrm{in}}\diagdown^{\times}(C) \cup {}^{\times}\diagup_{\mathrm{in}}(C)\right)} \cdot \boldsymbol{\omega} \\
= \; & \mathrm{coeff}_{\boldsymbol{\omega}}(C^{\mathrm{anti}}).
\end{aligned}
$$

Here (I) follows from Lemmas 6.4 and 6.5, while (II) follow from Lemma 6.6. The arguments for the other orientations are as for Proposition 4.15. ▷

The claim of the proposition follows than analogously to Proposition 4.15. □

Some rather dull lemmas needed for the proof of Proposition 4.19. Again, fix $\Lambda \in \mathtt{bl}^{\diamond}$.

Lemma 6.5. Let C be any circle in a stacked diagram D.

(a) If $\gamma \in {}_{\mathrm{ex}}\diagdown^{\times}(C)$, then $\mathrm{p}_\Lambda(\gamma) \equiv \mathrm{t}(C) + 1 \bmod 2$.
(b) If $\gamma \in {}^{\times}\diagup_{\mathrm{ex}}(C)$, then $\mathrm{p}_\Lambda(\gamma) \equiv \mathrm{t}(C) \bmod 2$.
(c) If $\gamma \in {}^{\times}\diagup_{\mathrm{in}}(C)$, then $\mathrm{p}_\Lambda(\gamma) \equiv \mathrm{t}(C) + 1 \bmod 2$.
(d) If $\gamma \in {}_{\mathrm{in}}\diagdown^{\times}(C)$, then $\mathrm{p}_\Lambda(\gamma) \equiv \mathrm{t}(C) \bmod 2$. ■

Proof. Recall that the symbol $\times$ counts as being of length 2. Hence, moving across parts in (a)-(d) preserves the parity. Thus, the claim follows as in Lemma 6.2. □

Lemma 6.6. Let C be any circle in a stacked diagram D. Then

$$
-\sum_{\gamma \in \cup_{\mathrm{ex}}(C)} s_\Lambda(\gamma) + \sum_{\gamma \in \cap_{\mathrm{ex}}(C)} s_\Lambda(\gamma) + \sum_{\gamma \in {}^{\times}\diagup_{\mathrm{ex}}(C)} 1 + \sum_{\gamma \in {}_{\mathrm{ex}}\diagdown^{\times}(C)} 1 = \tfrac{1}{4}\left(\mathrm{d}_\Lambda(C) - 2\right),
$$

$$
-\sum_{\gamma \in \cup_{\mathrm{in}}(C)} (s_\Lambda(\gamma) - 1) + \sum_{\gamma \in \cap_{\mathrm{in}}(C)} (s_\Lambda(\gamma) - 1) + \sum_{\gamma \in {}^{\times}\diagup_{\mathrm{in}}(C)} 1 + \sum_{\gamma \in {}_{\mathrm{in}}\diagdown^{\times}(C)} 1 = \tfrac{1}{4}\left(\mathrm{d}_\Lambda(C) - 2\right).
$$

Proof. This follows immediately by interpreting $\frac{1}{4}(\mathrm{d}_\Lambda(C) - 2)$ as the number of internal phantom edges of the circle as done in [18, Lemma 4.10]. □

Two other dull, yet important, lemmas.

Lemma 6.7. For a stacked diagram D^{or} (with orientation) and a circle C in it, we have

$$\overline{\mathrm{coeff}}_{\boldsymbol{\varepsilon}}(C,D^{\mathrm{or}})=\mathrm{coeff}_{\boldsymbol{\varepsilon}}(C,D^{\mathrm{or}})\cdot\chi_{\boldsymbol{\varepsilon}}(C),\qquad \overline{\mathrm{coeff}}_{\boldsymbol{\omega}}(C,D^{\mathrm{or}})=\mathrm{coeff}_{\boldsymbol{\omega}}(C,D^{\mathrm{or}})\cdot\chi_{\boldsymbol{\omega}}(C),$$

where

$$\begin{aligned}
\chi_{\boldsymbol{\varepsilon}}(C) &= \prod_{\gamma\in\smile_{\mathrm{ex}}(C)\cup\smile_{\mathrm{in}}(C)} \boldsymbol{\varepsilon}^{\mathrm{p}_\Lambda(\gamma)+s_\Lambda(\gamma)}\\
&= \prod_{\gamma\in\frown_{\mathrm{ex}}(C)\cup\frown_{\mathrm{in}}(C)} \boldsymbol{\varepsilon}^{\mathrm{p}_\Lambda(\gamma)+s_\Lambda(\gamma)}\cdot\boldsymbol{\varepsilon}^{\#\left(\diagdown^{\times}_{\mathrm{ex}}(C)\cup\diagup^{\times}_{\mathrm{ex}}(C)\cup\diagdown^{\times}_{\mathrm{in}}(C)\cup\diagup^{\times}_{\mathrm{in}}(C)\right)},\\
\chi_{\boldsymbol{\omega}}(C) &= \boldsymbol{\omega}^{\#\left(\smile_{\mathrm{ex}}(C)\cup\smile_{\mathrm{in}}(C)\right)}=\boldsymbol{\omega}^{\#\left(\frown_{\mathrm{ex}}(C)\cup\frown_{\mathrm{in}}(C)\right)}.
\end{aligned}$$

■

Proof. For $\overline{\mathrm{coeff}}_{\boldsymbol{\varepsilon}}(C,D^{\mathrm{or}})$, after rewriting all positions with respect to the rightmost point by using Lemmas 6.2 and 6.5 (all congruences below are modulo 2), the left side is:

$$\begin{aligned}
&\sum_{\gamma\in\smile_{\mathrm{in}}(C)} s_\Lambda(\gamma)\mathrm{t}(C)+\sum_{\gamma\in\frown_{\mathrm{in}}(C)}(s_\Lambda(\gamma)-1)(\mathrm{t}(C)+1)\\
&+\sum_{\gamma\in\diagdown^{\times}_{\mathrm{in}}(C)}(\mathrm{t}(C)+1)+\sum_{\gamma\in\diagup^{\times}_{\mathrm{in}}(C)}(\mathrm{t}(C)+1)\\
\equiv\ &\left(\sum_{\gamma\in\smile_{\mathrm{in}}(C)}(s_\Lambda(\gamma)-1)+\sum_{\gamma\in\frown_{\mathrm{in}}(C)}(s_\Lambda(\gamma)-1)+\#\left(\diagdown^{\times}_{\mathrm{in}}(C)\cup\diagup^{\times}_{\mathrm{in}}(C)\right)\right)\mathrm{t}(C)\\
&+\#(\smile_{\mathrm{in}}(C))\mathrm{t}(C)+\sum_{\gamma\in\frown_{\mathrm{in}}(C)}(s_\Lambda(\gamma)-1)+\#\left(\diagdown^{\times}_{\mathrm{in}}(C)\cup\diagup^{\times}_{\mathrm{in}}(C)\right)\\
\overset{6.6}{\equiv}\ &\left(\sum_{\gamma\in\smile_{\mathrm{ex}}(C)}s_\Lambda(\gamma)+\sum_{\gamma\in\frown_{\mathrm{ex}}(C)}s_\Lambda(\gamma)+\#\left(\diagdown^{\times}_{\mathrm{ex}}(C)\cup\diagup^{\times}_{\mathrm{ex}}(C)\right)\right)\mathrm{t}(C)\\
&+\#(\smile_{\mathrm{in}}(C))\mathrm{t}(C)+\sum_{\gamma\in\frown_{\mathrm{in}}(C)}(s_\Lambda(\gamma)-1)+\#\left(\diagdown^{\times}_{\mathrm{in}}(C)\cup\diagup^{\times}_{\mathrm{in}}(C)\right).
\end{aligned}$$

Collecting all terms that belong to $\mathrm{coeff}_{\boldsymbol{\varepsilon}}(C,D^{\mathrm{or}})$ we are left with

$$\begin{aligned}
&\#\left(\diagdown^{\times}_{\mathrm{ex}}(C)\cup\diagup^{\times}_{\mathrm{ex}}(C)\cup\diagdown^{\times}_{\mathrm{in}}(C)\cup\diagup^{\times}_{\mathrm{in}}(C)\right)\\
&+\#(\frown_{\mathrm{ex}}(C))\mathrm{t}(C)+\sum_{\gamma\in\frown_{\mathrm{ex}}(C)}s_\Lambda(\gamma)+\#(\frown_{\mathrm{in}}(C))\mathrm{t}(C)+\sum_{\gamma\in\frown_{\mathrm{in}}(C)}(s_\Lambda(\gamma)-1)\\
\overset{6.2}{\equiv}\ &\#\left(\diagdown^{\times}_{\mathrm{ex}}(C)\cup\diagup^{\times}_{\mathrm{ex}}(C)\cup\diagdown^{\times}_{\mathrm{in}}(C)\cup\diagup^{\times}_{\mathrm{in}}(C)\right)\\
&+\sum_{\gamma\in\frown_{\mathrm{ex}}(C)}(\mathrm{p}_\Lambda(\gamma)+s_\Lambda(\gamma))+\sum_{\gamma\in\frown_{\mathrm{in}}(C)}(\mathrm{p}_\Lambda(\gamma)+s_\Lambda(\gamma)).
\end{aligned}$$

That this can be rewritten with respect to cups instead is just an application of Lemma 6.6 to both sums in the first line above.

Next, the (easier) $\boldsymbol{\omega}$-term: $\overline{\mathrm{coeff}_{\boldsymbol{\omega}}}(C, D^{\mathrm{or}})$ can be rewritten as

$$
\begin{aligned}
& - \sum_{\gamma \in \smile_{\mathrm{in}}(C)} (s_{\Lambda}(\gamma) - 1) + \sum_{\gamma \in \frown_{\mathrm{in}}(C)} s_{\Lambda}(\gamma) + \#\left({}_{\mathrm{in}}\diagdown^{\times}(C) \cup {}^{\times}\diagup_{\mathrm{in}}(C) \right) \\
\overset{6.6}{=} & - \sum_{\gamma \in \smile_{\mathrm{ex}}(C)} s_{\Lambda}(\gamma) + \sum_{\gamma \in \frown_{\mathrm{ex}}(C)} (s_{\Lambda}(\gamma) - 1) + \#\left({}_{\mathrm{ex}}\diagdown^{\times}(C) \cup {}^{\times}\diagup_{\mathrm{ex}}(C) \right) \\
& + \#\left(\frown_{\mathrm{ex}}(C) \cup \frown_{\mathrm{in}}(C) \right).
\end{aligned}
$$

The first three summands are the powers in $\mathrm{coeff}_{\boldsymbol{\omega}}(C, D^{\mathrm{or}})$, while the last term is the power in $\chi_{\boldsymbol{\omega}}(C)$. That $\chi_{\boldsymbol{\omega}}(C)$ can be written in the two ways is evident. $\square$

Proof of Proposition 4.34. The proof of Proposition 4.34 given below uses the notation from Subsection 4.5.

Proof. Let us write $\mathfrak{W}[a]$ etc. respectively $\mathfrak{W}[b]$, if we are in the generic situation or in case (a) respectively in case (b). We also write $\mathfrak{W}[*]$ etc. if we mean both cases (for simplicity of notation we extend scalars to Q).

By Proposition 2.43, it suffices to show, that $\mathrm{Hom}_{\mathfrak{W}[*]}(\mathcal{W}[*](u), \mathcal{W}[*](v))$ is a free Q-module of finite rank (for any two webs u, v), and then calculate its rank. The first task is fairly easy: since the web bimodules are free Q-modules of finite rank by Corollary 2.39, the same holds for $\mathrm{Hom}_{\mathfrak{W}[*]}(\mathcal{W}[*](u), \mathcal{W}[*](v))$ as well. The main difficulty is to "control" the number of $\mathfrak{W}[*]$-bimodule homomorphisms. We do so by analyzing the (decomposition) structure of the web bimodules.

To this end, recall from Subsection 4.1 that, given a web u, then we can associate to it a $\vec{\Lambda}$-composite matching $\mathrm{a}(u)$ by erasing orientations and phantom edges. Here choose any presentation of the associated $\vec{\Lambda}$-composite matching $\mathrm{a}(u)$ in terms of the basic moves from (40). From this we obtain an $\mathfrak{A}[*]$-bimodule $\mathcal{A}[*](\mathrm{a}(u))$ associated to $\mathcal{W}[*](u)$ (the concerned reader might want to check that different choices in terms of basic moves give isomorphic $\mathfrak{A}[*]$-bimodules).

►**Case (a).** The main ingredient in order to control the number of $\mathfrak{W}[a]$-bimodule homomorphisms is to first use the results from Subsection 4.1 to identify $\mathfrak{W}[a]$ and its web bimodules with $\mathfrak{A}[a]$ and its arc bimodules. Then further identify $\mathfrak{A}[a]$ with $\mathfrak{A}[\mathbf{KBN}]$ and their arc bimodules by using the results from Subsection 4.2 and 4.3. Hence, we can use statements obtained in [7] and [8] as we explain below (where we note that these work, mutatis mutandis, in the generic case as well).

Now, recall from the proof of Proposition 2.43 that

$$
2\mathrm{Hom}_{\mathfrak{F}[a]}(u, v) \cong 2\mathrm{Hom}_{\mathfrak{F}[a]}(\mathbf{1}_{2\omega_\ell}, \mathrm{clap}(u)\mathrm{clap}(v)^*)\{d(\vec{k})\}
$$

as graded, free Q-modules. Thus, using the cup foam basis and the translation to the side of $\mathfrak{A}[a]$ from Lemma 4.5, the (graded) rank of $2\mathrm{Hom}_{\mathfrak{F}[a]}(u, v)$ is precisely given by all orientations of the composite matching for $\mathrm{a}(\mathrm{clap}(u)\mathrm{clap}(v)^*)$ (and their degrees). Thus, we have to show the same on the side of $\mathfrak{A}[\mathbf{KBN}]$:

$$
\begin{aligned}
& \mathrm{rank}_Q \left(\mathrm{Hom}_{\mathfrak{A}[\mathbf{KBN}]}(\mathcal{A}[\mathbf{KBN}](\mathrm{a}(u)), \mathcal{A}[\mathbf{KBN}](\mathrm{a}(v))\{s\}) \right) \\
& \quad \overset{!}{=} \#\{\text{orientations of } \mathrm{a}(\mathrm{clap}(u)\mathrm{clap}(v)^*) \text{ of degree } s\}.
\end{aligned}
\tag{86}
$$

Assume first that neither u nor v have internal circles. Then $\mathrm{a}(u)$ and $\mathrm{a}(v)$ fit into the framework from [7, Section 4], i.e. [7, Theorems 3.6 and 4.14] show that

$$
\mathcal{A}[\mathbf{KBN}](\mathrm{a}(u)) \text{ is indecomposable iff } \mathrm{a}(u) \text{ does not contain internal circles.}
$$

It follows now from [8, Theorem 3.5] that (86) holds in case $\boldsymbol{\alpha} = 0$ and $R = \mathbb{C}$. Scrutiny of the arguments used in [7] and [8] shows that these work under the circumstances of case (a) as well.

If now C is any circle in u (and thus, in $\mathrm{a}(u)$), then

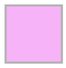
$$\mathcal{A}[\mathbf{KBN}](\mathrm{a}(u)) \cong \mathcal{A}[\mathbf{KBN}](\mathrm{a}(u){-}C)\{+1\} \oplus \mathcal{A}[\mathbf{KBN}](\mathrm{a}(u){-}C)\{-1\},$$

which follows as in [18, Example 3.22]. (Similarly for v.) By (26), the right-hand side of (86) behaves in the same way, i.e. for $w = u{-}C$ we have

$$\begin{aligned}&\#\{\text{orientations of } \mathrm{a}(\mathrm{clap}(u)\mathrm{clap}(v)^*) \text{ of degree } s\}\\ =&\#\{\text{orientations of } \mathrm{a}(\mathrm{clap}(w)\mathrm{clap}(v)^*) \text{ of degree } s+1\}\\ &+\#\{\text{orientations of } \mathrm{a}(\mathrm{clap}(w)\mathrm{clap}(v)^*) \text{ of degree } s-1\}.\end{aligned}$$

(Similarly for v.) The claim follows in case (a). $\triangleright$

►**Case (b).** As before, it suffices to study the case where u and v do not have internal circles. In this case $2\mathrm{End}_{\mathfrak{F}[b]}(u)$ has a basis which locally looks like

$\square$ and $\boxed{\bullet}$ and 

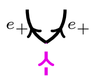
This can be shown by using the cup foam basis. Now, because of (9), (10) and (11), we do not need to worry about the phantom parts of any foam $f \in 2\mathrm{End}_{\mathfrak{F}[a]}(u)$, and we ignore these in what follows. A direct calculation shows that

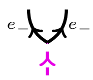
$$e_+ = \tfrac{1}{2}\left(\square + \sqrt{\mathsf{q}(\boldsymbol{\alpha})^{-1}} \cdot \boxed{\bullet}\right) \quad \text{and} \quad e_- = \tfrac{1}{2}\left(\square - \sqrt{\mathsf{q}(\boldsymbol{\alpha})^{-1}} \cdot \boxed{\bullet}\right)$$

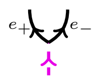
are idempotents satisfying $e_+e_- = 0 = e_-e_+$ and $1 = e_+ + e_-$. If u has c connected components (ignoring phantom edges, but counting both adjacent usual edges), then

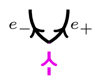
$$E = \{\vec{e} = (e_1, \dots, e_{2^c}) \mid e_i = e_\pm, i = 1, \dots, 2^c\} \setminus \{0\}$$

(some $\vec{e}$'s might be zero, see below) gives a complete set of pairwise orthogonal idempotents in $2\mathrm{End}_{\mathfrak{F}[b]}(u)$. Here the idempotents $\vec{e}$ are obtained by spreading the idempotents e_+ and e_- locally around a trivalent vertex as follows:

$\boldsymbol{\varepsilon} = 1$: 

and 

, $\boldsymbol{\varepsilon} = -1$: 

and 

This gives idempotents in the two different cases $\boldsymbol{\varepsilon} = \pm 1$ as one easily checks (all other possibilities of locally spreading the idempotents e_+ and e_- give zero). This shows that (with $w = \mathrm{clap}(u)\mathrm{clap}(v)^*$ being the "clapped web")

$$\begin{aligned}&\mathrm{rank}_Q(2\mathrm{Hom}_{\mathfrak{F}[b]}(u,v))\\ &= \#\{\text{number of non-zero "colorings" with idempotents } e_\pm \text{ of } w\}.\end{aligned} \tag{87}$$

Using E: as in [34, Proposition 3.13], one can show that $\mathfrak{W}[b]_{\vec{k}}$ is semisimple for all $\vec{k} \in \mathbb{bl}^\diamond$. In particular, a web bimodule $\mathcal{W}[b](u)$ for u having c connected components decomposes into pairwise non-isomorphic copies of Q. That is,

$$\mathcal{W}[b](u) \cong \bigoplus_{\vec{e} \in E} \vec{e}\,\mathcal{W}[b](u)\vec{e}.$$

Thus, the claim follows, since (with $w = \mathrm{clap}(u)\mathrm{clap}(v)^*$ being the "clapped web")

$$\begin{aligned}&\mathrm{rank}_Q(\mathrm{Hom}_{\mathfrak{W}[b]}(\mathcal{W}[b](u), \mathcal{W}[b](v)))\\ &= \#\{\text{number of non-zero "colorings" with idempotents } e_\pm \text{ of } w\}.\end{aligned}$$

Thus, by (87), $\mathrm{rank}_Q(2\mathrm{Hom}_{\mathfrak{F}[b]}(u,v)) = \mathrm{rank}_Q(\mathrm{Hom}_{\mathfrak{W}[b]}(\mathcal{W}[b](u), \mathcal{W}[b](v)))$. $\triangleright$

Altogether, this shows the claim. $\square$

Index of notation

For convenience we have listed our main notations used throughout.

KBN	**KBN** specializations
Ca	**Ca** specializations
CMW	**CMW** specializations
Bl	**Bl** specializations
$\mathtt{P}$	The set $\{\boldsymbol{\alpha},\boldsymbol{\tau}^{\pm 1},\boldsymbol{\omega}_{+}^{\pm 1},\boldsymbol{\omega}_{-}^{\pm 1}\}$
$\mathtt{Q}$	The set $\{\boldsymbol{\alpha},\boldsymbol{\varepsilon}^{\pm 1},\boldsymbol{\omega}^{\pm 1}\}$
$\boldsymbol{\alpha}$	The "two-dots parameter"
$\boldsymbol{\varepsilon}$	The "$\mathfrak{sl}_2$ vs. $\mathfrak{gl}_2$ parameter"
$\boldsymbol{\omega}$	The "gluing parameter"
P	The ring $\mathbb{Z}[\boldsymbol{\alpha},\boldsymbol{\tau}^{\pm 1},\boldsymbol{\omega}_{+}^{\pm 1},\boldsymbol{\omega}_{-}^{\pm 1}]$
Q	The ring $\mathbb{Z}[\boldsymbol{\alpha},\boldsymbol{\omega}^{\pm 1}]$
$\mathtt{p},\mathtt{q}$	Specialization maps
$\mathfrak{F}[\mathtt{P}]$	The general foam 2-category
$\mathbb{B}^{\circ}(\cdot)$	Various cup foam bases
$\mathfrak{F}_{\cdot}^{\cdot}[\cdot]$	Various specializations of $\mathfrak{F}[\mathtt{P}]$
$\mathfrak{W}[\mathtt{P}]$	The general web algebra
$\mathfrak{W}_{\cdot}^{\cdot}[\cdot]$	Various specializations of $\mathfrak{W}[\mathtt{P}]$
$\mathfrak{F}[\mathtt{Q}]$	The generic foam 2-category
$\mathfrak{W}[\mathtt{Q}]$	The generic web algebra
$\mathcal{W}(u)$	The web bimodules
$\mathfrak{W}[\cdot]$-$\mathbf{biMod}_{\mathrm{gr}}^{p}$	Web bimodule 2-category
$\oplus(\cdot)$	Additive closure
$\mathrm{p}_{\Lambda}(i)$	The position on arc diagrams
$s_{\Lambda}(\gamma)$	The saddle width
$\mathrm{d}_{\Lambda}(\cdot)$	The distance of e.g. an arc
$\mathbb{B}^{\circ}(\cdot)$	Various arc diagram bases
$\mathfrak{A}[\mathtt{Q}]$	The general arc algebra
$\mathfrak{A}_{\cdot}^{\cdot}[\cdot]$	Various specializations of $\mathfrak{A}[\mathtt{Q}]$
$\boldsymbol{\varepsilon}^{\mathrm{d}_{\Lambda}(\gamma_i^{\mathrm{dot}})}$	The dot moving sign
$\boldsymbol{\varepsilon}^{\frac{1}{4}(\mathrm{d}_{\Lambda}(C_{\mathrm{in}})-2)}$	The topological sign
$\boldsymbol{\varepsilon}^{s_{\Lambda}(\gamma)}$	The saddle sign
$\mathrm{t}(C)$	A rightmost point on a circle
$\mathcal{A}(\boldsymbol{\Lambda},\boldsymbol{t})$	The arc bimodules
$\mathfrak{A}[\cdot]$-$\mathbf{biMod}_{\mathrm{gr}}^{p}$	Arc bimodule 2-category
$\boldsymbol{\Phi}$ etc.	Various $\mathfrak{W}[\cdot]\xrightarrow{\cong}\mathfrak{A}[\cdot]$
$R[\boldsymbol{\alpha}]$	A ring with specialized $\boldsymbol{\varepsilon}$ and $\boldsymbol{\omega}$
$\boldsymbol{\Psi}$ etc.	Various $\mathfrak{A}[\mathtt{Q}]\xrightarrow{\cong}\mathfrak{A}[\cdot]$
$\mathrm{w}(\cdot)$	An associated web
$\mathfrak{W}[\cdot]^{\circ}$	Basic versions of the web algebras
$\cup^{\mathrm{ex}}(C)$	Cups "pushing inwards"
$\cap^{\mathrm{ex}}(C)$	Caps "pushing inwards"
$\cup^{\mathrm{in}}(C)$	Cups "pushing outwards"
$\cap^{\mathrm{in}}(C)$	Caps "pushing outwards"
$\mathrm{coeff}_{\cdot}^{\cdot}(\cdot)$	Various coefficient maps
$\#(\cdot)$	Various number of elements
${}_{\mathrm{ex}}\diagdown^{\times}(C)$	Left shift of $\times$ - left exterior
${}^{\times}\diagup_{\mathrm{ex}}(C)$	Right shift of $\times$ - right exterior
${}_{\mathrm{in}}\diagdown^{\times}(C)$	Left shift of $\times$ - left interior
${}^{\times}\diagup_{\mathrm{in}}(C)$	Right shift of $\times$ - right interior
$\overline{\mathrm{coeff}}_{\cdot}^{\cdot}(\cdot)$	Various reverse coefficient maps
$\mathfrak{C}_{\cdot}[\cdot]$	Cover(s) of arc algebra(s)
$\mathcal{O}^{p,q}$	Two block parabolic category $\mathcal{O}$
$[\![\cdot]\!]_{\cdot}^{\cdot}$	Various higher tangle invariants
1-$\mathbf{CC}$	Chain complex category
1-$\mathbf{HCC}$	Same, but up to homotopy
2-$\mathbf{CC}$, 2-$\mathbf{HCC}$	Their 2-versions
1-$\mathbf{Tan}$	The tangle category
2-$\mathbf{Tan}$	The tangle 2-category

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