CELLULAR STRUCTURES USING $\mathbf{U}_q$-TILTING MODULES

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Abstract. We use the theory of $\mathbf{U}_q$-tilting modules to construct cellular bases for centralizer algebras. Our methods are quite general and work for any quantum group $\mathbf{U}_q$ attached to a Cartan matrix and include the non-semisimple cases for q being a root of unity and ground fields of positive characteristic. Our approach also generalize to certain categories containing infinite-dimensional modules. As applications, we give a new semisimplicty criterion for centralizer algebras, and recover the cellularity of several known algebras (with partially new cellular bases) which all fit into our general setup.

Contents

1. Introduction

Fix a field $\mathbb{K}$ and set $\mathbb{K}^* = \mathbb{K} - \{0, -1\}$ if $\mathrm{char}(\mathbb{K}) > 2$ and $\mathbb{K}^* = \mathbb{K} - \{0\}$ otherwise. Let $\mathbf{U}_q(\mathfrak{g})$ be the quantum group for a fixed, arbitrary parameter $q \in \mathbb{K}^*$ associated to a simple Lie algebra $\mathfrak{g}$ over any field $\mathbb{K}$. The main result in this paper is the following.

Theorem. (**Cellularity of endomorphism algebras.**) Let T be a $\mathbf{U}_q(\mathfrak{g})$-tilting module. Then $\mathrm{End}_{\mathbf{U}_q(\mathfrak{g})}(T)$ is a cellular algebra in the sense of Graham and Lehrer [38]. □

It is important to note that cellular bases are not unique. In particular, a single algebra can have many cellular bases. As a concrete application, see Subsection 5.2, we construct (several) new cellular bases for the Temperley-Lieb algebra depending on the ground field

H.H.A. and D.T. were partially supported by the center of excellence grant "Centre for Quantum Geometry of Moduli Spaces (QGM)" from the "Danish National Research Foundation (DNRF)". C.S. was supported by the Max-Planck-Gesellschaft. D.T. was partially supported by a research funding of the "Deutsche Forschungsgemeinschaft (DFG)" during the last part of this work.

and the choice of deformation parameter. These bases differ therefore for instance from the construction in [38, Section 6] of cellular bases for the Temperley-Lieb algebras. Moreover, we also show that some of our bases for the Temperley-Lieb algebra can be equipped with a $\mathbb{Z}$-grading which is in contrast to Graham and Lehrer's bases. Our bases also depend heavily on the characteristic of $\mathbb{K}$ (and on $q \in \mathbb{K}^*$). Hence, they see more of the characteristic (and parameter) depended representation theory, but are also more difficult to construct explicitly.

We stress that the cellularity itself can be deduced from general theory. Namely, any $\mathbf{U}_q(\mathfrak{g})$-tilting module T is a summand of a full $\mathbf{U}_q(\mathfrak{g})$-tilting module $\tilde{T}$. By [72, Theorem 6] $\mathrm{End}_{\mathbf{U}_q(\mathfrak{g})}(\tilde{T})$ is quasi-hereditary and comes equipped with an involution as we explain in Subsection 3.3. Thus, it is cellular, see [55]. By [55, Theorem 4.3] this induces the cellularity of the idempotent truncation $\mathrm{End}_{\mathbf{U}_q(\mathfrak{g})}(T)$. In contrast, our approach provides the existence and a method of construction of many cellular bases. It generalizes to the infinite-dimensional Lie theory situation and has other nice consequences that we will explore in this paper. In particular, our results give a novel semisimplicity criterion for $\mathrm{End}_{\mathbf{U}_q(\mathfrak{g})}(T)$, see Theorem 4.13. This together with Jantzen sum formula give rise to a new way to obtain semisimplicity criteria for these algebras (we explain and explore this in [9] where we recover semisimplicity criteria for several algebras using the results of this paper). Here a crucial fact is that the tensor product of $\mathbf{U}_q$-tilting modules is again a $\mathbf{U}_q$-tilting module, see [68]. This implies that our results also vastly generalize [94] to the non-semisimple cases (where our main theorem is non-trivial).

The framework. Given any simple, complex Lie algebra $\mathfrak{g}$, we can assign to it a *quantum deformation* $\mathbf{U}_v = \mathbf{U}_v(\mathfrak{g})$ of its universal enveloping algebra by deforming its Serre presentation. (Here v is a generic parameter and $\mathbf{U}_v$ is an $\mathbb{Q}(v)$-algebra.) The representation theory of $\mathbf{U}_v$ shares many similarities with the one of $\mathfrak{g}$. In particular, the category[1] $\mathbf{U}_v$-**Mod** is semisimple.

But one can spice up the story drastically: the quantum group $\mathbf{U}_q = \mathbf{U}_q(\mathfrak{g})$ is obtained by specializing v to an arbitrary $q \in \mathbb{K}^*$. In particular, we can take q to be a root of unity[2]. In this case $\mathbf{U}_q$-**Mod** is not semisimple anymore, which makes the representation theory much more interesting. It has many connections and applications in different directions, e.g. the category has a neat combinatorics, is related to the corresponding almost-simple, simply connected algebraic group G over $\mathbb{K}$ with $\mathrm{char}(\mathbb{K})$ prime, see for example [4] or [60], to the representation theory of affine Kac-Moody algebras, see [49] or [87], and to (2+1)-TQFT's and the Witten-Reshetikhin-Turaev invariants of 3-manifolds, see for example [92].

Semisimplicity in light of our main result means the following. If we take $\mathbb{K} = \mathbb{C}$ and $q = \pm 1$, then our result says that the algebra $\mathrm{End}_{\mathbf{U}_q}(T)$ is cellular for any $\mathbf{U}_q$-module $T \in \mathbf{U}_q$-**Mod** because in this case all $\mathbf{U}_q$-modules are $\mathbf{U}_q$-tilting modules. This is no surprise: when T is a direct sum of simple $\mathbf{U}_q$-modules, then $\mathrm{End}_{\mathbf{U}_q}(T)$ is a direct sum of matrix algebras $M_n(\mathbb{K})$. Likewise, for any $\mathbb{K}$, if $q \in \mathbb{K}^* - \{1\}$ is not a root of unity, then $\mathbf{U}_q$-**Mod** is still semisimple and our result is (almost) standard. But even in the semisimple case we can say more: we get an Artin-Wedderburn basis as a cellular basis for $\mathrm{End}_{\mathbf{U}_q}(T)$, i.e. a basis realizing the decomposition of $\mathrm{End}_{\mathbf{U}_q}(T)$ into its matrix components, see Subsection 5.1.

[1] For any algebra A we denote by A-**Mod** the category of finite-dimensional, left A-modules. If not stated otherwise, all modules are assumed to be finite-dimensional, left modules.

[2] In our terminology: The two cases $q = \pm 1$ are special and do not count as roots of unity. Moreover, for technical reasons, we always exclude $q = -1$ in case $\mathrm{char}(\mathbb{K}) > 2$.

On the other hand, if $q = 1$ and $\mathrm{char}(\mathbb{K}) > 0$ or if $q \in \mathbb{K}^*$ is a root of unity, then $\mathbf{U}_q$-**Mod** is far away from being semisimple and our result gives a bunch of interesting cellular algebras.

For example, if $G = \mathrm{GL}(V)$ for some n-dimensional $\mathbb{K}$-vector space V, then $T = V^{\otimes d}$ is a G-tilting module for any $d \in \mathbb{Z}_{\geq 0}$. By Schur-Weyl duality we have

$$\Phi_{\mathrm{SW}} \colon \mathbb{K}[S_d] \twoheadrightarrow \mathrm{End}_G(T) \quad \text{and} \quad \Phi_{\mathrm{SW}} \colon \mathbb{K}[S_d] \xrightarrow{\cong} \mathrm{End}_G(T), \text{ if } n \geq d, \tag{1}$$

where $\mathbb{K}[S_d]$ is the group algebra of the symmetric group S_d in d letters. We can realize this as a special case in our framework by taking $q = 1$, $n \geq d$ and $\mathfrak{g} = \mathfrak{gl}_n$ (although $\mathfrak{gl}_n$ is not a simple, complex Lie algebra, our approach works fine for it as well). On the other hand, by taking q arbitrary in $\mathbb{K}^* - \{1\}$ and $n \geq d$, the group algebra $\mathbb{K}[S_d]$ is replaced by the type A_{d-1} *Iwahori-Hecke algebra* $\mathcal{H}_d(q)$ over $\mathbb{K}$ and our theorem gives cellular bases for this algebra as well. Note that one underlying fact why (1) stays true in the non-semisimple case is that $\dim(\mathrm{End}_G(T))$ is independent of the characteristic of $\mathbb{K}$ (and of the parameter q in the quantum case), since T is a G-tilting module.

Of course, both $\mathbb{K}[S_d]$ and $\mathcal{H}_d(q)$ are known to be cellular (these cases were one of the main motivations of Graham and Lehrer to introduce the notion of cellular algebras), but the point we want to make is, that they fit into our more general framework. The following known cellularity properties can also be recovered directly from our approach. And moreover: in most of the examples we either have no or only some mild restrictions on $\mathbb{K}$ and $q \in \mathbb{K}^*$.

- As sketched above: the algebras $\mathbb{K}[S_d]$ and $\mathcal{H}_d(q)$ and their quotients under Φ_{SW}.
- The *Temperley-Lieb algebras* $\mathcal{TL}_d(\delta)$ introduced in [88].
- Other less well-known endomorphism algebras for $\mathfrak{sl}_2$-related tilting modules appearing in more recent work, e.g. [5], [10] or [73].
- *Spider algebras* in the sense of Kuperberg [56].
- Quotients of the group algebras of $\mathbb{Z}/r\mathbb{Z} \wr S_d$ and its quantum version $\mathcal{H}_{d,r}(q)$, the *Ariki-Koike algebras* introduced in [12]. This includes the Ariki-Koike algebras themselves and thus, the Hecke algebras of type B. This also includes Martin and Saleur's *blob algebras* $\mathcal{BL}_d(q,m)$ from [64] and *(quantized) rook monoid algebras* (also called *Solomon algebras*) $\mathcal{R}_d(q)$ in the spirit of [85].
- *Brauer algebras* $\mathcal{B}_d(\delta)$ introduced in the context of classical invariant theory [15] and related algebras, e.g. the *walled Brauer algebras* $\mathcal{B}_{r,s}(\delta)$ as in [54] and [91], and the *Birman-Murakami-Wenzl* algebras $\mathcal{BMW}_d(\delta)$, in the sense of [14] and [66].

Note that our methods also apply for some categories containing infinite-dimensional modules. For example, with a little bit more care, one could allow T to be a non-necessary finite-dimensional $\mathbf{U}_q$-tilting module. Moreover, our methods also include the *BGG category* $\mathcal{O}$, its *parabolic subcategories* $\mathcal{O}^{\mathfrak{p}}$ and its quantum cousin $\mathcal{O}_q$ from [6]. For example, using the "big projective tilting" in the principal block, we get a cellular basis for the coinvariant algebra of the Weyl group associated to $\mathfrak{g}$. In fact, we get a vast generalization of this, e.g. we can fit *generalized Khovanov arc algebras* (see e.g. [19]), $\mathfrak{sl}_n$*-web algebras* (see e.g. [62]), *cyclotomic Khovanov-Lauda and Rouquier algebras* of type A (see [51] and [52] or [74]), for which we obtain cellularity via the connection to cyclotomic quotients of the degenerate affine Hecke algebra, see [16], cyclotomic $\mathbb{W}_d$*-algebras* (see e.g. [33]) and *cyclotomic quotients of affine Hecke algebras* $\mathbf{H}^s_{\mathbb{K},d}$ (see e.g. [75]) into our framework as well, see Subsection 5.1. However, we will for simplicity focus mostly on the finite-dimensional world. (Here we provide all necessary tools and arguments in great detail. Sometimes, for brevity, only in an extra file [8]). See also Remark 1.1.

Following Graham and Lehrer's approach, our cellular bases for $\mathrm{End}_{\mathbf{U}_q}(T)$ provide also $\mathrm{End}_{\mathbf{U}_q}(T)$-cell modules, classification of simple $\mathrm{End}_{\mathbf{U}_q}(T)$-modules etc. We give an interpretation of this in our setting as well, see Section 4. For instance, we deduce a new criterion for semisimplicity of $\mathrm{End}_{\mathbf{U}_q}(T)$, see Theorem 4.13.

Remark 1.1. Instead of working with the infinite-dimensional algebra $\mathbf{U}_q$, we could also work with a finite-dimensional, quasi-hereditary algebra (with a suitable anti-involution). By using results summarized in [30, Appendix], our constructions will go through very much in the same spirit as for $\mathbf{U}_q$. However, using $\mathbf{U}_q$ has some advantages. For example, we can construct an abundance of cellular bases (for the explicit construction of our basis we need "weight spaces" such that e.g. (2) or Lemma 3.4 work). Having several cellular bases is certainly an advantage, although calculating these is in general a non-trivial task. (For example, getting an explicit understanding of the endomorphisms giving rise to the cellular basis is a tough challenge. But see [70] for some crucial steps in this direction.) As a direct consequence of the existence of many cellular bases: most of the algebras appearing in our list of examples above can be additionally equipped with a $\mathbb{Z}$-grading. The basis elements from Theorem 3.9 can be chosen such that our approach leads to a $\mathbb{Z}$-graded cellular basis in the sense of [41]. We make this more precise in case of the Temperley-Lieb algebras, but one could for instance also recover the $\mathbb{Z}$-graded cellular bases of the Brauer algebras from [34] from our approach. We stress that in both cases the cellular bases in [38, Sections 4 and 6] are not $\mathbb{Z}$-graded. To keep the paper within reasonable boundaries, we do not treat the graded setup in this paper in detail. ▲

Acknowledgements. We like to thank Ben Cooper, Michael Ehrig, Matt Hogancamp, Johannes Kübel, Gus Lehrer, Paul Martin, Andrew Mathas, Volodymyr Mazorchuk, Steen Ryom-Hansen and Paul Wedrich for helpful comments and discussions, and the referees for further useful comments. H.H.A. would like to thank the Institut Mittag-Leffler for the hospitality he enjoyed there during the final stages of this work. C.S. is very grateful to the Max-Planck Institute in Bonn for the extraordinary support and excellent working conditions. A large part of her research was worked out during her stay there. D.T. likes to thank the dark Danish winter for very successfully limiting his non-work options and the Australian long blacks for pushing him forward.

2. Quantum groups, their representations and tilting modules

We briefly recall some facts that we need in this paper. Details can be found e.g. in [7] and [46], or [30] and [47]. For notations and arguments adopted to our situation see [8]. See also [72] and [29] for the classical treatment of tilting modules (in the modular case). As in the introduction, we fix a field $\mathbb{K}$ over which we work throughout.

2.1. **The quantum group $\mathbf{U}_q$.** Let Φ be a finite *root system* in an Euclidean space E. We fix a choice of *positive roots* $\Phi^+ \subset \Phi$ and *simple roots* $\Pi \subset \Phi^+$. We assume that we have n simple roots that we denote by $\alpha_1, \ldots, \alpha_n$. For each $\alpha \in \Phi$, we denote by $\alpha^\vee \in \Phi^\vee$ the corresponding *coroot*. Then $\mathbf{A} = (\langle \alpha_i, \alpha_j^\vee \rangle)_{i,j=1}^n$ is called the *Cartan matrix*.

By the set of *(integral) weights* we understand $X = \{\lambda \in E \mid \langle \lambda, \alpha_i^\vee \rangle \in \mathbb{Z} \text{ for all } \alpha_i \in \Pi\}$. The *dominant (integral) weights* X^+ are those $\lambda \in X$ such that $\langle \lambda, \alpha_i^\vee \rangle \geq 0$ for all $\alpha_i \in \Pi$.

Recall that there is a *partial ordering* on X given by $\mu \leq \lambda$ iff $\lambda - \mu$ is an $\mathbb{Z}_{\geq 0}$-valued linear combination of the simple roots, that is, $\lambda - \mu = \sum_{i=1}^n a_i \alpha_i$ with $a_i \in \mathbb{Z}_{\geq 0}$.

We denote by $\mathbf{U}_q = \mathbf{U}_q(\mathbf{A})$ the *quantum enveloping algebra* attached to a Cartan matrix $\mathbf{A}$ and specialized at $q \in \mathbb{K}^*$, where we follow [7] with our conventions for these. Note that $\mathbf{U}_q$ always means the quantum group over $\mathbb{K}$ defined via Lusztig's divided power construction, see e.g. [7]. (Thus, we have generators K_i, E_i and F_i for all $i = 1, \ldots, n$ as well as divided power generators.) We have a decomposition $\mathbf{U}_q = \mathbf{U}_q^- \mathbf{U}_q^0 \mathbf{U}_q^+$, with subalgebras generated by F's, K's and E's respectively (and some divided power generators, see e.g. [7, Section 1]). Note that we can recover the generic case $\mathbf{U}_v = \mathbf{U}_v(\mathbf{A})$ by choosing $\mathbb{K} = \mathbb{Q}(v)$ and $q = v$.

It is worth noting that $\mathbf{U}_q$ is a Hopf algebra, so its module category is a monoidal category with duals. We denote by $\mathbf{U}_q$-**Mod** the category of finite-dimensional $\mathbf{U}_q$-modules (of type 1, see [7, Subsection 1.4]). We consider only such $\mathbf{U}_q$-modules in what follows.

Recall that there is a contravariant, character-preserving *duality functor* $\mathcal{D}$ that is defined on the $\mathbb{K}$-vector space level via $\mathcal{D}(M) = M^*$ (the $\mathbb{K}$-linear dual of M) and an action of $\mathbf{U}_q$ on $\mathcal{D}(M)$ is defined as follows. Let $\omega \colon \mathbf{U}_q \to \mathbf{U}_q$ be the automorphism of $\mathbf{U}_q$ which interchanges E_i and F_i and interchanges K_i and K_i^{-1} (see e.g. [46, Lemma 4.6], which extends to our setup without difficulties). Then define $uf = m \mapsto f(\omega(S(u))m)$ for $u \in \mathbf{U}_q, f \in \mathcal{D}(M), m \in M$. Given any $\mathbf{U}_q$-homomorphism f between $\mathbf{U}_q$-modules, we also write $\mathrm{i}(f) = \mathcal{D}(f)$. This duality gives rise to the involution in our cellular datum from Subsection 3.3.

Assumption 2.1. If q is a root of unity, then, to *avoid technicalities*, we assume that q is a primitive root of unity of odd order l. A treatment of the even case, that can be used to repeat everything in this paper in the case where l is even, can be found in [3]. Moreover, in case of type G_2 we additionally assume that l is prime to 3. ▲

For each $\lambda \in X^+$ there is a *Weyl* $\mathbf{U}_q$*-module* $\Delta_q(\lambda)$ and a *dual Weyl* $\mathbf{U}_q$*-module* $\nabla_q(\lambda)$ satisfying $\mathcal{D}(\Delta_q(\lambda)) \cong \nabla_q(\lambda)$. The $\mathbf{U}_q$-module $\Delta_q(\lambda)$ has a unique simple head $L_q(\lambda)$ which is the unique simple socle of $\nabla_q(\lambda)$. Thus, there is a (up to scalars) unique $\mathbf{U}_q$-homomorphism

$$c^\lambda \colon \Delta_q(\lambda) \to \nabla_q(\lambda) \quad \text{(mapping head to socle)}. \tag{2}$$

This relies on the fact that $\Delta_q(\lambda)$ and $\nabla_q(\lambda)$ both have one-dimensional λ-weight spaces. The same fact implies that $\mathrm{End}_{\mathbf{U}_q}(L_q(\lambda)) \cong \mathbb{K}$ for all $\lambda \in X^+$, see [7, Corollary 7.4]. This last property fails for quasi-hereditary algebras in general when $\mathbb{K}$ is not algebraically closed.

Theorem 2.2. (**Ext-vanishing.**) We have for all $\lambda, \mu \in X^+$ that

$$\mathrm{Ext}^i_{\mathbf{U}_q}(\Delta_q(\lambda), \nabla_q(\mu)) \cong \begin{cases} \mathbb{K}c^\lambda, & \text{if } i = 0 \text{ and } \lambda = \mu, \\ 0, & \text{else.} \end{cases}$$ □

We have to enlarge the category $\mathbf{U}_q$-**Mod** by non-necessarily finite-dimensional $\mathbf{U}_q$-modules to have enough injectives such that the $\mathrm{Ext}^i_{\mathbf{U}_q}$-functors make sense by using q-analogous arguments as in [47, Part I, Chapter 3]. However, $\mathbf{U}_q$-**Mod** has enough injectives in characteristic zero, see [1, Proposition 5.8] for a treatment of the non-semisimple cases.

Proof. Similar to the modular analogon treated in [47, Proposition II.4.13] (a proof in our notation can be found in [8]). ■

2.2. **Tilting modules and Ext-vanishing.** We say that a $\mathbf{U}_q$-module M has a Δ_q-*filtration* if there exists some $k \in \mathbb{Z}_{\geq 0}$ and a finite descending sequence of $\mathbf{U}_q$-submodules

$$M = M_0 \supset M_1 \supset \cdots \supset M_{k'} \supset \cdots \supset M_{k-1} \supset M_k = 0,$$

such that $M_{k'}/M_{k'+1} \cong \Delta_q(\lambda_{k'})$ for all $k' = 0, \dots, k-1$ and some $\lambda_{k'} \in X^+$. A ∇_q-*filtration* is defined similarly, but using a finite ascending sequence of $\mathbf{U}_q$-submodules and $\nabla_q(\lambda)$'s instead of $\Delta_q(\lambda)$'s. We denote by $(M : \Delta_q(\lambda))$ and $(N : \nabla_q(\lambda))$ the corresponding multiplicities, which are well-defined by Corollary 2.3. Note that a $\mathbf{U}_q$-module M has a Δ_q-filtration iff its dual $\mathcal{D}(M)$ has a ∇_q-filtration.

A corollary of the Ext-vanishing theorem is the following, whose proof is left to the reader or can be found in [8]. (Note that the proof of Corollary 2.3 in [8] gives, in principle, a method to find and construct bases of $\mathrm{Hom}_{\mathbf{U}_q}(M, \nabla_q(\lambda))$ and $\mathrm{Hom}_{\mathbf{U}_q}(\Delta_q(\lambda), N)$ respectively.)

Corollary 2.3. Let $M, N \in \mathbf{U}_q\text{-}\mathbf{Mod}$ and $\lambda \in X^+$. Assume that M has a Δ_q-filtration and N has a ∇_q-filtration. Then

$$\dim(\mathrm{Hom}_{\mathbf{U}_q}(M, \nabla_q(\lambda))) = (M : \Delta_q(\lambda)) \quad \text{and} \quad \dim(\mathrm{Hom}_{\mathbf{U}_q}(\Delta_q(\lambda), N)) = (N : \nabla_q(\lambda)).$$

In particular, $(M : \Delta_q(\lambda))$ and $(N : \nabla_q(\lambda))$ are independent of the choice of filtrations. ■

Proposition 2.4. (**Donkin's Ext-criteria.**) The following are equivalent.

(a) An $M \in \mathbf{U}_q\text{-}\mathbf{Mod}$ has a Δ_q-filtration (respectively $N \in \mathbf{U}_q\text{-}\mathbf{Mod}$ has a ∇_q-filtration).
(b) We have $\mathrm{Ext}^i_{\mathbf{U}_q}(M, \nabla_q(\lambda)) = 0$ (respectively $\mathrm{Ext}^i_{\mathbf{U}_q}(\Delta_q(\lambda), N) = 0$) for all $\lambda \in X^+$ and all $i > 0$.
(c) We have $\mathrm{Ext}^1_{\mathbf{U}_q}(M, \nabla_q(\lambda)) = 0$ (respectively $\mathrm{Ext}^1_{\mathbf{U}_q}(\Delta_q(\lambda), N) = 0$) for all $\lambda \in X^+$. □

Proof. As in [47, Proposition II.4.16]. A proof in our notation can be found in [8]. ■

A $\mathbf{U}_q$-module T which has both, a Δ_q- and a ∇_q-filtration, is called a $\mathbf{U}_q$-*tilting module*. Following Donkin [29], we are now ready to define the *category of* $\mathbf{U}_q$-*tilting modules* that we denote by $\boldsymbol{\mathcal{T}}$. This category is our main object of study.

Definition 2.5. (**Category of $\mathbf{U}_q$-tilting modules.**) The category $\boldsymbol{\mathcal{T}}$ is the full subcategory of $\mathbf{U}_q\text{-}\mathbf{Mod}$ whose objects are given by all $\mathbf{U}_q$-tilting modules. ▲

From Proposition 2.4 we obtain directly an important statement.

Corollary 2.6. Let $T \in \mathbf{U}_q\text{-}\mathbf{Mod}$. Then

$$T \in \boldsymbol{\mathcal{T}} \quad \text{iff} \quad \mathrm{Ext}^1_{\mathbf{U}_q}(T, \nabla_q(\lambda)) = 0 = \mathrm{Ext}^1_{\mathbf{U}_q}(\Delta_q(\lambda), T) \quad \text{for all } \lambda \in X^+.$$

When $T \in \boldsymbol{\mathcal{T}}$, the corresponding higher Ext-groups vanish as well. ■

The indecomposable $\mathbf{U}_q$-modules in $\boldsymbol{\mathcal{T}}$, that we denote by $T_q(\lambda)$, are indexed by $\lambda \in X^+$. The $\mathbf{U}_q$-tilting module $T_q(\lambda)$ is determined by the property that it is indecomposable with λ as its unique maximal weight. In fact, $(T_q(\lambda) : \Delta_q(\lambda)) = 1$, and $(T_q(\lambda) : \Delta_q(\mu)) \neq 0$ only if $\mu \leq \lambda$. (Dually for ∇_q-filtrations.)

Note that the duality functor $\mathcal{D}$ from above restricts to $\boldsymbol{\mathcal{T}}$. Moreover, as a consequence of the classification of indecomposable $\mathbf{U}_q$-modules in $\boldsymbol{\mathcal{T}}$, we have $\mathcal{D}(T) \cong T$ for $T \in \boldsymbol{\mathcal{T}}$. In particular, we have for all $\lambda \in X^+$ that

$$(T : \Delta_q(\lambda)) = \dim(\mathrm{Hom}_{\mathbf{U}_q}(T, \nabla_q(\lambda))) = \dim(\mathrm{Hom}_{\mathbf{U}_q}(\Delta_q(\lambda), T)) = (T : \nabla_q(\lambda)).$$

It is known that $\boldsymbol{\mathcal{T}}$ is a Krull-Schmidt category, closed under finite direct sums, taking summands and finite tensor products (the latter is a non-trivial fact, see [68, Theorem 3.3]).

For a fixed $\lambda \in X^+$ we have $\mathbf{U}_q$-homomorphisms

$$\Delta_q(\lambda) \xhookrightarrow{\iota^\lambda} T_q(\lambda) \xrightarrow{\pi^\lambda} \nabla_q(\lambda),$$

where ι^λ is the inclusion of the first $\mathbf{U}_q$-submodule in a Δ_q-filtration of $T_q(\lambda)$ and π^λ is the surjection onto the last quotient in a ∇_q-filtration of $T_q(\lambda)$. Note that these are only defined up to scalars and we fix scalars in the following such that $\pi^\lambda \circ \iota^\lambda = c^\lambda$ (where c^λ is again the $\mathbf{U}_q$-homomorphism from (2)).

Remark 2.7. Let $T \in \boldsymbol{\mathcal{T}}$. An easy argument (based on Theorem 2.2) shows the following crucial fact:

$$\text{(3)} \quad \mathrm{Ext}^1_{\mathbf{U}_q}(\Delta_q(\lambda), T) = 0 = \mathrm{Ext}^1_{\mathbf{U}_q}(T, \nabla_q(\lambda)) \Rightarrow \mathrm{Ext}^1_{\mathbf{U}_q}(\mathrm{coker}(\iota^\lambda), T) = 0 = \mathrm{Ext}^1_{\mathbf{U}_q}(T, \ker(\pi^\lambda))$$

for all $\lambda \in X^+$. Consequently, we see that any $\mathbf{U}_q$-homomorphism $g\colon \Delta_q(\lambda) \to T$ extends to a $\mathbf{U}_q$-homomorphism $\overline{g}\colon T_q(\lambda) \to T$ whereas any $\mathbf{U}_q$-homomorphism $f\colon T \to \nabla_q(\lambda)$ factors through $T_q(\lambda)$ via some $\overline{f}\colon T \to T_q(\lambda)$. ▲

Remark 2.8. In [8] it is described in detail how to compute $(T_q(\lambda) : \Delta_q(\mu))$ for $\lambda, \mu \in X^+$. This can be done algorithmically in case q is a complex, primitive l-th root of unity, i.e. one can use Soergel's version of the *affine parabolic Kazhdan-Lusztig polynomials*. For brevity, we do not recall the definition of these polynomials here, but refer to [84, Section 3] where the relevant polynomials are denoted $n_{y,x}$ (and where all the other relevant notions are defined). The main point for us is the following theorem due to Soergel, see [81, Theorem 5.12] (see also [84, Conjecture 7.1]): Suppose $\mathbb{K} = \mathbb{C}$ and q is a complex, primitive l-th root of unity. For each pair $\lambda, \mu \in X^+$ with λ being an l-regular $\mathbf{U}_q$-weight (that is, $T_q(\lambda)$ belongs to a regular block of $\boldsymbol{\mathcal{T}}$) we have (with $n_{\mu\lambda}$ equal to the relevant $n_{y,x}$)

$$(T_q(\lambda) : \Delta_q(\mu)) = n_{\mu\lambda}(1) = (T_q(\lambda) : \nabla_q(\mu)).$$

From this one obtains a method to find the indecomposable summands of $\mathbf{U}_q$-tilting modules with known characters (e.g. tensor products of minuscule representations). ▲

3. Cellular structures on endomorphism algebras

In this section we give our construction of cellular bases for endomorphism rings $\mathrm{End}_{\mathbf{U}_q}(T)$ of $\mathbf{U}_q$-tilting modules T and prove our main result, that is, Theorem 3.9.

The main tool is Theorem 3.1. The proof of the latter needs several ingredients which we establish in form of separate lemmas collected in Subsection 3.2.

3.1. **The basis theorem.** As before, we consider the category $\mathbf{U}_q$-**Mod**. Moreover, we fix two $\mathbf{U}_q$-modules M, N, where we assume that M has a Δ_q-filtration and N has a ∇_q-filtration. Then, by Corollary 2.3, we have

$$\text{(4)} \qquad \dim(\mathrm{Hom}_{\mathbf{U}_q}(M, N)) = \sum_{\lambda \in X^+} (M : \Delta_q(\lambda))(N : \nabla_q(\lambda)).$$

We point out that the sum in (4) is actually finite since $(M : \Delta_q(\lambda)) \neq 0$ for only a finite number of $\lambda \in X^+$. (Dually, $(N : \nabla_q(\lambda)) \neq 0$ for only finitely many $\lambda \in X^+$.)

Given $\lambda \in X^+$, we define for $(N : \nabla_q(\lambda)) > 0$ respectively for $(M : \Delta_q(\lambda)) > 0$ the two sets

$$\mathcal{I}^\lambda = \{1, \dots, (N : \nabla_q(\lambda))\} \quad \text{and} \quad \mathcal{J}^\lambda = \{1, \dots, (M : \Delta_q(\lambda))\}.$$

By convention, $\mathcal{I}^\lambda = \emptyset$ and $\mathcal{J}^\lambda = \emptyset$ if $(N : \nabla_q(\lambda)) = 0$ respectively if $(M : \Delta_q(\lambda)) = 0$.

We can fix a basis of $\mathrm{Hom}_{\mathbf{U}_q}(M, \nabla_q(\lambda))$ indexed by $\mathcal{J}^\lambda$. We denote this fixed basis by $F^\lambda = \{f_j^\lambda \colon M \to \nabla_q(\lambda) \mid j \in \mathcal{J}^\lambda\}$. By Proposition 2.4 and (3), we see that all elements of F^λ factor through the $\mathbf{U}_q$-tilting module $T_q(\lambda)$, i.e. we have commuting diagrams

$$\begin{tikzcd} M \arrow[r, dotted, "\exists\, \overline{f}_j^\lambda"] \arrow[dr, "f_j^\lambda"'] & T_q(\lambda) \arrow[d, two heads, "\pi^\lambda"] \\ & \nabla_q(\lambda). \end{tikzcd}$$

We call $\overline{f}_j^\lambda$ a *lift* of f_j^λ. (Note that a lift $\overline{f}_j^\lambda$ is not unique.) Dually, we can choose a basis of $\mathrm{Hom}_{\mathbf{U}_q}(\Delta_q(\lambda), N)$ as $G^\lambda = \{g_i^\lambda \colon \Delta_q(\lambda) \to N \mid i \in \mathcal{I}^\lambda\}$, which extends to give (a non-unique) lifts $\overline{g}_i^\lambda \colon T_q(\lambda) \to N$ such that $\overline{g}_i^\lambda \circ \iota^\lambda = g_i^\lambda$ for all $i \in \mathcal{I}^\lambda$.

We can use this setup to define a basis for $\mathrm{Hom}_{\mathbf{U}_q}(M, N)$ which, when $M = N$, turns out to be a cellular basis, see Theorem 3.9. For each $\lambda \in X^+$ and all $i \in \mathcal{I}^\lambda, j \in \mathcal{J}^\lambda$ set

$$c_{ij}^\lambda = \overline{g}_i^\lambda \circ \overline{f}_j^\lambda \in \mathrm{Hom}_{\mathbf{U}_q}(M, N).$$

Our main result here is now the following.

Theorem 3.1. (**Basis theorem.**) For any choice of F^λ and G^λ as above and any choice of lifts of the f_j^λ's and the g_i^λ's (for all $\lambda \in X^+$), the set

$$GF = \{c_{ij}^\lambda \mid \lambda \in X^+,\ i \in \mathcal{I}^\lambda, j \in \mathcal{J}^\lambda\}$$

is a basis of $\mathrm{Hom}_{\mathbf{U}_q}(M, N)$. □

Proof. This follows from Proposition 3.3 combined with Lemmas 3.6 and 3.7 from below. ■

The basis GF for $\mathrm{Hom}_{\mathbf{U}_q}(M, N)$ can be illustrated in a commuting diagram as

$$\begin{tikzcd} & \Delta_q(\lambda) \arrow[d, hook, "\iota^\lambda"'] \arrow[dr, "g_i^\lambda"] & \\ M \arrow[r, dotted, "\overline{f}_j^\lambda"] \arrow[dr, "f_j^\lambda"'] & T_q(\lambda) \arrow[r, dotted, "\overline{g}_i^\lambda"'] \arrow[d, two heads, "\pi^\lambda"] & N . \\ & \nabla_q(\lambda) & \end{tikzcd}$$

Since $\mathbf{U}_q$-tilting modules have both a Δ_q- and a ∇_q-filtration, we get as an immediate consequence a key result for our purposes.

Corollary 3.2. Let $T \in \mathcal{T}$. Then GF is, for any choices involved, a basis of $\mathrm{End}_{\mathbf{U}_q}(T)$. ■

3.2. **Proof of the basis theorem.** We first show that, given lifts $\overline{f}_j^\lambda$, there is a consistent choice of lifts $\overline{g}_i^\lambda$ such that GF is a basis of $\mathrm{Hom}_{\mathbf{U}_q}(M,N)$.

Proposition 3.3. (**Basis theorem - dependent version.**) For any choice of F^λ and any choice of lifts of the f_j^λ's (for all $\lambda \in X^+$) there exist a choice of a basis G^λ and a choice of lifts of the g_i^λ's such that $GF = \{c_{ij}^\lambda \mid \lambda \in X^+,\ i \in \mathcal{I}^\lambda, j \in \mathcal{J}^\lambda\}$ is a basis of $\mathrm{Hom}_{\mathbf{U}_q}(M,N)$.$\square$

The corresponding statement with the roles of f's and g's swapped clearly holds as well.

Proof. We will construct GF inductively. For this purpose, let

$$0 = N_0 \subset N_1 \subset \cdots \subset N_{k-1} \subset N_k = N$$

be a ∇_q-filtration of N, i.e. $N_{k'+1}/N_{k'} \cong \nabla_q(\lambda_{k'})$ for some $\lambda_{k'} \in X^+$ and all $k' = 0,\ldots,k-1$.

Let $k=1$ and $\lambda_1 = \lambda$. Then $N_1 = \nabla_q(\lambda)$ and $\{c^\lambda \colon \Delta_q(\lambda) \to \nabla_q(\lambda)\}$ gives a basis of $\mathrm{Hom}_{\mathbf{U}_q}(\Delta_q(\lambda), \nabla_q(\lambda))$, where c^λ is again the $\mathbf{U}_q$-homomorphism chosen in (2). Set $g_1^\lambda = c^\lambda$ and observe that $\overline{g}_1^\lambda = \pi^\lambda$ satisfies $\overline{g}_1^\lambda \circ \iota^\lambda = g_1^\lambda$. Thus, we have a basis and a corresponding lift. This clearly gives a basis of $\mathrm{Hom}_{\mathbf{U}_q}(M, N_1)$, since, by assumption, we have that F^λ gives a basis of $\mathrm{Hom}_{\mathbf{U}_q}(M, \nabla_q(\lambda))$ and $\pi^\lambda \circ \overline{f}_j^\lambda = f_j^\lambda$.

Hence, it remains to consider the case $k > 1$. Set $\lambda_k = \lambda$ and observe that we have a short exact sequence of the form

$$0 \longrightarrow N_{k-1} \xhookrightarrow{\text{inc}} N_k \xrightarrow{\text{pro}} \nabla_q(\lambda) \longrightarrow 0. \tag{5}$$

By Theorem 2.2 (and the usual implication as in (3)) this leads to a short exact sequence

$$0 \longrightarrow \mathrm{Hom}_{\mathbf{U}_q}(M, N_{k-1}) \xhookrightarrow{\text{inc}_*} \mathrm{Hom}_{\mathbf{U}_q}(M, N_k) \xrightarrow{\text{pro}_*} \mathrm{Hom}_{\mathbf{U}_q}(M, \nabla_q(\lambda)) \longrightarrow 0. \tag{6}$$

By induction, we get from (6) for all $\mu \in X^+$ a basis of $\mathrm{Hom}_{\mathbf{U}_q}(\Delta_q(\mu), N_{k-1})$ consisting of g_i^μ's with lifts $\overline{g}_i^\mu$ such that

$$\{c_{ij}^\mu = \overline{g}_i^\mu \circ \overline{f}_j^\mu \mid \mu \in X^+,\ i \in \mathcal{I}_{k-1}^\mu, j \in \mathcal{J}^\mu\} \tag{7}$$

is a basis of $\mathrm{Hom}_{\mathbf{U}_q}(M, N_{k-1})$ (here we use $\mathcal{I}_{k-1}^\mu = \{1,\ldots,(N_{k-1} : \nabla_q(\mu))\}$). We define $g_i^\mu(N_k) = \text{inc} \circ g_i^\mu$ and $\overline{g}_i^\mu(N_k) = \text{inc} \circ \overline{g}_i^\mu$ for each $\mu \in X^+$ and each $i \in \mathcal{I}_{k-1}^\mu$.

We now have to consider two cases, namely $\lambda \neq \mu$ and $\lambda = \mu$. In the first case we see that $\mathrm{Hom}_{\mathbf{U}_q}(\Delta_q(\mu), \nabla_q(\lambda)) = 0$, so that, by using (5) and the usual implication from (3),

$$\mathrm{Hom}_{\mathbf{U}_q}(\Delta_q(\mu), N_{k-1}) \cong \mathrm{Hom}_{\mathbf{U}_q}(\Delta_q(\mu), N_k).$$

Thus, our basis from (7) gives a basis of $\mathrm{Hom}_{\mathbf{U}_q}(\Delta_q(\mu), N_k)$ and also gives the corresponding lifts. On the other hand, if $\lambda = \mu$, then $(N_k : \nabla_q(\lambda)) = (N_{k-1} : \nabla_q(\lambda)) + 1$. By Theorem 2.2 (and the corresponding implication as in (3)), we can choose $g^\lambda \colon \Delta_q(\lambda) \to N_k$ such that $\text{pro} \circ g^\lambda = c^\lambda$. Then any choice of a lift $\overline{g}^\lambda$ of g^λ will satisfy $\text{pro} \circ \overline{g}^\lambda = \pi^\lambda$.

Adjoining g^λ to the basis from (7) gives a basis of $\mathrm{Hom}_{\mathbf{U}_q}(\Delta_q(\lambda), N_k)$ which satisfies the lifting property. Note that we know from the case $k=1$ that

$$\{\text{pro} \circ \overline{g}^\lambda \circ \overline{f}_j^\lambda = \pi^\lambda \circ \overline{f}_j^\lambda \mid j \in \mathcal{J}^\lambda\}$$

is a basis of $\mathrm{Hom}_{\mathbf{U}_q}(M, \nabla_q(\lambda))$. Combining everything: we have that

$$\{c_{ij}^{\lambda} = \overline{g}_i^{\lambda}(N_k) \circ \overline{f}_j^{\lambda} \mid \lambda \in X^+,\ i \in \mathcal{I}^{\lambda}, j \in \mathcal{J}^{\lambda}\}$$

is a basis of $\mathrm{Hom}_{\mathbf{U}_q}(M, N_k)$ (by enumerating $\overline{g}^{\lambda}_{(N:\nabla_q(\lambda))}(N_k) = \overline{g}^{\lambda}$ in the $\lambda = \mu$ case). ■

We assume in the following that we have fixed some choices as in Proposition 3.3.

Let $\lambda \in X^+$. Given $\varphi \in \mathrm{Hom}_{\mathbf{U}_q}(M, N)$, we denote by $\varphi_\lambda \in \mathrm{Hom}_{\mathbf{U}_q^0}(M_\lambda, N_\lambda)$ the induced $\mathbf{U}_q^0$-homomorphism (that is, $\mathbb{K}$-linear maps) between the λ-weight spaces M_λ and N_λ. In addition, we denote by $\mathrm{Hom}_{\mathbb{K}}(M_\lambda, N_\lambda)$ the $\mathbb{K}$-linear maps between these λ-weight spaces.

Lemma 3.4. For any $\lambda \in X^+$ the induced set $\{(c_{ij}^{\lambda})_\lambda \mid c_{ij}^{\lambda} \in GF\}$ is a linearly independent subset of $\mathrm{Hom}_{\mathbb{K}}(M_\lambda, N_\lambda)$. □

Proof. We proceed as in the proof of Proposition 3.3.

If $N = \nabla_q(\lambda)$ (this was $k = 1$ above), then $c_{1j}^{\lambda} = \pi^{\lambda} \circ \overline{f}_j^{\lambda} = f_j^{\lambda}$ and the c_{1j}^{λ}'s form a basis of $\mathrm{Hom}_{\mathbf{U}_q}(M, \nabla_q(\lambda))$. By the q-Frobenius reciprocity from [7, Proposition 1.17] we have

$$\mathrm{Hom}_{\mathbf{U}_q}(M, \nabla_q(\lambda)) \cong \mathrm{Hom}_{\mathbf{U}_q^-\mathbf{U}_q^0}(M, \mathbb{K}_\lambda) \subset \mathrm{Hom}_{\mathbf{U}_q^0}(M, \mathbb{K}_\lambda) = \mathrm{Hom}_{\mathbb{K}}(M_\lambda, \mathbb{K}).$$

Hence, because $N_\lambda = \mathbb{K}$ in this case, we have the base of the induction.

Assume now $k > 1$. The construction of $\{c_{ij}^{\mu}(N_k)\}_{\mu,i,j}$ in the proof of Proposition 3.3 shows that this set consists of two separate parts: one being the bases from (7) coming from a basis for $\mathrm{Hom}_{\mathbf{U}_q}(M, N_{k-1})$ and the second part (which only occurs when $\lambda = \mu$) mapping to a basis of $\mathrm{Hom}_{\mathbf{U}_q}(M, \nabla_q(\lambda))$ (the case $k = 1$).

By (6) there is a short exact sequence

$$0 \longrightarrow \mathrm{Hom}_{\mathbb{K}}(M_\lambda, (N_{k-1})_\lambda) \xhookrightarrow{\mathrm{inc}_*} \mathrm{Hom}_{\mathbb{K}}(M_\lambda, (N_k)_\lambda) \xrightarrow{\mathrm{pro}_*} \mathrm{Hom}_{\mathbb{K}}(M_\lambda, \mathbb{K}) \longrightarrow 0.$$

Thus, we can proceed as in the proof of Proposition 3.3. ■

We need another piece of notation: we define for each $\lambda \in X^+$

$$\mathrm{Hom}_{\mathbf{U}_q}(M, N)^{\leq \lambda} = \{\varphi \in \mathrm{Hom}_{\mathbf{U}_q}(M, N) \mid \varphi_\mu = 0 \text{ unless } \mu \leq \lambda\}.$$

In words: a $\mathbf{U}_q$-homomorphism $\varphi \in \mathrm{Hom}_{\mathbf{U}_q}(M, N)$ belongs to $\mathrm{Hom}_{\mathbf{U}_q}(M, N)^{\leq \lambda}$ iff φ vanishes on all $\mathbf{U}_q$-weight spaces M_μ with $\mu \nleq \lambda$. In addition to the notation above, we use the evident notation $\mathrm{Hom}_{\mathbf{U}_q}(M, N)^{< \lambda}$. We arrive at the following.

Lemma 3.5. For any fixed $\lambda \in X^+$ the sets

$$\{c_{ij}^{\mu} \mid c_{ij}^{\mu} \in GF,\ \mu \leq \lambda\} \quad \text{and} \quad \{c_{ij}^{\mu} \mid c_{ij}^{\mu} \in GF,\ \mu < \lambda\}$$

are bases of $\mathrm{Hom}_{\mathbf{U}_q}(M, N)^{\leq \lambda}$ and $\mathrm{Hom}_{\mathbf{U}_q}(M, N)^{< \lambda}$ respectively. □

Proof. As c_{ij}^{μ} factors through $T_q(\mu)$ and $T_q(\mu)_\nu = 0$ unless $\nu \leq \mu$ (which follows using the classification of indecomposable $\mathbf{U}_q$-tilting modules), we see that $(c_{ij}^{\mu})_\nu = 0$ unless $\nu \leq \mu$. Moreover, by Lemma 3.4, each $(c_{ij}^{\mu})_\mu$ is non-zero. Thus, $c_{ij}^{\mu} \in \mathrm{Hom}_{\mathbf{U}_q}(M, N)^{\leq \lambda}$ iff $\mu \leq \lambda$. Now choose any $\varphi \in \mathrm{Hom}_{\mathbf{U}_q}(M, N)^{\leq \lambda}$. By Proposition 3.3 we may write

$$\varphi = \sum_{\mu,i,j} a_{ij}^{\mu} c_{ij}^{\mu}, \quad a_{ij}^{\mu} \in \mathbb{K}. \tag{8}$$

Choose $\mu \in X^+$ maximal with the property that there exist $i \in \mathcal{I}^\lambda, j \in \mathcal{J}^\lambda$ such that $a_{ij}^\mu \neq 0$.

We claim that $a_{ij}^\nu (c_{ij}^\nu)_\mu = 0$ whenever $\nu \neq \mu$. This is true because, as observed above, $(c_{ij}^\nu)_\mu = 0$ unless $\mu \leq \nu$, and for $\mu < \nu$ we have $a_{ij}^\nu = 0$ by the maximality of μ. We conclude $\varphi_\mu = \sum_{i,j} a_{ij}^\mu (c_{ij}^\mu)_\mu$ and thus, $\varphi_\mu \neq 0$ by Lemma 3.4. Hence, $\mu \leq \lambda$, which gives by (8) that $\varphi \in \mathrm{span}_{\mathbb{K}}\{c_{ij}^\mu \mid c_{ij}^\mu \in GF,\ \mu \leq \lambda\}$ as desired. This shows that $\{c_{ij}^\mu \mid c_{ij}^\mu \in GF,\ \mu \leq \lambda\}$ spans $\mathrm{Hom}_{\mathbf{U}_q}(M,N)^{\leq\lambda}$. Since it is clearly a linear independent set, it is a basis.

The second statement follows analogously, and therefore the details are omitted. ■

We need the following two lemmas to prove that all choices in Proposition 3.3 lead to bases of $\mathrm{Hom}_{\mathbf{U}_q}(M,N)$. As before we assume that we have, as in Proposition 3.3, constructed $\{g_i^\lambda, i \in \mathcal{I}^\lambda\}$ and the corresponding lifts $\overline{g}_i^\lambda$ for all $\lambda \in X^+$.

Lemma 3.6. Suppose that we have other $\mathbf{U}_q$-homomorphisms $\tilde{g}_i^\lambda \colon T_q(\lambda) \to N$ such that $\tilde{g}_i^\lambda \circ \iota^\lambda = g_i^\lambda$. Then the following set is also a basis of $\mathrm{Hom}_{\mathbf{U}_q}(M,N)$:

$$\{\tilde{c}_{ij}^\lambda = \tilde{g}_i^\lambda \circ \overline{f}_j^\lambda \mid \lambda \in X^+,\ i \in \mathcal{I}^\lambda, j \in \mathcal{J}^\lambda\}. \qquad \square$$

Proof. As $(\overline{g}_i^\lambda - \tilde{g}_i^\lambda) \circ \iota^\lambda = 0$, we see that $\overline{g}_i^\lambda - \tilde{g}_i^\lambda \in \mathrm{Hom}_{\mathbf{U}_q}(T_q(\lambda),N)^{<\lambda}$. Hence, we have $c_{ij}^\lambda - \tilde{c}_{ij}^\lambda \in \mathrm{Hom}_{\mathbf{U}_q}(M,N)^{<\lambda}$. Thus, by Lemma 3.5, there is a unitriangular change-of-basis matrix between $\{c_{ij}^\lambda\}_{\lambda,i,j}$ and $\{\tilde{c}_{ij}^\lambda\}_{\lambda,i,j}$. ■

Now assume that we have chosen another basis $\{h_i^\lambda \mid i \in \mathcal{I}^\lambda\}$ of the spaces $\mathrm{Hom}_{\mathbf{U}_q}(\Delta_q(\lambda),N)$ for each $\lambda \in X^+$ and the corresponding lifts $\overline{h}_i^\lambda$ as well.

Lemma 3.7. The following set is also a basis of $\mathrm{Hom}_{\mathbf{U}_q}(M,N)$:

$$\{d_{ij}^\lambda = \overline{h}_i^\lambda \circ \overline{f}_j^\lambda \mid \lambda \in X^+,\ i \in \mathcal{I}^\lambda, j \in \mathcal{J}^\lambda\}. \qquad \square$$

Proof. Write $g_i^\lambda = \sum_{k=1}^{(N:\nabla_q(\lambda))} b_{ik}^\lambda h_k^\lambda$ with $b_{ik}^\lambda \in \mathbb{K}$ and set $\tilde{g}_i^\lambda = \sum_{k=1}^{(N:\nabla_q(\lambda))} b_{ik}^\lambda \overline{h}_k^\lambda$. Then the $\tilde{g}_i^\lambda$'s are lifts of the g_i^λ's. Hence, by Lemma 3.6, the elements $\tilde{g}_i^\lambda \circ \overline{f}_j^\lambda$ form a basis of $\mathrm{Hom}_{\mathbf{U}_q}(M,N)$. Thus, this proves the lemma, since, by construction, $\{d_{ij}^\lambda\}_{\lambda,i,j}$ is related to this basis by the invertible change-of-basis matrix $(b_{ik}^\lambda)_{i,k=1;\lambda\in X^+}^{(N:\nabla_q(\lambda))}$. ■

In total, we established Proposition 3.3.

3.3. **Cellular structures on endomorphism algebras of $\mathbf{U}_q$-tilting modules.** This subsection finally contains the statement and proof of our main theorem. We keep on working over a field $\mathbb{K}$ instead of a ring as for example Graham and Lehrer [38] do. (This avoids technicalities, e.g. the theory of indecomposable $\mathbf{U}_q$-tilting modules over rings is much more subtle than over fields. See e.g. [29, Remark 1.7].)

Definition 3.8. (**Cellular algebras.**) Suppose A is a finite-dimensional $\mathbb{K}$-algebra. A *cell datum* is an ordered quadruple $(\mathcal{P},\mathcal{I},\mathcal{C},\mathrm{i})$, where $(\mathcal{P},\leq)$ is a finite poset, $\mathcal{I}^\lambda$ is a finite set for all $\lambda \in \mathcal{P}$, i is a $\mathbb{K}$-linear anti-involution of A and $\mathcal{C}$ is an injection

$$\mathcal{C} \colon \coprod_{\lambda\in\mathcal{P}} \mathcal{I}^\lambda \times \mathcal{I}^\lambda \to A,\ (i,j) \mapsto c_{ij}^\lambda.$$

The whole data should be such that the c_{ij}^{λ}'s form a basis of A with $\mathrm{i}(c_{ij}^{\lambda}) = c_{ji}^{\lambda}$ for all $\lambda \in \mathcal{P}$ and all $i, j \in \mathcal{I}^{\lambda}$. Moreover, for all $a \in A$ and all $\lambda \in \mathcal{P}$ we have

$$ac_{ij}^{\lambda} = \sum_{k \in \mathcal{I}^{\lambda}} r_{ik}(a) c_{kj}^{\lambda} \ (\mathrm{mod}\ A^{<\lambda}) \quad \text{for all } i, j \in \mathcal{I}^{\lambda}. \tag{9}$$

Here $A^{<\lambda}$ is the subspace of A spanned by the set $\{c_{ij}^{\mu} \mid \mu < \lambda \text{ and } i, j \in \mathcal{I}(\mu)\}$ and the scalars $r_{ik}(a) \in \mathbb{K}$ are supposed to be independent of j.

An algebra A with such a quadruple is called a *cellular algebra* and the c_{ij}^{λ} are called a *cellular basis* of A (with respect to the $\mathbb{K}$-linear anti-involution i). ▲

Let us fix $T \in \boldsymbol{\mathcal{T}}$ in the following. We will now construct cellular bases of $\mathrm{End}_{\mathbf{U}_q}(T)$ in the semisimple as well as in the non-semisimple case.

To this end, we need to specify the cell datum. Set

$$(\mathcal{P}, \leq) = (\{\lambda \in X^+ \mid (T : \nabla_q(\lambda)) = (T : \Delta_q(\lambda)) \neq 0\}, \leq),$$

where $\leq$ is the usual partial ordering on X^+, see at the beginning of Subsection 2.1. Note that $\mathcal{P}$ is finite since T is finite-dimensional. Moreover, motivated by Theorem 3.1, for each $\lambda \in \mathcal{P}$ define $\mathcal{I}^{\lambda} = \{1, \ldots, (T : \nabla_q(\lambda))\} = \{1, \ldots, (T : \Delta_q(\lambda))\} = \mathcal{J}^{\lambda}$.

Recalling that we write $\mathrm{i}(\cdot) = \mathcal{D}(\cdot)$ (for $\mathcal{D}$ being the duality functor from Subsection 2.1 that exchanges Weyl and dual Weyl $\mathbf{U}_q$-modules and fixes all $\mathbf{U}_q$-tilting modules), the assignment $\mathrm{i}\colon \mathrm{End}_{\mathbf{U}_q}(T) \to \mathrm{End}_{\mathbf{U}_q}(T), \phi \mapsto \mathcal{D}(\phi)$ is clearly a $\mathbb{K}$-linear anti-involution. Choose any basis G^{λ} of $\mathrm{Hom}_{\mathbf{U}_q}(\Delta_q(\lambda), T)$ as above and any lifts $\overline{g}_i^{\lambda}$. Then $\mathrm{i}(G^{\lambda})$ is a basis of $\mathrm{Hom}_{\mathbf{U}_q}(T, \nabla_q(\lambda))$ and $\mathrm{i}(\overline{g}_i^{\lambda})$ is a lift of $\mathrm{i}(g_i^{\lambda})$. By Corollary 3.2 we see that

$$\{c_{ij}^{\lambda} = \overline{g}_i^{\lambda} \circ \mathrm{i}(\overline{g}_j^{\lambda}) = \overline{g}_i^{\lambda} \circ \overline{f}_j^{\lambda} \mid \lambda \in \mathcal{P},\ i, j \in \mathcal{I}^{\lambda}\}$$

is a basis of $\mathrm{End}_{\mathbf{U}_q}(T)$. Finally let $\mathcal{C}\colon \mathcal{I}^{\lambda} \times \mathcal{I}^{\lambda} \to \mathrm{End}_{\mathbf{U}_q}(T)$ be given by $(i, j) \mapsto c_{ij}^{\lambda}$.

Now we are ready to state and prove our main theorem.

Theorem 3.9. (**A cellular basis for** $\mathrm{End}_{\mathbf{U}_q}(T)$**.**) The quadruple $(\mathcal{P}, \mathcal{I}, \mathcal{C}, \mathrm{i})$ defined above is a cell datum for $\mathrm{End}_{\mathbf{U}_q}(T)$. □

Proof. As mentioned above, the sets $\mathcal{P}$ and $\mathcal{I}^{\lambda}$ are finite for all $\lambda \in \mathcal{P}$. Moreover, i is a $\mathbb{K}$-linear anti-involution of $\mathrm{End}_{\mathbf{U}_q}(T)$ and the c_{ij}^{λ}'s form a basis of $\mathrm{End}_{\mathbf{U}_q}(T)$ by Corollary 3.2. Because the functor $\mathcal{D}(\cdot)$ is contravariant, we see that $\mathrm{i}(c_{ij}^{\lambda}) = \mathrm{i}(\overline{g}_i^{\lambda} \circ \mathrm{i}(\overline{g}_j^{\lambda})) = \overline{g}_j^{\lambda} \circ \mathrm{i}(\overline{g}_i^{\lambda}) = c_{ji}^{\lambda}$.

Thus, only the condition (9) remains to be proven. For this purpose, let $\varphi \in \mathrm{End}_{\mathbf{U}_q}(T)$. Since $\varphi \circ \overline{g}_i^{\lambda} \circ \iota^{\lambda} = \varphi \circ g_i^{\lambda} \in \mathrm{Hom}_{\mathbf{U}_q}(\Delta_q(\lambda), T)$, we have coefficients $r_{ik}^{\lambda}(\varphi) \in \mathbb{K}$ such that

$$\varphi \circ g_i^{\lambda} = \sum_{k \in \mathcal{I}^{\lambda}} r_{ik}^{\lambda}(\varphi) g_k^{\lambda}, \tag{10}$$

because we know that the g_i^{λ}'s form a basis of $\mathrm{Hom}_{\mathbf{U}_q}(\Delta_q(\lambda), T)$. But this implies then that $\varphi \circ \overline{g}_i^{\lambda} - \sum_{k \in \mathcal{I}^{\lambda}} r_{ik}^{\lambda}(\varphi) \overline{g}_k^{\lambda} \in \mathrm{Hom}_{\mathbf{U}_q}(T_q(\lambda), T)^{<\lambda}$, so that

$$\varphi \circ \overline{g}_i^{\lambda} \circ \overline{f}_j^{\lambda} - \sum_{k \in \mathcal{I}^{\lambda}} r_{ik}^{\lambda}(\varphi) \overline{g}_k^{\lambda} \circ \overline{f}_j^{\lambda} \in \mathrm{Hom}_{\mathbf{U}_q}(T, T)^{<\lambda} = \mathrm{End}_{\mathbf{U}_q}(T)^{<\lambda},$$

which proves (9). The theorem follows. ■

4. The cellular structure and $\mathrm{End}_{\mathbf{U}_q}(T)$-**Mod**

The goal of this section is to present the representation theory of cellular algebras for $\mathrm{End}_{\mathbf{U}_q}(T)$ from the viewpoint of $\mathbf{U}_q$-tilting theory. In fact, most of the results in this section are not new and have been proved for general cellular algebras, see e.g. [38, Section 3]. However, they take a nice and easy form in our setup. The last theorem, the semisimplicity criterion from Theorem 4.13, is new and has potentially many applications, see for example [9].

4.1. **Cell modules for** $\mathrm{End}_{\mathbf{U}_q}(T)$.

We study now the representation theory for $\mathrm{End}_{\mathbf{U}_q}(T)$ via the cellular structure we have found for it. We denote its module category by $\mathrm{End}_{\mathbf{U}_q}(T)$-**Mod**.

Definition 4.1. (**Cell modules.**) Let $\lambda \in \mathcal{P}$. The *cell module* associated to λ is the left $\mathrm{End}_{\mathbf{U}_q}(T)$-module given by $C(\lambda) = \mathrm{Hom}_{\mathbf{U}_q}(\Delta_q(\lambda), T)$. The right $\mathrm{End}_{\mathbf{U}_q}(T)$-module given by $C(\lambda)^* = \mathrm{Hom}_{\mathbf{U}_q}(T, \nabla_q(\lambda))$ is called the *dual cell module* associated to λ. ▲

The link to the definition of cell modules from [38, Definition 2.1] is given via our choice of basis $\{g_i^\lambda\}_{i\in\mathcal{I}^\lambda}$. In this basis the action of $\mathrm{End}_{\mathbf{U}_q}(T)$ on $C(\lambda)$ is given by

$$\varphi \circ g_i^\lambda = \sum_{k\in\mathcal{I}^\lambda} r_{ik}^\lambda(\varphi) g_k^\lambda, \quad \varphi \in \mathrm{End}_{\mathbf{U}_q}(T), \tag{11}$$

see (10). Here the coefficients are the same as those appearing when we consider the left action of $\mathrm{End}_{\mathbf{U}_q}(T)$ on itself in terms of the cellular basis $\{c_{ij}^\lambda\}_{i,j\in\mathcal{I}^\lambda}^{\lambda\in\mathcal{P}}$, that is,

$$\varphi \circ c_{ij}^\lambda = \sum_{k\in\mathcal{I}^\lambda} r_{ik}^\lambda(\varphi) c_{kj}^\lambda \ (\mathrm{mod}\ \mathrm{End}_{\mathbf{U}_q}(T)^{<\lambda}), \quad \varphi \in \mathrm{End}_{\mathbf{U}_q}(T). \tag{12}$$

In a completely similar fashion: the dual cell module $C(\lambda)^*$ has a basis consisting of $\{f_j^\lambda\}_{j\in\mathcal{I}^\lambda}$ with $f_j^\lambda = \mathrm{i}(g_j^\lambda)$. In this basis the right action of $\mathrm{End}_{\mathbf{U}_q}(T)$ is given via

$$f_j^\lambda \circ \varphi = \sum_{k\in\mathcal{I}^\lambda} r_{kj}^\lambda(\mathrm{i}(\varphi)) f_k^\lambda, \quad \varphi \in \mathrm{End}_{\mathbf{U}_q}(T). \tag{13}$$

We can use the unique $\mathbf{U}_q$-homomorphism from (2) and the duality functor $\mathcal{D}(\cdot)$ to define the *cellular pairing* in the spirit of Graham and Lehrer [38, Definition 2.3].

Definition 4.2. (**Cellular pairing.**) Let $\lambda \in \mathcal{P}$. Then we denote by ϑ^λ the $\mathbb{K}$-bilinear form $\vartheta^\lambda\colon C(\lambda)\otimes C(\lambda) \to \mathbb{K}$ determined by the property

$$\mathrm{i}(h) \circ g = \vartheta^\lambda(g,h) c^\lambda, \quad g,h \in C(\lambda) = \mathrm{Hom}_{\mathbf{U}_q}(\Delta_q(\lambda), T).$$

We call ϑ^λ the *cellular pairing* associated to $\lambda \in \mathcal{P}$. ▲

Lemma 4.3. The cellular pairing ϑ^λ is well-defined, symmetric and contravariant. □

Proof. That ϑ^λ is well-defined follows directly from the uniqueness of c^λ.

Applying i to the defining equation of ϑ^λ gives

$$\vartheta^\lambda(g,h)\mathrm{i}(c^\lambda) = \mathrm{i}(\vartheta^\lambda(g,h)c^\lambda) = \mathrm{i}(\mathrm{i}(h)\circ g) = \mathrm{i}(g)\circ h = \vartheta^\lambda(h,g)c^\lambda,$$

and thus, $\vartheta^\lambda(g,h) = \vartheta^\lambda(h,g)$ because $c^\lambda = \mathrm{i}(c^\lambda)$. (Recall that $c^\lambda\colon \Delta_q(\lambda) \to \nabla_q(\lambda)$ is unique up to scalars. Hence, we can fix scalars accordingly such that $c^\lambda = \mathrm{i}(c^\lambda)$.) Similarly, contravariance of $\mathcal{D}(\cdot)$ gives

$$\vartheta^\lambda(\varphi\circ g, h) = \vartheta^\lambda(g, \mathrm{i}(\varphi)\circ h), \quad \varphi \in \mathrm{End}_{\mathbf{U}_q}(T),\ g,h \in C(\lambda),$$

which shows contravariance of the cellular pairing. ■

Proposition 4.4. Let $\lambda \in \mathcal{P}$. Then $T_q(\lambda)$ is a summand of T iff $\vartheta^\lambda \neq 0$. □

Proof. (See also [2, Proposition 1.5].) Assume $T \cong T_q(\lambda) \oplus \text{rest}$. We denote by $\overline{g}\colon T_q(\lambda) \to T$ and by $\overline{f}\colon T \to T_q(\lambda)$ the corresponding inclusion and projection respectively. As usual, set $g = \overline{g} \circ \iota^\lambda$ and $f = \pi^\lambda \circ \overline{f}$. Then we have $f \circ g\colon \Delta_q(\lambda) \hookrightarrow T_q(\lambda) \hookrightarrow T \twoheadrightarrow T_q(\lambda) \twoheadrightarrow \nabla_q(\lambda) = c^\lambda$ (mapping head to socle), giving $\vartheta^\lambda(g, \mathrm{i}(f)) = 1$. This shows that $\vartheta^\lambda \neq 0$.

Conversely, assume that there exist $g, h \in C(\lambda)$ with $\vartheta^\lambda(g, h) \neq 0$. Then the commuting "bow tie diagram", i.e.

$$\begin{tikzcd}
\Delta_q(\lambda) \arrow[d, hook, "\iota^\lambda"'] \arrow[dr, "g"] & & \\
T_q(\lambda) \arrow[r, dotted, "\overline{g}"'] & T \arrow[r, dotted, "\overline{\mathrm{i}(h)}"] \arrow[dr, "\mathrm{i}(h)"'] & T_q(\lambda), \arrow[d, two heads, "\pi^\lambda"] \\
& & \nabla_q(\lambda)
\end{tikzcd}$$

shows that $\overline{\mathrm{i}(h)} \circ \overline{g}$ is non-zero on the λ-weight space of $T_q(\lambda)$, because $\mathrm{i}(h) \circ g = \vartheta^\lambda(g, h) c^\lambda$. Thus, $\overline{\mathrm{i}(h)} \circ \overline{g}$ must be an isomorphism (because $T_q(\lambda)$ is indecomposable and has therefore only invertible or nilpotent elements in $\mathrm{End}_{\mathbf{U}_q}(T_q(\lambda))$) showing that $T \cong T_q(\lambda) \oplus \text{rest}$. ■

In view of Proposition 4.4, it makes sense to study the set

$$\mathcal{P}_0 = \{\lambda \in \mathcal{P} \mid \vartheta^\lambda \neq 0\} \subset \mathcal{P}. \tag{14}$$

Hence, if $\lambda \in \mathcal{P}_0$, then we have $T \cong T_q(\lambda) \oplus \text{rest}$ for some $\mathbf{U}_q$-tilting module called rest. Note also that $\mathrm{End}_{\mathbf{U}_q}(T)$ is quasi-hereditary iff $\mathcal{P} = \mathcal{P}_0$, see e.g. [38, Remark 3.10].

4.2. **The structure of $\mathrm{End}_{\mathbf{U}_q}(T)$ and its cell modules.** Recall that, for any $\lambda \in \mathcal{P}$, we have that $\mathrm{End}_{\mathbf{U}_q}(T)^{\leq \lambda}$ and $\mathrm{End}_{\mathbf{U}_q}(T)^{<\lambda}$ are two-sided ideals in $\mathrm{End}_{\mathbf{U}_q}(T)$ (this follows from (9) and its right-handed version obtained by applying i), as in any cellular algebra. In our case we can also see this as follows. If $\varphi \in \mathrm{End}_{\mathbf{U}_q}(T)^{\leq \lambda}$, then $\varphi_\mu = 0$ unless $\mu \leq \lambda$. Hence, for any $\varphi, \psi \in \mathrm{End}_{\mathbf{U}_q}(T)$ we have $(\varphi \circ \psi)_\mu = \varphi_\mu \circ \psi_\mu = 0 = \psi_\mu \circ \varphi_\mu = (\psi \circ \varphi)_\mu$ unless $\mu \leq \lambda$. As a consequence, $\mathrm{End}_{\mathbf{U}_q}(T)^\lambda = \mathrm{End}_{\mathbf{U}_q}(T)^{\leq \lambda} / \mathrm{End}_{\mathbf{U}_q}(T)^{<\lambda}$ is an $\mathrm{End}_{\mathbf{U}_q}(T)$-bimodule.

Recall that, for any $g \in C(\lambda)$ and any $f \in C(\lambda)^*$, we denote by $\overline{g}\colon T_q(\lambda) \to T$ and $\overline{f}\colon T \to T_q(\lambda)$ a choice of lifts which satisfy $\overline{g} \circ \iota^\lambda = g$ and $\pi^\lambda \circ \overline{f} = f$, respectively.

Lemma 4.5. Let $\lambda \in \mathcal{P}$. Then the pairing map

$$\langle \cdot, \cdot \rangle^\lambda \colon C(\lambda) \otimes C(\lambda)^* \to \mathrm{End}_{\mathbf{U}_q}(T)^\lambda, \quad \langle g, f \rangle^\lambda = \overline{g} \circ \overline{f} + \mathrm{End}_{\mathbf{U}_q}(T)^{<\lambda}, g \in C(\lambda), f \in C(\lambda)^*$$

is an isomorphism of $\mathrm{End}_{\mathbf{U}_q}(T)$-bimodules. □

Proof. First we note that $\overline{g} \circ \overline{f} + \mathrm{End}_{\mathbf{U}_q}(T)^{<\lambda}$ does not depend on the choices for the lifts $\overline{f}, \overline{g}$, because the change-of-basis matrix from Lemma 3.6 is unitriangular (and works for swapped roles of f's and g's as well). This makes the pairing well-defined.

Note that the pairing $\langle \cdot, \cdot \rangle^\lambda$ takes, by birth, the basis $(g_i^\lambda \otimes f_j^\lambda)_{i,j \in \mathcal{I}^\lambda}$ of $C(\lambda) \otimes C(\lambda)^*$ to the basis $\{c_{ij}^\lambda + \mathrm{End}_{\mathbf{U}_q}(T)^{<\lambda}\}_{i,j \in \mathcal{I}^\lambda}$ of $\mathrm{End}_{\mathbf{U}_q}(T)^\lambda$ (where the latter is a basis by Lemma 3.5).

So we only need to check that $\langle \varphi \circ g_i^\lambda, f_j^\lambda \circ \psi \rangle^\lambda = \varphi \circ c_{ij}^\lambda \circ \psi \pmod{\mathrm{End}_{\mathbf{U}_q}(T)^{<\lambda}}$ for any $\varphi, \psi \in \mathrm{End}_{\mathbf{U}_q}(T)$. But this is a direct consequence of (11), (12) and (13). ■

The next lemma is straightforward by Lemma 4.5. Details are left to the reader.

Lemma 4.6. We have the following.

(a) There is an isomorphism of $\mathbb{K}$-vector spaces $\mathrm{End}_{\mathbf{U}_q}(T) \cong \bigoplus_{\lambda \in \mathcal{P}} \mathrm{End}_{\mathbf{U}_q}(T)^\lambda$.

(b) If $\varphi \in \mathrm{End}_{\mathbf{U}_q}(T)^{\leq\lambda}$, then we have $r_{ik}^\mu(\varphi) = 0$ for all $\mu \not\leq \lambda, i, k \in \mathcal{I}(\mu)$. Equivalently, $\mathrm{End}_{\mathbf{U}_q}(T)^{\leq\lambda} C(\mu) = 0$ unless $\mu \leq \lambda$. ■

In the following we assume that $\lambda \in \mathcal{P}_0$ as in (14). Define m_λ via

$$T \cong T_q(\lambda)^{\oplus m_\lambda} \oplus T', \tag{15}$$

where T' is a $\mathbf{U}_q$-tilting module containing no summands isomorphic to $T_q(\lambda)$.

Choose now a basis of $C(\lambda) = \mathrm{Hom}_{\mathbf{U}_q}(\Delta_q(\lambda), T)$ as follows. Let $\overline{g}_i^\lambda$ for $i = 1, \ldots, m_\lambda$ be the inclusion of $T_q(\lambda)$ into the i-th summand of $T_q(\lambda)^{\oplus m_\lambda}$ and set $g_i^\lambda = \overline{g}_i^\lambda \circ \iota^\lambda$. Then extend $\{g_1^\lambda, \ldots, g_{m_\lambda}^\lambda\}$ to a basis of the cell module $C(\lambda)$ by adding an arbitrary basis of $\mathrm{Hom}_{\mathbf{U}_q}(\Delta_q(\lambda), T')$. Thus, in our usual notation, we have $c_{ij}^\lambda = \overline{g}_i^\lambda \circ \overline{f}_j^\lambda$ with $\overline{f}_j^\lambda = \mathrm{i}(\overline{g}_j^\lambda)$.

In particular, $\overline{f}_j^\lambda$ projects onto the j-th summand in $T_q(\lambda)^{\oplus m_\lambda}$ for $j = 1, \ldots, m_\lambda$. Thus, the c_{ii}^λ's for $i \leq m_\lambda$ are idempotents in $\mathrm{End}_{\mathbf{U}_q}(T)$ corresponding to the i-th summand in $T_q(\lambda)^{\oplus m_\lambda}$. Since $\lambda \in \mathcal{P}_0$ (which implies $1 \leq m_\lambda$), c_{11}^λ is always such an idempotent. This is crucial for the following lemma, which will play an important role in the proof of Proposition 4.8.

Lemma 4.7. In the above notation:

(a) We have $c_{i1}^\lambda \circ g_1^\lambda = g_i^\lambda$ for all $i \in \mathcal{I}^\lambda$.

(b) We have $c_{ij}^\lambda \circ g_1^\lambda = 0$ for all $i, j \in \mathcal{I}^\lambda$ with $j \neq 1$. □

Proof. We have $\overline{f}_1^\lambda \circ g_1^\lambda = \overline{f}_1^\lambda \circ \overline{g}_1^\lambda \circ \iota^\lambda = \iota^\lambda$. This implies $c_{i1}^\lambda \circ g_1^\lambda = \overline{g}_i^\lambda \circ \iota^\lambda = g_i^\lambda$. Next, if $j \neq 1$, then $\overline{f}_j^\lambda \circ g_1^\lambda = 0$, since $\overline{f}_j^\lambda$ is zero on $T_q(\lambda)$. Thus, $c_{ij}^\lambda \circ g_1^\lambda = 0$ for all $i, j \in \mathcal{I}^\lambda$ with $j \neq 1$. ■

Proposition 4.8. (**Homomorphism criterion.**) Let $\lambda \in \mathcal{P}_0$ and fix $M \in \mathrm{End}_{\mathbf{U}_q}(T)$-**Mod**. Then there is an isomorphism of $\mathbb{K}$-vector spaces

$$\mathrm{Hom}_{\mathrm{End}_{\mathbf{U}_q}(T)}(C(\lambda), M) \cong \{m \in M \mid \mathrm{End}_{\mathbf{U}_q}(T)^{<\lambda} m = 0 \text{ and } c_{11}^\lambda m = m\}. \tag{16}$$
□

Proof. Let $\psi \in \mathrm{Hom}_{\mathrm{End}_{\mathbf{U}_q}(T)}(C(\lambda), M)$. Then $\psi(g_1^\lambda)$ belongs to the right-hand side, because, by (b) of Lemma 4.6, we get $\mathrm{End}_{\mathbf{U}_q}(T)^{<\lambda} C(\lambda) = 0$, and we have $c_{11}^\lambda \circ g_1^\lambda = g_1^\lambda$ by (a) of Lemma 4.7. Conversely, if $m \in M$ belongs to the right-hand side in (16), then we may define $\psi \in \mathrm{Hom}_{\mathrm{End}_{\mathbf{U}_q}(T)}(C(\lambda), M)$ by $\psi(g_i^\lambda) = c_{i1}^\lambda m$, $i \in \mathcal{I}^\lambda$. Moreover, the fact that this definition gives an $\mathrm{End}_{\mathbf{U}_q}(T)$-homomorphism follows from (10), (11) and (12) via a direct computation, since $\mathrm{End}_{\mathbf{U}_q}(T)^{<\lambda} m = 0$. Clearly these two operations are mutually inverses. ■

Corollary 4.9. Let $\lambda \in \mathcal{P}_0$. Then $C(\lambda)$ has a unique simple head, denoted by $L(\lambda)$. □

Proof. Set $\mathrm{Rad}(\lambda) = \{g \in C(\lambda) \mid \vartheta^\lambda(g, C(\lambda)) = 0\}$. As the cellular pairing ϑ^λ from Definition 4.2 is contravariant by Lemma 4.3, we see that $\mathrm{Rad}(\lambda)$ is an $\mathrm{End}_{\mathbf{U}_q}(T)$-submodule of

$C(\lambda)$. Since $\vartheta^{\lambda} \neq 0$ for $\lambda \in \mathcal{P}_0$, we have $\mathrm{Rad}(\lambda) \subsetneq C(\lambda)$. We claim that $\mathrm{Rad}(\lambda)$ is the unique maximal proper $\mathrm{End}_{\mathbf{U}_q}(T)$-submodule of $C(\lambda)$.

Let $g \in C(\lambda) - \mathrm{Rad}(\lambda)$. Moreover, choose $h \in C(\lambda)$ with $\vartheta^{\lambda}(g,h) = 1$. Then $\mathrm{i}(h) \circ g = c^{\lambda}$ so that $\overline{\mathrm{i}(h)} \circ g = \iota^{\lambda}$ (mod $\mathrm{End}_{\mathbf{U}_q}(T)^{<\lambda}$). Therefore, $g' = \overline{g}' \circ \overline{\mathrm{i}(h)} \circ g$ (mod $\mathrm{End}_{\mathbf{U}_q}(T)^{<\lambda}$) for all $g' \in C(\lambda)$. This implies $C(\lambda) = \mathrm{End}_{\mathbf{U}_q}(T)^{\leq\lambda} g$. Thus, any proper $\mathrm{End}_{\mathbf{U}_q}(T)$-submodule of $C(\lambda)$ is contained in $\mathrm{Rad}(\lambda)$ which implies the desired statement. ■

Corollary 4.10. Let $\lambda \in \mathcal{P}_0, \mu \in \mathcal{P}$ and assume that $\mathrm{Hom}_{\mathrm{End}_{\mathbf{U}_q}(T)}(C(\lambda), M) \neq 0$ for some $\mathrm{End}_{\mathbf{U}_q}(T)$-module M isomorphic to a subquotient of $C(\mu)$. Then we have $\mu \leq \lambda$. In particular, all composition factors $L(\lambda)$ of $C(\mu)$ satisfy $\mu \leq \lambda$. □

Proof. By Proposition 4.8 the assumption in the corollary implies the existence of an element $m \in M$ with $c^{\lambda}_{11} m = m$. But if $\mu \not\leq \lambda$, then c^{λ}_{11} vanishes on the $\mathbf{U}_q$-weight space T_μ and hence, $c^{\lambda}_{11} g$ kills the highest weight vector in $\Delta_q(\mu)$ for all $g \in C(\mu)$. This makes the existence of such an $m \in M$ impossible unless $\mu \leq \lambda$. ■

4.3. **Simple $\mathrm{End}_{\mathbf{U}_q}(T)$-modules and semisimplicity of $\mathrm{End}_{\mathbf{U}_q}(T)$.** Let $\lambda \in \mathcal{P}_0$. Note that Corollary 4.9 shows that $C(\lambda)$ has a unique simple head $L(\lambda)$. We then arrive at the following classification of all simple modules in $\mathrm{End}_{\mathbf{U}_q}(T)$-**Mod**.

Theorem 4.11. (**Classification of simple $\mathrm{End}_{\mathbf{U}_q}(T)$-modules.**) The set $\{L(\lambda) \mid \lambda \in \mathcal{P}_0\}$ forms a complete set of pairwise non-isomorphic, simple $\mathrm{End}_{\mathbf{U}_q}(T)$-modules. □

Proof. We have to show three statements, namely that the $L(\lambda)$'s are simple, that they are pairwise non-isomorphic and that every simple $\mathrm{End}_{\mathbf{U}_q}(T)$-module is one of the $L(\lambda)$'s.

Since the first statement follows directly from the definition of $L(\lambda)$ (see Corollary 4.9), we start by showing the second. Thus, assume that $L(\lambda) \cong L(\mu)$ for some $\lambda, \mu \in \mathcal{P}_0$. Then

$$\mathrm{Hom}_{\mathrm{End}_{\mathbf{U}_q}(T)}(C(\lambda), C(\mu)/\mathrm{Rad}(\mu)) \neq 0 \neq \mathrm{Hom}_{\mathrm{End}_{\mathbf{U}_q}(T)}(C(\mu), C(\lambda)/\mathrm{Rad}(\lambda)).$$

By Corollary 4.10, we get $\mu \leq \lambda$ and $\lambda \leq \mu$ from the left and right-hand side. Thus, $\lambda = \mu$.

Suppose that $L \in \mathrm{End}_{\mathbf{U}_q}(T)$-**Mod** is simple. Then we can choose $\lambda \in \mathcal{P}$ minimal such that (recall that $\mathrm{End}_{\mathbf{U}_q}(T)^{\leq\lambda}$ is a two-sided ideal)

$$\mathrm{End}_{\mathbf{U}_q}(T)^{<\lambda} L = 0 \quad \text{and} \quad \mathrm{End}_{\mathbf{U}_q}(T)^{\leq\lambda} L = L. \tag{17}$$

We claim that $\lambda \in \mathcal{P}_0$. Indeed, if not, then, by Proposition 4.4, we see that $T_q(\lambda)$ is not a summand of T. Hence, in our usual notation, all $\overline{f}^{\lambda}_j \circ \overline{g}^{\lambda}_{i'}$ vanish on the λ-weight space. It follows that $c^{\lambda}_{ij} c^{\lambda}_{i'j'}$ also vanish on the λ-weight space for all $i, j, i', j' \in \mathcal{I}^{\lambda}$. This means that we have $\mathrm{End}_{\mathbf{U}_q}(T)^{\leq\lambda} \mathrm{End}_{\mathbf{U}_q}(T)^{\leq\lambda} \subset \mathrm{End}_{\mathbf{U}_q}(T)^{<\lambda}$ making (17) impossible.

For $\lambda \in \mathcal{P}_0$ we see by Lemma 4.7 that

$$c^{\lambda}_{i1} c^{\lambda}_{1j} = c^{\lambda}_{ij} \ (\mathrm{mod}\ \mathrm{End}_{\mathbf{U}_q}(T)^{<\lambda}). \tag{18}$$

Hence, by (17), there exist $i, j \in \mathcal{I}^{\lambda}$ such that $c^{\lambda}_{ij} L \neq 0$. By (18) we also have $c^{\lambda}_{i1} L \neq 0 \neq c^{\lambda}_{1j} L$. This in turn (again by (18)) ensures that $c^{\lambda}_{11} L \neq 0$. Take then $m \in c^{\lambda}_{11} L - \{0\}$ and observe that $c^{\lambda}_{11} m = m$. Hence, by Proposition 4.8, there is a non-zero $\mathrm{End}_{\mathbf{U}_q}(T)$-homomorphism $C(\lambda) \to L$. The conclusion follows now from Corollary 4.9. ■

Recall from Subsection 4.2 the notation m_λ (the multiplicity of $T_q(\lambda)$ in T) and the choice of basis for $C(\lambda)$ (in the paragraphs before Lemma 4.7). Then we get the following connection between the decomposition of T as in (15) and the simple $\mathrm{End}_{\mathbf{U}_q}(T)$-modules $L(\lambda)$.

Theorem 4.12. (**Dimension formula.**) If $\lambda \in \mathcal{P}_0$, then $\dim(L(\lambda)) = m_\lambda$. □

Note that this result is implicit in [38] and has also been observed in e.g. [37] and [82].

Proof. We use the notation from Subsection 4.2. Since T' has no summands isomorphic to $T_q(\lambda)$, we see that $\mathrm{Hom}_{\mathbf{U}_q}(\Delta_q(\lambda), T') \subset \mathrm{Rad}(\lambda)$ (see the proof of Corollary 4.9). On the other hand, $g_i^\lambda \notin \mathrm{Rad}(\lambda)$ for $1 \leq i \leq m_\lambda$ because for these i we have $f_i^\lambda \circ g_i^\lambda = c^\lambda$ by construction. Thus, the statement follows. ■

Theorem 4.13. (**Semisimplicity criterion for** $\mathrm{End}_{\mathbf{U}_q}(T)$**.**) The cellular algebra $\mathrm{End}_{\mathbf{U}_q}(T)$ is semisimple iff T is a semisimple $\mathbf{U}_q$-module. □

Proof. Note that the $T_q(\lambda)$'s are simple iff $T_q(\lambda) \cong \Delta_q(\lambda) \cong L_q(\lambda) \cong \nabla_q(\lambda)$. Hence, T is semisimple as a $\mathbf{U}_q$-module iff $T = \bigoplus_{\lambda \in \mathcal{P}_0} \Delta_q(\lambda)^{\oplus m_\lambda}$ with m_λ as above.

Thus, we see that, if T decomposes into simple $\mathbf{U}_q$-modules, then $\mathrm{End}_{\mathbf{U}_q}(T)$ is semisimple by the Artin-Wedderburn theorem (since $\mathrm{End}_{\mathbf{U}_q}(T)$ will decompose into a direct sum of matrix algebras in this case).

On the other hand, if $\mathrm{End}_{\mathbf{U}_q}(T)$ is semisimple, then we know, by Corollary 4.9, that the cell modules $C(\lambda)$ are simple, i.e. $C(\lambda) = L(\lambda)$ for all $\lambda \in \mathcal{P}_0$. Then we have

$$T \cong \bigoplus_{\lambda \in \mathcal{P}_0} T_q(\lambda)^{\oplus m_\lambda}, \quad m_\lambda = \dim(L(\lambda)) = \dim(C(\lambda)) = \dim(\mathrm{Hom}_{\mathbf{U}_q}(\Delta_q(\lambda), T)) \tag{19}$$

by Theorem 4.12. Assume now that there exists a summand $T_q(\lambda')$ of T as in (19) with $T_q(\lambda') \ncong \Delta_q(\lambda')$ and choose $\lambda' \in \mathcal{P}_0$ minimal with this property.

Then there exists a $\mu < \lambda'$ such that $\mathrm{Hom}_{\mathbf{U}_q}(\Delta_q(\mu), T_q(\lambda')) \neq 0$. Choose also μ minimal among those. By our usual construction this then gives in turn a non-zero $\mathbf{U}_q$-homomorphism $\overline{g} \circ \overline{f} \colon T_q(\lambda') \to T_q(\mu) \to T_q(\lambda')$. By (19), we can extend $\overline{g} \circ \overline{f}$ to an element of $\mathrm{End}_{\mathbf{U}_q}(T)$ by defining it to be zero on all other summands.

Clearly, by construction, $(\overline{g} \circ \overline{f})C(\mu') = 0$ for $\mu' \in \mathcal{P}_0$ with $\mu' \neq \lambda'$ and $\mu' \nleq \mu$. If $\mu' \leq \mu$, then consider $\varphi \in C(\mu')$. Then $(\overline{g} \circ \overline{f}) \circ \varphi = 0$ unless φ has some non-zero component $\varphi' \colon \Delta_q(\mu') \to T_q(\lambda')$. This forces $\mu' = \mu$ by minimality of μ. But since $\Delta_q(\mu') \cong T_q(\mu')$, by minimality of λ', we conclude that $\overline{f} \circ \varphi = 0$ (otherwise $T_q(\mu')$ would be a summand of $T_q(\lambda')$).

Hence, the non-zero element $\overline{g} \circ \overline{f} \in \mathrm{End}_{\mathbf{U}_q}(T)$ kills all $C(\mu')$ for $\mu' \in \mathcal{P}_0$. This contradicts the semisimplicity of $\mathrm{End}_{\mathbf{U}_q}(T)$: as noted above, $C(\lambda) = L(\lambda)$ for all $\lambda \in \mathcal{P}_0$ which implies $\mathrm{End}_{\mathbf{U}_q}(T) \cong \bigoplus_{\lambda \in \mathcal{P}_0} C(\lambda)^{\oplus k_\lambda}$ for some $k_\lambda \in \mathbb{Z}_{\geq 0}$. ■

5. Cellular structures: examples and applications

In this section we provide many examples of cellular algebras arising from our main theorem. This includes several renowned examples where cellularity is known (but usually proved case by case spread over the literature and with cellular bases which differ in general from ours), and also new ones. In the first subsection we give a full treatment of the semisimple case and describe how to obtain all the examples from the introduction using our methods. In the second subsection we focus on the Temperley-Lieb algebras $\mathcal{TL}_d(\delta)$ and give a detailed account how to apply our results to these.

5.1. **Cellular structures using $\mathbf{U}_q$-tilting modules: several examples.** In the following let ω_i for $i = 1, \ldots, n$ denote the fundamental weights (of the corresponding type).

5.1.1. *The semisimple case.* Suppose the category $\mathbf{U}_q$-**Mod** is semisimple, that is, q is not a root of unity in $\mathbb{K}^* - \{1\}$ or $q = \pm 1 \in \mathbb{K}$ with $\mathrm{char}(\mathbb{K}) = 0$.

In this case $\boldsymbol{\mathcal{T}} = \mathbf{U}_q$-**Mod** and any $T \in \boldsymbol{\mathcal{T}}$ has a decomposition $T \cong \bigoplus_{\lambda \in X^+} \Delta_q(\lambda)^{\oplus m_\lambda}$ with the multiplicities $m_\lambda = (T : \Delta_q(\lambda))$. This induces an Artin-Wedderburn decomposition

$$\mathrm{End}_{\mathbf{U}_q}(T) \cong \bigoplus_{\lambda \in X^+} M_{m_\lambda}(\mathbb{K}) \tag{20}$$

into matrix algebras. A natural choice of basis for $\mathrm{Hom}_{\mathbf{U}_q}(\Delta_q(\lambda), T)$ is

$$G^\lambda = \{g_1^\lambda, \ldots, g_{m_\lambda}^\lambda \mid g_i^\lambda \colon \Delta_q(\lambda) \hookrightarrow T \text{ is the inclusion into the i-th summand}\}.$$

Then our cellular basis consisting of the c_{ij}^λ's as in Subsection 3.3 (no lifting is needed in this case) is an Artin-Wedderburn basis, that is, a basis of $\mathrm{End}_{\mathbf{U}_q}(T)$ that realizes the decomposition as in (20) in the following sense. The basis element c_{ij}^λ is the matrix $\mathbf{E}_{ij}^\lambda$ (in the λ-summand on the right hand side in (20)) which has all entries zero except one entry equals 1 in the i-th row and j-th column. Note that, as expected in this case, $\mathrm{End}_{\mathbf{U}_q}(T)$ has, by the Theorems 4.11 and 4.12, one simple $\mathrm{End}_{\mathbf{U}_q}(T)$-module $L(\lambda)$ of dimension m_λ for all summands $\Delta_q(\lambda)$ of T.

5.1.2. *The symmetric group and the Iwahori-Hecke algebra.* Let us fix $d \in \mathbb{Z}_{\geq 0}$ and let us denote by S_d the symmetric group in d letters and by $\mathcal{H}_d(q)$ its associated Iwahori-Hecke algebra. We note that $\mathbb{K}[S_d] \cong \mathcal{H}_d(1)$. Moreover, let $\mathbf{U}_q = \mathbf{U}_q(\mathfrak{gl}_n)$. The vector representation of $\mathbf{U}_q$, which we denote by $V = \mathbb{K}^n = \Delta_q(\omega_1)$, is a $\mathbf{U}_q$-tilting module (since ω_1 is minimal in X^+). Set $T = V^{\otimes d}$, which is again a $\mathbf{U}_q$-tilting module. Quantum Schur-Weyl duality (see [32, Theorem 6.3] for surjectivity and use Ext-vanishing for the fact that $\dim(\mathrm{End}_{\mathbf{U}_q}(T))$ is obtained via base change from $\mathbb{Z}[v, v^{-1}]$ to $\mathbb{K}$ for all $\mathbb{K}$ and $q \in \mathbb{K}^*$) states that

$$\Phi_{q\mathrm{SW}} \colon \mathcal{H}_d(q) \twoheadrightarrow \mathrm{End}_{\mathbf{U}_q}(T) \quad \text{and} \quad \Phi_{q\mathrm{SW}} \colon \mathcal{H}_d(q) \xrightarrow{\cong} \mathrm{End}_{\mathbf{U}_q}(T), \text{ if } n \geq d. \tag{21}$$

Thus, our main result implies that $\mathcal{H}_d(q)$, and in particular $\mathbb{K}[S_d]$, are cellular for any $q \in \mathbb{K}^*$ and any field $\mathbb{K}$ (by taking $n \geq d$).

In this case the cell modules for $\mathrm{End}_{\mathbf{U}_q}(T)$ are usually called Specht modules $S_{\mathbb{K}}^\lambda$ and our Theorem 4.12 gives the following (see also [37]).

- If $q = 1$ and $\mathrm{char}(\mathbb{K}) = 0$, then the dimension $\dim(S_{\mathbb{K}}^\lambda)$ is equal to the multiplicity of the simple $\mathbf{U}_1$-module $\Delta_1(\lambda) \cong L_1(\lambda)$ in $V^{\otimes d}$ for all $\lambda \in \mathcal{P}^0$. These numbers are given by known formulas (e.g. the hook length formula).
- If $q = 1$ and $\mathrm{char}(\mathbb{K}) > 0$, then the dimension of the simple head of $S_{\mathbb{K}}^\lambda$, usually denoted $D_{\mathbb{K}}^\lambda$, is the multiplicity with which $T_1(\lambda)$ occurs as a summand in $V^{\otimes d}$ for all $\lambda \in \mathcal{P}_0$. It is a wide open problem to determine these numbers. (See however [70].)
- If q is a complex, primitive root of unity, then we can compute the dimension of the simple $\mathcal{H}_d(q)$-modules by using the algorithm as in [8]. In particular, this connects with the LLT algorithm from [57].

- If q is a root of unity and $\mathbb{K}$ is arbitrary, then not much is known. Still, our methods apply and we get a way to calculate the dimensions of the simple $\mathcal{H}_d(q)$-modules, if we can decompose T into its indecomposable summands.

5.1.3. *The Temperley-Lieb algebra and other $\mathfrak{sl}_2$-related algebras.* Let $\mathbf{U}_q = \mathbf{U}_q(\mathfrak{sl}_2)$ and let T be as in 5.1.2 with $n = 2$. For any $d \in \mathbb{Z}_{\geq 0}$ we have $\mathcal{TL}_d(\delta) \cong \mathrm{End}_{\mathbf{U}_q}(T)$ by Schur-Weyl duality, where $\mathcal{TL}_d(\delta)$ is the Temperley-Lieb algebra in d-strands with parameter $\delta = q + q^{-1}$. This works for all $\mathbb{K}$ and all $q \in \mathbb{K}^*$ (this can be deduced from, for example, [32, Theorem 6.3]). Hence, $\mathcal{TL}_d(\delta)$ is always cellular. We discuss this case in more detail in Subsection 5.2.

Furthermore, if we are in the semisimple case, then $\Delta_q(i)$ is a $\mathbf{U}_q$-tilting module for all $i \in \mathbb{Z}_{\geq 0}$ and so is $T = \Delta_q(i_1) \otimes \cdots \otimes \Delta_q(i_d)$. Thus, we obtain that $\mathrm{End}_{\mathbf{U}_q}(T)$ is cellular.

The algebra $\mathrm{End}_{\mathbf{U}_q}(T)$ is known to give a diagrammatic presentation of the full category of $\mathbf{U}_q$-modules, gives rise to colored Jones-polynomials (see for example [73] and the references therein) and was studied[3] from a diagrammatical point of view in [73].

If $q \in \mathbb{K}$ is a root of unity and l is the order of q^2, then, for any $0 \leq i < l$, $\Delta_q(i)$ is a $\mathbf{U}_q$-tilting module (since its simple) and so is $T = \Delta_q(i)^{\otimes d}$. The endomorphism algebra $\mathrm{End}_{\mathbf{U}_q}(T)$ is cellular. This reproves parts of [5, Theorem 1.1] using our general approach.

In characteristic 0: Another family of $\mathbf{U}_q$-tilting modules was studied in [10]. That is, for any $d \in \mathbb{Z}_{\geq 0}$, fix any $\lambda_0 \in \{0, \ldots, l-2\}$ and consider $T = T_q(\lambda_0) \oplus \cdots \oplus T_q(\lambda_d)$ where λ_k is the unique integer $\lambda_k \in \{kl, \ldots, (k+1)l - 2\}$ linked to λ_0. We again obtain that $\mathrm{End}_{\mathbf{U}_q}(T)$ is cellular. Note that $\mathrm{End}_{\mathbf{U}_q}(T)$ can be identified with Khovanov-Seidel's algebra so-called (type A) zig-zag algebra A_d, see [10, Proposition 3.9], introduced in [53] in their study of Floer homology. These algebras are naturally graded making $\mathrm{End}_{\mathbf{U}_q}(T)$ into a graded cellular algebra in the sense of [41] and are special examples arising from the family of generalized Khovanov arc algebras whose cellularity is studied in [19].

5.1.4. *Spider algebras.* Let $\mathbf{U}_q = \mathbf{U}_q(\mathfrak{sl}_n)$ (or, alternatively, $\mathbf{U}_q(\mathfrak{gl}_n)$). One has for any $q \in \mathbb{K}^*$ that all $\mathbf{U}_q$-representations $\Delta_q(\omega_i)$ are $\mathbf{U}_q$-tilting modules (because the ω_i's are minimal in X^+). Hence, for any $k_i \in \{1, \ldots, n-1\}$, $T = \Delta_q(\omega_{k_1}) \otimes \cdots \otimes \Delta_q(\omega_{k_d})$ is a $\mathbf{U}_q$-tilting module. Thus, $\mathrm{End}_{\mathbf{U}_q}(T)$ is cellular. These algebras are related to type A_{n-1} spider algebras introduced in [56], are connected to the Reshetikhin-Turaev $\mathfrak{sl}_n$-link polynomials and give a diagrammatic description of the representation theory of $\mathfrak{sl}_n$, see [23], providing a link from our work to low-dimensional topology and diagrammatic algebra. Note that cellular bases (which, in this case, coincide with our cellular bases) of these, in the semisimple cases, were found in [36, Theorem 2.57].

More general: In any type we have that $\Delta_q(\lambda)$ are $\mathbf{U}_q(\mathfrak{g})$-tilting modules for minuscule $\lambda \in X^+$, see [47, Part II, Chapter 2, Section 15]. Moreover, if q is a root of unity "of order l big enough" (ensuring that the ω_i's are in the closure of the fundamental alcove), then the $\Delta_q(\omega_i)$ are $\mathbf{U}_q(\mathfrak{g})$-tilting modules by the linkage principle (see [3, Corollaries 4.4 and 4.6]). So in these cases we can generalize the above results to other types.

Still more general: we may take (for any type and any $q \in \mathbb{K}^*$) arbitrary $\lambda_j \in X^+$ (for $j = 1, \ldots, d$) and obtain a cellular structure on $\mathrm{End}_{\mathbf{U}_q}(T)$ for $T = T_q(\lambda_1) \otimes \cdots \otimes T_q(\lambda_d)$.

[3] As a category instead of an algebra. We abuse language here and also for some of the other algebras below.

5.1.5. *The Ariki-Koike algebra and related algebras.* Take $\mathfrak{g} = \mathfrak{gl}_{m_1} \oplus \cdots \oplus \mathfrak{gl}_{m_r}$ (which can be easily fit into our context) with $m_1 + \cdots + m_r = m$ and let V be the vector representation of $\mathbf{U}_1(\mathfrak{gl}_m)$ restricted to $\mathbf{U}_1 = \mathbf{U}_1(\mathfrak{g})$. This is again a $\mathbf{U}_1$-tilting module and so is $T = V^{\otimes d}$. Then we have a cyclotomic analogon of (21), namely

$$\Phi_{\mathrm{cl}} \colon \mathbb{C}[\mathbb{Z}/r\mathbb{Z} \wr S_d] \twoheadrightarrow \mathrm{End}_{\mathbf{U}_1}(T) \quad \text{and} \quad \Phi_{\mathrm{cl}} \colon \mathbb{C}[\mathbb{Z}/r\mathbb{Z} \wr S_d] \xrightarrow{\cong} \mathrm{End}_{\mathbf{U}_1}(T), \text{ if } m \geq d, \tag{22}$$

where $\mathbb{C}[\mathbb{Z}/r\mathbb{Z} \wr S_d]$ is the group algebra of the complex reflection group $\mathbb{Z}/r\mathbb{Z} \wr S_d \cong (\mathbb{Z}/r\mathbb{Z})^d \rtimes S_d$, see [65, Theorem 9]. Thus, we can apply our main theorem and obtain a cellular basis for these quotients of $\mathbb{C}[\mathbb{Z}/r\mathbb{Z} \wr S_d]$. If $m \geq d$, then (22) is an isomorphism (see [65, Lemma 11]) and we obtain that $\mathbb{C}[\mathbb{Z}/r\mathbb{Z} \wr S_d]$ itself is a cellular algebra for all r, d. In the extremal case $m_1 = m-1$ and $m_2 = 1$, the resulting quotient of (22) is known as Solomon's algebra introduced in [85] (also called algebra of the inverse semigroup or rook monoid algebra) and we obtain that Solomon's algebra is cellular. In the extremal case $m_1 = m_2 = 1$, the resulting quotient is a specialization of the blob algebra $\mathcal{BL}_d(1,2)$ (in the notation used in [77]). To see this, note that both algebras are quotients of $\mathbb{C}[\mathbb{Z}/r\mathbb{Z} \wr S_d]$. The kernel of the quotient to $\mathcal{BL}_d(1,2)$ is described explicitly by Ryom-Hansen in [77, (1)] and is by [65, Lemma 11] contained in the kernel of Φ_{cl} from (22). Since both algebras have the same dimensions, they are isomorphic.

Let $\mathbf{U}_q = \mathbf{U}_q(\mathfrak{g})$. We get in the quantized case (for $q \in \mathbb{C} - \{0\}$ not a root of unity)

$$\Phi_{q\mathrm{cl}} \colon \mathcal{H}_{d,r}(q) \twoheadrightarrow \mathrm{End}_{\mathbf{U}_q}(T) \quad \text{and} \quad \Phi_{q\mathrm{cl}} \colon \mathcal{H}_{d,r}(q) \xrightarrow{\cong} \mathrm{End}_{\mathbf{U}_q}(T), \text{ if } m \geq d, \tag{23}$$

where $\mathcal{H}_{d,r}(q)$ is the Ariki-Koike algebra introduced in [12]. A proof of (23) can for example be found in [78, Theorem 4.1]. Thus, as before, our main theorem applies and we obtain: the Ariki-Koike algebra $\mathcal{H}_{d,r}(q)$ is cellular (by taking $m \geq d$), the quantized rook monoid algebra $\mathcal{R}_d(q)$ from [39] is cellular and the blob algebra $\mathcal{BL}_d(q,m)$ is cellular (which follows as above). Note that the cellularity of $\mathcal{H}_{d,r}(q)$ was obtained in [28], the cellularity of the quantum rook monoid algebras and of the blob algebra can be found in [67] and in [76] respectively.

In fact, (23) is still true in the non-semisimple cases, see [43, Theorem 1.10 and Lemma 2.12] as long as $\mathbb{K}$ satisfies a certain separation condition (which implies that the algebra in question has the right dimension, see [11]). Again, our main theorem applies.

5.1.6. *The Brauer algebras and related algebras.* Consider $\mathbf{U}_q = \mathbf{U}_q(\mathfrak{g})$ where $\mathfrak{g}$ is either an orthogonal $g = \mathfrak{o}_{2n}$ and $g = \mathfrak{o}_{2n+1}$ or the symplectic $g = \mathfrak{sp}_{2n}$ Lie algebra. Let $V = \Delta_q(\omega_1)$ be the quantized version of the corresponding vector representation. In both cases, V is a $\mathbf{U}_q$-tilting module (for type B and $q = 1$ this requires $\mathrm{char}(\mathbb{K}) \neq 2$, see [45, Page 20]) and hence, so is $T = V^{\otimes d}$. We first take $q = 1$ and set $\delta = 2n$ in case $\mathfrak{g} = \mathfrak{o}_{2n}$, and $\delta = 2n+1$ in case $\mathfrak{g} = \mathfrak{o}_{2n+1}$ and $\delta = -2n$ in case $\mathfrak{g} = \mathfrak{sp}_{2n}$ respectively. Then (see [26, Theorem 1.4] and [31, Theorem 1.2] for infinite $\mathbb{K}$, or [35, Theorem 5.5] for $\mathbb{K} = \mathbb{C}$)

$$\Phi_{\mathrm{Br}} \colon \mathcal{B}_d(\delta) \twoheadrightarrow \mathrm{End}_{\mathbf{U}_1}(T) \quad \text{and} \quad \Phi_{\mathrm{Br}} \colon \mathcal{B}_d(\delta) \xrightarrow{\cong} \mathrm{End}_{\mathbf{U}_1}(T), \text{ if } n > d, \tag{24}$$

where $\mathcal{B}_d(\delta)$ is the Brauer algebra in d strands (for $\mathfrak{g} \neq \mathfrak{o}_{2n}$ the isomorphism in (24) already holds for $n = d$). Thus, we get cellularity of $\mathcal{B}_d(\delta)$ by observing that in characteristic p we can always assume that n is large because $\mathcal{B}_d(\delta) = \mathcal{B}_d(\delta + p)$.

Similarly, let $\mathbf{U}_q = \mathbf{U}_q(\mathfrak{gl}_n)$, $q \in \mathbb{K}^*$ be arbitrary and set $T = \Delta_q(\omega_1)^{\otimes r} \otimes \Delta_q(\omega_{n-1})^{\otimes s}$. By [27, Theorem 7.1 and Corollary 7.2] we have

$$\Phi_{\mathrm{wBr}} \colon \mathcal{B}^n_{r,s}([n]) \twoheadrightarrow \mathrm{End}_{\mathbf{U}_q}(T) \quad \text{and} \quad \Phi_{\mathrm{wBr}} \colon \mathcal{B}^n_{r,s}([n]) \xrightarrow{\cong} \mathrm{End}_{\mathbf{U}_q}(T), \text{ if } n \geq r+s. \tag{25}$$

Here $\mathcal{B}^n_{r,s}([n])$ the quantized walled Brauer algebra for $[n] = q^{1-n} + \cdots + q^{n-1}$. Since T is a $\mathbf{U}_q$-tilting module, we get from (25) cellularity of $\mathcal{B}^n_{r,s}([n])$ and of its quotients under Φ_{wBr}.

The walled Brauer algebra $\mathcal{B}^n_{r,s}(\delta)$ over $\mathbb{K} = \mathbb{C}$ for arbitrary parameter $\delta \in \mathbb{Z}$ appears as centralizer $\mathrm{End}_{\mathfrak{gl}(m|n)}(T)$ for $T = V^{\otimes r} \otimes (V^*)^{\otimes s}$ where V is the vector representation of the superalgebra $\mathfrak{gl}(m|n)$ with $\delta = m - n$. That is, we have

$$(26)\quad \Phi_{\mathrm{s}}\colon \mathcal{B}^n_{r,s}(\delta) \twoheadrightarrow \mathrm{End}_{\mathfrak{gl}(m|n)}(T) \;\text{ and }\; \Phi_{\mathrm{s}}\colon \mathcal{B}^n_{r,s}(\delta) \xrightarrow{\cong} \mathrm{End}_{\mathfrak{gl}(m|n)}(T),\; \text{if } (m+1)(n+1) \geq r+s,$$

see [18, Theorem 7.8]. It can be shown that T is a $\mathfrak{gl}(m|n)$-tilting module and thus, our main theorem applies and hence, by (26), $\mathcal{B}^n_{r,s}(\delta)$ is cellular. Similarly for the quantized version.

Quantizing the Brauer case, taking $q \in \mathbb{K}^*$, $\mathfrak{g}$, $V = \Delta_q(\omega_1)$ and T as before (without the restriction $\mathrm{char}(\mathbb{K}) \neq 2$ for type B) gives us a cellular structure on $\mathrm{End}_{\mathbf{U}_q}(T)$. The algebra $\mathrm{End}_{\mathbf{U}_q}(T)$ is a quotient of the Birman-Murakami-Wenzl algebra $\mathcal{BMW}_d(\delta)$ (for appropriate parameters), see [58, (9.6)] for the orthogonal case (which works for any $q \in \mathbb{C} - \{0, \pm 1\}$) and [40, Theorem 1.5] for the symplectic case (which works for any $q \in \mathbb{K}^* - \{1\}$ and infinite $\mathbb{K}$). Again, taking $n \geq d$ (or $n > d$), we recover the cellularity of $\mathcal{BMW}_d(\delta)$.

5.1.7. *Infinite-dimensional modules - highest weight categories.* Observe that our main theorem does not use the specific properties of $\mathbf{U}_q$-$\mathbf{Mod}$, but works for any $\mathrm{End}_{A\text{-}\mathbf{Mod}}(T)$ where T is an A-tilting module for some finite-dimensional, quasi-hereditary algebra A over $\mathbb{K}$ or $T \in \mathcal{C}$ for some highest weight category $\mathcal{C}$ in the sense of [24]. For the explicit construction of our basis we however need a notion like "weight spaces" such that Lemma 3.4 makes sense.

The most famous example of such a category is the BGG category $\mathcal{O} = \mathcal{O}(\mathfrak{g})$ attached to a complex semisimple or reductive Lie algebra $\mathfrak{g}$ with a corresponding Cartan $\mathfrak{h}$ and fixed Borel subalgebra $\mathfrak{b}$. We denote by $\Delta(\lambda) \in \mathcal{O}$ the Verma module attached to $\lambda \in \mathfrak{h}^*$. In the same vein, pick a parabolic $\mathfrak{p} \supset \mathfrak{b}$ and denote for any $\mathfrak{p}$-dominant weight λ the corresponding parabolic Verma module by $\Delta^{\mathfrak{p}}(\lambda)$. It is the unique quotient of the Verma module $\Delta(\lambda)$ which is locally $\mathfrak{p}$-finite, i.e. contained in the parabolic category $\mathcal{O}^{\mathfrak{p}} = \mathcal{O}^{\mathfrak{p}}(\mathfrak{g}) \subset \mathcal{O}$ (see e.g. [44]).

There is a contravariant, character preserving duality functor ${}^{\vee}\colon \mathcal{O}^{\mathfrak{p}} \to \mathcal{O}^{\mathfrak{p}}$ which allows us to set $\nabla^{\mathfrak{p}}(\lambda) = \Delta^{\mathfrak{p}}(\lambda)^{\vee}$. Hence, we can play the same game again since the $\mathcal{O}$-tilting theory works in a very similar fashion as for $\mathbf{U}_q$-$\mathbf{Mod}$ (see [44, Chapter 11] and the references therein). In particular, we have indecomposable $\mathcal{O}$-tilting modules $T(\lambda)$ for any $\lambda \in \mathfrak{h}^*$. Similarly for $\mathcal{O}^{\mathfrak{p}}$ giving an indecomposable $\mathcal{O}^{\mathfrak{p}}$-tilting module $T(\lambda)$ for any $\mathfrak{p}$-dominant $\lambda \in \mathfrak{h}^*$.

We shall give a few examples where our approach leads to cellular structures on interesting algebras. For this purpose, let $\mathfrak{p} = \mathfrak{b}$ and $\lambda = 0$. Then $T(0)$ has Verma factors of the form $\Delta(w.0)$ (for $w \in W$, where W is the Weyl group associated to $\mathfrak{g}$). Each of these appears with multiplicity 1. Hence, $\dim(\mathrm{End}_{\mathcal{O}}(T(0))) = |W|$ by the analogon of (4). Then we have $\mathrm{End}_{\mathcal{O}}(T(0)) \cong S(\mathfrak{h}^*)/S^W_+$. The algebra $S(\mathfrak{h}^*)/S^W_+$ is called coinvariant algebra. (For the notation, the conventions and the result see [83] - this is Soergel's famous Endomorphismensatz.) Hence, our main theorem implies that $S(\mathfrak{h}^*)/S^W_+$ is cellular, which is no big surprise since all finite-dimensional, commutative algebras are cellular, see [55, Proposition 3.5].

There is also a quantum version of this result: replace $\mathcal{O}$ by its quantum cousin $\mathcal{O}_q$ from [6] (which is the analogon of $\mathcal{O}$ for $\mathbf{U}_q(\mathfrak{g})$). This works over any field $\mathbb{K}$ with $\mathrm{char}(\mathbb{K}) = 0$ and any $q \in \mathbb{K}^* - \{1\}$ (which can be deduced from [6, Section 6]). There is furthermore a characteristic p version of this result: consider the G-tilting module $T(p\rho)$ in the category of finite-dimensional G-modules (here G is an almost simple, simply connected algebraic group over $\mathbb{K}$ with $\mathrm{char}(\mathbb{K}) = p$). Its endomorphism algebra is isomorphic to for the corresponding coinvariant algebra over $\mathbb{K}$, see [4, Proposition 19.8].

Returning to $\mathbb{K} = \mathbb{C}$: the example of the coinvariant algebra can be generalized. To this end, note that, if T is an $\mathcal{O}^{\mathfrak{p}}$-tilting module, then so is $T \otimes M$ for any finite-dimensional $\mathfrak{g}$-module M, see [44, Proposition 11.1 and Section 11.8] (and the references therein). Thus, $\mathrm{End}_{\mathcal{O}^{\mathfrak{p}}}(T \otimes M)$ is cellular by our main theorem.

A special case is: $\mathfrak{g}$ is of classical type, $T = \Delta^{\mathfrak{p}}(\lambda)$ is simple (hence, $\mathcal{O}^{\mathfrak{p}}$-tilting), V is the vector representation of $\mathfrak{g}$ and $M = V^{\otimes d}$. Let first $\mathfrak{g} = \mathfrak{gl}_n$ with standard Borel $\mathfrak{b}$ and parabolic $\mathfrak{p}$ of block size $(n_1, \dots, n_\ell)$. Then one can find a certain $\mathfrak{p}$-dominant weight λ_{I}, called Irving-weight, such that $T = \Delta^{\mathfrak{p}}(\lambda_{\mathrm{I}})$ is $\mathcal{O}^{\mathfrak{p}}$-tilting. Moreover, $\mathrm{End}_{\mathcal{O}^{\mathfrak{p}}}(T \otimes V^{\otimes d})$ is isomorphic to a sum of blocks of cyclotomic quotients of the degenerate affine Hecke algebra $\mathcal{H}_d/\Pi_{i=1}^{\ell}(x_i - n_i)$, see [17, Theorem 5.13]. In the special case of level $\ell = 2$, these algebras can be explicitly described in terms of generalizations of Khovanov's arc algebra (which Khovanov introduced in [50] to give an algebraic structure underlying Khovanov homology and which categorifies the Temperley-Lieb algebra $\mathcal{TL}_d(\delta)$) and have an interesting representation theory, see [19], [20], [21] and [22]. A consequence of this is: using the results from [79, Theorem 6.9] and [80, Theorem 1.1], one can realize the walled Brauer algebra from 5.1.6 for arbitrary parameter $\delta \in \mathbb{Z}$ as endomorphism algebras of some $\mathcal{O}^{\mathfrak{p}}$-tilting module and hence, using our main theorem, deduce cellularity again.

If $\mathfrak{g}$ is of another classical type, then the role of the (cyclotomic quotients of the) degenerate affine Hecke algebra is played by (cyclotomic quotients of) degenerate BMW algebras or so-called (cyclotomic quotients of) $\mathbb{W}_d$-algebras (also called Nazarov-Wenzl algebras). These are still poorly understood and technically quite involved, see [13]. In [33] special examples of level $\ell = 2$ quotients were studied and realized as endomorphism algebras of some $\mathcal{O}^{\mathfrak{p}}(\mathfrak{so}_{2n})$-tilting module $\Delta^{\mathfrak{p}}(\underline{\delta}) \otimes V \in \mathcal{O}^{\mathfrak{p}}(\mathfrak{so}_{2n})$ where V is the vector representation of $\mathfrak{so}_{2n}$, $\underline{\delta} = \frac{\delta}{2}\sum_{i=1}^{n} \epsilon_i$ and $\mathfrak{p}$ is a maximal parabolic subalgebra of type A (see [33, Theorem B]). Hence, our theorem implies cellularity of these algebras. Soergel's theorem is therefore just a shadow of a rich world of endomorphism algebras whose cellularity can be obtained from our approach.

Our methods also apply to (parabolic) category $\mathcal{O}^{\mathfrak{p}}(\hat{\mathfrak{g}})$ attached to an affine Kac-Moody algebra $\hat{\mathfrak{g}}$ over $\mathbb{K}$ and related categories. In particular, one can consider a (level-dependent) quotient $\hat{\mathfrak{g}}_\kappa$ of $\mathbf{U}(\hat{\mathfrak{g}})$ and a category, denoted by $\mathbf{O}_{\mathbb{K},\tau}^{\nu,\kappa}$, attached to it (we refer the reader to [75, Subsections 5.2 and 5.3] for the details). Then there is a subcategory $\mathbf{A}_{\mathbb{K},\tau}^{\nu,\kappa} \subset \mathbf{O}_{\mathbb{K},\tau}^{\nu,\kappa}$ and a $\mathbf{A}_{\mathbb{K},\tau}^{\nu,\kappa}$-tilting module $\mathbf{T}_{\mathbb{K},d}$ defined in [75, Subsection 5.5] such that

$$\Phi_{\mathrm{aff}} \colon \mathbf{H}_{\mathbb{K},d}^{s} \to \mathrm{End}_{\mathbf{A}_{\mathbb{K},\tau}^{\nu,\kappa}}(\mathbf{T}_{\mathbb{K},d}) \quad \text{and} \quad \Phi_{\mathrm{aff}} \colon \mathbf{H}_{\mathbb{K},d}^{s} \xrightarrow{\cong} \mathrm{End}_{\mathbf{A}_{\mathbb{K},\tau}^{\nu,\kappa}}(\mathbf{T}_{\mathbb{K},d}), \text{ if } \nu_p \geq d, p = 1, \dots N,$$

see [75, Theorem 5.37 and Proposition 8.1]. Here $\mathbf{H}_{\mathbb{K},d}^{s}$ denotes an appropriate cyclotomic quotient of the affine Hecke algebra. Again, our main theorem applies for $\mathbf{H}_{\mathbb{K},d}^{s}$ in case $\nu_p \geq d$.

5.1.8. *Graded cellular structures.* A striking property which arises in the context of (parabolic) category $\mathcal{O}$ (or $\mathcal{O}^{\mathfrak{p}}$) is that all the endomorphism algebras from 5.1.7 can be equipped with a $\mathbb{Z}$-grading as in [86] arising from the Koszul grading of category $\mathcal{O}$ (or of $\mathcal{O}^{\mathfrak{p}}$). We might choose our cellular basis compatible with this grading and obtain a grading on the endomorphism algebras turning them into graded cellular algebras in the sense of [41, Definition 2.1].

For the cyclotomic quotients this grading is non-trivial and in fact is the type A KL-R grading in the spirit of Khovanov and Lauda and independently Rouquier (see [51] and [52] or [74]), which can be seen as a grading on cyclotomic quotients of degenerate affine Hecke algebras, see [16]. See [21] for level $\ell = 2$ and [42] for all levels where the authors construct explicit graded cellular basis. For gradings on (cyclotomic quotients of) $\mathbb{W}_d$-algebras see Section 5 in [33] and for gradings on Brauer algebras see [34] or [59].

In the same spirit, it should be possible to obtain the higher level analoga of the generalizations of Khovanov's arc algebra, known as $\mathfrak{sl}_n$-web (or, alternatively, $\mathfrak{gl}_n$-web) algebras (see [62] and [61]), from our setup as well using the connections from cyclotomic KL-R algebras to these algebras in [89] and [90]. Although details still need to be worked out, this can be seen as the categorification of the connections to the spider algebras from 5.1.4: the spiders provide the setup to study the corresponding Reshetikhin-Turaev $\mathfrak{sl}_n$-link polynomials; the $\mathfrak{sl}_n$-web algebras provide the algebraic setup to study the Khovanov-Rozansky $\mathfrak{sl}_n$-link homologies. This would emphasize the connection between our work and low-dimensional topology.

5.2. **(Graded) cellular structures and the Temperley-Lieb algebras: a comparison.** Finally we want to present one explicit example, the Temperley-Lieb algebras, which is of particular interest in low-dimensional topology and categorification. Our main goal is to construct new (graded) cellular bases, and use our approach to establish semisimplicity conditions, and construct and compute the dimensions of its simple modules in new ways.

We start by briefly recalling the necessary definitions. The reader unfamiliar with these algebras might consider for example [38, Section 6] (or [8], where we recall the basics in detail using the usual Temperley-Lieb diagrams and our notation).

Fix $\delta = q + q^{-1}$ for $q \in \mathbb{K}^*$.[4] Recall that the *Temperley-Lieb algebra* $\mathcal{TL}_d(\delta)$ in d strands with parameter δ is the free diagram algebra over $\mathbb{K}$ with basis consisting of all possible non-intersecting tangle diagrams with d bottom and top boundary points modulo boundary preserving isotopy and the local relation for evaluating circles given by the parameter[5] δ.

Recall from 5.1.3 (whose notation we use now) that, by quantum Schur-Weyl duality, we can use Theorem 3.9 to obtain cellular bases of $\mathcal{TL}_d(\delta) \cong \mathrm{End}_{\mathbf{U}_q}(T)$ (we fix the isomorphism coming from quantum Schur-Weyl duality from now on). The aim now is to compare our cellular bases to the one given by Graham and Lehrer in [38, Theorem 6.7], where we point out that we do not obtain their cellular basis: our cellular basis depends for instance on whether $\mathcal{TL}_d(\delta)$ is semisimple or not. In the non-semisimple case, at least for $\mathbb{K} = \mathbb{C}$, we obtain a non-trivially $\mathbb{Z}$-graded cellular basis in the sense of [41, Definition 2.1], see Proposition 5.9.

Before stating our cellular basis, we provide a criterion which tells precisely whether $\mathcal{TL}_d(\delta)$ is semisimple or not. Recall that there is a known criteria for which Weyl modules $\Delta_q(i)$ are simple, see e.g. [10, Proposition 2.7].

Proposition 5.1. (**Semisimplicity criterion for $\mathcal{TL}_d(\delta)$.**) We have the following.

(a) Let $\delta \neq 0$. Then $\mathcal{TL}_d(\delta)$ is semisimple iff $[i] = q^{1-i} + \cdots + q^{i-1} \neq 0$ for all $i = 1, \ldots, d$ iff q is not a root of unity with $d < l = \mathrm{ord}(q^2)$, or $q = 1$ and $\mathrm{char}(\mathbb{K}) > d$.

(b) Let $\mathrm{char}(\mathbb{K}) = 0$. Then $\mathcal{TL}_d(0)$ is semisimple iff d is odd (or $d = 0$).

[4] The $\mathfrak{sl}_2$ case works with any $q \in \mathbb{K}^*$, including even roots of unity, see e.g. [10, Definition 2.3].

[5] We point out that there are two different conventions about circle evaluations in the literature: evaluating to δ or to $-\delta$. We use the first convention because we want to stay close to the cited literature.

(c) Let $\mathrm{char}(\mathbb{K}) = p > 0$. Then $\mathcal{TL}_d(0)$ is semisimple iff $d \in \{0, 1, 3, 5, \dots, 2p-1\}$. □

Proof. (a): We want to show that $T = V^{\otimes d}$ decomposes into simple $\mathbf{U}_q$-modules iff $d < l$, or $q = 1$ and $\mathrm{char}(\mathbb{K}) > d$, which is clearly equivalent to the non-vanishing of the $[i]$'s.

Assume that $d < l$. Since the maximal $\mathbf{U}_q$-weight of $V^{\otimes d}$ is d and since all Weyl $\mathbf{U}_q$-modules $\Delta_q(i)$ for $i < l$ are simple, we see that all indecomposable summands of $V^{\otimes d}$ are simple.

Otherwise, if $l \leq d$, then $T_q(d)$ (or $T_q(d-2)$ in the case $d \equiv -1 \bmod l$) is a non-simple, indecomposable summand of $V^{\otimes d}$ (note that this arguments fails if $l = 2$, i.e. $\delta = 0$).

The case $q = 1$ works similar, and we can now use Theorem 4.13 to finish the proof of (a).

(b): Since $\delta = 0$ iff $q = \pm\sqrt[2]{-1}$, we can use the linkage from e.g. [10, Theorem 2.23] in the case $l = 2$ to see that $T = V^{\otimes d}$ decomposes into a direct sum of simple $\mathbf{U}_q$-modules iff d is odd (or $d = 0$). This implies that $\mathcal{TL}_d(0)$ is semisimple iff d is odd (or $d = 0$) by Theorem 4.13.

(c): If $\mathrm{char}(\mathbb{K}) = p > 0$ and $\delta = 0$ (for $p = 2$ this is equivalent to $q = 1$), then we have $\Delta_q(i) \cong L_q(i)$ iff $i = 0$ or $i \in \{2ap^n - 1 \mid n \in \mathbb{Z}_{\geq 0}, 1 \leq a < p\}$. In particular, this means that for $d \geq 2$ we have that either $T_q(d)$ or $T_q(d-2)$ is a simple $\mathbf{U}_q$-module iff $d \in \{3, 5, \dots, 2p-1\}$. Hence, using the same reasoning as above, we see that $T = V^{\otimes d}$ is semisimple iff $d \in \{0, 1, 3, 5, \dots, 2p-1\}$. By Theorem 4.13 we see that $\mathcal{TL}_d(0)$ is semisimple iff $d \in \{0, 1, 3, 5, \dots, 2p-1\}$. ■

Example 5.2. We have that $[k] \neq 0$ for all $k = 1, 2, 3$ is satisfied iff q is not a forth or a sixth root of unity. By Proposition 5.1 we see that $\mathcal{TL}_3(\delta)$ is semisimple as long as q is not one of these values from above. The other way around is only true for q being a sixth root of unity (the conclusion from semisimplicity to non-vanishing of the quantum numbers above does not work in the case $q = \pm\sqrt[2]{-1}$). ▲

Remark 5.3. The semisimplicity criterion for $\mathcal{TL}_d(\delta)$ was already already found, using quite different methods, in [95, Section 5] in the case $\delta \neq 0$, and in the case $\delta = 0$ in [63, Chapter 7] or [71, above Proposition 4.9]. For us it is an easy application of Theorem 4.13. ▲

A direct consequence of Proposition 5.1 is that the Temperley-Lieb algebra $\mathcal{TL}_d(\delta)$ for $q \in \mathbb{K}^* - \{1\}$ not a root of unity is semisimple (or $q = \pm 1$ and $\mathrm{char}(\mathbb{K}) = 0$), regardless of d.

5.2.1. *Temperley-Lieb algebra: the semisimple case.* Assume that $q \in \mathbb{K}^* - \{1\}$ is not a root of unity (or $q = \pm 1$ and $\mathrm{char}(\mathbb{K}) = 0$). Thus, we are in the semisimple case.

Let us compare our cell datum $(\mathcal{P}, \mathcal{I}, \mathcal{C}, \mathrm{i})$ to the one of Graham and Lehrer (indicated by a subscript GL) from [38, Section 6]. They have the poset $\mathcal{P}_{\mathrm{GL}}$ consisting of all length-two partitions of d, and we have the poset $\mathcal{P}$ consisting of all $\lambda \in X^+$ such that $\Delta_q(\lambda)$ is a factor of T. The two sets are clearly the same: an element $\lambda = (\lambda_1, \lambda_2) \in \mathcal{P}_{\mathrm{GL}}$ corresponds to $\lambda_1 - \lambda_2 \in \mathcal{P}$. Similarly, an inductive reasoning shows that $\mathcal{I}_{\mathrm{GL}}$ (standard fillings of the Young diagram associated to λ) is also the same as our $\mathcal{I}$ (to see this one can use the facts listed in [10, Section 2]). One directly checks that the $\mathbb{K}$-linear anti-involution i_{GL} (turning diagrams upside-down) is also our involution i. Thus, except for $\mathcal{C}$ and $\mathcal{C}_{\mathrm{GL}}$, the cell data agree.

In order to state how our cellular bases for $\mathcal{TL}_d(\delta)$ look like, recall that the so-called *generalized Jones-Wenzl projectors* $JW_{\vec{\epsilon}}$ are indexed by d-tuples (with $d > 0$) of the form $\vec{\epsilon} = (\epsilon_1, \dots, \epsilon_d) \in \{\pm 1\}^d$ such that $\sum_{j=1}^{k} \epsilon_j \geq 0$ for all $k = 1, \dots, d$, see e.g. [25, Section 2]. In case $\vec{\epsilon} = (1, \dots, 1)$, one recovers the usual Jones-Wenzl projectors introduced by Jones in [48] and then further studied by Wenzl in [93].

Now, in [25, Proposition 2.19 and Theorem 2.20] it is shown that there exists non-zero scalars $a_{\vec{\epsilon}} \in \mathbb{K}$ such that $JW'_{\vec{\epsilon}} = a_{\vec{\epsilon}} JW_{\vec{\epsilon}}$ are well-defined idempotents forming a complete set of mutually orthogonal, primitive idempotents in $\mathcal{TL}_d(\delta)$. (The authors of [25] work over $\mathbb{C}$, but as long as $q \in \mathbb{K}^* - \{1\}$ is not a root of unity their arguments work in our setup as well.) These project to the summands of $T = V^{\otimes d}$ of the form $\Delta_q(i)$ for $i = \sum_{j=1}^{k} \epsilon_j$. In particular, the usual Jones-Wenzl projectors project to the highest weight summand $\Delta_q(d)$ of $T = V^{\otimes d}$.

Proposition 5.4. (**(New) cellular bases.**) The datum given by the quadruple $(\mathcal{P}, \mathcal{I}, \mathcal{C}, \mathrm{i})$ for $\mathcal{TL}_d(\delta) \cong \mathrm{End}_{\mathbf{U}_q}(T)$ is a cell datum for $\mathcal{TL}_d(\delta)$. Moreover, $\mathcal{C} \neq \mathcal{C}_{\mathrm{GL}}$ for all $d > 1$ and all choices involved in the definition of $\mathrm{im}(\mathcal{C})$. In particular, there is a choice such that all generalized Jones-Wenzl projectors $JW'_{\vec{\epsilon}}$ are part of $\mathrm{im}(\mathcal{C})$. □

Proof. That we get a cell datum as stated follows from Theorem 3.9 and the discussion above.

That our cellular basis $\mathcal{C}$ will never be $\mathcal{C}_{\mathrm{GL}}$ for $d > 1$ is due to the fact that Graham and Lehrer's cellular basis always contains the identity (which corresponds to the unique standard filling of the Young diagram associated to $\lambda = (d, 0)$).

In contrast, let $\lambda_k = (d-k, k)$ for $0 \leq k \leq \lfloor \frac{d}{2} \rfloor$. Then

$$T = V^{\otimes d} \cong \Delta_q(d) \oplus \bigoplus_{0 < k \leq \lfloor \frac{d}{2} \rfloor} \Delta_q(d-2k)^{\oplus m_{\lambda_k}} \tag{27}$$

for some multiplicities $m_{\lambda_k} \in \mathbb{Z}_{>0}$, we see that for $d > 1$ the identity is never part of any of our bases: all the $\Delta_q(i)$'s are simple $\mathbf{U}_q$-modules and each c_{ij}^k factors only through $\Delta_q(k)$. In particular, the basis element c_{11}^{λ} for $\lambda = \lambda_d$ has to be (a scalar multiple) of $JW_{(1,\dots,1)}$.

As in 5.1.1 we can choose for $\mathcal{C}$ an Artin-Wedderburn basis of $\mathcal{TL}_d(\delta) \cong \mathrm{End}_{\mathbf{U}_q}(T)$. Hence, by the above, the corresponding basis consists of the projectors $JW_{\vec{\epsilon}}$. ■

Note the following classification result (see for example [71, Corollary 5.2] for $\mathbb{K} = \mathbb{C}$).

Corollary 5.5. We have a complete set of pairwise non-isomorphic, simple $\mathcal{TL}_d(\delta)$-modules $L(\lambda)$, where $\lambda = (\lambda_1, \lambda_2)$ is a length-two partition of d. Moreover, $\dim(L(\lambda)) = |\mathrm{Std}(\lambda)|$, where $\mathrm{Std}(\lambda)$ is the set of all standard tableaux of shape λ. □

Proof. Directly from Proposition 5.4 and Theorems 4.11 and 4.12 because $m_\lambda = |\mathrm{Std}(\lambda)|$. ■

5.2.2. *Temperley-Lieb algebra: the non-semisimple case.* Let us assume that we have fixed $q \in \mathbb{K}^* - \{1, \pm\sqrt[2]{-1}\}$ to be a critical value such that $[k] = 0$ for some $k = 1, \dots, d$. Then, by Proposition 5.1, the algebra $\mathcal{TL}_d(\delta)$ is no longer semisimple. In particular, to the best of our knowledge, there is no diagrammatic analogon of the Jones-Wenzl projectors in general.

Proposition 5.6. (**(New) cellular basis - the second.**) The datum $(\mathcal{P}, \mathcal{I}, \mathcal{C}, \mathrm{i})$ with $\mathcal{C}$ as in Theorem 3.9 for $\mathcal{TL}_d(\delta) \cong \mathrm{End}_{\mathbf{U}_q}(T)$ is a cell datum for $\mathcal{TL}_d(\delta)$. Moreover, $\mathcal{C} \neq \mathcal{C}_{\mathrm{GL}}$ for all $d > 1$ and all choices involved in the definition of our basis. Thus, there is a choice such that all generalized, non-semisimple Jones-Wenzl projectors are part of $\mathrm{im}(\mathcal{C})$. □

Proof. As in the proof of Proposition 5.4 and left to the reader. ■

Hence, directly from Proposition 5.6 and Theorems 4.11 and 4.12, we obtain:

Corollary 5.7. We have a complete set of pairwise non-isomorphic, simple $\mathcal{TL}_d(\delta)$-modules $L(\lambda)$, where $\lambda = (\lambda_1, \lambda_2)$ is a length-two partition of d. Moreover, $\dim(L(\lambda)) = m_\lambda$, where m_λ is the multiplicity of $T_q(\lambda_1 - \lambda_2)$ as a summand of $T = V^{\otimes d}$. ■

Note that we can do better: one gets a decompositions

$$\boldsymbol{\mathcal{T}} \cong \boldsymbol{\mathcal{T}}_{-1} \oplus \boldsymbol{\mathcal{T}}_0 \oplus \boldsymbol{\mathcal{T}}_1 \oplus \cdots \oplus \boldsymbol{\mathcal{T}}_{l-3} \oplus \boldsymbol{\mathcal{T}}_{l-2} \oplus \boldsymbol{\mathcal{T}}_{l-1}, \tag{28}$$

where the blocks $\boldsymbol{\mathcal{T}}_{-1}$ and $\boldsymbol{\mathcal{T}}_{l-1}$ are semisimple if $\mathbb{K} = \mathbb{C}$. (This follows from the linkage principle. For notation and the statement see [10, Section 2].)

Fix $\mathbb{K} = \mathbb{C}$. As explained in [10, Section 3.5] each block in the decomposition (28) can be equipped with a non-trivial $\mathbb{Z}$-grading coming from Khovanov and Seidel's algebra from [53]. Hence, we have the following.

Lemma 5.8. The $\mathbb{C}$-algebra $\mathrm{End}_{\mathbf{U}_q}(T)$ can be equipped with a non-trivial $\mathbb{Z}$-grading. Thus, $\mathcal{TL}_d(\delta)$ over $\mathbb{C}$ can be equipped with a non-trivial $\mathbb{Z}$-grading. □

Proof. The second statement follows directly from the first using quantum Schur-Weyl duality. Hence, we only need to show the first.

Note that $T = V^{\otimes d}$ decomposes as in (27), but with $T_q(k)$'s instead of $\Delta_q(k)$'s, and we can order this decomposition by blocks. Each block carries a $\mathbb{Z}$-grading coming from Khovanov and Seidel's algebra (as explained in [10, Section 3]). In particular, we can choose the basis elements c_{ij}^λ in such a way that we get the $\mathbb{Z}$-graded basis obtained in [10, Corollary 4.23]. Since there is no interaction between different blocks, the statement follows. ■

Recall from [41, Definition 2.1] that a $\mathbb{Z}$-graded cell datum of a $\mathbb{Z}$-graded algebra is a cell datum for the algebra together with an additional *degree function* $\deg\colon \coprod_{\lambda\in\mathcal{P}} \mathcal{I}^\lambda \to \mathbb{Z}$, such that $\deg(c_{ij}^\lambda) = \deg(i) + \deg(j)$. For us the choice of $\deg(\cdot)$ is as follows.

If $\lambda \in \mathcal{P}$ is in one of the semisimple blocks, then we simply set $\deg(i) = 0$ for all $i \in \mathcal{I}^\lambda$.

Assume that $\lambda \in \mathcal{P}$ is not in the semisimple blocks. It is known that every $T_q(\lambda)$ has precisely two Weyl factors. The g_i^λ that map $\Delta_q(\lambda)$ into a higher $T_q(\mu)$ should be indexed by a 1-colored i whereas the g_i^λ mapping $\Delta_q(\lambda)$ into $T_q(\lambda)$ should have 0-colored i. Similarly for the f_j^λ's. Then the degree of the elements $i \in \mathcal{I}^\lambda$ should be the corresponding color. We get the following. (Here $\mathcal{C}$ is as in Theorem 3.9.)

Proposition 5.9. (**Graded cellular basis.**) The datum $(\mathcal{P}, \mathcal{I}, \mathcal{C}, \mathrm{i})$ supplemented with the function $\deg(\cdot)$ from above is a $\mathbb{Z}$-graded cell datum for the $\mathbb{C}$-algebra $\mathcal{TL}_d(\delta) \cong \mathrm{End}_{\mathbf{U}_q}(T)$.□

Proof. The hardest part is cellularity which directly follows from Theorem 3.9. That the quintuple $(\mathcal{P}, \mathcal{I}, \mathcal{C}, \mathrm{i}, \deg)$ gives a $\mathbb{Z}$-graded cell datum follows from the construction. ■

Remark 5.10. Our grading and the one found by Plaza and Ryom-Hansen in [69] agree (up to a shift of the indecomposable summands). To see this, note that our algebra is isomorphic to the algebra $K_{1,n}$ studied in [19] which is by [19, (4.8)] and [21, Theorem 6.3] a quotient of some particular cyclotomic KL-R algebra (the compatibility of the grading follows for example from [42, Corollary B.6]). The same holds, by construction, for the grading in [69]. ▲

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