

CELLULAR STRUCTURES USING \mathbf{U}_q -TILTING MODULES

HENNING HAAHR ANDERSEN, CATHARINA STROPPEL, AND DANIEL TUBBENHAUER

ABSTRACT. We use the theory of \mathbf{U}_q -tilting modules to construct cellular bases for centralizer algebras. Our methods are quite general and work for any quantum group \mathbf{U}_q attached to a Cartan matrix and include the non semi-simple cases for q being a root of unity and ground fields of positive characteristic. Our approach also generalize to certain categories containing infinite dimensional modules. As an application, we recover several known cellular structures (which can all be fit into our general set-up) as we illustrate in a list of examples.

CONTENTS

1. Introduction	2
1.1. The framework	2
1.2. Outline of the paper	4
2. Quantum groups and their representations	5
2.1. The quantum groups \mathbf{U}_v and \mathbf{U}_q	5
2.2. Representation theory of \mathbf{U}_v : the generic, semi-simple case	7
2.3. Representation theory of \mathbf{U}_q : the non semi-simple case	9
3. Tilting modules	12
3.1. \mathbf{U}_q -modules with a Δ_q - and a ∇_q -filtration	12
3.2. \mathbf{U}_q -tilting modules	13
3.3. The characters of indecomposable \mathbf{U}_q -tilting modules	16
4. Cellular structures on endomorphism algebras	22
4.1. Hom-spaces between \mathbf{U}_q -modules with a Δ_q - and a ∇_q -filtration	22
4.2. Proof of the basis theorem	24
4.3. Cellular structures on endomorphism algebras of \mathbf{U}_q -tilting modules	26
5. The cellular structure and $\text{End}_{\mathbf{U}_q}(T)\text{-Mod}$	28
5.1. Cell modules for $\text{End}_{\mathbf{U}_q}(T)$ and their contravariant pairings	28
5.2. The structure of $\text{End}_{\mathbf{U}_q}(T)$ and its cell modules	30
5.3. Simple $\text{End}_{\mathbf{U}_q}(T)$ -modules and semi-simplicity of $\text{End}_{\mathbf{U}_q}(T)$	32
6. Cellular structures: examples and applications	34
6.1. Cellular structures using \mathbf{U}_q -tilting modules: several examples	34
6.2. (Graded) cellular structures and the Temperley-Lieb algebras: a comparison	39
References	47

H.H.A. and D.T. were partially supported by the center of excellence grant “Centre for Quantum Geometry of Moduli Spaces (QGM)” from the “Danish National Research Foundation (DNRF)”. C.S. was supported by the Max-Planck-Gesellschaft. D.T. was partially supported by a research funding of the “Deutsche Forschungsgemeinschaft (DFG)” during the last part of this work.

1. INTRODUCTION

Let $\mathbf{U}_q(\mathfrak{g})$ be the quantum group with a fixed, arbitrary parameter $q \in \mathbb{K} - \{0\}$ associated to a simple Lie algebra \mathfrak{g} over any field \mathbb{K} . The main result in this paper is the following.

Theorem. (Cellularity of endomorphism algebras) Let T be a $\mathbf{U}_q(\mathfrak{g})$ -tilting module. Then $\text{End}_{\mathbf{U}_q(\mathfrak{g})}(T)$ is a cellular algebra in the sense of Graham and Lehrer [34].

This can also be deduced from general theory: any $\mathbf{U}_q(\mathfrak{g})$ -tilting module T is a summand of a full $\mathbf{U}_q(\mathfrak{g})$ -tilting module \tilde{T} . By Theorem 6 in [72] $\text{End}_{\mathbf{U}_q(\mathfrak{g})}(\tilde{T})$ is quasi-hereditary and thus, cellular, see [52]. By Theorem 4.3 in [52] this induces the cellularity of $\text{End}_{\mathbf{U}_q(\mathfrak{g})}(T)$. In contrast, our approach provides a method to construct cellular bases, generalizes to the infinite dimensional world and has other nice consequences that we will explore in this paper. Our results together with the Jantzen sum formula can be used to give as an application a straightforward proof of the known simisimplicity criterion for Iwahori-Hecke algebras of the symmetric group at roots of unities. By general theory, [69], a tensor product of tilting modules is again tilting. This in particular applies, see Corollary 3.15, to the tensor product $V^{\otimes d}$ of the vector representation of $\mathbf{U}_q = \mathbf{U}_q(\mathfrak{g})$ for \mathfrak{g} of type A , C or D from which we then can recover the cellularity of many known algebras. We are also able to show directly that $V^{\otimes d}$ is a tilting module using arguments which are much more elementary than the ones from the general proof in [69].

1.1. The framework. Recall that, given any simple, complex Lie algebra \mathfrak{g} , we can assign to it a *quantum deformation* $\mathbf{U}_v = \mathbf{U}_v(\mathfrak{g})$ of its universal enveloping algebra for a *generic parameter* v by deforming its Serre presentation. The representation theory of \mathbf{U}_v shares many similarities with the one of \mathfrak{g} . In particular, the category¹ $\mathbf{U}_v\text{-Mod}$ is semi-simple.

But one can spice up the story drastically: the quantum group $\mathbf{U}_q = \mathbf{U}_q(\mathfrak{g})$ is the *specialization* of \mathbf{U}_v at an arbitrary $q \in \mathbb{K} - \{0\}$. In particular, we can take q to be a root of unity². In this case $\mathbf{U}_q\text{-Mod}$ is not semi-simple anymore. This makes the root of unity case interesting with many connections and applications in different directions: the category has a neat combinatorics, is, for example, related to the corresponding almost simple, simply connected algebraic group G over \mathbb{K} with $\text{char}(\mathbb{K})$ prime, see for example [3] or [59], to the representation theory of the associated affine Kac-Moody algebra, see [45] or [88], and to 2 + 1-TQFT's and the Witten-Reshetikhin-Turaev invariants of 3-manifolds, see for example [92].

Semi-simplicity with respect to our main result means the following. If we take $\mathbb{K} = \mathbb{C}$ and $q = \pm 1$, then our result says that the algebra $\text{End}_{\mathbf{U}_q}(T)$ is cellular for any \mathbf{U}_q -module $T \in \mathbf{U}_q\text{-Mod}$ because in this case all \mathbf{U}_q -modules are \mathbf{U}_q -tilting modules. This is no surprise: when T is a direct sum of simple \mathbf{U}_q -modules, then $\text{End}_{\mathbf{U}_q}(T)$ is a direct sum of matrix algebras $M_n(\mathbb{K})$. Likewise, for any \mathbb{K} , if $q \in \mathbb{K} - \{0, \pm 1\}$ is not a root of unity, then $\mathbf{U}_q\text{-Mod}$ is still semi-simple and our result is (almost) standard. But even in the semi-simple case we can say more: we get as a cellular basis for $\text{End}_{\mathbf{U}_q}(T)$ an Artin-Wedderburn basis, i.e. a basis realizing the decomposition of $\text{End}_{\mathbf{U}_q}(T)$ into its matrix components, see Subsection 6.1.

¹For any algebra A we denote by $A\text{-Mod}$ the category of finite dimensional, left A -modules. If not stated otherwise, all modules are assumed to be finite dimensional, left modules.

²In our terminology: the two cases $q = \pm 1$ are special and do not count as roots of unities.

However, if $q = \pm 1$ and $\text{char}(\mathbb{K}) > 0$ or if $q \in \mathbb{K} - \{0\}$ is a root of unity, then $U_q\text{-Mod}$ is far away from being semi-simple and our result gives a bunch of interesting cellular algebras which come equipped with a cellular basis.

For example, if $G = \text{GL}(V)$ for some finite dimensional \mathbb{K} -vector space V of dimension n , then $T = V^{\otimes d}$ is a G -tilting module for any $d \in \mathbb{N}$. By Schur-Weyl duality we have

$$(1) \quad \Phi_{\text{SW}}: \mathbb{K}[S_d] \twoheadrightarrow \text{End}_G(T) \quad \text{and} \quad \Phi_{\text{SW}}: \mathbb{K}[S_d] \xrightarrow{\cong} \text{End}_G(T), \text{ if } n \geq d,$$

where $\mathbb{K}[S_d]$ is the group algebra of the symmetric group S_d in d letters. We can realize this as a special case in our framework by taking $q = 1$, $n \geq d$ and $\mathfrak{g} = \mathfrak{gl}_n$ (although \mathfrak{gl}_n is not a simple, complex Lie algebra, our approach works fine for it as well). On the other hand, by taking q arbitrary in $\mathbb{K} - \{0, \pm 1\}$ and $n \geq d$, the group algebra $\mathbb{K}[S_d]$ is replaced by the *Iwahori-Hecke algebra* $\mathcal{H}_d(q)$ over \mathbb{K} of type A_{d-1} and we obtain a cellular basis for it as well. Note that one underlying fact why (1) stays true in non semi-simple case is that $\dim(\text{End}_G(T))$ is independent of the characteristic of \mathbb{K} (and of the parameter q in the quantum case), since T is a G -tilting module.

Of course, both $\mathbb{K}[S_d]$ and $\mathcal{H}_d(q)$ are known to be cellular (these cases were one of the main motivations of Graham and Lehrer to introduce the notion of cellular algebras), but the point we want to make is, that they fit into our more general framework. The following known important algebras can also be recovered from our approach. We point out: in most of the examples we either have *no or only some mild restrictions* on \mathbb{K} and $q \in \mathbb{K} - \{0\}$.

- As sketched above: the algebras $\mathbb{K}[S_d]$ and $\mathcal{H}_d(q)$ and their quotients under Φ_{SW} .
- The *Temperley-Lieb algebras* $\mathcal{TL}_d(\delta)$ introduced in [89].
- Other less well-known endomorphism algebras for \mathfrak{sl}_2 -related tilting modules appearing in more recent work, e.g. [4], [8] or [73].
- *Spider algebras* in the sense of Kuperberg [53].
- Quotients of the group algebras of $\mathbb{Z}/r\mathbb{Z} \wr S_d$ and its quantum version $\mathcal{H}_{d,r}(q)$, the *Ariki-Koike algebras* introduced in [10]. This includes the Ariki-Koike algebras itself and thus, the Hecke algebras of type B . This includes also Martin and Saleur's *blob algebras* $\mathcal{BL}_d(q, m)$ from [64] and (*quantized*) *rook monoid algebras* (also called *Solomon algebras*) $\mathcal{R}_d(q)$ in the spirit of [85].
- *Brauer algebras* $\mathcal{B}_d(\delta)$ introduced in the context of classical invariant theory [14] and related algebras, e.g. *walled Brauer algebras* $\mathcal{B}_{r,s}(\delta)$ introduced in [51] and [91], and *Birman-Murakami-Wenzl algebras* $\mathcal{BMW}_d(\delta)$ (in the sense of [12] and [67]).

Note that our methods *also* apply for some categories containing infinite dimensional modules. This includes the *BGG category* \mathcal{O} , its *parabolic subcategories* \mathcal{O}^p and its *quantum cousin* \mathcal{O}_q from [5]. For example, using the “big projective tilting” in the principal block, we get a cellular basis for the coinvariant algebra of the Weyl group associated to \mathfrak{g} . In fact, we get a vast generalization of this, e.g. we can fit *generalized Khovanov arc algebras* (see e.g. [17]), *\mathfrak{sl}_n -web algebras* (see e.g. [62]), *cyclotomic Khovanov-Lauda and Rouquier algebras* of type A (see [48], [49] and [47] or [74]), *cyclotomic \mathbb{W}_d -algebras* (see e.g. [32]) and *cyclotomic quotients of affine Hecke algebras* $\mathbf{H}_{\mathbb{K},d}^s$ (see e.g. [75]) into our framework as well, see Subsection 6.1.

We will for simplicity focus on the finite dimensional world and provide all necessary tools and arguments in great detail (the infinite dimensional story follows in the same vein). In particular, since we believe that much of the literature on $U_q\text{-Mod}$ and U_q -tilting modules is

less familiar and less documented, we have included all necessary statements for our purpose as well as some of the proofs (sometimes, for brevity, only in an extra file [7]).

Following Graham and Lehrer’s approach, our cellular basis for $\text{End}_{\mathbf{U}_q}(T)$ provides also $\text{End}_{\mathbf{U}_q}(T)$ -cell modules, classification of simple $\text{End}_{\mathbf{U}_q}(T)$ -modules etc. We give an interpretation of this in our framework as well, see Section 5. For instance, we obtain a formula for the dimensions of simple $\text{End}_{\mathbf{U}_q}(T)$ -modules by counting decomposition multiplicities of T , see Theorem 5.12. We use this to deduce a criterion for semi-simplicity of $\text{End}_{\mathbf{U}_q}(T)$ in terms of the decomposition of T into indecomposable \mathbf{U}_q -modules, see Theorem 5.13.

Our approach gives a general method to construct cellular bases. Many of the algebras appearing in our examples can be additionally equipped with a \mathbb{Z} -grading. The basis elements from Theorem 4.11 can be chosen such that our approach leads to a graded cellular basis in the sense of [37]. To keep the paper in reasonable boundaries, we do not want to treat the graded set-up in this paper. We however show in the explicit example of $\mathcal{TL}_d(\delta)$ how our approach leads to a graded cellular structure.

We give plenty of examples as well to help the reader to find her/his way through the text.

1.2. Outline of the paper. The outline of the paper is as follows.

- In Section 2 we recall the basic definitions and facts about \mathbf{U}_v and its specialization \mathbf{U}_q . We also recall Lusztig’s \mathcal{A} -form $\mathbf{U}_{\mathcal{A}}$ which is the main ingredient to define \mathbf{U}_q .
- Section 3 contains a short account of the theory \mathbf{U}_q -tilting modules. In addition, we recall the important *Ext-vanishing theorem*, see Theorem 3.1, which is crucial for our construction of cellular bases for $\text{End}_{\mathbf{U}_q}(T)$. Moreover, we indicate in Subsection 3.3 how one can combinatorially work with \mathbf{U}_q -tilting modules. We note that we have an additional eprint [7] containing most of the proofs in Section 3 adapted to our notation. We hope that this makes the paper reasonably self-contained.
- Then, in Section 4, we state and prove our main theorem³, that is, Theorem 4.11.
- We give in Section 5 a short account of the representation theory of $\text{End}_{\mathbf{U}_q}(T)$ with respect to our cellular basis in the usual sense, including the classification of all simple $\text{End}_{\mathbf{U}_q}(T)$ -modules, see Theorem 5.11. In addition, we derive some further consequences for the category $\mathbf{U}_q\text{-Mod}$, see Proposition 5.4 and Theorems 5.12 and 5.13.
- Finally, in Section 6 we have collected a list of examples that fit into our framework of cellularity. In fact, we shortly sketch how to obtain cellular structures on all the examples mentioned before. This includes one example of a graded cellular structure, namely the Temperley-Lieb algebra $\mathcal{TL}_d(\delta)$ which we discuss in detail.

Acknowledgements. We like to thank Ben Cooper, Michael Ehrig, Matt Hogancamp, Johannes Kübel, Gus Lehrer, Paul Martin, Andrew Mathas, Volodymyr Mazorchuk, Steen Ryom-Hansen and Paul Wedrich for helpful comments and discussions. H.H.A. would like to thank the Institut Mittag-Leffler for the hospitality he enjoyed there during the final stages of this work. C.S. is very grateful to the Max-Planck institute in Bonn for the extraordinary support and excellent working conditions. A large part of her research was worked out during her stay there. D.T. likes to thank the dark Danish winter for very successfully limiting his non-work options and the Australian long blacks for pushing him forward.

³A “trick” enables us to get the \mathbf{U}_v case as well: take $\mathbb{K} = \mathbb{Q}(v)$ and “specialize q to v ”.

2. QUANTUM GROUPS AND THEIR REPRESENTATIONS

In the present section we recall the definitions and results about quantum groups and their representation theory in the semi-simple and the non semi-simple case.

2.1. The quantum groups U_v and U_q . Let Φ be a finite *root system* in an Euclidean space E . We fix a choice of *positive roots* $\Phi^+ \subset \Phi$ and *simple roots* $\Pi \subset \Phi^+$. We assume that we have n simple roots that we denote by $\alpha_1, \dots, \alpha_n$. For each $\alpha \in \Phi$, we denote by $\alpha^\vee \in \Phi^\vee$ the corresponding *co-root*. Let $a_{ij} = \langle \alpha_i, \alpha_j^\vee \rangle$ for $i, j = 1, \dots, n$. Then the matrix $\mathbf{A} = (a_{ij})_{i,j=1}^n$ is called the *Cartan matrix*. As usual, we need to symmetrize \mathbf{A} and we do so by choosing for $i = 1, \dots, n$ minimal $d_i \in \mathbb{N}$ such that $(d_i a_{ij})_{i,j=1}^n$ is symmetric (the Cartan matrix \mathbf{A} is already symmetric in most of our examples and thus, $d_i = 1$ for all $i = 1, \dots, n$).

By the set of (*integral*) *weights* we understand $X = \{\lambda \in E \mid \langle \lambda, \alpha_i^\vee \rangle \in \mathbb{Z} \text{ for all } \alpha_i \in \Pi\}$. The *dominant (integral) weights* X^+ are those $\lambda \in X$ such that $\langle \lambda, \alpha_i^\vee \rangle \geq 0$ for all $\alpha_i \in \Pi$.

The *fundamental weights*, denoted by $\omega_i \in X$ for $i = 1, \dots, n$, are characterized by

$$\langle \omega_i, \alpha_j^\vee \rangle = \delta_{ij} \quad \text{for all } j = 1, \dots, n.$$

Recall that there is a *partial ordering* on X given by $\mu \leq \lambda$ iff $\lambda - \mu$ is an \mathbb{N} -valued linear combination of the simple roots, that is, $\lambda - \mu = \sum_{i=1}^n a_i \alpha_i$ with $a_i \in \mathbb{N}$.

Example 2.1. One of the most important examples is the standard choice of Cartan datum $(\mathbf{A}, \Pi, \Phi, \Phi^+)$ associated with the Lie algebra $\mathfrak{g} = \mathfrak{sl}_{n+1}$ for $n \geq 1$. Here $E = \mathbb{R}^{n+1}$ and $\Pi = \{\alpha_i = e_i - e_{i+1} \mid i = 1, \dots, n\}$, where the e_i denote the standard basis of E . The positive roots are $\Phi^+ = \{e_i - e_j \mid 1 \leq i < j \leq n+1\}$ with maximal root $\alpha_0 = e_1 - e_{n+1}$. Moreover,

$$\rho = \sum_{i=1}^n (n - i + 1) e_i.$$

The set of fundamental weights is $\{\omega_i = e_1 + \dots + e_i \mid 1 \leq i \leq n\}$. For explicit calculations one often identifies

$$\lambda = \sum_{i=1}^n a_i \omega_i \in X^+$$

with the partition $\lambda = (\lambda_1 \geq \dots \geq \lambda_n \geq 0)$ given by $\lambda_k = \sum_{i=k}^n a_i$ for $k = 1, \dots, n$. ■

We now assign a quantum group $U_v = U_v(\mathbf{A})$ to a given Cartan matrix \mathbf{A} . Before giving the definition, we point out that v in our notation always means a *generic parameter*, while $q \in \mathbb{K} - \{0\}$ will always mean a *specialization*.

Definition 2.2. (Quantum enveloping algebra) Given a Cartan matrix \mathbf{A} , then the *quantum enveloping algebra* $U_v = U_v(\mathbf{A})$ associated to it is the associative, unital $\mathbb{Q}(v)$ -algebra

generated by $K_1^{\pm 1}, \dots, K_n^{\pm 1}$ and $E_1, F_1, \dots, E_n, F_n$, where n is the size of \mathbf{A} , subject to

$$\begin{aligned} K_i K_j &= K_j K_i, & K_i K_i^{-1} &= K_i^{-1} K_i = 1, & K_i E_j &= v^{d_i a_{ij}} E_j K_i, & K_i F_j &= v^{-d_i a_{ij}} F_j K_i, \\ E_i F_j - F_j E_i &= \delta_{i,j} \frac{K_i - K_i^{-1}}{v^{d_i} - v^{-d_i}}, \\ \sum_{r+s=1-a_{ij}} (-1)^s \begin{bmatrix} 1-a_{ij} \\ s \end{bmatrix}_{d_i} E_i^r E_j E_i^s &= 0, & \text{if } i \neq j, \\ \sum_{r+s=1-a_{ij}} (-1)^s \begin{bmatrix} 1-a_{ij} \\ s \end{bmatrix}_{d_i} F_i^r F_j F_i^s &= 0, & \text{if } i \neq j. \end{aligned}$$

Here we use, as usual, the conventions that, for $a \in \mathbb{Z}$ and $b, d \in \mathbb{N}$, $[a]_d$ denotes the a -quantum integer (with $[0]_d = 1$), $[b]_d!$ denotes the b -quantum factorial, that is,

$$[a]_d = \frac{v^{ad} - v^{-ad}}{v^d - v^{-d}}, \quad [a] = [a]_1 \quad \text{and} \quad [b]_d! = [0]_d [1]_d \cdots [b-1]_d [b]_d,$$

and

$$\begin{bmatrix} a \\ b \end{bmatrix}_d = \frac{[a]_d [a-1]_d \cdots [a-b+2]_d [a-b+1]_d}{[b]_d!}, \quad \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}_1$$

denotes the (a, b) -quantum binomial. Observe that $[-a]_d = -[a]_d$.

It is worth noting that \mathbf{U}_v is a Hopf algebra with coproduct Δ given by

$$\Delta(E_i) = E_i \otimes 1 + K_i \otimes E_i, \quad \Delta(F_i) = F_i \otimes K_i^{-1} + 1 \otimes F_i \quad \text{and} \quad \Delta(K_i) = K_i \otimes K_i.$$

The antipode S and the counit ε are given by

$$S(E_i) = -K_i^{-1} E_i, \quad S(F_i) = -F_i K_i, \quad S(K_i) = K_i^{-1}, \quad \varepsilon(E_i) = \varepsilon(F_i) = 0 \quad \text{and} \quad \varepsilon(K_i) = 1.$$

We are interested in the root of unity case. Thus, we want to “specialize” the generic parameter v of \mathbf{U}_v to be, for example, a root of unity $q \in \mathbb{K} - \{0\}$. In order to do so, we consider Lusztig’s \mathcal{A} -form $\mathbf{U}_{\mathcal{A}} = \mathbf{U}_{\mathcal{A}}(\mathbf{A})$ introduced in [60]. Let $\mathcal{A} = \mathbb{Z}[v, v^{-1}]$.

Definition 2.3. (**Lusztig’s \mathcal{A} -form $\mathbf{U}_{\mathcal{A}}$**) Define for all $j \in \mathbb{N}$ the j -th divided powers

$$E_i^{(j)} = \frac{E_i^j}{[j]_{d_i}!} \quad \text{and} \quad F_i^{(j)} = \frac{F_i^j}{[j]_{d_i}!}.$$

Then $\mathbf{U}_{\mathcal{A}}$ is defined as the \mathcal{A} -subalgebra of \mathbf{U}_v generated by $K_i, K_i^{-1}, E_i^{(j)}$ and $F_i^{(j)}$ for $i = 1, \dots, n$ and $j \in \mathbb{N}$.

Lusztig’s \mathcal{A} -form is designed to allow specializations. To this end, we fix a field \mathbb{K} of arbitrary characteristic in the following.

Definition 2.4. (**Quantum enveloping algebras at roots of unity**) Fix an arbitrary element $q \in \mathbb{K} - \{0\}$. Consider \mathbb{K} as an \mathcal{A} -module by specializing v to q . Define

$$\mathbf{U}_q = \mathbf{U}_q(\mathbf{A}) = \mathbf{U}_{\mathcal{A}} \otimes_{\mathcal{A}} \mathbb{K}.$$

Abusing notation, we will usually abbreviate $E_i^{(j)} \otimes 1 \in \mathbf{U}_q$ with $E_i^{(j)}$. Analogously for the other generators of \mathbf{U}_q .

Example 2.5. In the \mathfrak{sl}_2 case, the $\mathbb{Q}(v)$ -algebra $\mathbf{U}_v(\mathfrak{sl}_2)$ is generated by K and K^{-1} and E, F subject to the relations

$$\begin{aligned} KK^{-1} &= K^{-1}K = 1, \\ EF - FE &= \frac{K - K^{-1}}{v - v^{-1}}, \\ KE &= v^2EK \quad \text{and} \quad KF = v^{-2}FK. \end{aligned}$$

We point out that $\mathbf{U}_v(\mathfrak{sl}_2)$ already contains the divided powers since no quantum number vanishes in $\mathbb{Q}(v)$. Let q be a complex, primitive third root of unity. Thus, $q + q^{-1} = [2] = -1$, $q^2 + 1 + q^{-2} = [3] = 0$ and $q^3 + q^1 + q^{-1} + q^{-3} = [4] = 1$. More general,

$$[a] = i \in \{0, +1, -1\}, \quad i \equiv a \pmod{3}.$$

Hence, $\mathbf{U}_q(\mathfrak{sl}_2)$ is generated by $K, K^{-1}, E^{(3)}$ and $F^{(3)}$ subject to the relations as above (here $E^{(3)}, F^{(3)}$ are extra generators since $E^3 = [3]!E^{(3)} = 0$ because of $[3] = 0$). This is precisely the convention used in Chapter 1 of [42], but specialized at q . \blacksquare

It is easy to check that $\mathbf{U}_{\mathcal{A}}$ is a Hopf subalgebra of \mathbf{U}_v , see Proposition 4.8 in [58]. Thus, \mathbf{U}_q inherits a Hopf algebra structure from \mathbf{U}_v .

Moreover, it is known that all three algebras, that is, \mathbf{U}_v , $\mathbf{U}_{\mathcal{A}}$ and \mathbf{U}_q , have a triangular decomposition

$$\mathbf{U}_v = \mathbf{U}_v^- \mathbf{U}_v^0 \mathbf{U}_v^+ \quad , \quad \mathbf{U}_{\mathcal{A}} = \mathbf{U}_{\mathcal{A}}^- \mathbf{U}_{\mathcal{A}}^0 \mathbf{U}_{\mathcal{A}}^+ \quad , \quad \mathbf{U}_q = \mathbf{U}_q^- \mathbf{U}_q^0 \mathbf{U}_q^+,$$

where $\mathbf{U}_v^-, \mathbf{U}_{\mathcal{A}}^-, \mathbf{U}_q^-$ denote the subalgebras generated only by the F_i 's (or, in addition, the divided powers for $\mathbf{U}_{\mathcal{A}}^-$ and \mathbf{U}_q^-) and $\mathbf{U}_v^+, \mathbf{U}_{\mathcal{A}}^+, \mathbf{U}_q^+$ denote the subalgebras generated only by the E_i 's (or, in addition, the divided powers for $\mathbf{U}_{\mathcal{A}}^+$ and \mathbf{U}_q^+). The Cartan part \mathbf{U}_v^0 is as usual generated by K_i, K_i^{-1} for $i = 1, \dots, n$. For the Cartan part $\mathbf{U}_{\mathcal{A}}^0$ one needs to be a little bit more careful, since it is generated by

$$(2) \quad \tilde{K}_{i,t} = \begin{bmatrix} K_i \\ t \end{bmatrix} = \prod_{s=1}^t \frac{K_i v^{d_i(1-s)} - K_i^{-1} v^{-d_i(1-s)}}{v^{d_i s} - v^{-d_i s}}$$

for $i = 1, \dots, n$ and $t \in \mathbb{N}$ in addition to the usual generators K_i, K_i^{-1} . Similarly for \mathbf{U}_q^0 . Roughly: the triangular decomposition can be proven by ordering F 's to the left and E 's to the right using the relations from Definition 2.2 (the hard part is to show linear independence). Details can, for example, be found in Chapter 4, Section 17 of [42] for the generic case and in Theorem 8.3 (iii) of [60] for the other cases.

Note that, if $q = 1$, then \mathbf{U}_q modulo the ideal generated by $\{K_i - 1 \mid i = 1, \dots, n\}$ can be identified with the hyperalgebra of the semi-simple algebraic group G over \mathbb{K} associated to the Cartan matrix, see Part I, Chapter 7.7 in [43].

2.2. Representation theory of \mathbf{U}_v : the generic, semi-simple case. Let $\lambda \in X$ be a \mathbf{U}_v -weight. As usual, we identify λ with a *character* of \mathbf{U}_v^0 (an algebra homomorphism to $\mathbb{Q}(v)$) via

$$\lambda: \mathbf{U}_v^0 = \mathbb{Q}(v)[K_1^{\pm}, \dots, K_n^{\pm}] \rightarrow \mathbb{Q}(v), \quad K_i^{\pm} \mapsto v^{\pm d_i \langle \lambda, \alpha_i^\vee \rangle}, \quad i = 1, \dots, n.$$

Abusing notation, we use the same symbols for the \mathbf{U}_v -weights λ and the characters λ .

Moreover, if $\underline{\epsilon} = (\epsilon_1, \dots, \epsilon_n) \in \{\pm 1\}^n$, then this can be viewed as a character of \mathbf{U}_v^0 via

$$\underline{\epsilon}: \mathbf{U}_v^0 = \mathbb{Q}(v)[K_1^\pm, \dots, K_n^\pm] \rightarrow \mathbb{Q}(v), \quad K_i^\pm \mapsto \pm \epsilon_i, \quad i = 1, \dots, n.$$

This extends to a character of \mathbf{U}_v by setting $\underline{\epsilon}(E_i) = \underline{\epsilon}(F_i) = 0$.

Every finite dimensional \mathbf{U}_v -module M can be decomposed into

$$(3) \quad M = \bigoplus_{\lambda, \underline{\epsilon}} M_{\lambda, \underline{\epsilon}},$$

where $M_{\lambda, \underline{\epsilon}} = \{m \in M \mid um = \lambda(u)\underline{\epsilon}(u)m, u \in \mathbf{U}_v^0\}$ and the direct sum above runs over all \mathbf{U}_v -weights $\lambda \in X$ and all $\underline{\epsilon} \in \{\pm 1\}^n$, see Chapter 5, Section 2 in [42].

Set $M_1 = \bigoplus_{\lambda} M_{\lambda, (1, \dots, 1)}$ and call a \mathbf{U}_v -module M a \mathbf{U}_v -module of type 1 if $M_1 = M$.

Given a \mathbf{U}_v -module M satisfying (3), we have $M \cong \bigoplus_{\underline{\epsilon}} M_1 \otimes \underline{\epsilon}$. Thus, morally it suffices to study \mathbf{U}_v -modules of type 1, which we will do in this paper. From now on, *all appearing modules are assumed to be of type 1* (and we suppress to mention this in the following).

Example 2.6. If $\mathfrak{g} = \mathfrak{sl}_2$, then the $\mathbf{U}_v(\mathfrak{sl}_2)$ -modules of type 1 are precisely those where K has eigenvalues v^k for $k \in \mathbb{Z}$ whereas type -1 means that K has eigenvalues $-v^k$. \blacksquare

Proposition 2.7. (Semi-simplicity: the generic case) The category $\mathbf{U}_v\text{-Mod}$ of finite dimensional \mathbf{U}_v -modules is semi-simple.

Proof. This is Theorem 5.17 in [42]. \square

The simple modules in $\mathbf{U}_v\text{-Mod}$ can be constructed as follows. For each $\lambda \in X^+$ set

$$\nabla_v(\lambda) = \text{Ind}_{\mathbf{U}_v^- \mathbf{U}_v^0}^{\mathbf{U}_v} \mathbb{Q}(v)_\lambda,$$

called the *dual Weyl* \mathbf{U}_v -module associated to $\lambda \in X^+$. Here $\mathbb{Q}(v)_\lambda$ is the 1-dimensional $\mathbf{U}_v^- \mathbf{U}_v^0$ -module determined by the character λ (and extended to $\mathbf{U}_v^- \mathbf{U}_v^0$ via $\lambda(F_i) = 0$) and $\text{Ind}_{\mathbf{U}_v^- \mathbf{U}_v^0}^{\mathbf{U}_v}(\cdot)$ is the induction functor from Section 2 in [6], i.e. the functor

$$\text{Ind}_{\mathbf{U}_v^- \mathbf{U}_v^0}^{\mathbf{U}_v}: \mathbf{U}_v^- \mathbf{U}_v^0\text{-Mod} \rightarrow \mathbf{U}_v\text{-Mod}, \quad M \mapsto \mathcal{F}(\text{Hom}_{\mathbf{U}_v^- \mathbf{U}_v^0}(\mathbf{U}_v, M))$$

obtained by using the evident embedding of $\mathbf{U}_v^- \mathbf{U}_v^0$ into \mathbf{U}_v (and a certain functor \mathcal{F}).

It turns out that the $\nabla_v(\lambda)$ for $\lambda \in X^+$ form a complete set of non-isomorphic, simple \mathbf{U}_v -modules, see Theorem 5.10 in [42]. For example, we see that the category $\mathbf{U}_v(\mathfrak{g})\text{-Mod}$ is equivalent to the well-studied category of finite dimensional \mathfrak{g} -modules.

By construction, the \mathbf{U}_v -modules $\nabla_v(\lambda)$ satisfy the *Frobenius reciprocity*, that is, we have

$$(4) \quad \text{Hom}_{\mathbf{U}_v}(M, \nabla_v(\lambda)) \cong \text{Hom}_{\mathbf{U}_v^- \mathbf{U}_v^0}(M, \mathbb{Q}(v)_\lambda) \quad \text{for all } M \in \mathbf{U}_v\text{-Mod}.$$

Moreover, if we let $\text{ch}(M)$ denote the (*formal*) character of $M \in \mathbf{U}_v\text{-Mod}$, that is,

$$\text{ch}(M) = \sum_{\lambda \in X} (\dim(M_\lambda)) e^\lambda \in \mathbb{Z}[X],$$

where $M_\lambda = \{m \in M \mid um = \lambda(u)m, u \in \mathbf{U}_v^0\}$ (recall that the group algebra $\mathbb{Z}[X]$, where we regard X to be the free abelian group generated by the dominant (integral) \mathbf{U}_q -weights X^+ , is known as the *character ring*). Then we have

$$(5) \quad \text{ch}(\nabla_v(\lambda)) = \chi(\lambda) \in \mathbb{Z}[X] \quad \text{for all } \lambda \in X^+.$$

Here $\chi(\lambda)$ is the so-called *Weyl character* (that is, the classical character obtained from Weyl's character formula in the non-quantum case). A proof of (5) can be found in Theorem 5.15 of [42].

In addition, we have a contravariant, character-preserving *duality functor*

$$(6) \quad \mathcal{D}: \mathbf{U}_v\text{-Mod} \rightarrow \mathbf{U}_v\text{-Mod}$$

that is defined on the $\mathbb{Q}(v)$ -vector space level via $\mathcal{D}(M) = M^*$ (the $\mathbb{Q}(v)$ -linear dual of M) and an action of \mathbf{U}_v on $\mathcal{D}(M)$ is defined by

$$uf = m \mapsto f(\omega(S(u)m)), \quad m \in M, u \in \mathbf{U}_v, f \in \mathcal{D}(M).$$

Here $\omega: \mathbf{U}_v \rightarrow \mathbf{U}_v$ is the automorphism of \mathbf{U}_v which interchanges E_i and F_i and interchanges K_i and K_i^{-1} (see for example Lemma 4.6 in [42]). Note that the \mathbf{U}_v -weights of M and $\mathcal{D}(M)$ coincide. In particular, we have $\mathcal{D}(\nabla_v(\lambda)) \cong \Delta_v(\lambda)$, where the latter \mathbf{U}_q -module is called the *Weyl \mathbf{U}_v -module* associated to $\lambda \in X^+$. Thus, the Weyl and dual Weyl \mathbf{U}_v -modules are related by duality, since clearly $\mathcal{D}^2 \cong \text{id}_{\mathbf{U}_v\text{-Mod}}$.

Example 2.8. If we have $\mathfrak{g} = \mathfrak{sl}_2$, then the dominant (integral) \mathfrak{sl}_2 -weights X^+ can be identified with \mathbb{N} . Then the i -th Weyl module $\Delta_v(i)$ is the $i+1$ -dimensional $\mathbb{Q}(v)$ -vector space with a basis given by m_0, \dots, m_i and an $\mathbf{U}_v(\mathfrak{sl}_2)$ -action defined by

$$(7) \quad Km_k = v^{i-2k}m_k, \quad E^{(j)}m_k = \begin{bmatrix} i-k+j \\ j \end{bmatrix} m_{k-j} \quad \text{and} \quad F^{(j)}m_k = \begin{bmatrix} k+j \\ j \end{bmatrix} m_{k+j},$$

with the convention that $m_{<0} = m_{>i} = 0$. For example, for $i = 3$ we can visualize $\Delta_v(3)$ as

$$(8) \quad \begin{array}{ccccccc} \binom{v^{-3}}{2} & [1] & \binom{v^{-1}}{2} & [2] & \binom{v^+}{2} & [3] & \binom{v^+3}{2} \\ m_3 & \xleftrightarrow{[3]} & m_2 & \xleftrightarrow{[2]} & m_1 & \xleftrightarrow{[1]} & m_0 \end{array}$$

where the action of E points to the right, the action of F to the left and K acts as a loop.

Note that the $\mathbf{U}_v(\mathfrak{sl}_2)$ -action from (7) is already defined by the action of the generators $E, F, K^{\pm 1}$. For $\mathbf{U}_q(\mathfrak{sl}_2)$ the situation is different, see Example 2.12. \blacksquare

2.3. Representation theory of \mathbf{U}_q : the non semi-simple case. As before in Subsection 2.1, we let q denote a fixed, non-zero element of \mathbb{K} .

Let $\lambda \in X$ be a \mathbf{U}_q -weight. As above, we can identify λ with a character of $\mathbf{U}_{\mathcal{A}}^0$ via

$$\lambda: \mathbf{U}_{\mathcal{A}}^0 \rightarrow \mathcal{A}, \quad K_i^{\pm} \mapsto v^{\pm d_i \langle \lambda, \alpha_i^{\vee} \rangle}, \quad \tilde{K}_{i,t} \mapsto \begin{bmatrix} \langle \lambda, \alpha_i^{\vee} \rangle \\ t \end{bmatrix}_{d_i}, \quad i = 1, \dots, n, \quad t \in \mathbb{N},$$

which then also gives a character of \mathbf{U}_q^0 . Here we use the definition of $\tilde{K}_{i,t}$ from (2). Abusing notation again, we use the same symbols for the \mathbf{U}_q -weights λ and the characters λ .

It is still true that any finite dimensional \mathbf{U}_q -module M is a direct sum of its \mathbf{U}_q -weight spaces, see Theorem 9.2 in [6]. Thus, if we denote by $\mathbf{U}_q\text{-Mod}$ the category of finite dimensional \mathbf{U}_q -modules, then

$$M = \bigoplus_{\lambda \in X} M_{\lambda} = \bigoplus_{\lambda \in X} \{m \in M \mid um = \lambda(u)m, u \in \mathbf{U}_q^0\} \quad \text{for } M \in \mathbf{U}_q\text{-Mod}.$$

In particular, $L_q(3)$ is the $\mathbf{U}_q(\mathfrak{sl}_2)$ -module obtained from $\Delta_q(3)$ as in (9) by modding out the \mathbb{C} -span of the set $\{m_1, m_2\}$. The latter can be seen to be isomorphic to $L_q(1)$.

We encourage the reader to work out its dual case $\nabla_q(3)$ (note that the zero \mathbf{U}_q -action arrows from above turn around). It turns out that $L_q(1)$ is a \mathbf{U}_q -submodule of $\Delta_q(3)$ and $L_q(3)$ is a \mathbf{U}_q -submodule of $\nabla_q(3)$ and these can be visualized as

$$L_q(1) \cong \begin{array}{ccc} \begin{array}{c} q^{-1} \\ \downarrow \\ m_2 \end{array} & \begin{array}{c} \xrightarrow{-1} \\ \xleftarrow{-1} \end{array} & \begin{array}{c} q^{+1} \\ \downarrow \\ m_1 \end{array} \end{array} \quad \text{and} \quad L_q(3) \cong \begin{array}{ccc} \begin{array}{c} q^{-3} \\ \downarrow \\ m_3^* \end{array} & \begin{array}{c} \xrightarrow{+1} \\ \xleftarrow{+1} \end{array} & \begin{array}{c} q^{+3} \\ \downarrow \\ m_0^* \end{array} \end{array},$$

where for $L_q(3)$ the displayed actions are via $E^{(3)}$ (to the right) and $F^{(3)}$ (to the left) instead of E, F as before. Note that $L_q(1)$ and $L_q(3)$ have both dimension 2. This has no analogon in the generic \mathfrak{sl}_2 case where all simple \mathbf{U}_v -modules $L_v(i)$ have different dimensions. ■

A non-trivial fact (which relies on the q -version of the so-called *Kempf's vanishing theorem*, see Theorem 5.5 in [76]) is that the characters of the $\nabla_q(\lambda)$'s are still given by Weyl's character formula as in (5) (by duality, similar for the $\Delta_q(\lambda)$'s). On the other hand, the characters of the simple modules $L_q(\lambda)$ are only known if $\text{char}(\mathbb{K}) = 0$ (and "big enough" l). In that case, certain *Kazhdan-Lusztig polynomials* determine $\text{ch}(L_q(\lambda))$, see for example Theorem 6.4 and 7.1 in [88] and the references therein.

3. TILTING MODULES

In the present section we recall a few facts from the theory of \mathbf{U}_q -tilting modules. In the semi-simple case all \mathbf{U}_q -modules in $\mathbf{U}_q\text{-Mod}$ are \mathbf{U}_q -tilting modules. Hence, the theory of \mathbf{U}_q -tilting modules is kind of redundant in this case. In the non semi-simple case however the theory of \mathbf{U}_q -tilting modules is extremely rich and a source of neat combinatorics. For brevity, we only provide proofs if we need the arguments of the proofs in what follows. For more details see for example [28] and for arguments adopted to precisely our situation see [7].

3.1. \mathbf{U}_q -modules with a Δ_q - and a ∇_q -filtration. Recall that $\nabla_q(\lambda)$ has a simple socle and $\Delta_q(\lambda)$ has a simple head, both isomorphic to $L_q(\lambda)$. Thus, there is an (up to scalars) unique \mathbf{U}_q -homomorphism

$$(10) \quad c^\lambda: \Delta_q(\lambda) \rightarrow \nabla_q(\lambda)$$

which sends the head to the socle. To see this, note that we have, by Frobenius reciprocity from (4) (to be more precise, the q -version of it which can be found in Proposition 2.12 of [6]),

$$\mathrm{Hom}_{\mathbf{U}_q}(\Delta_q(\lambda), \nabla_q(\lambda)) \cong \mathrm{Hom}_{\mathbf{U}_q^- \mathbf{U}_q^0}(\Delta_q(\lambda), \mathbb{K}_\lambda)$$

which gives $\dim(\mathrm{Hom}_{\mathbf{U}_q}(\Delta_q(\lambda), \nabla_q(\lambda))) = 1$, since, by construction, $\Delta_q(\lambda)_\lambda \cong \mathbb{K}$.

This gives us the following (if $\mathrm{char}(\mathbb{K}) > 0$, then we have to enlarge the category $\mathbf{U}_q\text{-Mod}$ by non-necessarily finite dimensional \mathbf{U}_q -modules to have enough injectives such that the $\mathrm{Ext}_{\mathbf{U}_q}^i$ -functors make sense by using q -analogous arguments as in Part I, Chapter 3 of [43]).

Theorem 3.1. (Ext-vanishing) We have for all $\lambda, \mu \in X^+$ that

$$\mathrm{Ext}_{\mathbf{U}_q}^i(\Delta_q(\lambda), \nabla_q(\mu)) \cong \begin{cases} \mathbb{K}c^\lambda, & \text{if } i = 0 \text{ and } \lambda = \mu, \\ 0, & \text{else.} \end{cases}$$

Proof. Similar to the modular analogon treated in Part 2, Chapter H.15 of [43] (a proof in our notation can be found in [7]). \square

Definition 3.2. (Δ_q - and ∇_q -filtration) We say that a \mathbf{U}_q -module M has a Δ_q -filtration if there exists some $k \in \mathbb{N}$ and a descending sequence of \mathbf{U}_q -submodules

$$M = M_0 \supset M_1 \supset \cdots \supset M_{k'} \supset \cdots \supset M_{k-1} \supset M_k = 0,$$

such that for all $k' = 0, \dots, k-1$ we have $M_{k'}/M_{k'+1} \cong \Delta_q(\lambda_{k'})$ for some $\lambda_{k'} \in X^+$.

A ∇_q -filtration is defined similarly, but using $\nabla_q(\lambda)$ instead of $\Delta_q(\lambda)$ and an ascending sequence of \mathbf{U}_q -submodules, that is,

$$0 = M_0 \subset M_1 \subset \cdots \subset M_{k'} \subset \cdots \subset M_{k-1} \subset M_k = M,$$

such that for all $k' = 0, \dots, k-1$ we have $M_{k'+1}/M_{k'} \cong \nabla_q(\lambda_{k'})$ for some $\lambda_{k'} \in X^+$.

Clearly a \mathbf{U}_q -module M has a Δ_q -filtration iff its dual $\mathcal{D}(M)$ has a ∇_q -filtration.

Example 3.3. The simple \mathbf{U}_q -module $L_q(\lambda)$ has a Δ_q -filtration iff $L_q(\lambda) \cong \Delta_q(\lambda)$. In that case we have also $L_q(\lambda) \cong \nabla_q(\lambda)$ and thus, $L_q(\lambda)$ has a ∇_q -filtration as well. \blacksquare

A corollary of the Ext-vanishing theorem is the following.

Corollary 3.4. Let $M, N \in \mathbf{U}_q\text{-Mod}$ and $\lambda \in X^+$. Assume that M has a Δ_q -filtration and N has a ∇_q -filtration.

- (a) We have $\dim(\text{Hom}_{\mathbf{U}_q}(M, \nabla_q(\lambda))) = (M : \Delta_q(\lambda)) = |\{k' \mid \lambda_{k'} = \lambda\}|$.
- (b) We have $\dim(\text{Hom}_{\mathbf{U}_q}(\Delta_q(\lambda), N)) = (N : \nabla_q(\lambda)) = |\{k' \mid \lambda_{k'} = \lambda\}|$.

Here the \mathbf{U}_q -weights $\lambda_{k'}$ are as in Definition 3.2. In particular, the multiplicities $(M : \Delta_q(\lambda))$ and $(N : \nabla_q(\lambda))$ are independent of the choice of filtration.

Note that the proof of Corollary 3.4 below gives a method to find and construct bases of $\text{Hom}_{\mathbf{U}_q}(M, \nabla_q(\lambda))$ and $\text{Hom}_{\mathbf{U}_q}(\Delta_q(\lambda), N)$ respectively.

Proof. Let k be the length of the Δ_q -filtration of M . If $k = 1$, then

$$(11) \quad \dim(\text{Hom}_{\mathbf{U}_q}(M, \nabla_q(\lambda))) = (M : \Delta_q(\lambda))$$

follows from the uniqueness of c^λ from (10). Otherwise, we take the short exact sequence

$$0 \longrightarrow M' \hookrightarrow M \twoheadrightarrow \Delta_q(\mu) \longrightarrow 0$$

for some $\mu \in X^+$. Since both sides of (11) are additive with respect to short exact sequences by Theorem 3.1, the claim in (a) follows by induction. Similarly for (b) by duality. \square

In fact, following Donkin [27] who obtained the result below in the modular case, we can state two useful consequences of the Ext-vanishing theorem. These are very useful criteria to determine if given \mathbf{U}_q -modules M or N have a Δ_q - or ∇_q -filtration respectively.

Proposition 3.5. (Ext-criteria) Let $M, N \in \mathbf{U}_q\text{-Mod}$. Then the following are equivalent.

- (a) The \mathbf{U}_q -module M has a Δ_q -filtration (respectively N has a ∇_q -filtration).
- (b) We have $\text{Ext}_{\mathbf{U}_q}^i(M, \nabla_q(\lambda)) = 0$ (respectively $\text{Ext}_{\mathbf{U}_q}^i(\Delta_q(\lambda), N) = 0$) for all $\lambda \in X^+$ and all $i > 0$.
- (c) We have $\text{Ext}_{\mathbf{U}_q}^1(M, \nabla_q(\lambda)) = 0$ (respectively $\text{Ext}_{\mathbf{U}_q}^1(\Delta_q(\lambda), N) = 0$) for all $\lambda \in X^+$.

Proof. As in Proposition II.4.16 in [43]. A proof in our notation can be found in [7]. \square

Example 3.6. Let us come back to our favorite example, i.e. q being a complex, primitive third root of unity for $\mathbf{U}_q(\mathfrak{sl}_2)$. The simple \mathbf{U}_q -module $L_q(3)$ does neither have a Δ_q - nor a ∇_q -filtration (compare Example 2.12 with Example 3.3). This can also be seen with Proposition 3.5, because $\text{Ext}_{\mathbf{U}_q}^1(L_q(3), L_q(1))$ is not trivial: by Example 2.12 from above we have $\Delta_q(1) \cong L_q(1) \cong \nabla_q(1)$, but

$$0 \longrightarrow L_q(1) \hookrightarrow \Delta_q(3) \twoheadrightarrow L_q(3) \longrightarrow 0$$

does not split. Analogously, $\text{Ext}_{\mathbf{U}_q}^1(L_q(1), L_q(3)) \neq 0$ by duality. \blacksquare

3.2. \mathbf{U}_q -tilting modules. Following Donkin [27], we are now ready to define the *category of \mathbf{U}_q -tilting modules* that we denote by \mathcal{T} . This category is our main object of study.

Definition 3.7. (Category of \mathbf{U}_q -tilting modules) The category \mathcal{T} is the full subcategory of $\mathbf{U}_q\text{-Mod}$ whose objects are all \mathbf{U}_q -tilting modules, that is, \mathbf{U}_q -modules T which have both, a Δ_q - and a ∇_q -filtration.

From Proposition 3.5 we obtain directly an important statement.

Corollary 3.8. Let $T \in \mathbf{U}_q\text{-Mod}$. Then

$$T \in \mathcal{T} \quad \text{iff} \quad \text{Ext}_{\mathbf{U}_q}^1(T, \nabla_q(\lambda)) = 0 = \text{Ext}_{\mathbf{U}_q}^1(\Delta_q(\lambda), T) \quad \text{for all } \lambda \in X^+.$$

Moreover, the corresponding higher Ext-groups vanish as well. \square

Recall the contravariant, character preserving functor $\mathcal{D}: \mathbf{U}_q\text{-Mod} \rightarrow \mathbf{U}_q\text{-Mod}$ from (6). Clearly, by Corollary 3.8, $T \in \mathcal{T}$ iff $\mathcal{D}(T) \in \mathcal{T}$. Thus, $\mathcal{D}(\cdot)$ restricts to a functor $\mathcal{D}: \mathcal{T} \rightarrow \mathcal{T}$. In fact, we show below in Corollary 3.13, that the functor $\mathcal{D}(\cdot)$ restricts to (a functor isomorphic to) the identity functor on objects of \mathcal{T} .

Example 3.9. The $L_q(\lambda)$ are \mathbf{U}_q -tilting modules iff $\Delta_q(\lambda) \cong L_q(\lambda) \cong \nabla_q(\lambda)$.

Coming back to our favourite example, that is the case $\mathfrak{g} = \mathfrak{sl}_2$ and q is a complex, primitive third root of unity: a direct computation using similar reasoning as in Example 2.12 (that is, the appearance of some actions equals zero as in (9)) shows that $L_q(i)$ is a \mathbf{U}_q -tilting module iff $i = 0, 1$ or $i \equiv -1 \pmod{3}$. More general: if q is a complex, primitive l -th root of unity, then $L_q(i)$ is a \mathbf{U}_q -tilting module iff $i = 0, \dots, l-1$ or $i \equiv -1 \pmod{l}$. \blacksquare

Proposition 3.10. \mathcal{T} is a Krull-Schmidt category, closed under duality $\mathcal{D}(\cdot)$ and under finite direct sums. Furthermore, \mathcal{T} is closed under finite tensor products.

Proof. That \mathcal{T} is a Krull-Schmidt is immediate. By Corollary 3.8 we see that \mathcal{T} is closed under duality $\mathcal{D}(\cdot)$ and under finite direct sums. That \mathcal{T} is closed under finite tensor products is non-trivial and a proof can be found in Theorem 3.3 of [69] (and a proof of a less general version sufficient for most of our purposes in [7]). \square

The indecomposable \mathbf{U}_q -modules in \mathcal{T} , that we denote by $T_q(\lambda)$, are indexed by the dominant (integral) \mathbf{U}_q -weights $\lambda \in X^+$ (see Proposition 3.11 below). The \mathbf{U}_q -tilting module $T_q(\lambda)$ is determined by the property that it is indecomposable with λ as its unique maximal weight. Then λ appears in fact with multiplicity 1.

The following classification is, in the modular case, due to Ringel [72] and Donkin [27].

Proposition 3.11. (Classification of the indecomposable \mathbf{U}_q -tilting modules) For each $\lambda \in X^+$ there exists an indecomposable \mathbf{U}_q -tilting module, that we denote by $T_q(\lambda)$, with \mathbf{U}_q -weight spaces $T_q(\lambda)_\mu = 0$ unless $\mu \leq \lambda$. Moreover, $T_q(\lambda)_\lambda \cong \mathbb{K}$. In addition, given any indecomposable \mathbf{U}_q -tilting module $T \in \mathcal{T}$, then there exists $\lambda \in X^+$ such that $T \cong T_q(\lambda)$.

Thus, the $T_q(\lambda)$'s form a complete set of non-isomorphic indecomposables of \mathcal{T} and all indecomposable \mathbf{U}_q -tilting modules $T_q(\lambda)$ are uniquely determined by their maximal weight $\lambda \in X^+$, that is,

$$\{\text{indecomposable } \mathbf{U}_q\text{-tilting modules}\} \xleftrightarrow{1:1} X^+.$$

Proof. Analogously to Lemma II.E.3 and Proposition II.E.6 in [43] and left to the reader. We only point out that one uses the q -version of the Frobenius reciprocity from (4). A detailed proof in our notation can be found in [7]. \square

Remark 3.12. For a fixed $\lambda \in X$ we have \mathbf{U}_q -homomorphisms

$$\Delta_q(\lambda) \xrightarrow{\iota^\lambda} T_q(\lambda) \xrightarrow{\pi^\lambda} \nabla_q(\lambda)$$

where ι^λ is the inclusion of the first \mathbf{U}_q -submodule in a Δ_q -filtration of $T_q(\lambda)$ and π^λ is the surjection onto the last quotient in a ∇_q -filtration of $T_q(\lambda)$. Note that these are only defined

up to scalars and we fix scalars in the following such that $\pi^\lambda \circ \iota^\lambda = c^\lambda$ (where c^λ is again the \mathbf{U}_q -homomorphism from (10)).

Take any \mathbf{U}_q -tilting module $T \in \mathcal{T}$. An easy argument shows (see also the proof of Proposition 3.5 in [7]) the following crucial fact:

$$(12) \quad \text{Ext}_{\mathbf{U}_q}^1(\Delta_q(\lambda), T) = 0 = \text{Ext}_{\mathbf{U}_q}^1(T, \nabla_q(\lambda)) \Rightarrow \text{Ext}_{\mathbf{U}_q}^1(\text{coker}(\iota^\lambda), T) = 0 = \text{Ext}_{\mathbf{U}_q}^1(T, \ker(\pi^\lambda))$$

for all $\lambda \in X^+$. Hence, we see that any \mathbf{U}_q -homomorphism $g: \Delta_q(\lambda) \rightarrow M$ extends to an \mathbf{U}_q -homomorphism $\bar{g}: T_q(\lambda) \rightarrow M$ whereas any \mathbf{U}_q -homomorphism $f: M \rightarrow \nabla_q(\lambda)$ factors through $T_q(\lambda)$ via $\bar{f}: M \rightarrow T_q(\lambda)$.

Corollary 3.13. We have $\mathcal{D}(T) \cong T$ for $T \in \mathcal{T}$, that is, all \mathbf{U}_q -tilting modules T are self-dual. In particular, we have for all $\lambda \in X^+$ that

$$(T : \Delta_q(\lambda)) = \dim(\text{Hom}_{\mathbf{U}_q}(T, \nabla_q(\lambda))) = \dim(\text{Hom}_{\mathbf{U}_q}(\Delta_q(\lambda), T)) = (T : \nabla_q(\lambda)).$$

Proof. By the Krull-Schmidt property it suffices to show the statement for the indecomposable \mathbf{U}_q -tilting modules $T_q(\lambda)$. Since \mathcal{D} preserves characters, we see that $\mathcal{D}(T_q(\lambda))$ has λ as unique maximal weight, therefore $\mathcal{D}(T_q(\lambda)) \cong T_q(\lambda)$ by Proposition 3.11. Moreover, the leftmost and the rightmost equalities follow directly from Corollary 3.4. Finally

$$(T_q(\lambda) : \Delta_q(\lambda)) = (\mathcal{D}(T_q(\lambda)) : \mathcal{D}(\Delta_q(\lambda))) = (\mathcal{D}(T_q(\lambda)) : \nabla_q(\lambda)) = (T_q(\lambda) : \nabla_q(\lambda))$$

by definition and $\mathcal{D}(T_q(\lambda)) \cong T_q(\lambda)$ from above, which settles also the middle equality. \square

Example 3.14. Let us go back to the \mathfrak{sl}_2 case again. Then we obtain the family $(T_q(i))_{i \in \mathbb{N}}$ of indecomposable \mathbf{U}_q -tilting modules as follows.

Start by setting $T_q(0) \cong \Delta_q(0) \cong L_q(0) \cong \nabla_q(0)$ and $T_q(1) \cong \Delta_q(1) \cong L_q(1) \cong \nabla_q(1)$. Then we denote by $m_0 \in T_q(1)$ any eigenvector for K with eigenvalue q . For each $i > 1$ we define $T_q(i)$ to be the indecomposable summand of $T_q(1)^{\otimes i}$ which contains the vector $m_0 \otimes \cdots \otimes m_0 \in T_q(1)^{\otimes i}$. The $\mathbf{U}_q(\mathfrak{sl}_2)$ -tilting module $T_q(1)^{\otimes i}$ is not indecomposable if $i > 1$: By Proposition 3.11 we have $(T_q(1)^{\otimes i} : \Delta_q(i)) = 1$ and

$$T_q(1)^{\otimes i} \cong T_q(i) \oplus \bigoplus_{k < i} T_q(k)^{\oplus \text{mult}_k} \quad \text{for some } \text{mult}_k \in \mathbb{N}.$$

In the case $l = 3$, we have for instance $T_q(1)^{\otimes 2} \cong T_q(2) \oplus T_q(0)$ since the tensor product $T_q(1) \otimes T_q(1)$ looks as follows (abbreviation $m_{ij} = m_i \otimes m_j$):

$$\begin{array}{c} \otimes \cdots \cdots \begin{array}{ccc} \begin{array}{c} \left. \begin{array}{c} q^{-1} \\ \downarrow \end{array} \right\} m_1 & \xleftrightarrow{1} & \begin{array}{c} \left. \begin{array}{c} q^{+1} \\ \downarrow \end{array} \right\} m_0 \\ \hline 1 \end{array} \end{array} \\ \vdots \\ \begin{array}{ccc} \begin{array}{c} \left. \begin{array}{c} q^{-1} \\ \downarrow \end{array} \right\} m_1 & \xleftrightarrow{1} & \begin{array}{c} \left. \begin{array}{c} q^0 \\ \downarrow \end{array} \right\} m_{01} \\ \hline 1 \end{array} \end{array} & \begin{array}{c} \left. \begin{array}{c} q^{-2} \\ \downarrow \end{array} \right\} m_{11} & \xleftrightarrow{1} & \begin{array}{c} \left. \begin{array}{c} q^0 \\ \downarrow \end{array} \right\} m_{01} \\ \hline 1 \end{array} \end{array} & \begin{array}{c} \left. \begin{array}{c} q^0 \\ \downarrow \end{array} \right\} m_{10} & \xleftrightarrow{1} & \begin{array}{c} \left. \begin{array}{c} q^{+2} \\ \downarrow \end{array} \right\} m_{00} \\ \hline 1 \end{array} \end{array} \\ \vdots \\ \begin{array}{ccc} \begin{array}{c} \left. \begin{array}{c} q^{-1} \\ \downarrow \end{array} \right\} m_1 & \xleftrightarrow{1} & \begin{array}{c} \left. \begin{array}{c} q^{+1} \\ \downarrow \end{array} \right\} m_0 \\ \hline 1 \end{array} \end{array} & \begin{array}{c} \left. \begin{array}{c} q^{-1} \\ \downarrow \end{array} \right\} m_1 & \xleftrightarrow{1} & \begin{array}{c} \left. \begin{array}{c} q^0 \\ \downarrow \end{array} \right\} m_{01} \\ \hline 1 \end{array} \end{array} & \begin{array}{c} \left. \begin{array}{c} q^{-2} \\ \downarrow \end{array} \right\} m_{11} & \xleftrightarrow{1} & \begin{array}{c} \left. \begin{array}{c} q^0 \\ \downarrow \end{array} \right\} m_{01} \\ \hline 1 \end{array} \end{array} & \begin{array}{c} \left. \begin{array}{c} q^0 \\ \downarrow \end{array} \right\} m_{10} & \xleftrightarrow{1} & \begin{array}{c} \left. \begin{array}{c} q^{+2} \\ \downarrow \end{array} \right\} m_{00} \\ \hline 1 \end{array} \end{array} \end{array} \end{array}$$

By construction, the indecomposable $\mathbf{U}_q(\mathfrak{sl}_2)$ -module $T_q(2)$ contains m_{00} and therefore has to be the \mathbb{C} -span of $\{m_{00}, q^{-1}m_{10} + m_{01}, m_{11}\}$ as indicated above. The remaining summand is the 1-dimensional \mathbf{U}_q -tilting module $T_q(0) \cong L_q(0)$ from before. \blacksquare

The following is interesting in its own right.

Corollary 3.15. Let $\mu \in X^+$ be a minuscule \mathbf{U}_q -weight. Then $T = \Delta_q(\mu)^{\otimes d}$ is a \mathbf{U}_q -tilting module for any $d \in \mathbb{N}$ and $\dim(\text{End}_{\mathbf{U}_q}(T))$ is independent of the field \mathbb{K} and of $q \in \mathbb{K} - \{0\}$ (and given by (21)). In particular, this holds for $\Delta_q(\omega_1)$ being the vector representation of $\mathbf{U}_q = \mathbf{U}_q(\mathfrak{g})$ for \mathfrak{g} of type A , C or D .

Proof. Since $\mu \in X^+$ is minuscule: $\Delta_q(\mu) \cong L_q(\mu)$ is a simple \mathbf{U}_q -tilting module for any field \mathbb{K} and any $q \in \mathbb{K} - \{0\}$. Thus, by Proposition 3.10 we see that T is a \mathbf{U}_q -tilting module for any $d \in \mathbb{N}$. Hence, by Corollaries 3.4 and 3.13, we have $\dim(\text{End}_{\mathbf{U}_q}(T)) = \sum_{\lambda \in X^+} (T : \Delta_q(\lambda))^2$. Now use the fact that $\chi(\Delta_q(\mu))$ is as in the classical case which implies the statement. \square

3.3. The characters of indecomposable \mathbf{U}_q -tilting modules. In this subsection we describe how to compute $(T_q(\lambda) : \Delta_q(\mu))$ for all $\lambda, \mu \in X^+$ (which can be done algorithmically in the case where q is a complex, primitive l -th root of unity). As an application, we illustrate how to decompose tensor products of \mathbf{U}_q -tilting modules. This shows that, in principle, our cellular basis for endomorphism rings $\text{End}_{\mathbf{U}_q}(T)$ of \mathbf{U}_q -tilting modules T (that we introduce in Section 4) can be made completely explicit.

Given an abelian category \mathcal{AB} , we denote its *Grothendieck group* by $K_0(\mathcal{AB})$ and its *split Grothendieck group* by $K_0^\oplus(\mathcal{AB})$. We point out that the notation of the split Grothendieck group also makes sense for a given additive category \mathcal{AD} that satisfies the Krull-Schmidt property where we use the same notation (we refer the reader unfamiliar with these and the notation we use to Section 1.2 in [65]).

By Propositions 2.9 and 2.11, a \mathbb{Z} -basis of the Grothendieck group $K_0(\mathbf{U}_q\text{-Mod})$ is given by isomorphism classes $\{[\Delta_q(\lambda)] \mid \lambda \in X^+\}$.

On the other hand, \mathcal{T} is not abelian (see Example 3.9), but additive and satisfies the Krull-Schmidt property. A \mathbb{Z} -basis of $K_0^\oplus(\mathcal{T})$ is, by Proposition 3.11, spanned by isomorphism classes $\{[T_q(\lambda)]_\oplus \mid \lambda \in X^+\}$.

Since both, $\mathbf{U}_q\text{-Mod}$ and \mathcal{T} , are closed under tensor products, $K_0(\mathbf{U}_q\text{-Mod})$ and $K_0^\oplus(\mathcal{T})$ get an (in fact isomorphic) induced ring structure.

Corollary 3.16. The inclusion of categories $\iota: \mathcal{T} \rightarrow \mathbf{U}_q\text{-Mod}$ induces an isomorphism

$$[\iota]: K_0^\oplus(\mathcal{T}) \rightarrow K_0(\mathbf{U}_q\text{-Mod}), \quad [T_q(\lambda)]_\oplus \mapsto [\Delta_q(\lambda)], \quad \lambda \in X^+$$

of rings.

Proof. The set $B = \{[T_q(\lambda)] \mid \lambda \in X^+\}$ forms a \mathbb{Z} -basis of $K_0^\oplus(\mathcal{T})$ by Proposition 3.11 and it is clear that $[\iota]$ is a well-defined ring homomorphism.

Moreover, we have

$$(13) \quad [T_q(\lambda)] = [\Delta_q(\lambda)] + \sum_{\mu < \lambda \in X^+} (T_q(\mu) : \Delta_q(\mu)) [\Delta_q(\mu)] \in K_0(\mathbf{U}_q\text{-Mod})$$

with $T_q(0) \cong \Delta_q(0)$ by Proposition 3.11. Hence, $[\iota](B)$ is also a \mathbb{Z} -basis of $K_0(\mathbf{U}_q\text{-Mod})$ since the $\Delta_q(\lambda)$'s form a \mathbb{Z} -basis and the claim follows. \square

Recall that $\mathbb{Z}[X]$ carries an action of the Weyl group W associated to the Cartan datum (see below). Thus, we can look at the invariant part of this action, denoted by $\mathbb{Z}[X]^W$, which is known as *Weyl's character ring*.

We obtain the following (known) categorification result.

Corollary 3.17. The tilting category \mathcal{T} (naively) categorifies Weyl's character ring, that is,

$$K_0^\oplus(\mathcal{T}) \cong \mathbb{Z}[X]^W \quad \text{as rings.}$$

Proof. It is known that there is an isomorphism $K_0(\mathfrak{g}\text{-Mod}) \cong \mathbb{Z}[X]^W$ given by sending finite dimensional \mathfrak{g} -modules to their characters (which can be regarded as elements in $\mathbb{Z}[X]^W$).

Now the characters $\chi(\Delta_q(\lambda))$ of the $\Delta_q(\lambda)$'s are (as mentioned below Example 2.12) the same as in the classical case. Thus, we can adopt the isomorphism $K_0(\mathfrak{g}\text{-Mod}) \cong \mathbb{Z}[X]^W$ (non-quantized!). Details can, for example, be found in Chapter VIII, §7.7 of [13].

Then the statement follows from Corollary 3.16. \square

For each simple root $\alpha_i \in \Pi$ let s_i be the reflection

$$s_i(\lambda) = \lambda - \langle \lambda, \alpha_i^\vee \rangle \alpha_i, \quad \text{for } \lambda \in E,$$

in the hyperplane $H_{\alpha_i^\vee}$ orthogonal to α_i . These reflections s_i generate a group W , called *Weyl group*, associated to our Cartan datum.

For any fixed $l \in \mathbb{N}$, the *affine Weyl group* $W_l \cong W \ltimes l\mathbb{Z}\Pi$ is the group generated by the reflections $s_{\beta,r}$ in the *affine hyperplanes* $H_{\beta^\vee,r} = \{x \in E \mid \langle x, \beta^\vee \rangle = r\}$ for $\beta \in \Phi$ and $r \in \mathbb{Z}$. Note that, if $l = 0$, then $W_0 \cong W$.

For $\beta \in \Phi$ there exists $w \in W$ such that $\beta = w(\alpha_i)$ for some $i = 1, \dots, n$. We set $l_\beta = l_i$ where $l_i = \frac{l}{\gcd(l, d_i)}$. Using this, we have the *dot-action* of W_l on the \mathbf{U}_q -weight lattice X via

$$s_{\beta,r} \cdot \lambda = s_\beta(\lambda + \rho) - \rho + rl_\beta \beta.$$

Note that the case $l = 1$ recovers the usual action of the affine Weyl group W_1 on X .

Definition 3.18. (Alcove combinatorics) The *fundamental alcove* \mathcal{A}_0 is

$$(14) \quad \mathcal{A}_0 = \{\lambda \in X \mid 0 < \langle \lambda + \rho, \alpha^\vee \rangle < l, \text{ for all } \alpha \in \Phi^+\} \subset X^+.$$

Its *closure* $\overline{\mathcal{A}}_0$ is given by

$$(15) \quad \overline{\mathcal{A}}_0 = \{\lambda \in X \mid 0 \leq \langle \lambda + \rho, \alpha^\vee \rangle \leq l, \text{ for all } \alpha \in \Phi^+\} \subset X^+ - \rho.$$

The *non-affine walls* of \mathcal{A}_0 are

$$\partial \mathcal{A}_0^i = \overline{\mathcal{A}}_0 \cap H_{\alpha_i^\vee, 0}, \quad i = 1, \dots, n, \quad \partial \mathcal{A}_0 = \bigcup_{i=1}^n \partial \mathcal{A}_0^i.$$

The set

$$\hat{\partial} \mathcal{A}_0 = \overline{\mathcal{A}}_0 \cap H_{\alpha_0^\vee, 1}$$

is called the *affine wall* of \mathcal{A}_0 . Here α_0 is the maximal short root. We call the union of all these walls the *boundary* $\partial \mathcal{A}_0$ of \mathcal{A}_0 . More generally, an *alcove* \mathcal{A} is a connected component of

$$E - \bigcup_{r \in \mathbb{Z}, \beta \in \Phi} H_{\beta^\vee, r}.$$

We denote the set of alcoves by \mathcal{AL} .

Note that the affine Weyl group W_l acts simply transitively on \mathcal{AL} . Thus, we can associate $1 \in W_l \mapsto \mathcal{A}(1) = \mathcal{A}_0 \in \mathcal{AL}$ and in general $w \in W_l \mapsto \mathcal{A}(w) \in \mathcal{AL}$.

Example 3.19. In the case $\mathfrak{g} = \mathfrak{sl}_2$ we have $\rho = \omega_1 = 1$. Consider for instance again $l = 3$. Then $k \in \mathbb{N} = X^+$ is contained in the fundamental alcove \mathcal{A}_0 iff $0 < k + 1 < 3$.

Moreover, $-\rho \in \check{\partial}\mathcal{A}_0$ and $2 \in \hat{\partial}\mathcal{A}_0$ are on the walls. Thus, $\overline{\mathcal{A}}_0$ can be visualized as

$$\overset{-\rho}{\bullet} \text{---} \overset{0}{\bullet} \text{---} \overset{1}{\bullet} \text{---} \overset{2}{\bullet}$$

where the affine wall on the right is indicated in red and the non-affine wall on the left is indicated in green. \blacksquare

Example 3.20. Let us leave our running \mathfrak{sl}_2 example for a second and do another example which is graphically more interesting.

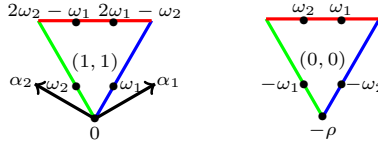
In the case $\mathfrak{g} = \mathfrak{sl}_3$ we have $\rho = \alpha_1 + \alpha_2 = \omega_1 + \omega_2 \in X^+$ and $\alpha_0 = \alpha_1 + \alpha_2$. Now consider again $l = 3$. The condition (14) means that \mathcal{A}_0 consists of those $\lambda = \lambda_1\omega_1 + \lambda_2\omega_2$ for which

$$0 < \langle \lambda_1\omega_1 + \lambda_2\omega_2 + \omega_1 + \omega_2, \alpha_i^\vee \rangle < 3 \quad \text{for } i = 1, 2, 0.$$

Thus, $0 < \lambda_1 + 1 < 3$, $0 < \lambda_2 + 1 < 3$ and $0 < \lambda_1 + \lambda_2 + 2 < 3$. Hence, only the $\mathbf{U}_q(\mathfrak{sl}_3)$ -weight $\lambda = (0, 0) \in X^+$ is in \mathcal{A}_0 . In addition, we have by condition (15) that

$$\check{\partial}\mathcal{A}_0 = \{-\rho, -\omega_1, -\omega_2\} \quad , \quad \hat{\partial}\mathcal{A}_0 = \{\omega_1, \omega_2\}.$$

Hence, $\overline{\mathcal{A}}_0$ can be visualized as (displayed without the $-\rho$ shift on the left)



where, as before, the affine wall at the top is indicated in red, the hyperplane orthogonal to α_1 on the left in green and the hyperplane orthogonal to α_2 on the right in blue. \blacksquare

We say $\lambda \in X^+ - \rho$ is *linked* to $\mu \in X^+$ if there exists $w \in W_l$ such that $w.\lambda = \mu$. We note the following theorem, called *the linkage principle*, where we, by convention, set $T_q(\lambda) = \Delta_q(\lambda) = \nabla_q(\lambda) = L_q(\lambda) = 0$ for $\lambda \in \check{\partial}\mathcal{A}_0$.

Theorem 3.21. (The linkage principle) All composition factors of $T_q(\lambda)$ have maximal weights μ linked to λ . Moreover, $T_q(\lambda)$ is a simple \mathbf{U}_q -module if $\lambda \in \overline{\mathcal{A}}_0$.

In addition, if λ is linked to an element of \mathcal{A}_0 , then $T_q(\lambda)$ is a simple \mathbf{U}_q -module iff $\lambda \in \mathcal{A}_0$.

Proof. This is a slight reformulation of the Corollaries 4.4 and 4.6 in [2]. \square

The linkage principle gives us now a decomposition into a direct sum of categories

$$\mathcal{T} \cong \bigoplus_{\lambda \in \mathcal{A}_0} \mathcal{T}_\lambda \oplus \bigoplus_{\lambda \in \partial\mathcal{A}_0} \mathcal{T}_\lambda,$$

where each \mathcal{T}_λ consists of all $T \in \mathcal{T}$ whose indecomposable summands are all of the form $T_q(\mu)$ for $\mu \in X^+$ lying in the W_l -dot-orbit of $\lambda \in \mathcal{A}_0$ (or of $\lambda \in \partial\mathcal{A}_0$). We call these categories *blocks* to stress that they are homologically unconnected (although they might be decomposable). Moreover, if $\lambda \in \mathcal{A}_0$, then we call \mathcal{T}_λ a *regular block*, while the \mathcal{T}_λ 's with $\lambda \in \partial\mathcal{A}_0$ are called *singular blocks*.

In fact, by Theorem 3.21, the \mathbf{U}_q -weights labelling the indecomposable \mathbf{U}_q -tilting modules are only the dominant (integral) weights $\lambda \in X^+$. They correspond blockwise precisely to the alcoves

$$\mathcal{AL}^+ = \mathcal{AL} \cap \{x \in E \mid \langle x, \beta^\vee \rangle \geq 0, \beta \in \Phi\}$$

contained in the dominant chamber. That is, they correspond to the set of coset representatives of minimal length in $\{wW_0 \mid w \in W_1\}$. In formulas,

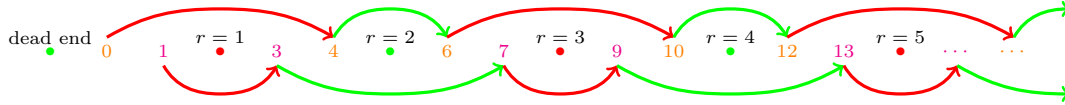
$$(16) \quad T_q(w.\lambda) \in \mathcal{T}_\lambda \iff \mathcal{A}(w) \in \mathcal{AL}^+ \iff wW_0 \subset W_1,$$

for all $\lambda \in \mathcal{A}_0$.

Example 3.22. In our pet example with $\mathfrak{g} = \mathfrak{sl}_2$ and $l = 3$ we have, by Theorem 3.21 and Example 3.19 a block decomposition

$$\mathcal{T} \cong \mathcal{T}_{-1} \oplus \mathcal{T}_0 \oplus \mathcal{T}_1 \oplus \mathcal{T}_2.$$

The W_l -dot-orbit of $0 \in \mathcal{A}_0$ respectively $1 \in \mathcal{A}_0$ can be visualized as



Compare also to (2.4.1) from [8].

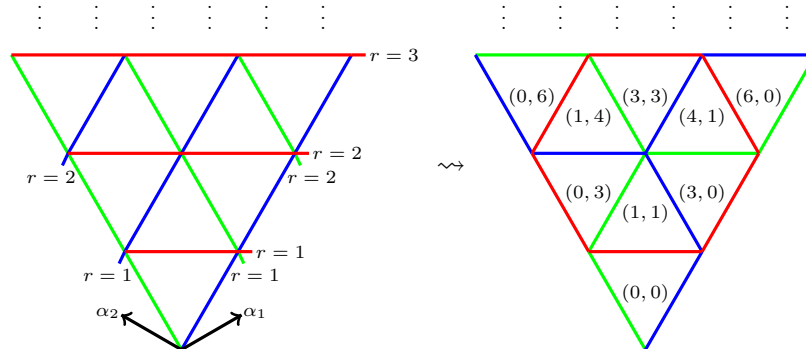
It turns out that, for $\mathbb{K} = \mathbb{C}$, both singular blocks \mathcal{T}_{-1} and \mathcal{T}_2 are semi-simple (in particular, these blocks decompose further), see Example 3.27 or Lemma 2.25 in [8]. ■

Example 3.23. In the \mathfrak{sl}_3 case with $l = 3$ we have the block decomposition

$$\mathcal{T} \cong \mathcal{T}_{-\rho} \oplus \mathcal{T}_{-\omega_2} \oplus \mathcal{T}_{-\omega_1} \oplus \mathcal{T}_{(0,0)} \oplus \mathcal{T}_{\omega_1} \oplus \mathcal{T}_{\omega_2}.$$

Note that the singular blocks are not necessarily semi-simple anymore (even when $\mathbb{K} = \mathbb{C}$).

The W_l -dot-orbit of the regular block looks as follows.



Here we reflect either in a red (that is, $\alpha_0 = (1, 1)$), green (that is, $\alpha_1 = (2, -1)$) or blue (that is, $\alpha_2 = (-1, 2)$) hyperplane. The r measures the hyperplane-distance from the origin (left picture above). In the right picture we have indicated the linkage.

Theorem 3.21 means now that $T_q((1, 1))$ satisfies

$$(T_q((1, 1)) : \Delta_q(\mu)) \neq 0 \implies \mu \in \{(0, 0), (1, 1)\}$$

and $T_q((3, 3))$ satisfies

$$(T_q((3, 3)) : \Delta_q(\mu)) \neq 0 \implies \mu \in \{(0, 0), (1, 1), (3, 0), (0, 3), (4, 1), (1, 4), (3, 3)\}.$$

We calculate the precise values later in Example 3.25. ■

In order to get our hands on the multiplicities, we need Soergel's version of the (*affine*) *parabolic Kazhdan-Lusztig polynomials*, which we denote by

$$(17) \quad n_{\mu\lambda}(t) \in \mathbb{Z}[v, v^{-1}], \quad \lambda, \mu \in X^+ - \rho.$$

For brevity, we do not recall the definition of these polynomials (which can be computed algorithmically) here, but refer to Section 3 of [84] where the relevant polynomial is denoted $n_{y,x}$ for $x, y \in W_l$ (which translates by (16) to our notation). The main point for us is the following theorem due to Soergel.

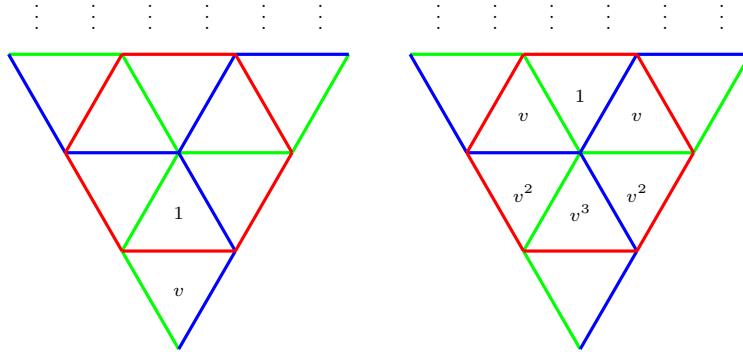
Theorem 3.24. (Multiplicity formula) Suppose $\mathbb{K} = \mathbb{C}$ and q is a complex, primitive l -th root of unity. For each pair $\lambda, \mu \in X^+$ with λ being an l -regular \mathbf{U}_q -weight (that is, $T_q(\lambda)$ belongs to a regular block of \mathcal{T}) we have

$$(T_q(\lambda) : \Delta_q(\mu)) = (T_q(\lambda) : \nabla_q(\mu)) = n_{\mu\lambda}(1).$$

In particular, if $\lambda, \mu \in X^+$ are not linked, then $n_{\mu\lambda}(v) = 0$.

Proof. This follows from Theorem 5.12 in [82] (see also Conjecture 7.1 in [84]). □

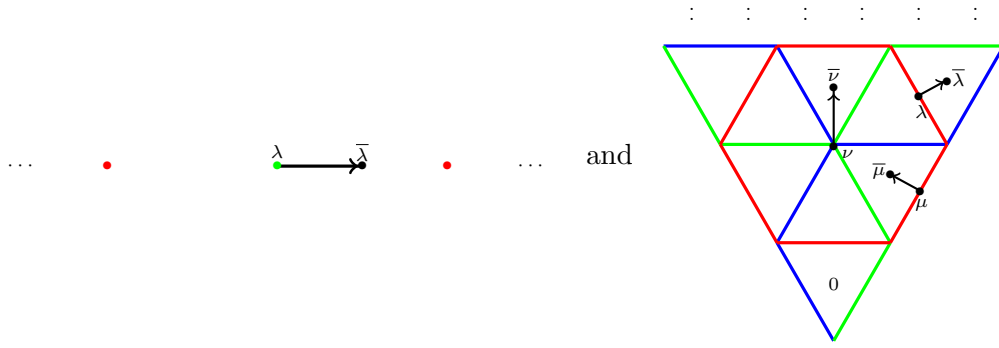
Example 3.25. Back to Example 3.23: for $\nu = \omega_1 + \omega_2 = (1, 1)$ we have $n_{\nu\nu}(v) = 1$ and $n_{\nu(0,0)}(v) = v$ as shown in the left picture below. Similarly, for $\xi = 3\omega_1 + 3\omega_2 = (3, 3)$ the only non-zero parabolic Kazhdan-Lusztig polynomials are $n_{\xi\xi}(v) = 1$, $n_{\xi(1,4)}(v) = v = n_{\xi(4,1)}(v)$, $n_{\xi(0,3)}(v) = v^2 = n_{\xi(3,0)}(v)$ and $n_{\xi\nu}(v) = v^3$ as illustrated on the right below.



Therefore, we have, by Theorem 3.24, that $(T_q(\nu) : \Delta_q(\mu)) = 1$ if $\mu \in \{(0, 0), (1, 1)\}$ and $(T_q(\nu) : \Delta_q(\mu)) = 0$ if $\mu \notin \{(0, 0), (1, 1)\}$. We encourage the reader to work out $(T_q(\xi) : \Delta_q(\mu))$ by using the above patterns and Example 3.23. For all patterns in rank 2 see [87]. ■

Remark 3.26. Assume $\lambda \in X^+$ to be not l -regular. Set $W_\lambda = \{w \in W_l \mid w.\lambda = \lambda\}$. Then we can find a unique l -regular $\bar{\lambda} \in W_l.0$ such that λ is in the closure of the alcove containing

$\bar{\lambda}$ and $\bar{\lambda}$ is maximal in $W_\lambda \cdot \bar{\lambda}$. Examples for $\mathfrak{g} = \mathfrak{sl}_2$ and $\mathfrak{g} = \mathfrak{sl}_3$ are



Assume now that $\lambda, \mu \in X^+$ are linked. Then

$$(18) \quad (T_q(\lambda) : \Delta_q(\mu)) = (T_q(\bar{\lambda}) : \Delta_q(\bar{\mu}))$$

and the latter is given by Theorem 3.24. Compare to Proposition 5.2 and Corollary 5.4 in [1]. The proof of (18) follows by using arguments involving translation functors. For brevity, we discuss the details only in [7].

Since the polynomials from (17) can be computed inductively, we can use Theorem 3.24 and Remark 3.26 in the case $\mathbb{K} = \mathbb{C}$ to *explicitly* calculate the decomposition of a tensor product of U_q -tilting modules $T = T_q(\lambda_1) \otimes \cdots \otimes T_q(\lambda_d)$ into its indecomposable summands:

- Calculate, by using Theorem 3.24 and Remark 3.26, $(T_q(\lambda_i) : \Delta_q(\mu))$ for $i = 1, \dots, d$.
- This gives the multiplicities of T , by the Corollary 3.16 and the fact that $\chi(\Delta_q(\lambda))$ are as in the classical case.
- Use (13) to recursively compute the decomposition of T (starting with any maximal U_q -weight of T).

Example 3.27. Let us come back to our favourite case, that is, $\mathfrak{g} = \mathfrak{sl}_2$, $\mathbb{K} = \mathbb{C}$ and $l = 3$. In the regular cases we have $T_q(k) \cong \Delta_q(k)$ for $k = 0, 1$ and the parabolic Kazhdan-Lusztig polynomials are

$$n_{jk}(v) = \begin{cases} 1, & \text{if } j = k, \\ v, & \text{if } j < k \text{ are separated by precisely one wall,} \\ 0, & \text{else,} \end{cases}$$

for $k > 1$. By (18) we obtain $T_q(k) \cong \Delta_q(k)$ for $k \in \mathbb{N}$ singular, hence, the two singular blocks \mathcal{T}_{-1} and \mathcal{T}_2 are semi-simple.

In Example 3.14 we have already calculated $T_q(1) \otimes T_q(1) \cong T_q(2) \oplus T_q(0)$. Let us go one step further now: $T_q(1) \otimes T_q(1) \otimes T_q(1)$ has only $(T_q(1)^{\otimes 3} : \Delta_q(3)) = 1$ and $(T_q(1)^{\otimes 3} : \Delta_q(1)) = 2$ as non-zero multiplicities. This means that $T_q(3)$ is a summand of $T_q(1) \otimes T_q(1) \otimes T_q(1)$. Since $T_q(3)$ has only $(T_q(3) : \Delta_q(3)) = 1$ and $(T_q(3) : \Delta_q(1)) = 1$ as non-zero multiplicities (by the calculation of the periodic Kazhdan-Lusztig polynomials), we have

$$(19) \quad T_q(1) \otimes T_q(1) \otimes T_q(1) \cong T_q(3) \oplus T_q(1) \in \mathcal{T}_1.$$

Moreover, we have (as we, as usual, encourage the reader to work out)

$$T_q(1) \otimes T_q(1) \otimes T_q(1) \otimes T_q(1) \cong (T_q(4) \oplus T_q(0)) \oplus (T_q(2) \oplus T_q(2) \oplus T_q(2)) \in \mathcal{T}_0 \oplus \mathcal{T}_2.$$

To illustrate how this decomposition depends on l : assume now that $l > 3$. Then, which can be verified similarly as in Example 3.19, the \mathbf{U}_q -tilting module $T_q(3)$ is in the fundamental alcove \mathcal{A}_0 . Thus, by Theorem 3.21, $T_q(3)$ is simple as in the “classical” case. Said otherwise, we have $T_q(3) \cong \Delta_q(3)$. Hence, in the same spirit as above, we obtain (as in the generic case)

$$(20) \quad T_q(1) \otimes T_q(1) \otimes T_q(1) \cong T_q(3) \oplus (T_q(1) \oplus T_q(1)) \in \mathcal{T}_3 \oplus \mathcal{T}_1$$

in contrast to the decomposition in (19). ■

4. CELLULAR STRUCTURES ON ENDOMORPHISM ALGEBRAS

In this section we give our construction of a cellular basis for endomorphism rings $\text{End}_{\mathbf{U}_q}(T)$ of \mathbf{U}_q -tilting modules T and prove our main result, that is, Theorem 4.11.

The main step in order to prove Theorem 4.11 is Theorem 4.1. The proof of the latter is rather involving and we had to split it into a collection of intermediate steps which we collected in Subsection 4.2.

4.1. Hom-spaces between \mathbf{U}_q -modules with a Δ_q - and a ∇_q -filtration. As before, we consider the category $\mathbf{U}_q\text{-Mod}$. Moreover, we fix two \mathbf{U}_q -modules M, N , where we assume that M has a Δ_q -filtration and N has a ∇_q -filtration. Then, by Corollary 3.4, we have

$$(21) \quad \dim(\text{Hom}_{\mathbf{U}_q}(M, N)) = \sum_{\lambda \in X^+} (M : \Delta_q(\lambda))(N : \nabla_q(\lambda)).$$

We point out that the sum in (21) is actually finite since $(M : \Delta_q(\lambda)) \neq 0$ for only a finite number of \mathbf{U}_q -weights $\lambda \in X^+$ because the Δ_q -filtration of M from Definition 3.2 is of finite length (dually, $(N : \nabla_q(\lambda)) \neq 0$ for only finitely many $\lambda \in X^+$).

Given $\lambda \in X^+$, we define for $(N : \nabla_q(\lambda)) > 0$ respectively for $(M : \Delta_q(\lambda)) > 0$ the two sets

$$\mathcal{I}^\lambda = \{1, \dots, (N : \nabla_q(\lambda))\} \quad \text{and} \quad \mathcal{J}^\lambda = \{1, \dots, (M : \Delta_q(\lambda))\}.$$

By convention, $\mathcal{I}^\lambda = \emptyset$ and $\mathcal{J}^\lambda = \emptyset$ if $(N : \nabla_q(\lambda)) = 0$ respectively if $(M : \Delta_q(\lambda)) = 0$. We note that Theorem 3.24 and Remark 3.26 give a way to calculate $|\mathcal{I}^\lambda|$ and $|\mathcal{J}^\lambda|$ in the case $\mathbb{K} = \mathbb{C}$ and $M = N$.

We can choose a basis of $\text{Hom}_{\mathbf{U}_q}(M, \nabla_q(\lambda))$ as $F^\lambda = \{f_j^\lambda : M \rightarrow \nabla_q(\lambda) \mid j \in \mathcal{J}^\lambda\}$ (by part (a) of Corollary 3.4). By Proposition 3.5 (which has an implication as in (12)) we see that all elements of F^λ factor through the \mathbf{U}_q -tilting module $T_q(\lambda)$, i.e. we have a commuting diagram

$$\begin{array}{ccc} & & T_q(\lambda) \\ & \nearrow \exists \bar{f}_j^\lambda & \downarrow \pi^\lambda \\ M & \xrightarrow{f_j^\lambda} & \nabla_q(\lambda). \end{array}$$

We call \bar{f}_j^λ a *lift* of f_j^λ . Note that a lift \bar{f}_j^λ is not unique. Dually, we can choose a basis of $\text{Hom}_{\mathbf{U}_q}(\Delta_q(\lambda), N)$ as $G^\lambda = \{g_i^\lambda : \Delta_q(\lambda) \rightarrow N \mid i \in \mathcal{I}^\lambda\}$, which extend to give $\bar{g}_i^\lambda : T_q(\lambda) \rightarrow N$ such that $\bar{g}_i^\lambda \circ \iota^\lambda = g_i^\lambda$ for all $i \in \mathcal{I}^\lambda$.

We can use this set-up to define a basis for $\mathrm{Hom}_{\mathbf{U}_q}(M, N)$ which, when $M = N$, turns out to be a cellular basis, see Theorem 4.11. To this end, set

$$c_{ij}^\lambda = \bar{g}_i^\lambda \circ \bar{f}_j^\lambda \in \mathrm{Hom}_{\mathbf{U}_q}(M, N)$$

for each $\lambda \in X^+$ and all $i \in \mathcal{I}^\lambda, j \in \mathcal{J}^\lambda$.

Theorem 4.1. (Basis theorem) Assume that M has a Δ_q - and N has a ∇_q -filtration. For any choice of F^λ and G^λ as above and any choice of lifts of the f_j^λ 's and the g_i^λ 's (for all $\lambda \in X^+$), the set

$$GF = \{c_{ij}^\lambda \mid \lambda \in X^+, i \in \mathcal{I}^\lambda, j \in \mathcal{J}^\lambda\}$$

is a basis of $\mathrm{Hom}_{\mathbf{U}_q}(M, N)$.

Proof. This follows from Proposition 4.4 combined with Lemmata 4.7 and 4.8 from below. \square

The basis GF for $\mathrm{Hom}_{\mathbf{U}_q}(M, N)$ can be illustrated in a commuting diagram as

$$\begin{array}{ccccc} & & \Delta_q(\lambda) & & \\ & & \downarrow \iota^\lambda & \searrow g_i^\lambda & \\ M & \xrightarrow{\bar{f}_j^\lambda} & T_q(\lambda) & \xrightarrow{\bar{g}_i^\lambda} & N \\ & \searrow f_j^\lambda & \downarrow \pi^\lambda & & \\ & & \nabla_q(\lambda) & & \end{array}$$

Since \mathbf{U}_q -tilting modules have both a Δ_q - and a ∇_q -filtration, we get as an immediate consequence a key result for our purposes.

Corollary 4.2. Let $T \in \mathcal{T}$. Then GF is, for any choices involved, a basis of $\mathrm{End}_{\mathbf{U}_q}(T)$. \square

Example 4.3. We return to our running example $\mathfrak{g} = \mathfrak{sl}_2$, $\mathbb{K} = \mathbb{C}$ and $l = 3$. By Theorem 3.24 and Example 3.27 we have

$$(T_q(3) : \Delta_q(i)) = \delta_{i3} + \delta_{i1} \quad \text{and} \quad (T_q(1) : \Delta_q(i)) = \delta_{i1}.$$

Thus, we get by (21) that $\dim(\mathrm{End}_{\mathbf{U}_q}(T_q(3))) = 2$. By Theorem 4.1 we obtain an explicit basis of $\mathrm{End}_{\mathbf{U}_q}(T_q(3))$ that can be illustrated via (recall that $T_q(1) \cong \Delta_q(1) \cong L_q(1) \cong \nabla_q(1)$)

$$(22) \quad \begin{array}{ccc} \Delta_q(3) & \begin{array}{c} \downarrow \iota^3 \\ T_q(3) \end{array} & \begin{array}{c} \searrow \iota^3 \\ T_q(3) \end{array} \\ T_q(3) \xrightarrow{\mathrm{id}} T_q(3) \xrightarrow{\mathrm{id}} T_q(3) & \text{and} & T_q(3) \xrightarrow{\pi_{\Delta_q}^1} T_q(1) \xrightarrow{\iota_{\nabla_q}^1} T_q(3) \\ \begin{array}{c} \searrow \pi^3 \\ \nabla_q(3) \end{array} & & \begin{array}{c} \downarrow \pi^1 \\ \nabla_q(1) \end{array} \end{array}$$

For the diagram on the right: we consider $T_q(1)$ simultaneously as $\Delta_q(1)$ and $\nabla_q(1)$. That is, the left \mathbf{U}_q -homomorphism $\pi_{\Delta_q}^1$ is the projection to $\Delta_q(1)$ and the right \mathbf{U}_q -homomorphism $\iota_{\nabla_q}^1$ is the inclusion of $\nabla_q(1)$. Thus, the right basis element takes the head to the socle, that is, we have $\mathrm{End}_{\mathbf{U}_q}(T_q(3)) \cong \mathbb{C}[X]/(X^2)$. Compare to Corollary 2.29 in [8]. \blacksquare

4.2. Proof of the basis theorem. We first show that, given lifts \bar{f}_j^λ , there is a consistent choice of lifts \bar{g}_i^λ such that GF is a basis of $\text{Hom}_{\mathbf{U}_q}(M, N)$.

Proposition 4.4. (Basis theorem - dependent version) Assume that M has a Δ_q - and N has a ∇_q -filtration. For any choice of F^λ and any choice of lifts of the f_j^λ 's (for all $\lambda \in X^+$) there exist a choice of a basis G^λ and a choice of lifts of the g_i^λ 's such that

$$GF = \{c_{ij}^\lambda \mid \lambda \in X^+, i \in \mathcal{I}^\lambda, j \in \mathcal{J}^\lambda\}$$

is a basis of $\text{Hom}_{\mathbf{U}_q}(M, N)$.

Proof. We will construct GF inductively. For this purpose, let

$$0 = N_0 \subset N_1 \subset \cdots \subset N_{k-1} \subset N_k = N$$

be a ∇_q -filtration of N , i.e. $N_{k'+1}/N_{k'} \cong \nabla_q(\lambda_{k'})$ for some $\lambda_{k'} \in X^+$ and all $k' = 0, \dots, k-1$.

Let $k = 1$ and $\lambda_1 = \lambda$. Then $N_1 = \nabla_q(\lambda)$ and $\{c^\lambda: \Delta_q(\lambda) \rightarrow \nabla_q(\lambda)\}$ gives a basis of $\text{Hom}_{\mathbf{U}_q}(\Delta_q(\lambda), \nabla_q(\lambda))$, where c^λ is again the unique \mathbf{U}_q -homomorphism from (10). Set $g_1^\lambda = c^\lambda$ and observe that $\bar{g}_1^\lambda = \pi^\lambda$ satisfies $\bar{g}_1^\lambda \circ \iota^\lambda = g_1^\lambda$. Thus, we have a basis and a corresponding lift. This clearly gives a basis of $\text{Hom}_{\mathbf{U}_q}(M, N_1)$, since, by assumption, we have that F^λ gives a basis of $\text{Hom}_{\mathbf{U}_q}(M, \nabla_q(\lambda))$ and $\pi^\lambda \circ \bar{f}_j^\lambda = f_j^\lambda$.

Hence, it remains to consider the case $k > 1$. Set $\lambda_k = \lambda$ and observe that we have a short exact sequence of the form

$$(23) \quad 0 \longrightarrow N_{k-1} \xrightarrow{\text{inc}} N_k \xrightarrow{\text{pro}} \nabla_q(\lambda) \longrightarrow 0.$$

By the Ext-vanishing theorem 3.1 (and the usual implication as in (12)) this leads to a short exact sequence

$$(24) \quad 0 \longrightarrow \text{Hom}_{\mathbf{U}_q}(M, N_{k-1}) \xrightarrow{\text{inc}_*} \text{Hom}_{\mathbf{U}_q}(M, N_k) \xrightarrow{\text{pro}_*} \text{Hom}_{\mathbf{U}_q}(M, \nabla_q(\lambda)) \longrightarrow 0.$$

By induction, we get from (24) for all $\mu \in X^+$ a basis of $\text{Hom}_{\mathbf{U}_q}(\Delta_q(\mu), N_{k-1})$ consisting of g_i^μ 's with lifts \bar{g}_i^μ such that

$$\{c_{ij}^\mu = \bar{g}_i^\mu \circ \bar{f}_j^\mu \mid \mu \in X^+, i \in \mathcal{I}_{k-1}^\mu, j \in \mathcal{J}^\mu\}$$

is a basis of $\text{Hom}_{\mathbf{U}_q}(M, N_{k-1})$ (here we use $\mathcal{I}_{k-1}^\mu = \{1, \dots, (N_{k-1} : \nabla_q(\mu))\}$). We define $g_i^\mu(N_k) = \text{inc} \circ g_i^\mu$ and $\bar{g}_i^\mu(N_k) = \text{inc} \circ \bar{g}_i^\mu$ for each $\mu \in X^+$ and each $i \in \mathcal{I}_{k-1}^\mu$.

We now have to consider two cases, namely $\lambda \neq \mu$ and $\lambda = \mu$. In the first case we see that $\text{Hom}_{\mathbf{U}_q}(\Delta_q(\mu), \nabla_q(\lambda)) = 0$, so that, by using (23) and the usual implication from (12),

$$\text{Hom}_{\mathbf{U}_q}(\Delta_q(\mu), N_{k-1}) \cong \text{Hom}_{\mathbf{U}_q}(\Delta_q(\mu), N_k).$$

Thus, our ‘‘included’’ basis from above gives a basis of $\text{Hom}_{\mathbf{U}_q}(\Delta_q(\mu), N_k)$ and also gives the corresponding lifts. On the other hand, if $\lambda = \mu$, then $(N_k : \nabla_q(\lambda)) = (N_{k-1} : \nabla_q(\lambda)) + 1$. By Theorem 3.1 (and the corresponding implication as in (12)), we can choose $g^\lambda: \Delta_q(\lambda) \rightarrow N_k$ such that $\text{pro} \circ g^\lambda = c^\lambda$. Then any choice of a lift \bar{g}^λ of g^λ will satisfy $\text{pro} \circ \bar{g}^\lambda = \pi^\lambda$.

Adjoining g^λ to the ‘‘included’’ basis from above gives a basis of $\text{Hom}_{\mathbf{U}_q}(\Delta_q(\lambda), N_k)$ which satisfies the lifting property. Note that we know from the case $k = 1$ that

$$\{\text{pro} \circ \bar{g}^\lambda \circ \bar{f}_j^\lambda = \pi^\lambda \circ \bar{f}_j^\lambda \mid j \in \mathcal{J}^\lambda\}$$

is a basis of $\mathrm{Hom}_{\mathbf{U}_q}(M, \nabla_q(\lambda))$. Combining everything: we have that

$$\{c_{ij}^\lambda = \bar{g}_i^\lambda(N_k) \circ \bar{f}_j^\lambda \mid \lambda \in X^+, i \in \mathcal{I}^\lambda, j \in \mathcal{J}^\lambda\}$$

is a basis of $\mathrm{Hom}_{\mathbf{U}_q}(M, N_k)$ (by enumerating $\bar{g}_{(N:\nabla_q(\lambda))}^\lambda(N_k) = \bar{g}^\lambda$ in the $\lambda = \mu$ case). \square

The corresponding statement with the roles of f 's and g 's swapped holds as well. We leave the details to the reader.

We assume in the following that we have fixed some choices as in Proposition 4.4.

Let $\lambda \in X^+$. Given $\varphi \in \mathrm{Hom}_{\mathbf{U}_q}(M, N)$, we denote by $\varphi_\lambda \in \mathrm{Hom}_{\mathbf{U}_q^0}(M_\lambda, N_\lambda)$ the induced \mathbf{U}_q^0 -homomorphism (that is, \mathbb{K} -linear maps) between the λ -weight spaces M_λ and N_λ . In addition, we denote by $\mathrm{Hom}_{\mathbb{K}}(M_\lambda, N_\lambda)$ the \mathbb{K} -linear maps between these λ -weight spaces.

Lemma 4.5. For any $\lambda \in X^+$ the induced set

$$\{(c_{ij}^\lambda)_\lambda \mid c_{ij}^\lambda \in GF\}$$

is a linearly independent subset of $\mathrm{Hom}_{\mathbb{K}}(M_\lambda, N_\lambda)$.

Proof. We proceed as in the proof of Proposition 4.4.

If $N = \nabla_q(\lambda)$ (this was $k = 1$ above), then $c_{1j}^\lambda = \pi^\lambda \circ \bar{f}_j^\lambda = f_j^\lambda$ and the c_{1j}^λ 's form a basis of $\mathrm{Hom}_{\mathbf{U}_q}(M, \nabla_q(\lambda))$. By the q -version of Frobenius reciprocity from (4) we have

$$\mathrm{Hom}_{\mathbf{U}_q}(M, \nabla_q(\lambda)) \cong \mathrm{Hom}_{\mathbf{U}_q^- \mathbf{U}_q^0}(M, \mathbb{K}_\lambda) \subset \mathrm{Hom}_{\mathbf{U}_q^0}(M, \mathbb{K}_\lambda) = \mathrm{Hom}_{\mathbb{K}}(M_\lambda, \mathbb{K}).$$

Hence, because $N_\lambda = \mathbb{K}$ in this case, we have our induction start.

Assume now $k > 1$. The construction of $\{c_{ij}^\mu(N_k)\}_{\mu, i, j}$ in the proof of Proposition 4.4 shows that this set consists of two separate parts: one being the ‘‘included bases’’ coming from a basis for $\mathrm{Hom}_{\mathbf{U}_q}(M, N_{k-1})$ and the second part (which only occurs when $\lambda = \mu$) mapping to a basis of $\mathrm{Hom}_{\mathbf{U}_q}(M, \nabla_q(\lambda))$ (the case $k = 1$).

By (24) there is a short exact sequence

$$0 \longrightarrow \mathrm{Hom}_{\mathbb{K}}(M_\lambda, (N_{k-1})_\lambda) \xrightarrow{\mathrm{inc}_*} \mathrm{Hom}_{\mathbb{K}}(M_\lambda, (N_k)_\lambda) \xrightarrow{\mathrm{pro}_*} \mathrm{Hom}_{\mathbb{K}}(M_\lambda, \mathbb{K}) \longrightarrow 0.$$

Thus, we can proceed as in the proof of Proposition 4.4. \square

We need another piece of notation: we define for each $\lambda \in X^+$

$$\mathrm{Hom}_{\mathbf{U}_q}(M, N)^{\leq \lambda} = \{\varphi \in \mathrm{Hom}_{\mathbf{U}_q}(M, N) \mid \varphi_\mu = 0 \text{ unless } \mu \leq \lambda\}.$$

In words: a \mathbf{U}_q -homomorphism $\varphi \in \mathrm{Hom}_{\mathbf{U}_q}(M, N)$ belongs to $\mathrm{Hom}_{\mathbf{U}_q}(M, N)^{\leq \lambda}$ iff φ vanishes on all \mathbf{U}_q -weight spaces M_μ with $\mu \not\leq \lambda$. In addition to the notation above, we use the evident notation $\mathrm{Hom}_{\mathbf{U}_q}(M, N)^{< \lambda}$. We arrive at the following.

Lemma 4.6. For any fixed $\lambda \in X^+$ the sets

$$\{c_{ij}^\mu \mid c_{ij}^\mu \in GF, \mu \leq \lambda\} \quad \text{and} \quad \{c_{ij}^\mu \mid c_{ij}^\mu \in GF, \mu < \lambda\}$$

are bases of $\mathrm{Hom}_{\mathbf{U}_q}(M, N)^{\leq \lambda}$ and $\mathrm{Hom}_{\mathbf{U}_q}(M, N)^{< \lambda}$ respectively.

Proof. As c_{ij}^μ factors through $T_q(\mu)$ and $T_q(\mu)_\nu = 0$ unless $\nu \leq \mu$ (by Proposition 3.11), we see that $(c_{ij}^\mu)_\nu = 0$ unless $\nu \leq \mu$. Moreover, by Lemma 4.5, each $(c_{ij}^\mu)_\mu$ is non-zero. Thus, $c_{ij}^\mu \in \text{Hom}_{\mathbf{U}_q}(M, N)^{\leq \lambda}$ iff $\mu \leq \lambda$. Now choose any $\varphi \in \text{Hom}_{\mathbf{U}_q}(M, N)^{\leq \lambda}$. By Proposition 4.4 we may write

$$(25) \quad \varphi = \sum_{\mu, i, j} a_{ij}^\mu c_{ij}^\mu, \quad a_{ij}^\mu \in \mathbb{K}.$$

Choose $\mu \in X^+$ maximal with the property that there exist $i \in \mathcal{I}^\lambda, j \in \mathcal{J}^\lambda$ such that $a_{ij}^\mu \neq 0$.

We claim that $a_{ij}^\nu (c_{ij}^\nu)_\mu = 0$ whenever $\nu \neq \mu$. This is true because, as observed above, $(c_{ij}^\nu)_\mu = 0$ unless $\mu \leq \nu$, and for $\mu < \nu$ we have $a_{ij}^\nu = 0$ by the maximality of μ . We conclude $\varphi_\mu = \sum_{i, j} a_{ij}^\mu (c_{ij}^\mu)_\mu$ and thus, $\varphi_\mu \neq 0$ by Lemma 4.5. Hence, $\mu \leq \lambda$, which gives by (25) that $\varphi \in \text{span}_{\mathbb{K}}\{c_{ij}^\mu \mid c_{ij}^\mu \in GF, \mu \leq \lambda\}$ as desired. This shows that $\{c_{ij}^\mu \mid c_{ij}^\mu \in GF, \mu \leq \lambda\}$ spans $\text{Hom}_{\mathbf{U}_q}(M, N)^{\leq \lambda}$. Since it is clearly a linear independent set, it is a basis.

The second statement follows analogously and we leave the details to the reader. \square

We need the following two lemmata to prove that the choices in Proposition 4.4 do not matter. As before we assume that we have, as in Proposition 4.4, constructed $\{g_i^\lambda, i \in \mathcal{I}^\lambda\}$ and the corresponding lifts \bar{g}_i^λ for all $\lambda \in X^+$.

Lemma 4.7. Suppose that we have other \mathbf{U}_q -homomorphisms $\tilde{g}_i^\lambda: T_q(\lambda) \rightarrow N$ such that $\tilde{g}_i^\lambda \circ \iota^\lambda = g_i^\lambda$. Then the following set is also a basis of $\text{Hom}_{\mathbf{U}_q}(M, N)$:

$$\{\tilde{c}_{ij}^\lambda = \tilde{g}_i^\lambda \circ \bar{f}_j^\lambda \mid \lambda \in X^+, i \in \mathcal{I}^\lambda, j \in \mathcal{J}^\lambda\}.$$

Proof. As $(\bar{g}_i^\lambda - \tilde{g}_i^\lambda) \circ \iota^\lambda = 0$, we see that $\bar{g}_i^\lambda - \tilde{g}_i^\lambda \in \text{Hom}_{\mathbf{U}_q}(T_q(\lambda), N)^{< \lambda}$. Hence, we have $c_{ij}^\lambda - \tilde{c}_{ij}^\lambda \in \text{Hom}_{\mathbf{U}_q}(M, N)^{< \lambda}$. Thus, by Lemma 4.6, there is a unitriangular change-of-basis matrix between $\{c_{ij}^\lambda\}_{\lambda, i, j}$ and $\{\tilde{c}_{ij}^\lambda\}_{\lambda, i, j}$. \square

Now assume that we have chosen another basis $\{h_i^\lambda \mid i \in \mathcal{I}^\lambda\}$ of the spaces $\text{Hom}_{\mathbf{U}_q}(\Delta_q(\lambda), N)$ for each $\lambda \in X^+$ and the corresponding lifts \bar{h}_i^λ as well.

Lemma 4.8. The following set is also a basis of $\text{Hom}_{\mathbf{U}_q}(M, N)$:

$$\{d_{ij}^\lambda = \bar{h}_i^\lambda \circ \bar{f}_j^\lambda \mid \lambda \in X^+, i \in \mathcal{I}^\lambda, j \in \mathcal{J}^\lambda\}.$$

Proof. Write $g_i^\lambda = \sum_{k=1}^{(N: \nabla_q(\lambda))} b_{ik}^\lambda h_k^\lambda$ with $b_{ik}^\lambda \in \mathbb{K}$ and set $\tilde{g}_i^\lambda = \sum_{k=1}^{(N: \nabla_q(\lambda))} b_{ik}^\lambda \bar{h}_k^\lambda$. Then the \tilde{g}_i^λ 's are lifts of the g_i^λ 's. Hence, by Lemma 4.7, the elements $\tilde{g}_i^\lambda \circ \bar{f}_j^\lambda$ form a basis of $\text{Hom}_{\mathbf{U}_q}(M, N)$. Thus, this proves the lemma, since, by construction, $\{d_{ij}^\lambda\}_{\lambda, i, j}$ is related to this basis by the invertible change-of-basis matrix $(b_{ik}^\lambda)_{i, k=1; \lambda \in X^+}^{(N: \nabla_q(\lambda))}$. \square

4.3. Cellular structures on endomorphism algebras of \mathbf{U}_q -tilting modules. This subsection finally contains the statement and proof of our main theorem: given any \mathbf{U}_q -tilting module $T \in \mathcal{T}$, then $\text{End}_{\mathbf{U}_q}(T)$ is a cellular algebra. Before we start, we recall the definition of a cellular algebra due to Graham and Lehrer [34]. For simplicity, we keep working over a field \mathbb{K} of arbitrary characteristic instead of, as they do, over a commutative ring with identity.

Definition 4.9. (Cellular algebras) Suppose A is a finite dimensional \mathbb{K} -algebra. A *cell datum* is an ordered quadruple $(\mathcal{P}, \mathcal{I}, \mathcal{C}, \mathfrak{i})$, where (\mathcal{P}, \leq) is a finite poset, \mathcal{I}^λ is a finite set for all $\lambda \in \mathcal{P}$, \mathfrak{i} is a \mathbb{K} -linear anti-involution of A and \mathcal{C} is an injection

$$\mathcal{C}: \coprod_{\lambda \in \mathcal{P}} \mathcal{I}^\lambda \times \mathcal{I}^\lambda \rightarrow A, (i, j) \mapsto c_{ij}^\lambda.$$

The whole data should be such that the c_{ij}^λ 's form a basis of A with $\mathfrak{i}(c_{ij}^\lambda) = c_{ji}^\lambda$ for all $\lambda \in \mathcal{P}$ and all $i, j \in \mathcal{I}^\lambda$. Moreover, for all $a \in A$ and all $\lambda \in \mathcal{P}$ we have

$$(26) \quad ac_{ij}^\lambda = \sum_{k \in \mathcal{I}^\lambda} r_{ik}(a)c_{kj}^\lambda \pmod{A^{<\lambda}} \quad \text{for all } i, j \in \mathcal{I}^\lambda.$$

Here $A^{<\lambda}$ is the subspace of A spanned by the set $\{c_{ij}^\mu \mid \mu < \lambda \text{ and } i, j \in \mathcal{I}(\mu)\}$ and the scalars $r_{ik}(a) \in \mathbb{K}$ are supposed to be independent of j .

An algebra A with such a quadruple is called a *cellular algebra* and the c_{ij}^λ are called a *cellular basis* of A (with respect to the \mathbb{K} -linear anti-involution \mathfrak{i}).

Example 4.10. Let $M_n(\mathbb{K})$ denote the algebra of all $n \times n$ -matrices with entries in \mathbb{K} . Moreover, let $B = \{\mathbf{E}_{ij} \mid 1 \leq i, j \leq n\}$ be the usual basis given by matrices \mathbf{E}_{ij} with only one non-zero entry 1 in row i and column j . The algebra $M_n(\mathbb{K})$ comes equipped with a \mathbb{K} -linear anti-involution $\mathfrak{i}: M_n(\mathbb{K}) \rightarrow M_n(\mathbb{K}), \mathbf{M} \mapsto \mathbf{M}^T$ given by transposing the matrices.

Since $\mathbf{E}_{ij}\mathbf{E}_{kl} = \delta_{j,k}\mathbf{E}_{il}$ and $\mathbf{E}_{ij}^T = \mathbf{E}_{ji}$, we see that $M_n(\mathbb{K})$ is a cellular algebra and B is a cellular basis of $M_n(\mathbb{K})$ with respect to \mathfrak{i} . In this case, \mathcal{P} consist of just one element λ and has a trivial ordering whereas $\mathcal{I}^\lambda = \{1, \dots, n\}$.

By using Artin-Wedderburn's theorem, we see that the same holds for any semi-simple algebra. Thus, without any further work to be done, we see by Proposition 2.7 that $\text{End}_{\mathbf{U}_v}(T)$ is a cellular algebra for any \mathbf{U}_v -module T . Similarly for $\text{End}_{\mathbf{U}_q}(T)$ if $q = \pm 1$ and $\text{char}(\mathbb{K}) = 0$ or $q \in \mathbb{K} - \{0, \pm 1\}$ is not a root of unity by Proposition 2.9. However, the cellular basis depends on the explicit realization as matrix algebras. \blacksquare

Let us fix any \mathbf{U}_q -tilting module $T \in \mathcal{T}$ in the following. We will construct now cellular bases of $\text{End}_{\mathbf{U}_q}(T)$ in the semi-simple as well as in the non semi-simple case.

To this end, we need to specify the cell datum. Set

$$(\mathcal{P}, \leq) = (\{\lambda \in X^+ \mid (T : \nabla_q(\lambda)) = (T : \Delta_q(\lambda)) \neq 0\}, \leq),$$

where \leq is the usual partial ordering on X^+ that we recalled at the beginning of Subsection 2.1. Note that \mathcal{P} is finite since T is finite dimensional. Moreover, motivated by Theorem 4.1, set $\mathcal{I}^\lambda = \{1, \dots, (T : \nabla_q(\lambda))\} = \{1, \dots, (T : \Delta_q(\lambda))\} = \mathcal{J}^\lambda$ for each $\lambda \in \mathcal{P}$.

By Corollary 3.13 we see that

$$\mathfrak{i}: \text{End}_{\mathbf{U}_q}(T) \rightarrow \text{End}_{\mathbf{U}_q}(T), \phi \mapsto \mathcal{D}(\phi)$$

is a \mathbb{K} -linear anti-involution. Choose any basis G^λ of $\text{Hom}_{\mathbf{U}_q}(\Delta_q(\lambda), T)$ as above and any lifts \bar{g}_i^λ . Then $\mathfrak{i}(G^\lambda)$ is a basis of $\text{Hom}_{\mathbf{U}_q}(T, \nabla_q(\lambda))$ and $\mathfrak{i}(\bar{g}_i^\lambda)$ is a lift of $\mathfrak{i}(g_i^\lambda)$.

By Corollary 4.2 we see that

$$\{c_{ij}^\lambda = \bar{g}_i^\lambda \circ \mathfrak{i}(\bar{g}_j^\lambda) = \bar{g}_i^\lambda \circ \bar{f}_j^\lambda \mid \lambda \in \mathcal{P}, i, j \in \mathcal{I}^\lambda\}$$

is a basis of $\text{End}_{\mathbf{U}_q}(T)$. Finally let $\mathcal{C}: \mathcal{I}^\lambda \times \mathcal{I}^\lambda \rightarrow \text{End}_{\mathbf{U}_q}(T)$ be given by $(i, j) \mapsto c_{ij}^\lambda$.

Now we are ready to state and prove our main theorem.

Theorem 4.11. (A cellular basis for $\text{End}_{\mathbf{U}_q}(T)$) The quadruple $(\mathcal{P}, \mathcal{I}, \mathcal{C}, \mathfrak{i})$ defined above is a cell datum for $\text{End}_{\mathbf{U}_q}(T)$.

Proof. As mentioned above, the sets \mathcal{P} and \mathcal{I}^λ are finite for all $\lambda \in \mathcal{P}$. Moreover, \mathfrak{i} is a \mathbb{K} -linear anti-involution of $\text{End}_{\mathbf{U}_q}(T)$ and the c_{ij}^λ 's form a basis of $\text{End}_{\mathbf{U}_q}(T)$ by Corollary 4.2. Because the functor $\mathcal{D}(\cdot)$ is contravariant, we see that

$$\mathfrak{i}(c_{ij}^\lambda) = \mathfrak{i}(\bar{g}_i^\lambda \circ \mathfrak{i}(\bar{g}_j^\lambda)) = \bar{g}_j^\lambda \circ \mathfrak{i}(\bar{g}_i^\lambda) = c_{ji}^\lambda.$$

Thus, only the condition (26) remains to be proven. For this purpose, let $\varphi \in \text{End}_{\mathbf{U}_q}(T)$. Since $\varphi \circ \bar{g}_i^\lambda \circ \iota^\lambda = \varphi \circ g_i^\lambda \in \text{Hom}_{\mathbf{U}_q}(\Delta_q(\lambda), T)$, we have coefficients $r_{ik}^\lambda(\varphi) \in \mathbb{K}$ such that

$$(27) \quad \varphi \circ g_i^\lambda = \sum_{k \in \mathcal{I}^\lambda} r_{ik}^\lambda(\varphi) g_k^\lambda,$$

because we know that the g_i^λ 's form a basis of $\text{Hom}_{\mathbf{U}_q}(\Delta_q(\lambda), T)$. But this implies then that $\varphi \circ \bar{g}_i^\lambda - \sum_{k \in \mathcal{I}^\lambda} r_{ik}^\lambda(\varphi) \bar{g}_k^\lambda \in \text{Hom}_{\mathbf{U}_q}(T_q(\lambda), T)^{<\lambda}$, so that

$$\varphi \circ \bar{g}_i^\lambda \circ \bar{f}_j^\lambda - \sum_{k \in \mathcal{I}^\lambda} r_{ik}^\lambda(\varphi) \bar{g}_k^\lambda \circ \bar{f}_j^\lambda \in \text{Hom}_{\mathbf{U}_q}(T, T)^{<\lambda} = \text{End}_{\mathbf{U}_q}(T)^{<\lambda},$$

which proves (26). The theorem follows. \square

Example 4.12. Coming back to Example 4.3: in the case of $\text{End}_{\mathbf{U}_q}(T_q(3))$ we have the poset $\mathcal{P} = \{1, 3\}$ (with $1 < 3$) and $\mathcal{I}^3 = \mathcal{I}^1 = \{1\}$ giving two basis elements c_{11}^3 and c_{11}^1 . Here c_{11}^3 is the identity \mathbf{U}_q -homomorphism from the left side of (22) and c_{11}^1 is the \mathbf{U}_q -homomorphism from the right side of (22) (that takes the head to the socle). We have

$$c_{11}^3 c_{11}^3 = c_{11}^3, \quad c_{11}^1 c_{11}^3 = c_{11}^1, \quad c_{11}^3 c_{11}^1 = c_{11}^1, \quad c_{11}^1 c_{11}^1 = 0.$$

This is a cellular structure on $\mathbb{C}[X]/(X^2)$. \blacksquare

5. THE CELLULAR STRUCTURE AND $\text{End}_{\mathbf{U}_q}(T)$ -Mod

We keep the notation from Section 4.

Cellular algebras have a nice representation theory akin to the representation theory of the Iwahori-Hecke algebras $\mathcal{H}_d(q)$ with their Kazhdan-Lusztig combinatorics. The goal of this section is to present this theory for $\text{End}_{\mathbf{U}_q}(T)$ which is cellular by Theorem 4.11. We denote its module category, as usual, by $\text{End}_{\mathbf{U}_q}(T)$ -Mod.

5.1. Cell modules for $\text{End}_{\mathbf{U}_q}(T)$ and their contravariant pairings. We study now the representation theory induced from the cellular structure.

Definition 5.1. (Cell modules) Let $\lambda \in \mathcal{P}$. The *cell module* associated to λ is the left $\text{End}_{\mathbf{U}_q}(T)$ -module given by

$$C(\lambda) = \text{Hom}_{\mathbf{U}_q}(\Delta_q(\lambda), T).$$

In addition, the right $\text{End}_{\mathbf{U}_q}(T)$ -module

$$C(\lambda)^* = \text{Hom}_{\mathbf{U}_q}(T, \nabla_q(\lambda))$$

is called the *dual cell module* associated to λ .

The link to the definition of cell modules from Definition 2.1 in [34] is given via our choice of basis $\{g_i^\lambda\}_{i \in \mathcal{I}^\lambda}$. In this basis the action of $\text{End}_{\mathbf{U}_q}(T)$ on $C(\lambda)$ is given by

$$(28) \quad \varphi \circ g_i^\lambda = \sum_{k \in \mathcal{I}^\lambda} r_{ik}^\lambda(\varphi) g_k^\lambda, \quad \varphi \in \text{End}_{\mathbf{U}_q}(T),$$

see (27). Here the coefficients are the same as those appearing when we consider the left action of $\text{End}_{\mathbf{U}_q}(T)$ on itself in terms of the cellular basis $\{c_{ij}^\lambda\}_{i,j \in \mathcal{I}^\lambda}^{\lambda \in \mathcal{P}}$, that is,

$$(29) \quad \varphi \circ c_{ij}^\lambda = \sum_{k \in \mathcal{I}^\lambda} r_{ik}^\lambda(\varphi) c_{kj}^\lambda \pmod{\text{End}_{\mathbf{U}_q}(T)^{<\lambda}}, \quad \varphi \in \text{End}_{\mathbf{U}_q}(T).$$

In a completely similar fashion: the dual cell module $C(\lambda)^*$ has a basis consisting of $\{f_j^\lambda\}_{j \in \mathcal{I}^\lambda}$ with $f_j^\lambda = i(g_j^\lambda)$. In this basis the right action of $\text{End}_{\mathbf{U}_q}(T)$ is given via

$$(30) \quad f_j^\lambda \circ \varphi = \sum_{k \in \mathcal{I}^\lambda} r_{kj}^\lambda(i(\varphi)) f_k^\lambda, \quad \varphi \in \text{End}_{\mathbf{U}_q}(T).$$

The reader familiar with the representation theory of the symmetric group can think of cell modules as analoga of the Specht modules. Roughly: Specht modules are simple over fields \mathbb{K} with $\text{char}(K) = 0$, have a simple head in general and can be used to classify simple modules. The goal of this section is to develop such results for our large class of cellular algebras.

In order to get started, recall that there is a unique \mathbf{U}_q -homomorphism $c^\lambda: \Delta_q(\lambda) \rightarrow \nabla_q(\lambda)$ for all $\lambda \in X^+$, see (10). Moreover, we also have a contravariant, duality functor $\mathcal{D}(\cdot)$, see (6), that determines the \mathbb{K} -linear anti-involution i from the cell datum.

We can use this unique \mathbf{U}_q -homomorphism and the duality functor to define the *cellular pairing* in the spirit of Graham and Lehrer's Definition 2.3 in [34].

Definition 5.2. (Cellular pairing) Let $\lambda \in \mathcal{P}$. Then we denote by ϑ^λ the \mathbb{K} -bilinear form $\vartheta^\lambda: C(\lambda) \otimes C(\lambda) \rightarrow \mathbb{K}$ determined by the property

$$i(h) \circ g = \vartheta^\lambda(g, h) c^\lambda, \quad g, h \in C(\lambda) = \text{Hom}_{\mathbf{U}_q}(\Delta_q(\lambda), T).$$

We call ϑ^λ the *cellular pairing* associated to $\lambda \in \mathcal{P}$.

Lemma 5.3. The cellular pairing ϑ^λ is well-defined, symmetric and contravariant.

Proof. That ϑ^λ is well-defined follows directly from the uniqueness of c^λ .

Applying i to the defining equation of ϑ^λ gives

$$\vartheta^\lambda(g, h) i(c^\lambda) = i(\vartheta^\lambda(g, h) c^\lambda) = i(i(h) \circ g) = i(g) \circ h = \vartheta^\lambda(h, g) c^\lambda,$$

and thus, $\vartheta^\lambda(g, h) = \vartheta^\lambda(h, g)$ because $c^\lambda = i(c^\lambda)$. Similarly, contravariance of $\mathcal{D}(\cdot)$ gives

$$\vartheta^\lambda(\varphi \circ g, h) = \vartheta^\lambda(g, i(\varphi) \circ h), \quad \varphi \in \text{End}_{\mathbf{U}_q}(T), \quad g, h \in C(\lambda),$$

which shows contravariance of the cellular pairing. \square

The following proposition is a neat way to check if some indecomposable \mathbf{U}_q -tilting module $T_q(\lambda)$ is a summand of T . To this end, let $\lambda \in \mathcal{P}$.

Proposition 5.4. (Summand criterion) $T_q(\lambda)$ is a summand of T iff $\vartheta^\lambda \neq 0$.

Proof. Assume $T \cong T_q(\lambda) \oplus \text{rest}$. Then we denote by $\bar{g}: T_q(\lambda) \rightarrow T$ and by $\bar{f}: T \rightarrow T_q(\lambda)$ the corresponding inclusion and projection respectively. As usual, set $g = \bar{g} \circ \iota^\lambda$ and $f = \pi^\lambda \circ \bar{f}$. Then we have

$$f \circ g: \Delta_q(\lambda) \hookrightarrow T_q(\lambda) \hookrightarrow T \twoheadrightarrow T_q(\lambda) \twoheadrightarrow \nabla_q(\lambda) = c^\lambda \quad (\text{head to socle}),$$

giving $\vartheta^\lambda(g, i(f)) = 1$. This shows that $\vartheta^\lambda \neq 0$.

Conversely, assume that there exist $g, h \in C(\lambda)$ with $\vartheta^\lambda(g, h) \neq 0$. Then the commuting “bow tie diagram”, i.e.

$$\begin{array}{ccccc} \Delta_q(\lambda) & & & & \\ \downarrow \iota^\lambda & \searrow g & & & \\ T_q(\lambda) & \xrightarrow{\bar{g}} & T & \xrightarrow{\overline{i(h)}} & T_q(\lambda) \\ & & \searrow i(h) & & \downarrow \pi^\lambda \\ & & & & \nabla_q(\lambda) \end{array} ,$$

shows that $\overline{i(h)} \circ \bar{g}$ is non-zero on the λ -weight space of $T_q(\lambda)$, because $i(h) \circ g = \vartheta^\lambda(g, h)c^\lambda$. Thus, $\overline{i(h)} \circ \bar{g}$ must be an isomorphism (because $T_q(\lambda)$ is indecomposable and has therefore only invertible or nilpotent elements in $\text{End}_{\mathbf{U}_q}(T_q(\lambda))$) showing that $T \cong T_q(\lambda) \oplus \text{rest}$. \square

In view of Proposition 5.4: it makes sense to study the set

$$(31) \quad \mathcal{P}_0 = \{\lambda \in \mathcal{P} \mid \vartheta^\lambda \neq 0\} \subset \mathcal{P}.$$

Hence, if $\lambda \in \mathcal{P}_0$, then we have $T \cong T_q(\lambda) \oplus \text{rest}$ for some \mathbf{U}_q -tilting module called rest (this is a \mathbf{U}_q -tilting module due to the Krull-Schmidt property of \mathcal{T}).

5.2. The structure of $\text{End}_{\mathbf{U}_q}(T)$ and its cell modules. Recall that, for any $\lambda \in \mathcal{P}$, we have that $\text{End}_{\mathbf{U}_q}(T)^{\leq \lambda}$ and $\text{End}_{\mathbf{U}_q}(T)^{< \lambda}$ are two-sided ideals in $\text{End}_{\mathbf{U}_q}(T)$ (this follows from (26) and its right handed version obtained by applying i) as in any cellular algebra. In our case we can also see this as follows: if $\varphi \in \text{End}_{\mathbf{U}_q}(T)^{\leq \lambda}$, then $\varphi_\mu = 0$ unless $\mu \leq \lambda$. Hence, for any $\varphi, \psi \in \text{End}_{\mathbf{U}_q}(T)$ we have

$$(\varphi \circ \psi)_\mu = \varphi_\mu \circ \psi_\mu = 0 = \psi_\mu \circ \varphi_\mu = (\psi \circ \varphi)_\mu \quad \text{unless } \mu \leq \lambda.$$

As a consequence, $\text{End}_{\mathbf{U}_q}(T)^\lambda = \text{End}_{\mathbf{U}_q}(T)^{\leq \lambda} / \text{End}_{\mathbf{U}_q}(T)^{< \lambda}$ is an $\text{End}_{\mathbf{U}_q}(T)$ -bimodule.

Recall that, for any $g \in C(\lambda)$ and any $f \in C(\lambda)^*$, we denote by $\bar{g}: T_q(\lambda) \rightarrow T$ and $\bar{f}: T \rightarrow T_q(\lambda)$ their lifts which satisfy $\bar{g} \circ \iota^\lambda = g$ and $\pi^\lambda \circ \bar{f} = f$ respectively.

Lemma 5.5. Let $\lambda \in \mathcal{P}$. Then the pairing map

$$\langle \cdot, \cdot \rangle^\lambda: C(\lambda) \otimes C(\lambda)^* \rightarrow \text{End}_{\mathbf{U}_q}(T)^\lambda, \quad \langle g, f \rangle^\lambda = \bar{g} \circ \bar{f} + \text{End}_{\mathbf{U}_q}(T)^{< \lambda}, \quad g \in C(\lambda), f \in C(\lambda)^*$$

is an isomorphism of $\text{End}_{\mathbf{U}_q}(T)$ -bimodules.

Proof. First we note that $\bar{g} \circ \bar{f} + \text{End}_{\mathbf{U}_q}(T)^{< \lambda}$ does not depend on the choices for the lifts, because the change-of-basis matrix from Lemma 4.7 is unitriangular (and works the “other way around” as well). This makes the pairing well-defined.

Note that the pairing $\langle \cdot, \cdot \rangle^\lambda$ takes, by birth, the basis $(g_i^\lambda \otimes f_j^\lambda)_{i,j \in \mathcal{I}^\lambda}$ of $C(\lambda) \otimes C(\lambda)^*$ to the basis $\{c_{ij}^\lambda + \text{End}_{\mathbf{U}_q}(T)^{<\lambda}\}_{i,j \in \mathcal{I}^\lambda}$ of $\text{End}_{\mathbf{U}_q}(T)^\lambda$ (where the latter is a basis by Lemma 4.6).

So we only need to check for any $\varphi, \psi \in \text{End}_{\mathbf{U}_q}(T)$ that

$$\langle \varphi \circ g_i^\lambda, f_j^\lambda \circ \psi \rangle^\lambda = \varphi \circ c_{ij}^\lambda \circ \psi \pmod{\text{End}_{\mathbf{U}_q}(T)^{<\lambda}}.$$

But this is a direct consequence of (28), (29) and (30). \square

Lemma 5.6. We have the following.

- (a) There is an isomorphism of \mathbb{K} -vector spaces $\text{End}_{\mathbf{U}_q}(T) \cong \bigoplus_{\lambda \in \mathcal{P}} \text{End}_{\mathbf{U}_q}(T)^\lambda$.
- (b) If $\varphi \in \text{End}_{\mathbf{U}_q}(T)^{\leq \lambda}$, then we have $r_{ik}^\mu(\varphi) = 0$ for all $\mu \not\leq \lambda, i, k \in \mathcal{I}(\mu)$. Equivalently, $\text{End}_{\mathbf{U}_q}(T)^{\leq \lambda} C(\mu) = 0$ unless $\mu \leq \lambda$.

Proof. This follows straightforwardly from Lemma 5.5. Details are left to the reader. \square

In the following we assume that $\lambda \in \mathcal{P}_0$ as in (31). Define m_λ via

$$(32) \quad T \cong T_q(\lambda)^{\oplus m_\lambda} \oplus T',$$

where T' is a \mathbf{U}_q -tilting module containing no summands isomorphic to $T_q(\lambda)$.

Choose now a basis of $C(\lambda) = \text{Hom}_{\mathbf{U}_q}(\Delta_q(\lambda), T)$ as follows. We let g_i^λ for $i = 1, \dots, m_\lambda$ be the inclusion of $T_q(\lambda)$ into the i -th summand of $T_q(\lambda)^{\oplus m_\lambda}$ and set $g_i^\lambda = \bar{g}_i^\lambda \circ \iota^\lambda$.

We then extend $\{g_1^\lambda, \dots, g_{m_\lambda}^\lambda\}$ to a basis of the cell module $C(\lambda)$ by adding an arbitrary basis of $\text{Hom}_{\mathbf{U}_q}(\Delta_q(\lambda), T')$. Thus, in our usual notation, we have $c_{ij}^\lambda = \bar{g}_i^\lambda \circ \bar{f}_j^\lambda$ with $\bar{f}_j^\lambda = i(\bar{g}_j^\lambda)$.

Note in particular that \bar{f}_j^λ for $j = 1, \dots, m_\lambda$ is the projection of T onto the j -th summand in $T_q(\lambda)^{\oplus m_\lambda}$. Hence, the elements c_{ii}^λ for $i = 1, \dots, m_\lambda$ are idempotents corresponding to the i -th summand in $T_q(\lambda)^{\oplus m_\lambda}$.

Lemma 5.7. In the above notation:

- (a) We have $c_{i1}^\lambda \circ g_1^\lambda = g_i^\lambda$ for all $i \in \mathcal{I}^\lambda$.
- (b) We have $c_{ij}^\lambda \circ g_1^\lambda = 0$ for all $i, j \in \mathcal{I}^\lambda$ with $j \neq 1$.

Proof. (a): We have $\bar{f}_1^\lambda \circ g_1^\lambda = \bar{f}_1^\lambda \circ \bar{g}_1^\lambda \circ \iota^\lambda = \iota^\lambda$. This implies $c_{i1}^\lambda \circ \bar{g}_1^\lambda = \bar{g}_i^\lambda \circ \iota^\lambda = g_i^\lambda$.

(b): If $j \neq 1$, then $\bar{f}_j^\lambda \circ g_1^\lambda = 0$ because, by construction, \bar{f}_j^λ is zero on $T_q(\lambda)$. Thus, $c_{ij}^\lambda \circ \bar{g}_1^\lambda = 0$ for all $i, j \in \mathcal{I}^\lambda$ with $j \neq 1$. \square

Proposition 5.8. (Homomorphism criterion) Let $\lambda \in \mathcal{P}_0$ and fix $M \in \text{End}_{\mathbf{U}_q}(T)\text{-Mod}$. Then there is an isomorphism of \mathbb{K} -vector spaces

$$\text{Hom}_{\text{End}_{\mathbf{U}_q}(T)}(C(\lambda), M) \cong \{m \in M \mid \text{End}_{\mathbf{U}_q}(T)^{<\lambda} m = 0 \text{ and } c_{11}^\lambda m = m\}.$$

Proof. Let $\psi \in \text{Hom}_{\text{End}_{\mathbf{U}_q}(T)}(C(\lambda), M)$. Then $\psi(g_1^\lambda)$ belongs to the right hand side, because, by (b) of Lemma 5.6, we have $\text{End}_{\mathbf{U}_q}(T)^{<\lambda} C(\lambda) = 0$ and, by (a) of Lemma 5.7, we have $c_{11}^\lambda \circ g_1^\lambda = g_1^\lambda$. Conversely, if $m \in M$ belongs to the right hand side from above, then we may define $\psi \in \text{Hom}_{\text{End}_{\mathbf{U}_q}(T)}(C(\lambda), M)$ by $\psi(g_i^\lambda) = c_{i1}^\lambda m$. Moreover, the fact that this definition gives an $\text{End}_{\mathbf{U}_q}(T)$ -homomorphism follows from (28) and (29) via direct computation.

Clearly these two operations reverse each other. \square

From the above we obtain two corollaries.

Corollary 5.9. Let $\lambda \in \mathcal{P}_0$. Then $C(\lambda)$ has a unique simple head which we denote by $L(\lambda)$.

Proof. Set

$$\text{Rad}(\lambda) = \{g \in C(\lambda) \mid \vartheta^\lambda(g, C(\lambda)) = 0\}.$$

As the cellular pairing ϑ^λ from Definition 5.2 is contravariant by Lemma 5.3, we see that $\text{Rad}(\lambda)$ is an $\text{End}_{\mathbf{U}_q}(T)$ -submodule of $C(\lambda)$. Since $\vartheta^\lambda \neq 0$ for $\lambda \in \mathcal{P}_0$, we have $\text{Rad}(\lambda) \subsetneq C(\lambda)$. We claim that $\text{Rad}(\lambda)$ is the unique, proper, maximal $\text{End}_{\mathbf{U}_q}(T)$ -submodule of $C(\lambda)$.

Let $g \in C(\lambda) - \text{Rad}(\lambda)$. Moreover, choose $h \in C(\lambda)$ with $\vartheta^\lambda(g, h) = 1$. Then $i(h) \circ g = c^\lambda$ so that $\overline{i(h)} \circ g = \iota^\lambda \pmod{\text{End}_{\mathbf{U}_q}(T)^{<\lambda}}$. Therefore, $g' = \overline{g'} \circ \overline{i(h)} \circ g \pmod{\text{End}_{\mathbf{U}_q}(T)^{<\lambda}}$ for all $g' \in C(\lambda)$. This implies $C(\lambda) = \text{End}_{\mathbf{U}_q}(T)^{\leq \lambda} g$. Thus, any proper $\text{End}_{\mathbf{U}_q}(T)$ -submodule of $C(\lambda)$ is contained in $\text{Rad}(\lambda)$ which implies the desired statement. \square

Corollary 5.10. Let $\lambda \in \mathcal{P}_0, \mu \in \mathcal{P}$ and assume that $\text{Hom}_{\text{End}_{\mathbf{U}_q}(T)}(C(\lambda), M) \neq 0$ for some subquotient M of $C(\mu)$. Then we have $\mu \leq \lambda$. In particular, all composition factors $L(\lambda)$ of $C(\mu)$ satisfy $\mu \leq \lambda$.

Proof. By Proposition 5.8 the assumption in the corollary implies the existence of a vector $m \in M$ with $c_{11}^\lambda m = m$. But if $\mu \not\leq \lambda$, then c_{11}^λ vanishes on the \mathbf{U}_q -weight space T_μ and hence, $c_{11}^\lambda g$ kills the highest weight vector in $\Delta_q(\mu)$ for all $g \in C(\mu)$. This makes the existence of such an $m \in M$ impossible unless $\mu \leq \lambda$. \square

5.3. Simple $\text{End}_{\mathbf{U}_q}(T)$ -modules and semi-simplicity of $\text{End}_{\mathbf{U}_q}(T)$. Let $\lambda \in \mathcal{P}_0$. Note that Corollary 5.9 shows that $C(\lambda)$ has a unique simple head $L(\lambda)$. We then arrive at the following classification of all simple modules in $\text{End}_{\mathbf{U}_q}(T)\text{-Mod}$.

Theorem 5.11. (Classification of simple $\text{End}_{\mathbf{U}_q}(T)$ -modules) The set

$$\{L(\lambda) \mid \lambda \in \mathcal{P}_0\}$$

forms a complete set of pairwise non-isomorphic, simple $\text{End}_{\mathbf{U}_q}(T)$ -modules.

Proof. We have to show three statements, namely that the $L(\lambda)$'s are simple, that they are pairwise non-isomorphic and that every simple $\text{End}_{\mathbf{U}_q}(T)$ -module is one of the $L(\lambda)$'s.

Since the first statement follows directly from the definition of $L(\lambda)$ (see Corollary 5.9), we start by showing the second. Thus, assume that $L(\lambda) \cong L(\mu)$ for some $\lambda, \mu \in \mathcal{P}_0$. Then

$$\text{Hom}_{\text{End}_{\mathbf{U}_q}(T)}(C(\lambda), C(\mu)/\text{Rad}(\mu)) \neq 0 \neq \text{Hom}_{\text{End}_{\mathbf{U}_q}(T)}(C(\mu), C(\lambda)/\text{Rad}(\lambda)).$$

By Corollary 5.10, we get $\mu \leq \lambda$ and $\lambda \leq \mu$ from the left and right hand side. Thus, $\lambda = \mu$.

Suppose that $L \in \text{End}_{\mathbf{U}_q}(T)\text{-Mod}$ is simple. Then we can choose $\lambda \in \mathcal{P}$ minimal such that (recall that $\text{End}_{\mathbf{U}_q}(T)^{\leq \lambda}$ is a two-sided ideal)

$$(33) \quad \text{End}_{\mathbf{U}_q}(T)^{<\lambda} L = 0 \quad \text{and} \quad \text{End}_{\mathbf{U}_q}(T)^{\leq \lambda} L = L.$$

We claim that $\lambda \in \mathcal{P}_0$. Indeed, if not, then, by Proposition 5.4, we see that $T_q(\lambda)$ is not a summand of T . Hence, in our usual notation, all $\overline{f}_j^\lambda \circ \overline{g}_i^\lambda$ vanish on the λ -weight space. It follows that $c_{ij}^\lambda c_{i'j'}^\lambda$ also vanish on the λ -weight space for all $i, j, i', j' \in \mathcal{I}^\lambda$. This means that we have $\text{End}_{\mathbf{U}_q}(T)^{\leq \lambda} \text{End}_{\mathbf{U}_q}(T)^{\leq \lambda} \subset \text{End}_{\mathbf{U}_q}(T)^{<\lambda}$ making (33) impossible.

For $\lambda \in \mathcal{P}_0$ we see by Lemma 5.7 that

$$(34) \quad c_{i1}^\lambda c_{1j}^\lambda = c_{ij}^\lambda \pmod{\text{End}_{\mathbf{U}_q}(T)^{<\lambda}}.$$

Hence, by (33), there exist $i, j \in \mathcal{I}^\lambda$ such that $c_{ij}^\lambda L \neq 0$. By (34) we also have $c_{i1}^\lambda L \neq 0 \neq c_{1j}^\lambda L$. This in turn (again by (34)) ensures that $c_{11}^\lambda L \neq 0$. Take then $m \in c_{11}^\lambda L - \{0\}$ and observe that $c_{11}^\lambda m = m$. Hence, by Proposition 5.8, there is a non-zero $\text{End}_{\mathbf{U}_q}(T)$ -homomorphism $C(\lambda) \rightarrow L$. The conclusion follows now from Corollary 5.9. \square

Recall from Subsection 5.2 the notation m_λ (the multiplicity of $T_q(\lambda)$ in T) and the choice of basis for $C(\lambda)$ (in the paragraph before Lemma 5.7). Then we get the following connection between the decomposition of T as in (32) and the simple $\text{End}_{\mathbf{U}_q}(T)$ -modules $L(\lambda)$.

Theorem 5.12. (Dimension formula) If $\lambda \in \mathcal{P}_0$, then $\dim(L(\lambda)) = m_\lambda$.

In particular, if $\mathbb{K} = \mathbb{C}$, then these dimensions can be inductively computed by using parabolic Kazhdan-Lusztig polynomials as in Subsection 3.3. Compare also to Example 3.27.

Proof. We use the notation from Subsection 5.2. Since T' has no summands isomorphic to $T_q(\lambda)$, we see that $\text{Hom}_{\mathbf{U}_q}(\Delta_q(\lambda), T') \subset \text{Rad}(\lambda)$ (see the proof of Corollary 5.9). On the other hand, $g_i^\lambda \notin \text{Rad}(\lambda)$ for $1 \leq i \leq m_\lambda$ because for these i we have $f_i^\lambda \circ g_i^\lambda = c^\lambda$ by construction. Thus, the statement follows. \square

We finish this subsection by giving a semi-simplicity criterion for $\text{End}_{\mathbf{U}_q}(T)$.

Theorem 5.13. (Semi-simplicity criterion for $\text{End}_{\mathbf{U}_q}(T)$) The cellular algebra $\text{End}_{\mathbf{U}_q}(T)$ is semi-simple iff T is a semi-simple \mathbf{U}_q -module.

Proof. Note that the $T_q(\lambda)$'s are simple iff $T_q(\lambda) \cong \Delta_q(\lambda) \cong L_q(\lambda) \cong \nabla_q(\lambda)$. Hence, T is semi-simple as a \mathbf{U}_q -module iff $T = \bigoplus_{\lambda \in \mathcal{P}_0} \Delta_q(\lambda)^{\oplus m_\lambda}$ with m_λ as above.

Thus, we see that, if T decomposes into simple \mathbf{U}_q -modules, then $\text{End}_{\mathbf{U}_q}(T)$ is semi-simple by the Artin-Wedderburn theorem (since $\text{End}_{\mathbf{U}_q}(T)$ will decompose into a direct sum of matrix algebras in this case).

On the other hand, if $\text{End}_{\mathbf{U}_q}(T)$ is semi-simple, then we know, by Corollary 5.9, that the cell $\text{End}_{\mathbf{U}_q}(T)$ -modules $C(\lambda)$ are simple, i.e. $C(\lambda) = L(\lambda)$ for all $\lambda \in \mathcal{P}_0$. Then we have

$$(35) \quad T \cong \bigoplus_{\lambda \in \mathcal{P}_0} T_q(\lambda)^{\oplus m_\lambda}, \quad m_\lambda = \dim(L(\lambda)) = \dim(C(\lambda)) = \dim(\text{Hom}_{\mathbf{U}_q}(\Delta_q(\lambda), T))$$

by Theorem 5.12. Assume now that there exists a summand $T_q(\lambda')$ of T as in (35) with $T_q(\lambda') \not\cong \Delta_q(\lambda')$ and choose $\lambda' \in \mathcal{P}_0$ minimal with this property.

Then there exists a $\mu < \lambda'$ such that $\text{Hom}_{\mathbf{U}_q}(\Delta_q(\mu), T_q(\lambda')) \neq 0$. Choose also μ minimal among those. By our usual construction this then gives in turn a non-zero \mathbf{U}_q -homomorphism $\bar{g} \circ \bar{f}: T_q(\lambda') \rightarrow T_q(\mu) \rightarrow T_q(\lambda')$. By (35), we can extend $\bar{g} \circ \bar{f}$ to an element of $\text{End}_{\mathbf{U}_q}(T)$ by defining it to be zero on all other summands.

Clearly, by construction, $(\bar{g} \circ \bar{f})C(\mu') = 0$ for $\mu' \in \mathcal{P}_0$ with $\mu' \neq \lambda'$ and $\mu' \not\leq \mu$. If $\mu' \leq \mu$, then consider $\varphi \in C(\mu')$. Then $(\bar{g} \circ \bar{f}) \circ \varphi = 0$ unless φ has some non-zero component $\varphi': \Delta_q(\mu') \rightarrow T_q(\lambda')$. This forces $\mu' = \mu$ by minimality of μ . But since $\Delta_q(\mu') \cong T_q(\mu')$, by minimality of λ' , we conclude that $\bar{f} \circ \varphi = 0$ (otherwise $T_q(\mu')$ would be a summand of $T_q(\lambda')$).

Hence, the non-zero element $\bar{g} \circ \bar{f} \in \text{End}_{\mathbf{U}_q}(T)$ kills all $C(\mu')$ for $\mu' \in \mathcal{P}_0$. This contradicts the semi-simplicity of $\text{End}_{\mathbf{U}_q}(T)$: as noted above, $C(\lambda) = L(\lambda)$ for all $\lambda \in \mathcal{P}_0$ which implies $\text{End}_{\mathbf{U}_q}(T) \cong \bigoplus_{\lambda \in \mathcal{P}_0} C(\lambda)^{\oplus k_\lambda}$ for some $k_\lambda \in \mathbb{N}$. \square

6. CELLULAR STRUCTURES: EXAMPLES AND APPLICATIONS

In this section we provide many examples of cellular algebras arising from our main theorem. This includes several renowned examples where the cellularity is known (but usually proved case by case spread over the literature), but also new ones. In the first subsection we give a full treatment of the semi-simple case and we describe how to obtain all the examples from the introduction using our methods. In the second subsection we focus on the Temperley-Lieb algebras $\mathcal{TL}_d(\delta)$ and give a detailed account how to apply our results to these.

6.1. Cellular structures using \mathbf{U}_q -tilting modules: several examples.

6.1.1. *The semi-simple case.* Suppose the category $\mathbf{U}_q\text{-Mod}$ is semi-simple, that is, $q = \pm 1$ and $\text{char}(\mathbb{K}) = 0$ or q is not a root of unity in $\mathbb{K} - \{0, \pm 1\}$, see Proposition 2.9.

In this case $\mathcal{T} = \mathbf{U}_q\text{-Mod}$ and any $T \in \mathcal{T}$ has a decomposition $T \cong \bigoplus_{\lambda \in X^+} \Delta_q(\lambda)^{\oplus m_\lambda}$ with the multiplicities $m_\lambda = (T : \Delta_q(\lambda))$. This induces an Artin-Wedderburn decomposition

$$(36) \quad \text{End}_{\mathbf{U}_q}(T) \cong \bigoplus_{\lambda \in X^+} M_{m_\lambda}(\mathbb{K})$$

into matrix algebras. A natural choice of basis for $\text{Hom}_{\mathbf{U}_q}(\Delta_q(\lambda), T)$ is

$$G^\lambda = \{g_1^\lambda, \dots, g_{m_\lambda}^\lambda \mid g_i^\lambda: \Delta_q(\lambda) \hookrightarrow T \text{ is the inclusion into the } i\text{-th summand}\}.$$

Then our cellular basis consisting of the c_{ij}^λ 's as in Subsection 4.3 is an Artin-Wedderburn basis, that is, a basis of $\text{End}_{\mathbf{U}_q}(T)$ that realizes the decomposition as in (36) in the following sense: the basis element c_{ij}^λ is the matrix \mathbf{E}_{ij}^λ (in the λ -summand on the right hand side in (36)) which has all entries zero except one entry equals 1 in the i -th row and j -th column. Note that, as expected in this case, $\text{End}_{\mathbf{U}_q}(T)$ has, by the Theorems 5.11 and 5.12, one simple $\text{End}_{\mathbf{U}_q}(T)$ -module $L(\lambda)$ of dimension m_λ for all summands $\Delta_q(\lambda)$ of T .

Of course, in this case $\text{End}_{\mathbf{U}_q}(T)$ has many cellular bases, but our basis is chosen to be adapted to the non semi-simple cases as well.

6.1.2. *The symmetric group and the Iwahori-Hecke algebra.* Let us fix $d \in \mathbb{N}$ and let us denote by S_d the symmetric group in d letters and by $\mathcal{H}_d(q)$ its associated Iwahori-Hecke algebra. We note that $\mathbb{K}[S_d] \cong \mathcal{H}_d(1)$. Moreover, let $\mathbf{U}_q = \mathbf{U}_q(\mathfrak{sl}_n)$. The vector representation of \mathbf{U}_q , which we denote by $V = \mathbb{K}^n = \Delta_q(\omega_1)$, is a \mathbf{U}_q -tilting module (since ω_1 is minimal in X^+). Set $T = V^{\otimes d}$. This is, by Proposition 3.10, again a \mathbf{U}_q -tilting module. Quantum Schur-Weyl duality (see for example Theorem 6.3 in [29] for surjectivity and Corollary 3.15 for the fact that $\dim(\text{End}_{\mathbf{U}_q}(T))$ is independent of \mathbb{K} and of $q \in \mathbb{K} - \{0\}$ and thus, the statement about the isomorphism follows as in the usual case) states that

$$(37) \quad \Phi_{q\text{SW}}: \mathcal{H}_d(q) \twoheadrightarrow \text{End}_{\mathbf{U}_q}(T) \quad \text{and} \quad \Phi_{q\text{SW}}: \mathcal{H}_d(q) \xrightarrow{\cong} \text{End}_{\mathbf{U}_q}(T), \text{ if } n \geq d.$$

Thus, our main result implies that $\mathcal{H}_d(q)$ and in particular $\mathbb{K}[S_d]$ are cellular for any $q \in \mathbb{K} - \{0\}$ and any field \mathbb{K} (by taking $n \geq d$).

In this case the cell modules for $\text{End}_{\mathbf{U}_q}(T)$ are usually called Specht modules $S_{\mathbb{K}}^\lambda$ and our Theorem 5.12 gives the following.

- If $q = 1$ and $\text{char}(\mathbb{K}) = 0$, then the dimension $\dim(S_{\mathbb{K}}^\lambda)$ is equal to the multiplicity of the simple \mathbf{U}_1 -module $\Delta_1(\lambda) \cong L_1(\lambda)$ in $V^{\otimes d}$ for all $\lambda \in \mathcal{P}^0$. These numbers are given by known formulas (e.g. the hook length formula).
- If $q = 1$ and $\text{char}(\mathbb{K}) > 0$, then the dimension of the simple head of $S_{\mathbb{K}}^\lambda$, usually denoted $D_{\mathbb{K}}^\lambda$, is the multiplicity with which $T_1(\lambda)$ occurs as a summand in $V^{\otimes d}$ for all $\lambda \in \mathcal{P}_0$. It is a wide open problem to determine these numbers.
- If q is a complex, primitive root of unity, then we can compute the dimension of the simple $\mathcal{H}_d(q)$ -modules by using the algorithm from Subsection 3.3. In particular, this connects with the LLT algorithm from [54].
- If q is a root of unity and \mathbb{K} is arbitrary, then not much is known. Still, our methods apply and we get a way to calculate the dimensions of the simple $\mathcal{H}_d(q)$ -modules, if we can decompose T into its indecomposable summands.

6.1.3. *The Temperley-Lieb algebra and other \mathfrak{sl}_2 -related algebras.* Let $\mathbf{U}_q = \mathbf{U}_q(\mathfrak{sl}_2)$ and let T be as in 6.1.2 with $n = 2$. Note that, for any $d \in \mathbb{N}$, we have by Schur-Weyl duality that $\mathcal{TL}_d(\delta) \cong \text{End}_{\mathbf{U}_q}(T)$, where $\mathcal{TL}_d(\delta)$ is the Temperley-Lieb algebra in d -strands with parameter $\delta = q + q^{-1}$. This works for all \mathbb{K} and all specializations $q \in \mathbb{K} - \{0\}$, see for example Theorem 6.3 in [29] (to be precise: they show a more general statement which implies the isomorphism we need by Corollary 3.4 or (21)). Hence, $\mathcal{TL}_d(\delta)$ is cellular for any d , any $q \in \mathbb{K} - \{0\}$ and any \mathbb{K} . We discuss this case in more details in Subsection 6.2.

Furthermore, if we are in the semi-simple case, then $\Delta_q(i)$ is a \mathbf{U}_q -tilting module for all $i \in \mathbb{N}$ and so is $T = \Delta_q(i_1) \otimes \cdots \otimes \Delta_q(i_d)$. Thus, we obtain that $\text{End}_{\mathbf{U}_q}(T)$ is cellular.

The algebra $\text{End}_{\mathbf{U}_q}(T)$ is known to give a diagrammatic presentation of the full category of \mathbf{U}_q -modules, gives rise to colored Jones-polynomials (see for example [73] and the references therein) and was studied⁴ from a diagrammatical point of view in [73].

If $q \in \mathbb{K}$ is a root of unity and l is the order of q^2 , then, for any $0 \leq i < l$, $\Delta_q(i)$ is a \mathbf{U}_q -tilting module (see Example 3.9) and so is $T = \Delta_q(i)^{\otimes d}$. The endomorphism algebra $\text{End}_{\mathbf{U}_q}(T)$ is cellular. This reproves parts of Theorem 1.1 in [4] using our general approach.

In characteristic 0: another family of \mathbf{U}_q -tilting modules was studied in [8]. That is, for any $d \in \mathbb{N}$, fix any $\lambda_0 \in \{0, \dots, l-2\}$ and consider $T = T_q(\lambda_0) \oplus \cdots \oplus T_q(\lambda_d)$ where λ_k is the unique integer $\lambda_k \in \{kl, \dots, (k+1)l-2\}$ linked to λ_0 , see Example 3.22. We again obtain that $\text{End}_{\mathbf{U}_q}(T)$ is cellular. Note that $\text{End}_{\mathbf{U}_q}(T)$ can be identified with Khovanov-Seidel's quiver algebra A_d , see Proposition 3.9 in [8], introduced in [50] in their study of Floer homology. These algebras are naturally graded making $\text{End}_{\mathbf{U}_q}(T)$ into a graded cellular algebra in the sense of [37] and are special examples arising from the family of generalized Khovanov arc algebras whose cellularity is studied in [17].

6.1.4. *Spider algebras.* Let $\mathbf{U}_q = \mathbf{U}_q(\mathfrak{sl}_n)$. As in Example 3.19 one has for any $q \in \mathbb{K} - \{0\}$ that the fundamental \mathbf{U}_q -representations $\Delta_q(\omega_i)$ are all \mathbf{U}_q -tilting modules (because the ω_i are minimal in X^+). Thus, for any $k_i \in \{1, \dots, n-1\}$, $T = \Delta_q(\omega_{k_1}) \otimes \cdots \otimes \Delta_q(\omega_{k_d})$ is a \mathbf{U}_q -tilting module. Hence, $\text{End}_{\mathbf{U}_q}(T)$ is cellular. These algebras are related to type A_{n-1} spider algebras introduced in [53], are connected to the Reshetikhin-Turaev \mathfrak{sl}_n -link polynomials and give a diagrammatic description of the representation theory of \mathfrak{sl}_n , see [24], providing a link from our work to low dimensional topology and diagrammatic algebra.

⁴As a category instead of an algebra. We abuse language here and also for some of the other algebras below.

More general: in any type we have that $\Delta_q(\lambda)$ are $\mathbf{U}_q(\mathfrak{g})$ -tilting modules for minuscule $\lambda \in X^+$, see Part II, Chapter 2, Section 15 in [43]. Moreover, if q is a root of unity “of order l big enough” (ensuring that the ω_i ’s are in the closure of the fundamental alcove), then the $\Delta_q(\omega_i)$ are $\mathbf{U}_q(\mathfrak{g})$ -tilting modules as well by Theorem 3.21. So in these cases we can generalize the above results to other types (which includes also spiders in other types).

Still more general: we may take (for any type and any $q \in \mathbb{K} - \{0\}$) arbitrary $\lambda_j \in X^+$ (for $j = 1, \dots, d$) and obtain a cellular structure on $\text{End}_{\mathbf{U}_q}(T)$ for $T = T_q(\lambda_1) \otimes \cdots \otimes T_q(\lambda_d)$.

6.1.5. *The Ariki-Koike algebra and related algebras.* Take $\mathfrak{g} = \mathfrak{gl}_{m_1} \oplus \cdots \oplus \mathfrak{gl}_{m_r}$ (which can be easily fit into our context) with $m_1 + \cdots + m_r = m$ and let V be the vector representation of $\mathbf{U}_1(\mathfrak{gl}_m)$ restricted to $\mathbf{U}_1 = \mathbf{U}_1(\mathfrak{g})$. This is again a \mathbf{U}_1 -tilting module and so is $T = V^{\otimes d}$. Then we have a cyclotomic analogon of (37), namely

$$(38) \quad \Phi_{\text{cl}}: \mathbb{C}[\mathbb{Z}/r\mathbb{Z} \wr S_d] \twoheadrightarrow \text{End}_{\mathbf{U}_1}(T) \quad \text{and} \quad \Phi_{\text{cl}}: \mathbb{C}[\mathbb{Z}/r\mathbb{Z} \wr S_d] \xrightarrow{\cong} \text{End}_{\mathbf{U}_1}(T), \text{ if } m \geq d,$$

where $\mathbb{C}[\mathbb{Z}/r\mathbb{Z} \wr S_d]$ is the group algebra of the complex reflection group $\mathbb{Z}/r\mathbb{Z} \wr S_d \cong (\mathbb{Z}/r\mathbb{Z})^d \rtimes S_d$, see Theorem 9 in [66]. Thus, we can apply our main theorem and obtain a cellular basis for these quotients of $\mathbb{C}[\mathbb{Z}/r\mathbb{Z} \wr S_d]$. In particular, if $m \geq d$, then (38) is an isomorphism (see Lemma 11 in [66]) and we obtain that $\mathbb{C}[\mathbb{Z}/r\mathbb{Z} \wr S_d]$ itself is a cellular algebra for all r, d . In the extremal case $m_1 = m - 1$ and $m_2 = 1$, the resulting quotient of (38) is known as Solomon’s algebra introduced in [85] (also called algebra of the inverse semigroup or rook monoid algebra) and we obtain that Solomon’s algebra is cellular. In the extremal case $m_1 = m_2 = 1$, the resulting quotient is a specialization of the blob algebra $\mathcal{BL}_d(1, 2)$ (in the notation used by Ryom-Hansen in [78]). To see this, note that both algebras are quotients of $\mathbb{C}[\mathbb{Z}/r\mathbb{Z} \wr S_d]$. The kernel of the quotient to $\mathcal{BL}_d(1, 2)$ is described explicitly by Ryom-Hansen in (1) of [78] and is by Lemma 11 of [66] contained in the kernel of Φ_{cl} from (38). Since both algebras have the same dimensions, i.e. $\sum_{i=0}^d \binom{d}{i}^2$, they are isomorphic.

Let $\mathbf{U}_q = \mathbf{U}_q(\mathfrak{g})$. We get in the quantized case (for $q \in \mathbb{C} - \{0\}$ being not a root of unity)

$$(39) \quad \Phi_{q\text{cl}}: \mathcal{H}_{d,r}(q) \twoheadrightarrow \text{End}_{\mathbf{U}_q}(T) \quad \text{and} \quad \Phi_{q\text{cl}}: \mathcal{H}_{d,r}(q) \xrightarrow{\cong} \text{End}_{\mathbf{U}_q}(T), \text{ if } m \geq d,$$

where $\mathcal{H}_{d,r}(q)$ is the Ariki-Koike algebra introduced in [10]. A proof of (39) can for example be found in Theorem 4.1 of [79]. Thus, as before, our main theorem applies and we obtain: the Ariki-Koike algebra $\mathcal{H}_{d,r}(q)$ is cellular (by taking $m \geq d$), the quantized rook monoid algebra $\mathcal{R}_d(q)$ from [35] is cellular and the blob algebra $\mathcal{BL}_d(q, m)$ is cellular (which follows as above). Note that the cellularity of $\mathcal{H}_{d,r}(q)$ was obtained in [26], the cellularity of the quantum rook monoid algebras and of the blob algebra can be found in [68] and in [77] respectively.

In fact, (39) is, in the non semi-simple cases, still true, see Theorem 1.10 and Lemma 2.12 in [39] as long as \mathbb{K} satisfies a certain separation condition (which implies that the algebra in question has the right dimension, see [9]). Our main theorem applies and provides a cellular basis for these cases as well.

6.1.6. *The Brauer algebras and related algebras.* Consider $\mathbf{U}_q = \mathbf{U}_q(\mathfrak{g})$ where \mathfrak{g} is either an orthogonal $g = \mathfrak{o}_{2n}$ and $g = \mathfrak{o}_{2n+1}$ or the symplectic $g = \mathfrak{sp}_{2n}$ Lie algebra. Let $V = \Delta_q(\omega_1)$ be the quantized version of the corresponding vector representation. In both cases, V is a \mathbf{U}_q -tilting module (for type B and $q = 1$ this requires $\text{char}(\mathbb{K}) \neq 2$, see page 20 in [41]) and hence, so is $T = V^{\otimes d}$. We can first take $q = 1$ and set $\delta = 2n$ in the case $\mathfrak{g} = \mathfrak{o}_{2n}$, and

$\delta = 2n + 1$ in case $\mathfrak{g} = \mathfrak{o}_{2n+1}$ and $\delta = -2n$ in the case $\mathfrak{g} = \mathfrak{sp}_{2n}$ respectively. Then we have (Theorem 6.38 in [33])

$$(40) \quad \Phi_{\text{Br}}: \mathcal{B}_d(\delta) \twoheadrightarrow \text{End}_{\mathbf{U}_1}(T) \quad \text{and} \quad \Phi_{\text{Br}}: \mathcal{B}_d(\delta) \xrightarrow{\cong} \text{End}_{\mathbf{U}_1}(T), \text{ if } n > d,$$

where $\mathcal{B}_d(\delta)$ is the Brauer algebra in d strands (for $\mathfrak{g} \neq \mathfrak{o}_{2n}$ the isomorphism in (40) already holds for $n = d$). Thus, we get cellularity of $\mathcal{B}_d(\delta)$ and of its quotients under Φ_{Br} .

Similarly, let $\mathbf{U}_q = \mathbf{U}_q(\mathfrak{gl}_n)$, $q \in \mathbb{K} - \{0\}$ be arbitrary and set $T = \Delta_q(\omega_1)^{\otimes r} \otimes \Delta_q(\omega_{n-1})^{\otimes s}$. By Theorem 7.1 and Corollary 7.2 in [25] we have

$$(41) \quad \Phi_{\text{wBr}}: \mathcal{B}_{r,s}^n([n]) \twoheadrightarrow \text{End}_{\mathbf{U}_q}(T) \quad \text{and} \quad \Phi_{\text{wBr}}: \mathcal{B}_{r,s}^n([n]) \xrightarrow{\cong} \text{End}_{\mathbf{U}_q}(T), \text{ if } n \geq r + s,$$

where $\mathcal{B}_{r,s}^n([n])$ the so-called quantized walled Brauer algebra $\mathcal{B}_{r,s}^n([n])$. Since T is a \mathbf{U}_q -tilting module, we get as usual from (41) cellularity of $\mathcal{B}_{r,s}^n([n])$ and of its quotients under Φ_{wBr} .

The walled Brauer algebra $\mathcal{B}_{r,s}^n(\delta)$ over $\mathbb{K} = \mathbb{C}$ for arbitrary parameter $\delta \in \mathbb{Z}$ appears as centralizer $\text{End}_{\mathfrak{gl}(m|n)}(T)$ for $T = V^{\otimes r} \otimes (V^*)^{\otimes s}$ where V is the vector representation of the superalgebra $\mathfrak{gl}(m|n)$ with $\delta = m - n$. That is, we have

$$(42) \quad \Phi_s: \mathcal{B}_{r,s}^n(\delta) \twoheadrightarrow \text{End}_{\mathfrak{gl}(m|n)}(T) \quad \text{and} \quad \Phi_s: \mathcal{B}_{r,s}^n(\delta) \xrightarrow{\cong} \text{End}_{\mathfrak{gl}(m|n)}(T), \text{ if } (m+1)(n+1) \geq r+s,$$

see Theorem 7.8 in [21]. By [30], T is a $\mathfrak{gl}(m|n)$ -tilting module and thus, our main theorem applies and hence, by (42), $\mathcal{B}_{r,s}^n(\delta)$ is cellular. See also [30] for the quantized version.

Quantizing the Brauer case: taking $q \in \mathbb{K} - \{0, \pm 1\}$, \mathfrak{g} , $V = \Delta_q(\omega_1)$ and T as before (without the restriction $\text{char}(\mathbb{K}) \neq 2$ for type B) gives us a cellular structure on $\text{End}_{\mathbf{U}_q}(T)$. The algebra $\text{End}_{\mathbf{U}_q}(T)$ is a quotient of the Birman-Murakami-Wenzl algebra $\mathcal{BMW}_d(\delta)$ (for appropriate parameters), see (9.6) in [55] for the orthogonal (which works for any $q \in \mathbb{C} - \{0, \pm 1\}$) and Theorem 1.5 in [36] for the symplectic case (which works for any $q \in \mathbb{K} - \{0, \pm 1\}$ and infinity \mathbb{K}). Again, taking $n \geq d$ (or $n \geq d - 1$), we recover the cellularity of $\mathcal{BMW}_d(\delta)$.

6.1.7. Infinite dimensional modules - highest weight categories. Observe that our main theorem does not use the specific properties of $\mathbf{U}_q\text{-Mod}$, but works for any $\text{End}_{A\text{-Mod}}(T)$ where T is an A -tilting module for some finite dimensional, quasi-hereditary algebra A over \mathbb{K} or $T \in \mathcal{C}$ for some highest weight category \mathcal{C} in the sense of [22]. For the explicit construction of our basis we however need a notion like “weight spaces” such that Lemma 4.5 makes sense.

The most famous example of such a category is the BGG category $\mathcal{O} = \mathcal{O}(\mathfrak{g})$ attached to a complex semi-simple or reductive Lie algebra \mathfrak{g} with a corresponding Cartan \mathfrak{h} and fixed Borel subalgebra \mathfrak{b} (the reader unfamiliar with it is referred to [40]). We denote by $\Delta(\lambda) \in \mathcal{O}$ the Verma module attached to $\lambda \in \mathfrak{h}^*$. In the same vein, pick a parabolic $\mathfrak{p} \supset \mathfrak{b}$ and denote for any \mathfrak{p} -dominant weight λ the corresponding parabolic Verma module by $\Delta^{\mathfrak{p}}(\lambda)$. It is the unique quotient of the Verma module $\Delta(\lambda)$ which is locally \mathfrak{p} -finite, i.e. contained in the parabolic category $\mathcal{O}^{\mathfrak{p}} = \mathcal{O}^{\mathfrak{p}}(\mathfrak{g}) \subset \mathcal{O}$ (we refer again to [40] for readers unfamiliar with $\mathcal{O}^{\mathfrak{p}}$).

There is a contravariant, character preserving duality functor ${}^{\vee}: \mathcal{O}^{\mathfrak{p}} \rightarrow \mathcal{O}^{\mathfrak{p}}$ which allows us to set $\nabla^{\mathfrak{p}}(\lambda) = \Delta^{\mathfrak{p}}(\lambda)^{\vee}$. Hence, we can play the same game again: the \mathcal{O} -tilting theory works in a very similar fashion as for $\mathbf{U}_q\text{-Mod}$ (see Chapter 11 in [40] and the references therein). In particular, we have indecomposable \mathcal{O} -tilting modules $T(\lambda)$ for any $\lambda \in \mathfrak{h}^*$. Similarly for $\mathcal{O}^{\mathfrak{p}}$ giving an indecomposable $\mathcal{O}^{\mathfrak{p}}$ -tilting module $T(\lambda)$ for any \mathfrak{p} -dominant $\lambda \in \mathfrak{h}^*$.

We shall give a few examples where our approach leads to cellular structures on interesting algebras. For this purpose, let $\mathfrak{p} = \mathfrak{b}$ and $\lambda = 0$. Then $T(0)$ has Verma factors of the form

$\Delta(w.0)$ (for $w \in W$, where W is the Weyl group associated to \mathfrak{g}). Each of these appears with multiplicity 1. Hence, $\dim(\text{End}_{\mathcal{O}}(T(0))) = |W|$ by the analogon of (21). By Soergel's Endomorphismensatz $\text{End}_{\mathcal{O}}(T(0)) \cong S(\mathfrak{h}^*)/S_+^W$. The algebra $S(\mathfrak{h}^*)/S_+^W$ is called coinvariant algebra (for the notation, the conventions and the result see [83] - this is Soergel's famous Endomorphismensatz). Hence, our main theorem implies that $S(\mathfrak{h}^*)/S_+^W$ is cellular.

There is also a quantum version of this result: replace \mathcal{O} by its quantum cousin \mathcal{O}_q from [5] (which is the analogon of \mathcal{O} for $\mathbf{U}_q(\mathfrak{g})$). This works over any field \mathbb{K} with $\text{char}(\mathbb{K}) = 0$ and any $q \in \mathbb{K} - \{0, \pm 1\}$ (which can be deduced from Section 6 in [5]). There is furthermore a characteristic p version of this result: consider the G -tilting module $T(p\rho)$ in the category of finite dimensional G -modules (here G is an almost simple, simply connected algebraic group over \mathbb{K} with $\text{char}(\mathbb{K}) = p$). Its endomorphism algebra is isomorphic to for the corresponding coinvariant algebra over \mathbb{K} , see Proposition 19.8 in [3].

Returning to $\mathbb{K} = \mathbb{C}$: the example of the coinvariant algebra can be generalized. To this end, note that, if T is an $\mathcal{O}^{\mathfrak{p}}$ -tilting module, then so is $T \otimes M$ for any finite dimensional \mathfrak{g} -module M , see Proposition 11.1 and Section 11.8 in [40] (and the references therein). Thus, $\text{End}_{\mathcal{O}^{\mathfrak{p}}}(T \otimes M)$ is cellular by our main theorem.

A special case is: \mathfrak{g} is of classical type, $T = \Delta^{\mathfrak{p}}(\lambda)$ is simple (hence, $\mathcal{O}^{\mathfrak{p}}$ -tilting), V is the vector representation of \mathfrak{g} and $M = V^{\otimes d}$. Let first $\mathfrak{g} = \mathfrak{gl}_n$ with standard Borel \mathfrak{b} and parabolic \mathfrak{p} of block size (n_1, \dots, n_ℓ) . Then one can find a certain \mathfrak{p} -dominant weight λ_{Γ} , called Irving-weight, such that $T = \Delta^{\mathfrak{p}}(\lambda_{\Gamma})$ is $\mathcal{O}^{\mathfrak{p}}$ -tilting. Moreover, $\text{End}_{\mathcal{O}^{\mathfrak{p}}}(T \otimes V^{\otimes d})$ is isomorphic to a sum of blocks of cyclotomic quotients of the degenerate affine Hecke algebra $\mathcal{H}_d/\prod_{i=1}^{\ell}(x_i - n_i)$, see Theorem 5.13 in [16]. In the special case of level $\ell = 2$, these algebras can be explicitly described in terms of generalizations of Khovanov's arc algebra (which Khovanov introduced in [46] to give an algebraic structure underlying Khovanov homology and which categorifies the Temperley-Lieb algebra $\mathcal{TL}_d(\delta)$) and have an interesting representation theory, see [17], [18], [19] and [20]. A consequence of this is: using the results from Theorem 6.9 in [80] and Theorem 1.1 in [81], one can realize the walled Brauer algebra from 6.1.6 for arbitrary parameter $\delta \in \mathbb{Z}$ as endomorphism algebras of some $\mathcal{O}^{\mathfrak{p}}$ -tilting module and hence, using our main theorem, deduce cellularity again.

If \mathfrak{g} is of another classical type, then the role of the (cyclotomic quotients of the) degenerate affine Hecke algebra is played by (cyclotomic quotients of) degenerate BMW algebras or so-called (cyclotomic quotients of) \mathbb{W}_d -algebras (also called Nazarov-Wenzl algebras). These are still poorly understood and technically quite involved, see [11]. In [32] special examples of level $\ell = 2$ quotients were studied and realized as endomorphism algebras of some $\mathcal{O}^{\mathfrak{p}}(\mathfrak{so}_{2n})$ -tilting module $\Delta^{\mathfrak{p}}(\underline{\delta}) \otimes V \in \mathcal{O}^{\mathfrak{p}}(\mathfrak{so}_{2n})$ where V is the vector representation of \mathfrak{so}_{2n} , $\underline{\delta} = \frac{\delta}{2} \sum_{i=1}^n \epsilon_i$ and \mathfrak{p} is a maximal parabolic subalgebra of type A (see Theorem B in [32]). Hence, our theorem implies cellularity of these algebras.

Soergel's theorem is therefore just a shadow of a rich world of interesting endomorphism algebras whose cellularity can be obtained from our approach.

Our methods also apply to (parabolic) category $\mathcal{O}^{\mathfrak{p}}(\hat{\mathfrak{g}})$ attached to an affine Kac-Moody algebra $\hat{\mathfrak{g}}$ over \mathbb{K} and related categories. In particular, one can consider a (level-dependent) quotient $\hat{\mathfrak{g}}_{\kappa}$ of $\mathbf{U}(\hat{\mathfrak{g}})$ and a category, denoted by $\mathbf{O}_{\mathbb{K}, \tau}^{\nu, \kappa}$, attached to it (we refer the reader to Subsections 5.2 and 5.3 in [75] for the details). Then there is a subcategory $\mathbf{A}_{\mathbb{K}, \tau}^{\nu, \kappa} \subset \mathbf{O}_{\mathbb{K}, \tau}^{\nu, \kappa}$ and

a $\mathbf{A}_{\mathbb{K},\tau}^{\nu,\kappa}$ -tilting module $\mathbf{T}_{\mathbb{K},d}$ defined in Subsection 5.5 of [75] such that

$$\Phi_{\text{aff}}: \mathbf{H}_{\mathbb{K},d}^s \rightarrow \text{End}_{\mathbf{A}_{\mathbb{K},\tau}^{\nu,\kappa}}(\mathbf{T}_{\mathbb{K},d}) \quad \text{and} \quad \Phi_{\text{aff}}: \mathbf{H}_{\mathbb{K},d}^s \xrightarrow{\cong} \text{End}_{\mathbf{A}_{\mathbb{K},\tau}^{\nu,\kappa}}(\mathbf{T}_{\mathbb{K},d}), \quad \text{if } \nu_p \geq d, p = 1, \dots, N,$$

see Theorem 5.37 and Proposition 8.15 in [75]. Here $\mathbf{H}_{\mathbb{K},d}^s$ denotes an appropriate cyclotomic quotient of the affine Hecke algebra. Again, our main theorem gives the cellularity of $\mathbf{H}_{\mathbb{K},d}^s$.

6.1.8. Graded cellular structures. A striking property and extra tool which arises in the context of (parabolic) category \mathcal{O} (or \mathcal{O}^p) is that all the endomorphism algebras from 6.1.7 can be equipped with a \mathbb{Z} -grading as in [86] arising from the Koszul grading of category \mathcal{O} (or on \mathcal{O}^p). In particular, we might choose our cellular basis compatible with this grading and obtain a grading on the endomorphism algebras turning them into graded cellular algebras in the sense of Definition 2.1 in [37].

For the cyclotomic quotients this grading is highly non-trivial and in fact is the type A KL-R grading in the spirit of Khovanov and Lauda and independently Rouquier (see [48], [49] and [47] or [74]), which can be seen as a grading on cyclotomic quotients of degenerate affine Hecke algebras, see [15]. See [19] for level $\ell = 2$ and [38] for all levels where the authors construct explicit graded cellular basis. For gradings on (cyclotomic quotients of) \mathbb{W}_d -algebras see Section 5 in [32] and for gradings on Brauer algebras see [31] or [56].

In the same spirit, it should be possible to obtain the higher level analoga of the generalizations of Khovanov's arc algebra, known as \mathfrak{sl}_n -web algebras (see [62] and [61]), from our set-up as well using the connections from cyclotomic KL-R algebras to these algebras in [90]. Although details still need to be worked out, this can be seen as the categorification of the connections to the spider algebras from 6.1.4: the spiders provide the algebraic set-up to study the corresponding Reshetikhin-Turaev \mathfrak{sl}_n -link polynomials, while the \mathfrak{sl}_n -web algebras provide the algebraic set-up to study the corresponding Khovanov-Rozansky \mathfrak{sl}_n -link homologies. This would emphasize the connection between our work and low dimensional topology.

6.2. (Graded) cellular structures and the Temperley-Lieb algebras: a comparison.

Finally we want to present one explicit example, the Temperley-Lieb algebras, which is of particular interest in low dimensional topology and categorification. We present new ways to establish semi-simplicity conditions, and construction and computation of the dimensions of the simple modules the Temperley-Lieb algebras.

Recall that the *Temperley-Lieb algebra* $\mathcal{TL}_d(\delta)$ in d -strands with a fixed, chosen parameter $\delta \in \mathbb{K} - \{0, \pm 1, \pm \sqrt[2]{-1}\}$ ⁵ is the free diagram algebra over \mathbb{K} with basis consisting of all possible non-intersecting tangle diagrams with d bottom and top boundary points modulo boundary preserving isotopy and the local relation for evaluating circles, that is⁶,

$$\bigcirc = \delta = q + q^{-1} \in \mathbb{K}.$$

⁵We exclude $q = \pm \sqrt[2]{-1}$ for technical reasons. But something can still be said in those cases, see Remark 6.5. Moreover, the \mathfrak{sl}_2 case works with any root of unity $q \in \mathbb{K}$ by setting $l = \text{ord}(q^2)$, see Definition 2.3 in [8].

⁶We point out that there are two different conventions about circle evaluations in the literature: evaluating to δ or to $-\delta$. We use the first convention because we want to stay close to the cited literature.

The algebra $\mathcal{TL}_d(\delta)$ is locally generated by

$$1 = \begin{array}{cccccc} 1 & i-1 & i & i+1 & i+2 & d \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & i-1 & i & i+1 & i+2 & d \end{array}, \quad U_i = \begin{array}{cccccc} 1 & i-1 & i & i+1 & i+2 & d \\ \vdots & \vdots & \cup & \cup & \vdots & \vdots \\ 1 & i-1 & i & i+1 & i+2 & d \end{array}$$

for $i = 1, \dots, d-1$ called *identity* 1 and *cap-cup* U_i (which takes place between the strand i and $i+1$). For simplicity, we suppress the boundary labels in the following.

The multiplication $y \circ x$ is giving by stacking diagram y on top of diagram x . For example

Recall from 6.1.3 that, by quantum Schur-Weyl duality, we can use our Theorem 4.11 to obtain a cellular basis of $\mathcal{TL}_d(\delta)$. The aim of this subsection is to compare our cellular basis to the one already identified by Graham and Lehrer, where we immediately point out that we *do not* obtain their cellular basis, but a “smarter” one: our cellular basis depends if the Temperley-Lieb algebra $\mathcal{TL}_d(\delta)$ is semi-simple or not. In the latter case, at least for $\mathbb{K} = \mathbb{C}$, we obtain a graded cellular basis in the sense of Hu and Mathas’ Definition 2.1 in [37].

We first recall Graham and Lehrer’s cell datum $(\mathcal{P}_{\text{TL}}, \mathcal{I}_{\text{TL}}, \mathcal{C}_{\text{TL}}, \text{i}_{\text{TL}})$ for $\mathcal{TL}_d(\delta)$, see Section 6 in [34]. The \mathbb{K} -linear anti-involution i_{TL} is given by “turning pictures upside down”. For example

For the insistent reader: more formally, the \mathbb{K} -linear anti-involution i_{TL} is the unique \mathbb{K} -linear anti-involution which fixes all U_i ’s for $i = 1, \dots, d-1$.

The data \mathcal{P}_{TL} and \mathcal{I}_{TL} are given combinatorially: \mathcal{P}_{TL} is the set $\Lambda^+(2, d)$ of all *Young diagrams with d nodes and at most two rows*. For example, the elements of $\Lambda^+(2, 3)$ are

$$(43) \quad \lambda = \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array}, \quad \mu = \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array},$$

where we point out that we use the English notation for Young diagrams. Now $\mathcal{I}_{\text{TL}}^\lambda$ is the set of all *standard tableaux of shape λ* , denoted by $\text{Std}(\lambda)$, that is, all fillings of λ with non-repeating numbers $1, \dots, d$ such that the entries strictly increase along rows and columns. For example, the elements of $\text{Std}(\mu)$ for μ as in (43) are

$$(44) \quad t_1 = \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & \\ \hline \end{array}, \quad t_2 = \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & \\ \hline \end{array}.$$

The set \mathcal{P}_{TL} is a poset where the order \leq is the so-called *dominance order* on Young diagrams. In the “at most two rows case” this is $\mu \leq \lambda$ iff μ has fewer columns (an example is (43) with the same notation).

The only thing missing is thus the parametrization of the cellular basis. This works as follows: fix $\lambda \in \Lambda^+(2, d)$ and assign to each $t \in \text{Std}(\lambda)$ a “half diagram” x_t via the rule that one “caps off” the strands whose numbers appear in the second row with the biggest possible candidate to the left of the corresponding number (going from left to right in the second row). Note that this is well-defined due to planarity. For example,

$$(45) \quad s = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 6 \\ \hline 4 & 5 & & \\ \hline \end{array} \rightsquigarrow x_s = \left| \begin{array}{c} \text{---} \\ \text{---} \end{array} \right|, \quad t = \begin{array}{|c|c|c|c|} \hline 1 & 3 & 4 & 5 \\ \hline 2 & 6 & & \\ \hline \end{array} \rightsquigarrow x_t = \left| \begin{array}{c} \text{---} \\ \text{---} \end{array} \right|$$

Then one obtains c_{st}^λ by “turning x_s upside down and stacking it on top of x_t ”. For example,

$$c_{st}^\lambda = \text{i}_{\text{TL}}(x_s) \circ x_t = \left| \begin{array}{c} \text{---} \\ \text{---} \end{array} \right| \circ \left| \begin{array}{c} \text{---} \\ \text{---} \end{array} \right| = \left| \begin{array}{c} \text{---} \\ \text{---} \end{array} \right|$$

for $\lambda \in \Lambda^+(2, 6)$ and $s, t \in \text{Std}(\lambda)$ as in (45). The map \mathcal{C}_{TL} sends $(s, t) \in \mathcal{I}_{\text{TL}}^\lambda \times \mathcal{I}_{\text{TL}}^\lambda$ to c_{st}^λ .

Theorem 6.1. (Cellular basis for $\mathcal{TL}_d(\delta)$ - the first) The quadruple $(\mathcal{P}_{\text{TL}}, \mathcal{I}_{\text{TL}}, \mathcal{C}_{\text{TL}}, \text{i}_{\text{TL}})$ is a cell datum for $\mathcal{TL}_d(\delta)$.

Proof. This is Theorem 6.7 in [34]. □

Example 6.2. For $\mathcal{TL}_3(\delta)$ we have five basis elements, namely

$$c_{cc}^\lambda = \left| \begin{array}{c} \text{---} \\ \text{---} \end{array} \right|, \quad c_{t_1 t_1}^\mu = \left| \begin{array}{c} \text{---} \\ \text{---} \end{array} \right|, \quad c_{t_1 t_2}^\mu = \left| \begin{array}{c} \text{---} \\ \text{---} \end{array} \right|, \quad c_{t_2 t_1}^\mu = \left| \begin{array}{c} \text{---} \\ \text{---} \end{array} \right|, \quad c_{t_2 t_2}^\mu = \left| \begin{array}{c} \text{---} \\ \text{---} \end{array} \right|$$

where we use the notation from (43) and (44) (and the “canonical” filling c for λ). ■

Before stating our cellular basis, we provide a criterion which tells precisely whether $\mathcal{TL}_d(\delta)$ is semi-simple or not (already found, using different methods, in [94]).

Proposition 6.3. (Semi-simplicity criterion for the Temperley-Lieb algebra $\mathcal{TL}_d(\delta)$) The Temperley-Lieb algebra $\mathcal{TL}_d(\delta)$ is semi-simple iff $[i] \neq 0$ for all $i = 1, \dots, d$ iff q is not a root of unity with $d < l = \text{ord}(q^2)$.

Proof. We have that $T = V^{\otimes d}$ decomposes into simple $U_q(\mathfrak{sl}_2)$ -modules iff $d < l$ (which is clearly equivalent to the non-vanishing of the quantum numbers).

To see this, assume that $d < l$. Since the maximal $U_q(\mathfrak{sl}_2)$ -weight of $V^{\otimes d}$ is d and since all Weyl $U_q(\mathfrak{sl}_2)$ -modules $\Delta_q(i)$ for $i < l$ are in $\overline{\mathcal{A}}_0$ (compare to Example 3.19), we see that all indecomposable summands of $V^{\otimes d}$ are simple by Theorem 3.21.

Otherwise, if $l \leq d$, then $T_q(d)$ (or $T_q(d-2)$ in the case $d \equiv -1 \pmod{l}$) is a non-simple, indecomposable summand of $V^{\otimes d}$ (note that this arguments fails if $l = 2$, i.e. $\delta = 0$).

We can now use Theorem 5.13 to finish the proof. □

Example 6.4. We have that $[k] \neq 0$ for all $k = 1, 2, 3$ is satisfied iff q is not a fourth or a sixth root of unity. By Proposition 6.3 we see that $\mathcal{TL}_3(\delta)$ is semi-simple as long as q is not one of these values from above. The other way around is only true for q being a sixth root of unity (the conclusion from semi-simplicity to non-vanishing of the quantum numbers above does not work in the case $q = \pm \sqrt[2]{-1}$, see also Remark 6.5). ■

Remark 6.5. In the special case where $\delta = 0 \in \mathbb{C}$, Westbury’s proof from Section 5 of [94] does not work anymore since he divides by δ at one point. In fact, the criterion from Proposition 6.3 is false in the case $\delta = 0 \in \mathbb{C}$.

We can do slightly better: since $\delta = 0 \in \mathbb{C}$ iff $q = \pm\sqrt[2]{-1}$, we can use the periodic patterns from Subsection 3.3 in the case $l = 2$ to see that $T = V^{\otimes d}$ decomposes into a direct sum of simple $\mathbf{U}_q(\mathfrak{sl}_2)$ -modules iff d is odd (or $d = 0$). This implies that $\mathcal{TL}_d(0)$ is semi-simple iff d is odd (or $d = 0$) by Theorem 5.13.

Thus, we can recover the known semi-simplicity statement at $\delta = 0 \in \mathbb{C}$ (see for example Chapter 7 in [63] or above Proposition 4.9 in [71]) from our approach as well.

We can do even better: if $\text{char}(\mathbb{K}) = p > 0$ and $\delta = 0$ (if $p = 2$ this is equivalent to $q = 1$), then we have $\Delta_q(i) \cong L_q(i)$ iff $i = 0$ or $i \in \{2ap^n - 1 \mid n \in \mathbb{N}, 1 \leq a < p\}$. In particular, this means that for $d \geq 2$ we have that either $T_q(d)$ or $T_q(d - 2)$ is a simple \mathbf{U}_q -module iff $d \in \{3, 5, \dots, 2p - 1\}$. Hence, using the same reasoning as in the proof of Proposition 6.3, we see that $T = V^{\otimes d}$ is semi-simple iff $d \in \{0, 1, 3, 5, \dots, 2p - 1\}$. By Theorem 5.13 we see that $\mathcal{TL}_d(0)$ is semi-simple iff $d \in \{0, 1, 3, 5, \dots, 2p - 1\}$.

6.2.1. *Temperley-Lieb algebra: the semi-simple case.* An easy to deduce consequence of Proposition 6.3 is that the Temperley-Lieb algebra $\mathcal{TL}_d(\delta)$ for $q \in \mathbb{K} - \{0, \pm 1\}$ *not* being a root of unity is semi-simple (or $q = \pm 1$ and $\text{char}(\mathbb{K}) = 0$) regardless of d . So assume that $q \in \mathbb{K} - \{0, \pm 1\}$ is not a root of unity (or $q = \pm 1$ and $\text{char}(\mathbb{K}) = 0$). As before we take V to be the vector representation of $\mathbf{U}_q(\mathfrak{sl}_2)$ and $T = V^{\otimes d}$. Since $\mathcal{TL}_d(\delta) \cong \text{End}_{\mathbf{U}_q(\mathfrak{sl}_2)}(T)$, we see that our basis consisting of c_{ij}^λ gives rise to a cellular basis of $\mathcal{TL}_d(\delta)$.

Let us first compare the cell datum of Graham and Lehrer with our cell datum. We have the poset \mathcal{P}_{TL} consisting of all $\lambda \in \Lambda^+(2, d)$ in Graham and Lehrer’s case and the poset \mathcal{P} consisting of all $\lambda \in X^+$ such that $\Delta_q(\lambda)$ is a factor of T in our case.

The two sets are the same: an element $\lambda = (\lambda_1, \lambda_2) \in \mathcal{P}_{\text{TL}}$ corresponds to $\lambda_1 - \lambda_2 \in \mathcal{P}$. This is clearly an injection of sets. Moreover, $\Delta_q(i) \otimes \Delta_q(1) \cong \Delta_q(i + 1) \oplus \Delta_q(i - 1)$ for $i > 0$ shows surjectivity. Two easy examples are

$$\lambda = (\lambda_1, \lambda_2) = (3, 0) = \begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \end{array} \in \mathcal{P}_{\text{TL}} \rightsquigarrow \lambda_1 - \lambda_2 = 3 \in \mathcal{P},$$

and

$$\mu = (\mu_1, \mu_2) = (2, 1) = \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \\ \hline \end{array} \in \mathcal{P}_{\text{TL}} \rightsquigarrow \mu_1 - \mu_2 = 1 \in \mathcal{P},$$

which fits to the decomposition as in (20).

Similarly, an inductive reasoning shows that there is a factor $\Delta_q(i)$ of T for any standard filling for the Young diagram that gives rise to i under the identification from above. Thus, \mathcal{I}_{TL} is also the same as our \mathcal{I} .

As an example, we encourage the reader to compare (43) and (44) with (20).

To see that the \mathbb{K} -linear anti-involution i_{TL} is also our involution i , we note that we build our basis from a “top” part g_i^λ and a “bottom” part f_j^λ and i switches top and bottom similarly as the \mathbb{K} -linear anti-involution i_{TL} .

To summarize: except for the map \mathcal{C}_{TL} , we have the same cell datum as Graham and Lehrer.

In order to state how our cellular basis for $\mathcal{TL}_d(\delta)$ looks like, recall the following definition of the (*generalized*) Jones-Wenzl projectors.

Definition 6.6. (Jones-Wenzl projectors) The d -th *Jones-Wenzl projectors*, which we denote by $JW_d \in \mathcal{TL}_d(\delta)$, is recursively defined via the recursion rule

$$\begin{array}{c} \dots \\ | \\ \boxed{JW_d} \\ | \\ \dots \end{array} = \begin{array}{c} \dots \\ | \\ \boxed{JW_{d-1}} \\ | \\ \dots \end{array} \left| - \frac{[d-1]}{[d]} \begin{array}{c} \dots \\ | \\ \boxed{JW_{d-1}} \\ \text{---} \\ \boxed{JW_{d-1}} \\ | \\ \dots \end{array}$$

where we assume that JW_1 is the identity diagram in one strand.

Note that the projector JW_d can be identified with the projection of $T = V^{\otimes d}$ onto its maximal weight summand. These projectors were introduced by Jones in [44] and then further studied by Wenzl in [93]. In fact, they can be generalized as follows.

Definition 6.7. (Generalized Jones-Wenzl projectors) Given any d -tuple (with $d > 0$) of the form $\vec{\epsilon} = (\epsilon_1, \dots, \epsilon_d) \in \{\pm 1\}^d$ such that $\sum_{j=1}^k \epsilon_j \geq 0$ for all $k = 1, \dots, d$. Set $i = \sum_{j=1}^d \epsilon_j$. We define recursively two certain “half-diagrams” $t_{(\epsilon_1, \dots, \epsilon_d, \pm 1)}$ via

$$t_{(\epsilon_1, \dots, \epsilon_d, +1)} = \begin{array}{c} \dots \\ | \\ \boxed{JW_{i+1}} \\ | \\ \dots \\ | \\ \boxed{t_{\vec{\epsilon}}} \\ | \\ \dots \end{array}, \quad t_{(\epsilon_1, \dots, \epsilon_d, -1)} = \begin{array}{c} \dots \\ | \\ \boxed{JW_{i-1}} \\ | \\ \dots \\ | \\ \boxed{t_{\vec{\epsilon}}} \\ | \\ \dots \end{array}$$

where $t_{(+1)} \in \mathcal{TL}_1(\delta)$ is defined to be the identity element. By convention, $t_{(\epsilon_1, \dots, \epsilon_d, -1)} = 0$ if $i - 1 < 0$. Note that $t_{(\epsilon_1, \dots, \epsilon_d, \pm 1)}$ has always $d + 1$ bottom boundary points, but $i \pm 1$ top boundary points.

Then we assign to any such $\vec{\epsilon}$ a *generalized Jones-Wenzl “projector”* $JW_{\vec{\epsilon}} \in \mathcal{TL}_d(\delta)$ via

$$JW_{\vec{\epsilon}} = i(t_{\vec{\epsilon}}) \circ t_{\vec{\epsilon}},$$

where i is, as above, the \mathbb{K} -linear anti-involution that “turns pictures upside down”.

Example 6.8. We point out again that the $t_{\vec{\epsilon}}$ are “half-diagrams”. For example, we have

$$t_{(+1)} = \left| \begin{array}{c} \dots \\ | \\ \dots \end{array} \right|, \quad t_{(+1,+1)} = \left| \begin{array}{c} \dots \\ | \\ \dots \\ | \\ \dots \end{array} \right| - \frac{1}{[2]} \begin{array}{c} \dots \\ | \\ \dots \\ | \\ \dots \end{array}, \quad t_{(+1,-1)} = \begin{array}{c} \dots \\ | \\ \dots \\ | \\ \dots \end{array}, \quad t_{(+1,-1,+1)} = \begin{array}{c} \dots \\ | \\ \dots \\ | \\ \dots \end{array}$$

where we can read off the top boundary points by summing the ϵ_i 's. ■

Note that Jones-Wenzl projectors are special cases of the construction in Definition 6.7, i.e. $JW_d = JW_{(1, \dots, 1)}$. Moreover, while we think about the Jones-Wenzl projectors as projecting to the maximal weight summand of $T = V^{\otimes d}$, the generalized Jones-Wenzl projectors $JW_{\vec{\epsilon}}$ project to the summands of $T = V^{\otimes d}$ of the form $\Delta_q(i)$ where i is as above $i = \sum_{j=1}^d \epsilon_j$. To be more precise, we have the following.

Proposition 6.9. (Diagrammatic projectors) There exists non-zero scalars $a_{\vec{\epsilon}} \in \mathbb{K}$ such that $JW'_{\vec{\epsilon}} = a_{\vec{\epsilon}} JW_{\vec{\epsilon}}$ are well-defined idempotents forming a complete set of mutually orthogonal, commuting, primitive idempotents in $\mathcal{TL}_d(\delta)$.

Proof. That they are well-defined follows from the fact that no quantum numbers vanish if $q \in \mathbb{K} - \{0, \pm 1\}$ is not a root of unity.

The other statements can be proven as in Proposition 2.19 and Theorem 2.20 in [23]. \square

Example 6.10. Recall from Example 3.27 that we have the following decompositions.

$$(46) \quad V^{\otimes 1} = \Delta_q(1), \quad V^{\otimes 2} \cong \Delta_q(2) \oplus \Delta_q(0), \quad V^{\otimes 3} \cong \Delta_q(3) \oplus \Delta_q(1) \oplus \Delta_q(1).$$

Moreover, there are the following $\vec{\epsilon}$ vectors. We have $\vec{\epsilon}_1 = (+1)$ and

$$\vec{\epsilon}_2 = (+1, +1), \quad \vec{\epsilon}_3 = (+1, -1), \quad \vec{\epsilon}_4 = (+1, +1, +1), \quad \vec{\epsilon}_5 = (+1, +1, -1), \quad \vec{\epsilon}_6 = (+1, -1, +1).$$

We point out that $(+1, -1, -1)$ does not satisfy the sum property from Definition 6.7 and thus, does not count.

By construction, $JW_{\vec{\epsilon}_1}$ is the identity strand in one variable and hence, is the projector on the unique factor in (46). Moreover, we have

$$JW_2 = JW_{\vec{\epsilon}_2} = \left| \left| -\frac{1}{[2]} \begin{array}{c} \cup \\ \cup \end{array} \right. \right., \quad JW_{\vec{\epsilon}_3} = \begin{array}{c} \cup \\ \cup \end{array}$$

where $JW_{\vec{\epsilon}_2}$ and $JW_{\vec{\epsilon}_3}$ are the (up to scalars) projectors onto the $\Delta_q(2)$ and the $\Delta_q(0)$ summand in (46) respectively. Furthermore, we have

$$JW_3 = JW_{\vec{\epsilon}_4} = \left| \left| \left| -\frac{[2]}{[3]} \left(\begin{array}{c} \cup \\ \cup \end{array} \right) \right| + \left| \begin{array}{c} \cup \\ \cup \end{array} \right) + \frac{1}{[3]} \left(\begin{array}{c} \cup \\ \cup \end{array} \right) \right.$$

is the projection to the $\Delta_q(3)$ summand in (46). The other two projectors are (up to scalars)

$$JW_{\vec{\epsilon}_5} = \left| \begin{array}{c} \cup \\ \cup \end{array} - \frac{1}{[2]} \left(\begin{array}{c} \cup \\ \cup \end{array} + \begin{array}{c} \cup \\ \cup \end{array} \right) + \frac{1}{[2]^2} \begin{array}{c} \cup \\ \cup \end{array} \right|, \quad JW_{\vec{\epsilon}_6} = \begin{array}{c} \cup \\ \cup \end{array} \left| \right.$$

as we invite the reader to check. \blacksquare

Proposition 6.11. (Cellular basis for $\mathcal{TL}_d(\delta)$ - the second) The datum given by the quadruple $(\mathcal{P}_{\text{TL}}, \mathcal{I}_{\text{TL}}, \mathcal{C}, \text{id}_{\text{TL}})$ with \mathcal{C} as in Theorem 4.11 for $\mathcal{TL}_d(\delta) \cong \text{End}_{\mathbf{U}_q(\mathfrak{sl}_2)}(T)$ is a cell datum for $\mathcal{TL}_d(\delta)$.

Moreover, $\mathcal{C}_{\text{TL}} \neq \mathcal{C}$ for all $d > 1$ and all choices involved in the definition of $\text{im}(\mathcal{C})$. In particular, there is a choice such that all generalized Jones-Wenzl projectors are part of $\text{im}(\mathcal{C})$.

Proof. That we get a cell datum as stated follows from Theorem 4.11 and the discussion above.

That our cellular basis \mathcal{C} will never be \mathcal{C}_{TL} for $d > 1$ is due to the fact that Graham and Lehrer's cellular basis always contains the identity (which corresponds to the unique standard filling of the Young diagram associated to $\lambda = d$).

In contrast, since

$$(47) \quad T = V^{\otimes d} \cong \Delta_q(d) \oplus \bigoplus_{0 < k \leq \lfloor \frac{d}{2} \rfloor} \Delta_q(d - 2k)^{\oplus m_k}$$

for some multiplicities $m_k \in \mathbb{N}$, we see that the identity diagram is for $d > 1$ never part of our basis: all the $\Delta_q(i)$'s are simple $\mathbf{U}_q(\mathfrak{sl}_2)$ -modules and each c_{ij}^k factors through $\Delta_q(k)$. In particular, the basis element c_{11}^λ for $\lambda = d$ has to be (a scalar multiple of) the d -th Jones-Wenzl projector JW_d from Definition 6.6.

As in 6.1.1 we can choose for \mathcal{C} an Artin-Wedderburn basis of $\text{End}_{U_q(\mathfrak{sl}_2)}(T) \cong \mathcal{TL}_d(\delta)$. By our construction, all basis elements c_{ij}^k are block matrices of the form

$$\begin{pmatrix} \mathbf{M}_d & 0 & \cdots & 0 \\ 0 & \mathbf{M}_{d-2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{M}_\varepsilon \end{pmatrix}$$

with $\varepsilon = 0$ if d is even and $\varepsilon = 1$ if d is odd (where we regard V as decomposed as in (47), the indices should indicate the summands and \mathbf{M}_{d-2k} is of size $m_k \times m_k$).

Clearly, the block matrices of the form \mathbf{E}_{ii}^k for $i = 1, \dots, m_k$ with only non-zero entry in the i -th column and row of \mathbf{M}_k form a set of mutually orthogonal, commuting, primitive idempotents. Hence, by Proposition 6.9, these have to be the generalized Jones-Wenzl projectors JW_ϵ for $k = \sum_{j=1}^k \epsilon_j$. \square

Example 6.12. Let us consider $\mathcal{TL}_3(\delta)$ as in Example 6.2 for any $q \in \mathbb{K} - \{0, \pm 1, \pm \sqrt[2]{-1}\}$ that is not a critical value as in Example 6.4. Then $\mathcal{TL}_3(\delta)$ is semi-simple by Proposition 6.3.

In particular, we have a decomposition of $V^{\otimes 3}$ as in (46). Fix the same order as in (46). Then we can choose five basis elements as

$$c_{cc}^\lambda = \mathbf{E}_{11}, \quad c_{t_1 t_1}^\mu = \mathbf{E}_{22}, \quad c_{t_1 t_2}^\mu = \mathbf{E}_{23}, \quad c_{t_2 t_1}^\mu = \mathbf{E}_{32}, \quad c_{t_2 t_2}^\mu = \mathbf{E}_{33},$$

where we use the notation from (43) and (44) (and the ‘‘canonical’’ filling c for λ) again.

Note that c_{cc}^λ corresponds to the Jones-Wenzl projector $JW_3 = JW_{(+1+1+1)}$, $c_{t_1 t_1}^\mu$ corresponds to $JW_{(+1+1-1)}$ and $c_{t_2 t_2}^\mu$ corresponds to $JW_{(+1-1+1)}$. Compare to Example 6.10. \blacksquare

Note the following (known) corollary (see for example Corollary 5.2 in [71] for $\mathbb{K} = \mathbb{C}$).

Corollary 6.13. We have a complete set of pairwise non-isomorphic, simple $\mathcal{TL}_d(\delta)$ -modules $L(\lambda)$ for $\lambda \in \Lambda^+(2, d)$ with

$$\dim(L(\lambda)) = |\text{Std}(\lambda)|,$$

where $\text{Std}(\lambda)$ is the set of all standard tableaux of shape λ .

Proof. Directly from Proposition 6.15 and Theorems 5.11 and 5.12 because $m_\lambda = |\text{Std}(\lambda)|$ (with the notation from Theorem 5.12). \square

Example 6.14. The Temperley-Lieb algebra $\mathcal{TL}_3(\delta)$ in the semi-simple case has

$$\dim(L(\square\square\square)) = 1, \quad \dim\left(L\left(\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array}\right)\right) = 2.$$

Compare to (44). \blacksquare

6.2.2. Temperley-Lieb algebra: the non semi-simple case. Let us assume that we have fixed $q \in \mathbb{K} - \{0, \pm 1, \pm \sqrt[2]{-1}\}$ to be a critical value such that $[k] = 0$ for some $k = 1, \dots, d$. Then, by Proposition 6.3, the algebra $\mathcal{TL}_d(\delta)$ is no longer semi-simple.

In particular, at least in the knowledge of the authors, there is *no diagrammatic* analogon of the Jones-Wenzl projectors in general (left aside the generalized Jones-Wenzl projectors).

Proposition 6.15. (Cellular basis for $\mathcal{TL}_d(\delta)$ - the third) The datum $(\mathcal{P}_{\text{TL}}, \mathcal{I}_{\text{TL}}, \mathcal{C}, i_{\text{TL}})$ with \mathcal{C} as in Theorem 4.11 for $\mathcal{TL}_d(\delta) \cong \text{End}_{\mathbf{U}_q(\mathfrak{sl}_2)}(T)$ is a cell datum for $\mathcal{TL}_d(\delta)$.

Moreover, $\mathcal{C}_{\text{TL}} \neq \mathcal{C}$ for all $d > 1$ and all choices involved in the definition of our basis. In particular, there is a choice such that all generalized non semi-simple Jones-Wenzl projectors are part of $\text{im}(\mathcal{C})$.

Proof. As in the proof of Proposition 6.11 and left to the reader. \square

Note that we can do better: as in Example 3.22 one gets a decomposition

$$(48) \quad \mathcal{T} \cong \mathcal{T}_{-1} \oplus \mathcal{T}_0 \oplus \mathcal{T}_1 \oplus \cdots \oplus \mathcal{T}_{l-3} \oplus \mathcal{T}_{l-2} \oplus \mathcal{T}_{l-1},$$

where the blocks \mathcal{T}_{-1} and \mathcal{T}_{l-1} are semi-simple if $\mathbb{K} = \mathbb{C}$. Compare also to Lemma 2.25 in [8].

If we fix $\mathbb{K} = \mathbb{C}$: as explained in Section 3.5 of [8] each block in the decomposition (48) can be equipped with a non-trivial \mathbb{Z} -grading coming from Khovanov and Seidel's quiver algebra from [50]. Hence, we have the following.

Lemma 6.16. The \mathbb{C} -algebra $\text{End}_{\mathbf{U}_q(\mathfrak{sl}_2)}(T)$ can be equipped with a non-trivial \mathbb{Z} -grading. Thus, the Temperley-Lieb algebra $\mathcal{TL}_d(\delta)$ over \mathbb{C} can be equipped with a non-trivial \mathbb{Z} -grading.

Proof. The second statement follows directly from the first using quantum Schur-Weyl duality. Hence, we only need to show the first.

Note that $T = V^{\otimes d}$ decomposes as in (47), we can order this decomposition by blocks. Each block carries a \mathbb{Z} -grading coming from Khovanov and Seidel's quiver algebra (as explained in Section 3 of [8]). In particular, we can choose the basis elements c_{ij}^λ in such a way that we get the \mathbb{Z} -graded basis obtained in Corollary 4.23 in [8]. Since there is no interaction between different blocks, the statement follows. \square

Recall from Definition 2.1 in [37] that a \mathbb{Z} -graded cell datum of a \mathbb{Z} -graded algebra is a cell datum for the algebra together with an additional *degree function* $\text{deg}: \coprod_{\lambda \in \mathcal{P}} \mathcal{I}^\lambda \rightarrow \mathbb{Z}$, such that $\text{deg}(c_{ij}^\lambda) = \text{deg}(i) + \text{deg}(j)$. For us the choice of $\text{deg}(\cdot)$ is as follows.

If $\lambda \in \mathcal{P}$ is in one of the semi-simple blocks, then we simply set $\text{deg}(i) = 0$ for all $i \in \mathcal{I}^\lambda$.

Assume that $\lambda \in \mathcal{P}$ is not in the semi-simple blocks. By Example 3.27 we know that every $T_q(\lambda)$ has precisely two Weyl factors. The g_i^λ that maps $\Delta_q(\lambda)$ into a higher $T_q(\mu)$ should be indexed by a 1-colored i whereas the g_i^λ mapping $\Delta_q(\lambda)$ into $T_q(\lambda)$ should have 0-colored i . Similarly for the f_j^λ 's. Then the degree of the elements $i \in \mathcal{I}^\lambda$ should be the corresponding color. We get the following.

Proposition 6.17. (Graded cellular basis for $\mathcal{TL}_d(\delta)$) The datum $(\mathcal{P}_{\text{TL}}, \mathcal{I}_{\text{TL}}, \mathcal{C}, i_{\text{TL}}, \text{deg})$ with \mathcal{C} as in Theorem 4.11 for the \mathbb{C} -algebra $\mathcal{TL}_d(\delta) \cong \text{End}_{\mathbf{U}_q(\mathfrak{sl}_2)}(T)$ is a \mathbb{Z} -graded cell datum for $\mathcal{TL}_d(\delta)$.

Proof. The hardest part is cellularity which directly follows from 4.11. That the quintuple $(\mathcal{P}_{\text{TL}}, \mathcal{I}_{\text{TL}}, \mathcal{C}, i_{\text{TL}}, \text{deg})$ gives a \mathbb{Z} -graded cell datum follows from the construction. \square

Example 6.18. Let us consider $\mathcal{TL}_3(\delta)$ as in Example 6.12, namely q being a complex, primitive third root of unity. Then $\mathcal{TL}_3(\delta)$ is non semi-simple by Proposition 6.3. In particular, we have a decomposition of $V^{\otimes 3}$ different from (46), namely as in (19). In this case $\mathcal{P} = \{1, 3\}$,

$\mathcal{I}^3 = \{1, 3\}$ and $\mathcal{I}^1 = \{1\}$. By our choice from above

$$\deg(i) = \begin{cases} 0, & \text{if } i = 1 \in \mathcal{I}^1 \text{ or } i = 3 \in \mathcal{I}^3, \\ 1, & \text{if } i = 1 \in \mathcal{I}^3. \end{cases}$$

As in Example 6.12 (if we use the ordering induced by the decomposition from (19)), we can choose basis elements as

$$c_{cc}^\lambda = \mathbf{E}_{11}, \quad c_{t_1 t_1}^\mu = \mathbf{E}_{22}, \quad c_{t_1 t_2}^\mu = \mathbf{E}_{21}, \quad c_{t_2 t_1}^\mu = \mathbf{E}_{12}, \quad c_{t_2 t_2}^\mu = \mathbf{E}_{33},$$

where we use the notation from (43) and (44) (and the ‘‘canonical’’ filling c for λ) again. These are of degrees 0, 1, 1, 2 and 0 respectively. We also note the difference to the basis in the semi-simple case from Example 6.12. ■

Remark 6.19. Our grading and the one found by Plaza and Ryom-Hansen in [70] agree (up to a shift of the indecomposable summands). To see this, note that our algebra is isomorphic to the algebra $K_{1,n}$ from [17] which is by (4.8) in [17] and Theorem 6.3 of [19] a quotient of some cyclotomic KL-R algebra (the compatibility of the grading follows for example from Corollary B.6 in [38]). The same holds, by construction, for the grading in [70].

Corollary 6.20. Let $\mathbb{K} = \mathbb{C}$. We have a complete set of pairwise non-isomorphic, simple $\mathcal{TL}_d(\delta)$ -modules $L(\lambda)$ for $\lambda \in \Lambda^+(2, d)$ such that $T_q(\lambda)$ is a summand of $T = V^{\otimes d}$ with

$$\dim(L(\lambda)) = m_\lambda,$$

where m_λ is the multiplicity of $T_q(\lambda)$ as a summand of $T = V^{\otimes d}$.

Proof. As in Corollary 6.13. □

Example 6.21. If q is a complex, primitive third root of unity, then $\mathcal{TL}_3(\delta)$ has

$$\dim(L(\begin{smallmatrix} \square & \square & \square \end{smallmatrix})) = 1, \quad \dim\left(L\left(\begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix}\right)\right) = 1.$$

Note the contrast to the semi-simple case from Example 6.14. ■

Remark 6.22. In the case $\mathbb{K} = \mathbb{C}$ we can give a dimension formula, namely

$$\dim(L(\lambda)) = m_\lambda = \begin{cases} |\text{Std}(\lambda)|, & \text{if } \lambda_1 - \lambda_2 \equiv -1 \pmod{l}, \\ \sum_{\mu=w.\lambda, \mu \geq \lambda \in \Lambda^+(2,d)} (-1)^{\ell(w)} |\text{Std}(\mu)|, & \text{if } \lambda_1 - \lambda_2 \not\equiv -1 \pmod{l}, \end{cases}$$

where $w \in W_l$ is the affine Weyl group and $\ell(w)$ is the length of a reduced word $w \in W_l$. This matches the formulas from, for example, Proposition 6.7 in [4] or Corollary 5.2 in [71]. In the case where $\text{char}(\mathbb{K}) > 0$ one can in principle also obtain a formula. But this time we do not encourage the reader to work out the (rather complicated) formula.

REFERENCES

- [1] H.H. Andersen, A sum formula for tilting filtrations, *J. Pure Appl. Algebra* 152-1-3 (2000), 17-40.
- [2] H.H. Andersen, The strong linkage principle for quantum groups at roots of 1, *J. Algebra* 260-1 (2003), 2-15.
- [3] H.H. Andersen, J.C. Jantzen and W. Soergel, *Representations of quantum groups at a p th root of unity and of semisimple groups in characteristic p : independence of p* , *Astérisque* 220 (1994), 321 pages.
- [4] H.H. Andersen, G.I. Lehrer and R. Zhang, Cellularity of certain quantum endomorphism algebras, to appear in *Pacific J. Math.*, online available arXiv:1303.0984.

- [5] H.H. Andersen and V. Mazorchuk, Category \mathcal{O} for quantum groups, *J. Eur. Math. Soc.* 17 (2015), 405431, online available arXiv:1105.5500.
- [6] H.H. Andersen, P. Polo and K. Wen, Representations of quantum algebras, *Invent. Math.* 104-1 (1991), 1-59.
- [7] H.H. Andersen, C. Stroppel and D. Tubbenhauer, Additional notes for the paper “Cellular structures using U_q -tilting modules”, eprint, online available <http://www.uni-math.gwdg.de/dtubben/cell-tilt-proofs.pdf>.
- [8] H.H. Andersen and D. Tubbenhauer, Diagram categories for U_q -tilting modules at roots of unity, online available arXiv:1409.2799.
- [9] S. Ariki, Cyclotomic q -Schur algebras as quotients of quantum algebras, *J. Reine Angew. Math.* 513 (1999), 53-69.
- [10] S. Ariki and K. Koike, A Hecke algebra of $(\mathbb{Z}/r\mathbb{Z}) \wr \mathfrak{S}_n$ and construction of its irreducible representations, *Adv. Math.* 106-2 (1994), 216-243.
- [11] S. Ariki, A. Mathas and H. Rui, Cyclotomic Nazarov-Wenzl algebras, *Nagoya Math. J.* 182 (2006), 47-134, online available arXiv:math/0506467.
- [12] J.S. Birman and H. Wenzl, Braids, link polynomials and a new algebra, *Trans. Amer. Math. Soc.* 313-1 (1989), 249-273.
- [13] N. Bourbaki, *Lie groups and Lie algebras*, Chapters 7-9, Translated from the 1975 and 1982 French originals, Elements of Mathematics, Springer-Verlag (2005).
- [14] R. Brauer, On algebras which are connected with the semisimple continuous groups, *Ann. of Math.* 38-4 (1937), 857-872.
- [15] J. Brundan and A. Kleshchev, Blocks of cyclotomic Hecke algebras and Khovanov-Lauda algebras, *Invent. Math.* 178-3 (2009), 451-484, online available arXiv:0808.2032.
- [16] J. Brundan and A. Kleshchev, Schur-Weyl duality for higher levels, *Selecta Math. (N.S.)* 14-1 (2008), 1-57, online available arXiv:math/0605217.
- [17] J. Brundan and C. Stroppel, Highest weight categories arising from Khovanov’s diagram algebra I: cellularity, *Mosc. Math. J.* 11-4 (2011), 685-722, online available arXiv:0806.1532.
- [18] J. Brundan and C. Stroppel, Highest weight categories arising from Khovanov’s diagram algebra II: Koszulity, *Transform. Groups* 15-1 (2010), 1-45, online available arXiv:0806.3472.
- [19] J. Brundan and C. Stroppel, Highest weight categories arising from Khovanov’s diagram algebra III: category \mathcal{O} , *Represent. Theory* 15 (2011), 170-243, online available arXiv:0812.1090.
- [20] J. Brundan and C. Stroppel, Highest weight categories arising from Khovanov’s diagram algebra IV: the general linear supergroup, *J. Eur. Math. Soc.* 14-2 (2012), 373-419, online available arXiv:0907.2543.
- [21] J. Brundan and C. Stroppel, Gradings on walled Brauer algebras and Khovanov’s arc algebra, *Adv. Math.* 231-2 (2012), 709-773, online available arXiv:1107.0999.
- [22] E. Cline, B. Parshall and L. Scott, Finite-dimensional algebras and highest weight categories, *J. Reine Angew. Math.* 391 (1988), 85-99.
- [23] B. Cooper and M. Hogancamp, An Exceptional Collection For Khovanov Homology, online available arXiv:1209.1002.
- [24] S. Cautis, J. Kamnitzer and S. Morrison, Webs and quantum skew Howe duality, *Math. Ann.* 360-1-2 (2014), 351-390, online available arXiv:1210.6437.
- [25] R. Dipper, S. Doty and F. Stoll, The quantized walled Brauer algebra and mixed tensor space, *Algebr. Represent. Theory* 17-2 (2014), 675-701, online available arXiv:0806.0264.
- [26] R. Dipper, G. James and A. Mathas, Cyclotomic q -Schur algebras, *Math. Z.* 229-3 (1998), 385-416.
- [27] S. Donkin, On tilting modules for algebraic groups, *Math. Z.* 212-1 (1993), 39-60.
- [28] S. Donkin, *The q -Schur algebra*, London Mathematical Society Lecture Note Series 253, Cambridge University Press (1998).
- [29] J. Du, B. Parshall and L. Scott, Quantum Weyl reciprocity and tilting modules, *Comm. Math. Phys.* 195-2 (1998), 321-352.
- [30] M. Ehrig, Quantized super Schur-Weyl dualities, in preparation.
- [31] M. Ehrig and C. Stroppel, Graded Brauer algebras, in preparation.
- [32] M. Ehrig and C. Stroppel, Nazarov-Wenzl algebras, coideal subalgebras and categorified skew Howe duality, online available arXiv:1310.1972.

- [33] M. Ehrig and C. Stroppel, Schur-Weyl duality for the Brauer algebra and the ortho-symplectic Lie super-algebra, online available arXiv:1412.7853.
- [34] J.J. Graham and G.I. Lehrer, Cellular algebras, *Invent. Math.* 123-1 (1996), 1-34.
- [35] T. Halverson and A. Ram, q -rook monoid algebras, Hecke algebras, and Schur-Weyl duality, *J. Math. Sci. (N.Y.)* 121-3 (2004), 2419-2436, online available arXiv:math/0401330.
- [36] J. Hu, BMW algebra, quantized coordinate algebra and type C Schur-Weyl duality, *Represent. Theory* 15 (2011), 1-62, online available arXiv:0708.3009.
- [37] J. Hu and A. Mathas, Graded cellular bases for the cyclotomic Khovanov-Lauda-Rouquier algebras of type A , *Adv. Math.* 225-2 (2010), 598-642, online available arXiv:0907.2985.
- [38] J. Hu and A. Mathas, Quiver Schur algebras I: linear quivers, online available arXiv:1110.1699.
- [39] J. Hu and F. Stoll, On double centralizer properties between quantum groups and Ariki-Koike algebras, *J. Algebra* 275-1 (2004), 397-418.
- [40] J.E. Humphreys, *Representations of semisimple Lie algebras in the BGG category \mathcal{O}* , Graduate Studies in Mathematics 94, American Mathematical Society (2008).
- [41] J.C. Jantzen, *Darstellungen halbeinfacher algebraischer Gruppen und zugeordnete kontravariante Formen*, Bonn. Math. Schr. No. 67 (1973), v+124 pp. (German).
- [42] J.C. Jantzen, *Lectures on quantum groups*, Graduate Studies in Mathematics 6, American Mathematical Society (1996).
- [43] J.C. Jantzen, *Representations of Algebraic Groups*, Mathematical Surveys and Monographs 107, Second edition, American Mathematical Society (2003).
- [44] V.F.R. Jones, Index for subfactors, *Invent. Math.* 72-1 (1983), 1-25.
- [45] D. Kazhdan and G. Lusztig, Tensor structures arising from affine Lie algebras I-IV, *J. Amer. Math. Soc.* 6-4 (1994), 905-947, 949-1011 and *J. Amer. Math. Soc.* 7-2 (1994), 335-381, 383-453.
- [46] M. Khovanov, A functor-valued invariant of tangles, *Algebr. Geom. Topol.* 2 (2002), 665-741, online available arXiv:math/0103190.
- [47] M. Khovanov and A.D. Lauda, A categorification of quantum \mathfrak{sl}_n , *Quantum Topol.* 2-1 (2010), 1-92, online available arXiv:0807.3250.
- [48] M. Khovanov and A.D. Lauda, A diagrammatic approach to categorification of quantum groups I, *Represent. Theor.* 13 (2009), 309-347, online available arXiv:0803.4121.
- [49] M. Khovanov and A.D. Lauda, A diagrammatic approach to categorification of quantum groups II, *Trans. Amer. Math. Soc.* 363-5 (2011), 2685-2700, online available arXiv:0804.2080.
- [50] M. Khovanov and P. Seidel, Quivers, Floer cohomology, and braid group actions, *J. Amer. Math. Soc.* 15-1 (2002), 203-271, online available arXiv:math/0006056.
- [51] K. Koike, On the decomposition of tensor products of the representations of the classical groups: by means of the universal characters, *Adv. Math.* 74-1 (1989), 57-86.
- [52] S. König and C. Xi, On the structure of cellular algebras, *Algebras and modules II*, CMS Conf. Proc. 24 (1996), 365-386.
- [53] G. Kuperberg, Spiders for rank 2 Lie algebras, *Comm. Math. Phys.* 180-1 (1996), 109-151, online available arXiv:q-alg/9712003.
- [54] A. Lascoux, B. Leclerc and J.-Y. Thibon, Hecke algebras at roots of unity and crystal bases of quantum affine algebras, *Comm. Math. Phys.* 181-1 (1996), 205-263.
- [55] G.I. Lehrer and R. Zhang, The second fundamental theorem of invariant theory for the orthogonal group, *Ann. of Math. (2)* 176-3 (2012), 2031-2054, online available arXiv:1102.3221.
- [56] G. Li, A KLR Grading of the Brauer Algebras, online available arXiv:1409.1195.
- [57] G. Lusztig, *Introduction to Quantum Groups*, Reprint of the 1994 edition, Modern Birkhäuser Classics, Birkhäuser/Springer (2010).
- [58] G. Lusztig, Finite-dimensional Hopf algebras arising from quantized universal enveloping algebra, *J. Amer. Math. Soc.* 3-1 (1990), 257-296.
- [59] G. Lusztig, Modular representations and quantum groups, *Contemp. Math.* 82 (1989), 59-77.
- [60] G. Lusztig, Quantum groups at roots of 1, *Geom. Dedicata* 35-1-3 (1990), 89-113.
- [61] M. Mackaay, The $\mathfrak{sl}(N)$ -web algebras and dual canonical bases, *J. Algebra* 409 (2014), 54-100, online available arXiv:1308.0566.

- [62] M. Mackaay, W. Pan and D. Tubbenhauer, The \mathfrak{sl}_3 -web algebra, *Math. Z.* 277-1-2 (2014), 401-479, online available arXiv:1206.2118.
- [63] P. Martin, *Potts models and related problems in statistical mechanics*, Series on Advances in Statistical Mechanics 5, World Scientific Publishing Co. (1991).
- [64] P. Martin and H. Saleur, The blob algebra and the periodic Temperley-Lieb algebra, *Lett. Math. Phys.* 30-3 (1994), 189-206, online available arXiv:hep-th/9302094.
- [65] V. Mazorchuk, *Lectures on algebraic categorification*, QGM Master Class Series, Eur. Math. Soc. (2012), online available arXiv:1011.0144.
- [66] V. Mazorchuk and C. Stroppel, $G(l, k, d)$ -modules via groupoids, online available arXiv:1412.4494.
- [67] J. Murakami, The Kauffman polynomial of links and representation theory, *Osaka J. Math.* 24-4 (1987), 745-758.
- [68] R. Paget, Representation theory of q -rook monoid algebras, *J. Algebraic Combin.* 24-3 (2006), 239-252.
- [69] J. Paradowski, Filtration of modules over the quantum algebra, *Proc. Sympos. Pure Math.* 56, part 2 (1994), 93-108.
- [70] D. Plaza and S. Ryom-Hansen, Graded cellular bases for Temperley-Lieb algebras of type A and B, *J. Algebraic Combin.* 40-1 (2014), 137-177, online available arXiv:1203.2592.
- [71] D. Ridout and Y. Saint-Aubin, Standard Modules, Induction and the Temperley-Lieb Algebra, *Adv. Theor. Math. Phys.* 18-5 (2014), 957-1041, online available arXiv:1204.4505.
- [72] C.M. Ringel, The category of modules with good filtrations over a quasi-hereditary algebra has almost split sequences, *Math. Z.* 208-2 (1991), 209-223.
- [73] D.E.V. Rose and D. Tubbenhauer, Symmetric webs, Jones-Wenzl recursions and q -Howe duality, online available arXiv:1501.00915.
- [74] R. Rouquier, 2-Kac-Moody algebras, online available arXiv:0812.5023.
- [75] R. Rouquier, P. Shan, M. Varagnolo and E. Vasserot, Categorifications and cyclotomic rational double affine Hecke algebras, online available arXiv:1305.4456.
- [76] S. Ryom-Hansen, A q -analogue of Kempf's vanishing theorem, *Mosc. Math. J.* 3-1 (2003), 173-187, online available arXiv:0905.0236.
- [77] S. Ryom-Hansen, Cell structures on the blob algebra, *Represent. Theory* 16 (2012), 540-567, online available arXiv:0911.1923.
- [78] S. Ryom-Hansen, The Ariki-Terasoma-Yamada tensor space and the blob-algebra, *J. Algebra* 324-10 (2010), 2658-2675, online available arXiv:math/0505278.
- [79] M. Sakamoto and T. Shoji, Schur-Weyl reciprocity for Ariki-Koike algebras, *J. Algebra* 221-1 (1999), 293-314.
- [80] A. Sartori, The degenerate affine walled Brauer algebra, *J. Algebra* 417 (2014), 198-233, online available arXiv:1305.2347.
- [81] A. Sartori and C. Stroppel, Walled Brauer algebras as idempotent truncations of level 2 cyclotomic quotients, online available arXiv:1411.2771.
- [82] W. Soergel, Charakterformeln für Kipp-Moduln über Kac-Moody-Algebren, *Represent. Theory* 1 (1997), 115-132 (German), *Represent. Theory* 2 (1998), 432-448 (English).
- [83] W. Soergel, Kategorie \mathcal{O} , perverse Garben und Moduln über den Koinvarianten zur Weylgruppe, *J. Amer. Math. Soc.* 3-2 (1990), 421-445 (German).
- [84] W. Soergel, Kazhdan-Lusztig-Polynome und eine Kombinatorik für Kipp-Moduln, *Represent. Theory* 1 (1997), 37-68 (German), *Represent. Theory* 1 (1997), 83-114 (English).
- [85] L. Solomon, The Bruhat decomposition, Tits system and Iwahori ring for the monoid of matrices over a finite field, *Geom. Dedicata* 36-1 (1990), 15-49.
- [86] C. Stroppel, Category \mathcal{O} : Gradings and translation functors, *J. Algebra* 268-1 (2003), 301-326.
- [87] C. Stroppel, Untersuchungen zu den parabolischen Kazhdan-Lusztig-Polynomen für affine Weyl-Gruppen, Diploma Thesis (1997), 74 pages (German), online available http://www.math.uni-bonn.de/ag/stroppel/arbeit_Stroppel.pdf.
- [88] T. Tanisaki, *Character formulas of Kazhdan-Lusztig type*, Representations of finite dimensional algebras and related topics in Lie theory and geometry, *Fields Inst. Commun.* 40, Amer. Math. Soc. (2004), 261-276.

- [89] H.N.V. Temperley and E.H. Lieb, Relations between the “percolation” and “colouring” problem and other graph-theoretical problems associated with regular planar lattices: some exact results for the “percolation” problem, Proc. Roy. Soc. London Ser. A 322-1549 (1971), 251-280.
- [90] D. Tubbenhauer, \mathfrak{sl}_n -webs, categorification and Khovanov-Rozansky homologies, online available arXiv:1404.5752.
- [91] V. Turaev, Operator invariants of tangles, and R -matrices, Math. USSR-Izv. 35-2 (1990), 411-444.
- [92] V. Turaev, *Quantum invariants of knots and 3-manifolds*, second revised edition, de Gruyter Studies in Mathematics 18, Walter de Gruyter & Co. (2010).
- [93] H. Wenzl, On sequences of projections, C. R. Math. Rep. Acad. Sci. Canada 9-1 (1987), 5-9.
- [94] B.W. Westbury, The representation theory of the Temperley-Lieb algebras, Math. Z. 219-4 (1995), 539-566.

H.H.A.: CENTRE FOR QUANTUM GEOMETRY OF MODULI SPACES, AARHUS UNIVERSITY, NY MUNKEGADE 118, BUILDING 1530, ROOM 327, 8000 AARHUS C, DENMARK
E-mail address: `mathha@qgm.au.dk`

C.S.: MATHEMATISCHES INSTITUT, UNIVERSITÄT BONN, ENDENICHER ALLEE 60, ROOM 4.007, 53115 BONN, GERMANY
E-mail address: `stroppel@math.uni-bonn.de`

D.T.: CENTRE FOR QUANTUM GEOMETRY OF MODULI SPACES, AARHUS UNIVERSITY, NY MUNKEGADE 118, BUILDING 1530, ROOM 316, 8000 AARHUS C, DENMARK
E-mail address: `dtubben@qgm.au.dk`