ADDITIONAL NOTES FOR THE PAPER "CELLULAR STRUCTURES USING $\mathbf{U}_q$-TILTING MODULES"

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This eprint contains additional notes for the paper [4]. In particular, we give some of the in [4] omitted proofs as well as the following two sections which are partially not part of [4].

2. Quantum groups and their representations

In the present section we recall the definitions and results about quantum groups and their representation theory in the semisimple and the non-semisimple case.

2.1. The quantum groups $\mathbf{U}_v$ and $\mathbf{U}_q$. Let Φ be a finite *root system* in an Euclidean space E. We fix a choice of *positive roots* $\Phi^+ \subset \Phi$ and *simple roots* $\Pi \subset \Phi^+$. We assume that we have n simple roots that we denote by $\alpha_1, \dots, \alpha_n$. For each $\alpha \in \Phi$, we denote by $\alpha^\vee \in \Phi^\vee$ the corresponding *co-root* and we let $\rho = \frac{1}{2}\sum_{\alpha\in\Phi^+}\alpha$ be the *half-sum of all positive roots*. Let $a_{ij} = \langle \alpha_i, \alpha_j^\vee\rangle$ for $i,j = 1,\dots,n$. Then the matrix $\mathbf{A} = (a_{ij})_{i,j=1}^n$ is called the *Cartan matrix*. As usual, we need to symmetrize $\mathbf{A}$ and we do so by choosing for $i = 1,\dots,n$ minimal $d_i \in \mathbb{N}$ such that $(d_i a_{ij})_{i,j=1}^n$ is symmetric (the Cartan matrix $\mathbf{A}$ is already symmetric in most of our examples and thus, $d_i = 1$ for all $i = 1,\dots,n$).

By the set of *(integral) weights* we understand $X = \{\lambda \in E \mid \langle \lambda, \alpha_i^\vee\rangle \in \mathbb{Z} \text{ for all } \alpha_i \in \Pi\}$. The *dominant (integral) weights* X^+ are those $\lambda \in X$ such that $\langle \lambda, \alpha_i^\vee\rangle \geq 0$ for all $\alpha_i \in \Pi$.

The *fundamental weights*, denoted by $\omega_i \in X$ for $i = 1,\dots,n$, are characterized by

$$\langle \omega_i, \alpha_j^\vee\rangle = \delta_{ij} \quad \text{for all } j = 1,\dots,n.$$

Recall that there is a *partial ordering* on X given by $\mu \leq \lambda$ iff $\lambda - \mu$ is an $\mathbb{N}$-valued linear combination of the simple roots, that is, $\lambda - \mu = \sum_{i=1}^n a_i\alpha_i$ with $a_i \in \mathbb{N}$.

Example 2.1. One of the most important examples is the standard choice of Cartan datum $(\mathbf{A}, \Pi, \Phi, \Phi^+)$ associated with the Lie algebra $\mathfrak{g} = \mathfrak{sl}_{n+1}$ for $n \geq 1$. Here $E = \mathbb{R}^{n+1}/(1,\dots,1)$ (which we identify with $\mathbb{R}^n$ in calculations) and $\Pi = \{\alpha_i = \varepsilon_i - \varepsilon_{i+1} \mid i = 1,\dots,n\}$, where the ε_i denote the standard basis of E. The positive roots are $\Phi^+ = \{\varepsilon_i - \varepsilon_j \mid 1 \leq i < j \leq n+1\}$ with maximal root $\alpha_0 = \varepsilon_1 - \varepsilon_{n+1}$. Moreover,

$$\rho = \tfrac{1}{2}\sum_{i=1}^{n+1}(n - 2(i-1))\varepsilon_i = \sum_{i=1}^{n+1}(n - i + 1)\varepsilon_i - \tfrac{1}{2}(n,\dots,n)$$

(seen as a $\mathfrak{sl}_{n+1}$-weight, i.e. we can drop the $\frac{1}{2}(n,\dots,n)$).

The set of fundamental weights is $\{\omega_i = \varepsilon_1 + \cdots + \varepsilon_i \mid 1 \leq i \leq n\}$. For explicit calculations one often identifies

$$\lambda = \sum_{i=1}^n a_i\omega_i \in X^+$$

with the partition $\lambda = (\lambda_1 \geq \cdots \geq \lambda_n \geq 0)$ given by $\lambda_k = \sum_{i=k}^{n} a_i$ for $k = 1, \ldots, n$. ▲

We now assign a quantum group $\mathbf{U}_v = \mathbf{U}_v(\mathbf{A})$ to a given Cartan matrix $\mathbf{A}$. Abusing notation, we also write $\mathbf{U}_v(\mathfrak{g})$ etc. if no confusion can arise. Before giving the definition, we point out that v in our notation always means a *generic parameter*, while $q \in \mathbb{K}^*$ will always mean a *specialization*.

Definition 2.2. (**Quantum enveloping algebra**) Given a Cartan matrix $\mathbf{A}$, then the *quantum enveloping algebra* $\mathbf{U}_v = \mathbf{U}_v(\mathbf{A})$ associated to it is the associative, unital $\mathbb{Q}(v)$-algebra generated by $K_1^{\pm 1}, \ldots, K_n^{\pm 1}$ and $E_1, F_1, \ldots, E_n, F_n$, where n is the size of $\mathbf{A}$, subject to

$$K_iK_j = K_jK_i, \quad K_iK_i^{-1} = K_i^{-1}K_i = 1, \quad K_iE_j = v^{d_ia_{ij}}E_jK_i, \quad K_iF_j = v^{-d_ia_{ij}}F_jK_i,$$

$$E_iF_j - F_jE_i = \delta_{i,j}\frac{K_i - K_i^{-1}}{v^{d_i} - v^{-d_i}},$$

$$\sum_{r+s=1-a_{ij}} (-1)^s \begin{bmatrix} 1-a_{ij} \\ s \end{bmatrix}_{d_i} E_i^rE_jE_i^s = 0, \quad \text{if} \quad i \neq j,$$

$$\sum_{r+s=1-a_{ij}} (-1)^s \begin{bmatrix} 1-a_{ij} \\ s \end{bmatrix}_{d_i} F_i^rF_jF_i^s = 0, \quad \text{if} \quad i \neq j.$$

Here we use, as usual, the conventions that, for $a \in \mathbb{Z}$ and $b, d \in \mathbb{N}$, $[a]_d$ denotes the *a-quantum integer* (with $[0]_d = 0$), $[b]_d!$ denotes the *b-quantum factorial*, that is,

$$[a]_d = \frac{v^{ad} - v^{-ad}}{v^d - v^{-d}}, \quad [a] = [a]_1 \quad \text{and} \quad [b]_d! = [1]_d \cdots [b-1]_d[b]_d, \quad [b]! = [b]_1!$$

(with $[0]_d! = 1$ by convention) and

$$\begin{bmatrix} a \\ b \end{bmatrix}_d = \frac{[a]_d[a-1]_d \cdots [a-b+2]_d[a-b+1]_d}{[b]_d!}, \quad \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}_1$$

denotes the (a,b)-*quantum binomial*. Observe that $[-a]_d = -[a]_d$. ▲

It is worth noting that $\mathbf{U}_v$ is a Hopf algebra with coproduct Δ given by

$$\Delta(E_i) = E_i \otimes 1 + K_i \otimes E_i, \;\; \Delta(F_i) = F_i \otimes K_i^{-1} + 1 \otimes F_i \;\; \text{and} \;\; \Delta(K_i) = K_i \otimes K_i.$$

The antipode S and the counit ε are given by

$$S(E_i) = -K_i^{-1}E_i, \;\; S(F_i) = -F_iK_i, \;\; S(K_i) = K_i^{-1}, \;\; \varepsilon(E_i) = \varepsilon(F_i) = 0 \;\; \text{and} \;\; \varepsilon(K_i) = 1.$$

We are interested in the root of unity case. Thus, we want to *"specialize"* the generic parameter v of $\mathbf{U}_v$ to be, for example, a root of unity $q \in \mathbb{K}^*$. In order to do so, we consider *Lusztig's $\mathcal{A}$-form* $\mathbf{U}_{\mathcal{A}} = \mathbf{U}_{\mathcal{A}}(\mathbf{A})$ introduced in [22]. To this end, let $\mathcal{A} = \mathbb{Z}[v, v^{-1}]$.

Definition 2.3. (**Lusztig's $\mathcal{A}$-form $\mathbf{U}_{\mathcal{A}}$**) Define for all $j \in \mathbb{N}$ the *j-th divided powers*

$$E_i^{(j)} = \frac{E_i^j}{[j]_{d_i}!} \quad \text{and} \quad F_i^{(j)} = \frac{F_i^j}{[j]_{d_i}!}.$$

Then $\mathbf{U}_{\mathcal{A}}$ is defined as the $\mathcal{A}$-subalgebra of $\mathbf{U}_v$ generated by K_i, K_i^{-1}, $E_i^{(j)}$ and $F_i^{(j)}$ for $i = 1, \ldots, n$ and $j \in \mathbb{N}$. ▲

Lusztig's $\mathcal{A}$-form is designed to allow specializations. For this purpose, we fix a field $\mathbb{K}$ of arbitrary characteristic and set $\mathbb{K}^* = \mathbb{K} - \{0, -1\}$ if $\mathrm{char}(\mathbb{K}) > 2$ and $\mathbb{K}^* = \mathbb{K} - \{0\}$ otherwise.

Definition 2.4. (**Quantum enveloping algebras at roots of unity**) Fix an arbitrary element $q \in \mathbb{K}^*$. Consider $\mathbb{K}$ as an $\mathcal{A}$-module by specializing v to q. Define

$$\mathbf{U}_q = \mathbf{U}_q(\mathbf{A}) = \mathbf{U}_{\mathcal{A}} \otimes_{\mathcal{A}} \mathbb{K}.$$

Abusing notation, we will usually abbreviate $E_i^{(j)} \otimes 1 \in \mathbf{U}_q$ with $E_i^{(j)}$. Analogously for the other generators of $\mathbf{U}_q$. ▲

Example 2.5. In the $\mathfrak{sl}_2$ case and the datum $\mathbf{A}$ as in Example 2.1 above, the $\mathbb{Q}(v)$-algebra $\mathbf{U}_v(\mathfrak{sl}_2) = \mathbf{U}_q(\mathbf{A})$ is generated by K and K^{-1} and E, F subject to the relations

$$KK^{-1} = K^{-1}K = 1,$$
$$EF - FE = \frac{K - K^{-1}}{v - v^{-1}},$$
$$KE = v^2 EK \quad \text{and} \quad KF = v^{-2} FK.$$

We point out that $\mathbf{U}_v(\mathfrak{sl}_2)$ already contains the divided powers since no quantum number vanishes in $\mathbb{Q}(v)$. Let q be a complex, primitive third root of unity. Thus, $q + q^{-1} = [2] = -1$, $q^2 + 1 + q^{-2} = [3] = 0$ and $q^3 + q^1 + q^{-1} + q^{-3} = [4] = 1$. More generally,

$$[a] = i \in \{0, +1, -1\}, \quad i \equiv a \bmod 3.$$

Hence, $\mathbf{U}_q(\mathfrak{sl}_2)$ is generated by $K, K^{-1}, E, F, E^{(3)}$ and $F^{(3)}$ subject to the relations as above (here $E^{(3)}, F^{(3)}$ are extra generators since $E^3 = [3]!E^{(3)} = 0$ because of $[3] = 0$). This is precisely the convention used in [15, Chapter 1], but specialized at q. ▲

It is easy to check that $\mathbf{U}_{\mathcal{A}}$ is a Hopf subalgebra of $\mathbf{U}_v$, see [20, Proposition 4.8]. Thus, $\mathbf{U}_q$ inherits a Hopf algebra structure from $\mathbf{U}_v$.

Moreover, it is known that all three algebras, that is, $\mathbf{U}_v$, $\mathbf{U}_{\mathcal{A}}$ and $\mathbf{U}_q$, have a triangular decomposition

$$\mathbf{U}_v = \mathbf{U}_v^- \mathbf{U}_v^0 \mathbf{U}_v^+, \qquad \mathbf{U}_{\mathcal{A}} = \mathbf{U}_{\mathcal{A}}^- \mathbf{U}_{\mathcal{A}}^0 \mathbf{U}_{\mathcal{A}}^+, \qquad \mathbf{U}_q = \mathbf{U}_q^- \mathbf{U}_q^0 \mathbf{U}_q^+,$$

where $\mathbf{U}_v^-, \mathbf{U}_{\mathcal{A}}^-, \mathbf{U}_q^-$ denote the subalgebras generated only by the F_i's (or, in addition, the divided powers for $\mathbf{U}_{\mathcal{A}}^-$ and $\mathbf{U}_q^-$) and $\mathbf{U}_v^+, \mathbf{U}_{\mathcal{A}}^+, \mathbf{U}_q^+$ denote the subalgebras generated only by the E_i's (or, in addition, the divided powers for $\mathbf{U}_{\mathcal{A}}^+$ and $\mathbf{U}_q^+$). The Cartan part $\mathbf{U}_v^0$ is as usual generated by K_i, K_i^{-1} for $i = 1, \ldots, n$. For the Cartan part $\mathbf{U}_{\mathcal{A}}^0$ one needs to be a little bit more careful, since it is generated by

$$\tilde{K}_{i,t} = \begin{bmatrix} K_i \\ t \end{bmatrix} = \prod_{s=1}^{t} \frac{K_i v^{d_i(1-s)} - K_i^{-1} v^{-d_i(1-s)}}{v^{d_i s} - v^{-d_i s}} \tag{1}$$

for $i = 1, \ldots, n$ and $t \in \mathbb{N}$ in addition to the usual generators K_i, K_i^{-1}. Similarly for $\mathbf{U}_q^0$. Roughly: the triangular decomposition can be proven by ordering F's to the left and E's to the right using the relations from Definition 2.2 (the hard part here is to show linear independence). Details can, for example, be found in [15, Chapter 4, Section 17] for the generic case and in [22, Theorem 8.3(iii)] for the other cases.

Note that, if $q=1$, then $\mathbf{U}_q$ modulo the ideal generated by $\{K_i-1 \mid i=1,\dots,n\}$ can be identified with the hyperalgebra of the semisimple algebraic group G over $\mathbb{K}$ associated to the Cartan matrix, see [16, Part I, Chapter 7.7].

2.2. **Representation theory of $\mathbf{U}_v$: the generic, semisimple case.** Let $\lambda \in X$ be a $\mathbf{U}_v$-weight. As usual, we identify λ with a *character of* $\mathbf{U}_v^0$ (an algebra homomorphism to $\mathbb{Q}(v)$) via

$$\lambda\colon \mathbf{U}_v^0 = \mathbb{Q}(v)[K_1^{\pm},\dots,K_n^{\pm}] \to \mathbb{Q}(v), \quad K_i^{\pm} \mapsto v^{\pm d_i\langle\lambda,\alpha_i^{\vee}\rangle}, \quad i=1,\dots,n.$$

Abusing notation, we use the same symbols for the $\mathbf{U}_v$-weights λ and the characters λ.

Moreover, if $\underline{\epsilon}=(\epsilon_1,\dots,\epsilon_n)\in\{\pm1\}^n$, then this can be viewed as a character of $\mathbf{U}_v^0$ via

$$\underline{\epsilon}\colon \mathbf{U}_v^0 = \mathbb{Q}(v)[K_1^{\pm},\dots,K_n^{\pm}] \to \mathbb{Q}(v), \quad K_i^{\pm} \mapsto \pm\epsilon_i, \quad i=1,\dots,n.$$

This extends to a character of $\mathbf{U}_v$ by setting $\underline{\epsilon}(E_i)=\underline{\epsilon}(F_i)=0$.

Every finite-dimensional $\mathbf{U}_v$-module M can be decomposed into

$$M = \bigoplus_{\lambda,\underline{\epsilon}} M_{\lambda,\underline{\epsilon}}, \tag{2}$$

where $M_{\lambda,\underline{\epsilon}} = \{m\in M \mid um=\lambda(u)\underline{\epsilon}(u)m, u\in\mathbf{U}_v^0\}$ and the direct sum above runs over all $\mathbf{U}_v$-weights $\lambda\in X$ and all $\underline{\epsilon}\in\{\pm1\}^n$, see [15, Chapter 5, Section 2].

Set $M_1=\bigoplus_\lambda M_{\lambda,(1,\dots,1)}$ and call a $\mathbf{U}_v$-module M a $\mathbf{U}_v$*-module of type 1* if $M_1=M$.

Example 2.6. If $\mathfrak{g}=\mathfrak{sl}_2$, then the $\mathbf{U}_v(\mathfrak{sl}_2)$-modules of type 1 are precisely those where K has eigenvalues v^k for $k\in\mathbb{Z}$ whereas type -1 means that K has eigenvalues $-v^k$. ▲

Given a $\mathbf{U}_v$-module M satisfying (2), we have $M\cong\bigoplus_{\underline{\epsilon}} M_1\otimes\underline{\epsilon}$. Thus, morally it suffices to study $\mathbf{U}_v$-modules of type 1, which we will do in this paper. From now on, *all appearing modules are assumed to be of type 1* (and we suppress to mention this in the following).

Proposition 2.7. (**Semisimplicity: the generic case**) The category $\mathbf{U}_v$-**Mod** consisting of finite-dimensional $\mathbf{U}_v$-modules is semisimple. □

Proof. This is [3, Corollary 7.7] or [15, Theorem 5.17]. ■

The simple modules in $\mathbf{U}_v$-**Mod** can be constructed as follows. For each $\lambda\in X^+$ set

$$\nabla_v(\lambda) = \mathrm{Ind}_{\mathbf{U}_v^-\mathbf{U}_v^0}^{\mathbf{U}_v}\mathbb{Q}(v)_\lambda,$$

called the *dual Weyl* $\mathbf{U}_v$-module associated to $\lambda\in X^+$. Here $\mathbb{Q}(v)_\lambda$ is the 1-dimensional $\mathbf{U}_v^-\mathbf{U}_v^0$-module determined by the character λ (and extended to $\mathbf{U}_v^-\mathbf{U}_v^0$ via $\lambda(F_i)=0$) and $\mathrm{Ind}_{\mathbf{U}_v^-\mathbf{U}_v^0}^{\mathbf{U}_v}(\cdot)$ is the induction functor from [3, Section 2], i.e. the functor

$$\mathrm{Ind}_{\mathbf{U}_v^-\mathbf{U}_v^0}^{\mathbf{U}_v}\colon \mathbf{U}_v^-\mathbf{U}_v^0\text{-}\mathbf{Mod}\to\mathbf{U}_v\text{-}\mathbf{Mod},\ M'\mapsto\mathcal{F}(\mathrm{Hom}_{\mathbf{U}_v^-\mathbf{U}_v^0}(\mathbf{U}_v,M'))$$

obtained by using the evident embedding of $\mathbf{U}_v^-\mathbf{U}_v^0$ into $\mathbf{U}_v$. Here the functor $\mathcal{F}$ (as given in [3, Subsection 2.2]) assigns to an arbitrary $\mathbf{U}_v$-module M the $\mathbf{U}_v$-module

$$\mathcal{F}(M) = \Big\{m\in\textstyle\bigoplus_{\lambda\in X}M_\lambda \mid E_i^{(r)}m=0=F_i^{(r)}m \quad \text{for all } i\in\mathbb{N} \text{ and for } r\gg0\Big\}$$

(which thus, defines $\mathcal{F}(M)$ for $M=\mathrm{Hom}_{\mathbf{U}_v^-\mathbf{U}_v^0}(\mathbf{U}_v,M')$).

It turns out that the $\nabla_v(\lambda)$ for $\lambda \in X^+$ form a complete set of non-isomorphic, simple $\mathbf{U}_v$-modules, see [15, Theorem 5.10]. For example, we see that the category $\mathbf{U}_v(\mathfrak{g})$-**Mod** is equivalent to the well-studied category of finite-dimensional $\mathfrak{g}$-modules.

By construction, the $\mathbf{U}_v$-modules $\nabla_v(\lambda)$ satisfy the *Frobenius reciprocity*, that is, we have

$$\mathrm{Hom}_{\mathbf{U}_v}(M, \nabla_v(\lambda)) \cong \mathrm{Hom}_{\mathbf{U}_v^- \mathbf{U}_v^0}(M, \mathbb{Q}(v)_\lambda) \quad \text{for all } M \in \mathbf{U}_v\text{-}\mathbf{Mod}. \tag{3}$$

Moreover, if we let $\mathrm{ch}(M)$ denote the *(formal) character of* $M \in \mathbf{U}_v$-**Mod**, that is,

$$\mathrm{ch}(M) = \sum_{\lambda \in X} (\dim(M_\lambda)) e^\lambda \in \mathbb{Z}[X],$$

where $M_\lambda = \{m \in M \mid um = \lambda(u)m, u \in \mathbf{U}_v^0\}$ (recall that the group algebra $\mathbb{Z}[X]$, where we regard X to be the free abelian group generated by the dominant (integral) $\mathbf{U}_q$-weights X^+, is known as the *character ring*). Then we have

$$\mathrm{ch}(\nabla_v(\lambda)) = \chi(\lambda) \in \mathbb{Z}[X] \quad \text{for all } \lambda \in X^+. \tag{4}$$

Here $\chi(\lambda)$ is the so-called *Weyl character* (that is, the classical character obtained from Weyl's character formula in the non-quantum case). A proof of the equation from (4) can be found in [3, Corollary 5.12 and the following remark], see also [15, Theorem 5.15].

In addition, we have a contravariant, character-preserving *duality functor*

$$\mathcal{D}\colon \mathbf{U}_v\text{-}\mathbf{Mod} \to \mathbf{U}_v\text{-}\mathbf{Mod} \tag{5}$$

that is defined on the $\mathbb{Q}(v)$-vector space level via $\mathcal{D}(M) = M^*$ (the $\mathbb{Q}(v)$-linear dual of M) and an action of $\mathbf{U}_v$ on $\mathcal{D}(M)$ is defined by

$$uf = m \mapsto f(\omega(S(u))m), \quad m \in M, u \in \mathbf{U}_v, f \in \mathcal{D}(M).$$

Here $\omega\colon \mathbf{U}_v \to \mathbf{U}_v$ is the automorphism of $\mathbf{U}_v$ which interchanges E_i and F_i and interchanges K_i and K_i^{-1} (see for example [15, Lemma 4.6]). Note that the $\mathbf{U}_v$-weights of M and $\mathcal{D}(M)$ coincide. In particular, we have $\mathcal{D}(\nabla_v(\lambda)) \cong \Delta_v(\lambda)$, where the latter $\mathbf{U}_q$-module is called the *Weyl* $\mathbf{U}_v$-module associated to $\lambda \in X^+$. Thus, the Weyl and dual Weyl $\mathbf{U}_v$-modules are related by duality, since clearly $\mathcal{D}^2 \cong \mathrm{id}_{\mathbf{U}_v\text{-}\mathbf{Mod}}$.

Example 2.8. If we have $\mathfrak{g} = \mathfrak{sl}_2$, then the dominant (integral) $\mathfrak{sl}_2$-weights X^+ can be identified with $\mathbb{N}$. Then the i-th Weyl module $\Delta_v(i)$ is the $i+1$-dimensional $\mathbb{Q}(v)$-vector space with a basis given by $m_0, \ldots, m_i$ and an $\mathbf{U}_v(\mathfrak{sl}_2)$-action defined by

$$Km_k = v^{i-2k} m_k, \;\; E^{(j)} m_k = \begin{bmatrix} i-k+j \\ j \end{bmatrix} m_{k-j} \quad \text{and} \quad F^{(j)} m_k = \begin{bmatrix} k+j \\ j \end{bmatrix} m_{k+j}, \tag{6}$$

with the convention that $m_{<0} = m_{>i} = 0$. For example, for $i = 3$ we can visualize $\Delta_v(3)$ as

$$\overset{v^{-3}\,\circlearrowright}{m_3} \underset{[3]}{\overset{[1]}{\rightleftarrows}} \overset{v^{-1}\,\circlearrowright}{m_2} \underset{[2]}{\overset{[2]}{\rightleftarrows}} \overset{v^{+1}\,\circlearrowright}{m_1} \underset{[1]}{\overset{[3]}{\rightleftarrows}} \overset{v^{+3}\,\circlearrowright}{m_0}, \tag{7}$$

where the action of E points to the right, the action of F to the left and K acts as a loop.

Note that the $\mathbf{U}_v(\mathfrak{sl}_2)$-action from (6) is already defined by the action of the generators $E, F, K^{\pm 1}$. For $\mathbf{U}_q(\mathfrak{sl}_2)$ the situation is different, see Example 2.12. ▲

2.3. **Representation theory of $\mathbf{U}_q$: the non-semisimple case.** As before in Subsection 2.1, we let q denote a fixed element of $\mathbb{K}^*$.

Let $\lambda \in X$ be a $\mathbf{U}_q$-weight. As above, we can identify λ with a character of $\mathbf{U}^0_{\mathcal{A}}$ via

$$\lambda\colon \mathbf{U}^0_{\mathcal{A}} \to \mathcal{A}, \quad K_i^{\pm} \mapsto v^{\pm d_i \langle \lambda, \alpha_i^{\vee} \rangle}, \quad \tilde{K}_{i,t} \mapsto \begin{bmatrix} \langle \lambda, \alpha_i^{\vee} \rangle \\ t \end{bmatrix}_{d_i}, \quad i = 1, \dots, n,\ t \in \mathbb{N},$$

which then also gives a character of $\mathbf{U}^0_q$. Here we use the definition of $\tilde{K}_{i,t}$ from (1). Abusing notation again, we use the same symbols for the $\mathbf{U}_q$-weights λ and the characters λ.

It is still true that any finite-dimensional $\mathbf{U}_q$-module M is a direct sum of its $\mathbf{U}_q$-weight spaces, see [3, Theorem 9.2]. Thus, if we denote by $\mathbf{U}_q$-**Mod** the category of finite-dimensional $\mathbf{U}_q$-modules, then

$$M = \bigoplus_{\lambda \in X} M_\lambda = \bigoplus_{\lambda \in X} \{m \in M \mid um = \lambda(u)m, u \in \mathbf{U}^0_q\} \quad \text{for } M \in \mathbf{U}_q\text{-}\mathbf{Mod}.$$

Hence, in complete analogy to the generic case discussed in Subsection 2.2, we can define the *(formal) character* $\chi(M)$ of $M \in \mathbf{U}_q$-**Mod** and the *(dual) Weyl* $\mathbf{U}_q$-module $\Delta_q(\lambda)$ (or $\nabla_q(\lambda)$) associated to $\lambda \in X^+$.

Using this notation, we arrive at the following which explains our main interest in the root of unity case. Note that we do not have any restrictions on the characteristic of $\mathbb{K}$ here.

Proposition 2.9. (**Semisimplicity: the specialized case**) The category $\mathbf{U}_q$-**Mod** consisting of finite-dimensional $\mathbf{U}_q$-modules is semisimple iff $q \in \mathbb{K}^* - \{1\}$ is not a root of unity or $q = \pm 1 \in \mathbb{K}$ with $\mathrm{char}(\mathbb{K}) = 0$. Moreover, if $\mathbf{U}_q$-**Mod** is semisimple, then the $\nabla_q(\lambda)$'s for $\lambda \in X^+$ form a complete set of pairwise non-isomorphic, simple $\mathbf{U}_q$-modules. □

Proof. For semisimplicity at non-roots of unity or $q = \pm 1, \mathrm{char}(\mathbb{K}) = 0$ see [3, Theorem 9.4] (or [21, Section 33.2] for $q = -1$). To see the converse: at roots of unity or in positive characteristic, most of the $\nabla_q(\lambda)$'s will not be semisimple (compare to Example 2.12). ■

In particular, if $\mathbb{K} = \mathbb{C}$, $q = 1$ and the Cartan datum comes from a simple Lie algebra $\mathfrak{g}$, then, $\mathbf{U}_1$-**Mod** is equivalent to the well-studied category of finite-dimensional $\mathfrak{g}$-modules.

Thus, Proposition 2.9 motivates the study of the case where q is a root of unity.

Assumption 2.10. If we want q to be a root of unity, then, to *avoid technicalities*, we assume that q is a primitive root of unity of odd order l (a treatment of the even case, that can be used to repeat everything in this paper in the case where l is even, can be found in [1]). Moreover, if we are in type G_2, then we, in addition, assume that l is prime to 3. ▲

In the root of unity case, by Proposition 2.9, our main category $\mathbf{U}_q$-**Mod** under study is no longer semisimple. In addition, the $\mathbf{U}_q$-modules $\nabla_q(\lambda)$ are in general not simple anymore, but they have a unique *simple socle* that we denote by $L_q(\lambda)$. By duality (note that the functor $\mathcal{D}(\cdot)$ from (5) carries over to $\mathbf{U}_q$-**Mod**), these are also the unique *simple heads* of the $\Delta_q(\lambda)$'s.

Proposition 2.11. (**Simple $\mathbf{U}_q$-modules: the non-semisimple case**) The socles $L_q(\lambda)$ of the $\nabla_q(\lambda)$'s are simple $\mathbf{U}_q$-modules $L_q(\lambda)$'s for $\lambda \in X^+$. They form a complete set of pairwise non-isomorphic, simple $\mathbf{U}_q$-modules in $\mathbf{U}_q$-**Mod**. □

Proof. See [3, Corollary 6.2 and Proposition 6.3]. ■

Example 2.12. With the same notation as in Example 2.8 but for q being a complex, primitive third root of unity, we have $[3] = 0$ and we can thus visualize $\Delta_q(3)$ as

$$\overset{q^{-3}\,\circlearrowleft}{m_3} \underset{0}{\overset{+1}{\rightleftarrows}} \overset{q^{-1}\,\circlearrowleft}{m_2} \underset{-1}{\overset{-1}{\rightleftarrows}} \overset{q^{+1}\,\circlearrowleft}{m_1} \underset{+1}{\overset{0}{\rightleftarrows}} \overset{q^{+3}\,\circlearrowleft}{m_0}, \qquad m_3 \overset{+1}{\longleftrightarrow} m_0 \tag{8}$$

where the action of E points to the right, the action of F to the left and K acts as a loop. In contrast to Example 2.8, the picture in (8) also shows the actions of the divided powers $E^{(3)}$ and $F^{(3)}$ as a long arrow connecting m_0 and m_3 (recall that these are additional generators of $\mathbf{U}_q(\mathfrak{sl}_2)$, see Example 2.5). Note also that, again in contrast to (7), some generators act on these basis vectors as zero. We also have $F^{(3)}m_1 = 0$ and $E^{(3)}m_2 = 0$. Thus, the $\mathbb{C}$-span of $\{m_1, m_2\}$ is now stable under the action of $\mathbf{U}_q(\mathfrak{sl}_2)$.

In particular, $L_q(3)$ is the $\mathbf{U}_q(\mathfrak{sl}_2)$-module obtained from $\Delta_q(3)$ as in (8) by taking the quotient of the $\mathbb{C}$-span of the set $\{m_1, m_2\}$. The latter can be seen to be isomorphic to $L_q(1)$.

We encourage the reader to work out its dual case $\nabla_q(3)$ (note that the zero $\mathbf{U}_q$-action arrows from above turn around). It turns out that $L_q(1)$ is a $\mathbf{U}_q$-submodule of $\Delta_q(3)$ and $L_q(3)$ is a $\mathbf{U}_q$-submodule of $\nabla_q(3)$ and these can be visualized as

$$L_q(1) \cong \overset{q^{-1}\,\circlearrowleft}{m_2} \underset{-1}{\overset{-1}{\rightleftarrows}} \overset{q^{+1}\,\circlearrowleft}{m_1} \quad \text{and} \quad L_q(3) \cong \overset{q^{-3}\,\circlearrowleft}{m_3^*} \underset{+1}{\overset{+1}{\rightleftarrows}} \overset{q^{+3}\,\circlearrowleft}{m_0^*},$$

where for $L_q(3)$ the displayed actions are via $E^{(3)}$ (to the right) and $F^{(3)}$ (to the left) instead of E, F as before. Note that $L_q(1)$ and $L_q(3)$ have both dimension 2. This has no analogon in the generic $\mathfrak{sl}_2$ case where all simple $\mathbf{U}_v$-modules $L_v(i)$ have different dimensions. ▲

A non-trivial fact (which relies on the q-version of the so-called *Kempf's vanishing theorem*, see [29, Theorem 5.5]) is that the characters of the $\nabla_q(\lambda)$'s are still given by Weyl's character formula as in (4) (by duality, similar for the $\Delta_q(\lambda)$'s). In particular, $\dim(\nabla_q(\lambda)_\lambda) = 1$ and $\dim(\nabla_q(\lambda)_\mu) = 0$ unless $\mu \leq \lambda$ (again similar for the $\Delta_q(\lambda)$'s). On the other hand, the characters of the simple modules $L_q(\lambda)$ are only known if $\mathrm{char}(\mathbb{K}) = 0$ (and "big enough" l). In that case, certain *Kazhdan-Lusztig polynomials* determine the character $\mathrm{ch}(L_q(\lambda))$, see for example [33, Theorem 6.4 and 7.1] and the references therein.

3. Tilting modules

In the present section we recall a few facts from the theory of $\mathbf{U}_q$-tilting modules. In the semisimple case all $\mathbf{U}_q$-modules in $\mathbf{U}_q$-**Mod** are $\mathbf{U}_q$-tilting modules. Hence, the theory of $\mathbf{U}_q$-tilting modules is kind of redundant in this case. In the non semisimple case however the theory of $\mathbf{U}_q$-tilting modules is extremely rich and a source of neat combinatorics. For brevity, we only provide proofs if we need the arguments of the proofs in what follows. For more details see for example [11].

3.1. **$\mathbf{U}_q$-modules with a Δ_q- and a ∇_q-filtration.** Recall that $\nabla_q(\lambda)$ has a simple socle and $\Delta_q(\lambda)$ has a simple head, both isomorphic to $L_q(\lambda)$. Thus, there is an (up to scalars) unique $\mathbf{U}_q$-homomorphism

$$c^\lambda \colon \Delta_q(\lambda) \to \nabla_q(\lambda) \tag{9}$$

which sends the head to the socle. To see this, note that we have, by Frobenius reciprocity from (3) (to be more precise, the q-version of it which can be found in [3, Proposition 2.12]),

$$\mathrm{Hom}_{\mathbf{U}_q}(\Delta_q(\lambda),\nabla_q(\lambda))\cong\mathrm{Hom}_{\mathbf{U}_q^-\mathbf{U}_q^0}(\Delta_q(\lambda),\mathbb{K}_\lambda)$$

which gives $\dim(\mathrm{Hom}_{\mathbf{U}_q}(\Delta_q(\lambda),\nabla_q(\lambda)))=1$, since, by construction, $\Delta_q(\lambda)_\lambda\cong\mathbb{K}$.

This gives us the following (we have to enlarge the category $\mathbf{U}_q$-**Mod** by non-necessarily finite-dimensional $\mathbf{U}_q$-modules to have enough injectives such that the $\mathrm{Ext}^i_{\mathbf{U}_q}$-functors make sense by using q-analogous arguments as in [16, Part I, Chapter 3]).

Theorem 3.1. (**Ext-vanishing**) We have for all $\lambda,\mu\in X^+$ that

$$\mathrm{Ext}^i_{\mathbf{U}_q}(\Delta_q(\lambda),\nabla_q(\mu))\cong\begin{cases}\mathbb{K}c^\lambda, & \text{if } i=0 \text{ and } \lambda=\mu,\\ 0, & \text{else.}\end{cases}$$ □

Proof. Denote by $\mathcal{C}^0$ and $\mathcal{C}^-$ the categories of integrable $\mathbf{U}_q^0$ and $\mathbf{U}_q^0\mathbf{U}_q^-$-modules respectively. Then a $\mathbf{U}_q^0$-module M belongs to $\mathcal{C}^0$ iff $M=\bigoplus_{\lambda\in X}M_\lambda$ and a $\mathbf{U}_q^0\mathbf{U}_q^-$-module N belongs to $\mathcal{C}^-$ iff $N\in\mathcal{C}^0$ and there exists for all $n\in N$ some $r\in\mathbb{N}$ such that $F_i^{(r)}n=0$ for all $i=1,\dots,n$. Moreover, let $\mathcal{C}$ denote the category of integrable $\mathbf{U}_q$-modules[1].

Below we shall need a certain induction functor. To this end, recall the functor $\mathcal{F}$ which to an arbitrary $\mathbf{U}_q^0\mathbf{U}_q^-$-module $M\in\mathcal{C}^-$ assigns

$$\mathcal{F}(M)=\{m\in\bigoplus_{\lambda\in X}M_\lambda\mid F_i^{(r)}m=0\quad\text{for all } i\in\mathbb{N}\text{ and for } r\gg 0\},$$

see Subsection 2.2 in [3]. Then

$$\mathrm{Ind}_{\mathcal{C}^0}^{\mathcal{C}^-}:\mathcal{C}^0\to\mathcal{C}^-,\ M\mapsto\mathcal{F}(\mathrm{Hom}_{\mathcal{C}^0}(\mathbf{U}_q^0\mathbf{U}_q^-,M))\tag{10}$$

obtained by using the evident embedding of $\mathbf{U}_q^0$ into $\mathbf{U}_q^0\mathbf{U}_q^-$, see Subsection 2.4 in [3].

Recall from Subsection 2.11 in [3] that this functor is exact and that

$$\mathrm{Ind}_{\mathcal{C}^0}^{\mathcal{C}^-}(M)=\bigoplus_{\lambda\in X}(M_\lambda\otimes\mathbb{K}[\mathbf{U}_q^-]_{-\lambda}).$$

Here $\mathbb{K}[\mathbf{U}_q^-]$ is the quantum coordinate algebra for $\mathbf{U}_q^-$ (see Subsection 1.8 in [3]). Note in particular, that the weights $\lambda\in X$ of $\mathbb{K}[\mathbf{U}_q^-]$ satisfy $\lambda\geq 0$ with $\lambda=0$ occurring with multiplicity 1.

If $\lambda\in X$, then we denote by $\mathbb{K}_\lambda\in\mathcal{C}^0$ the corresponding 1-dimensional $\mathbf{U}_q^0$-module. This modules extends to $\mathbf{U}_q^0\mathbf{U}_q^-$ by letting all $F_i^{(r)}$'s act trivially for $r>0$ and we, by abuse of notation, denote this $\mathbf{U}_q^0\mathbf{U}_q^-$-module also by $\mathbb{K}_\lambda$.

We claim that

$$\mathrm{Ext}^i_{\mathcal{C}^-}(\mathbb{K}_0,\mathbb{K}_\lambda)\cong\begin{cases}\mathbb{K}, & \text{if } i=0 \text{ and } \lambda=0,\\ 0, & \text{if } i>0 \text{ and } \lambda\not<0,\end{cases}\tag{11}$$

for all $\lambda\in X$.

[1]We need to go to the categories of integrable modules due to the fact that the injective modules we use are usually infinite-dimensional. Furthermore, we take $\mathbf{U}_q^0\mathbf{U}_q^-$ here instead of $\mathbf{U}_q^-\mathbf{U}_q^0$ as in Subsection 2.2 of [4] since we want to consider $\mathbf{U}_q^0\mathbf{U}_q^-$ as a left $\mathbf{U}_q^0$-module for the induction functor.

The $i = 0$ part of this claim is clear. To check the $i > 0$ part, we construct an injective resolution of $\mathbb{K}_\lambda$ as follows.

We set $I_0(\lambda) = \mathrm{Ind}_{\boldsymbol{\mathcal{C}}^0}^{\boldsymbol{\mathcal{C}}^-}(\mathbb{K}_\lambda)$. Note that $\mathbb{K}_\lambda$ is a $\mathbf{U}_q^0\mathbf{U}_q^-$-submodule of $I_0(\lambda)$. Thus, we may define the quotient $Q_1(\lambda) = I_0(\lambda)/Q_0(\lambda)$ by setting $Q_0(\lambda) = \mathbb{K}_\lambda$.

This pattern can be repeated: define for $k > 0$ recursively

$$I_k(\lambda) = \mathrm{Ind}_{\boldsymbol{\mathcal{C}}^0}^{\boldsymbol{\mathcal{C}}^-}(Q_k(\lambda)), \quad \text{with } Q_k(\lambda) = I_{k-1}(\lambda)/Q_{k-1}(\lambda)$$

and obtain

$$0 \hookrightarrow \mathbb{K}_\lambda \hookrightarrow I_0(\lambda) \longrightarrow I_1(\lambda) \longrightarrow \cdots . \tag{12}$$

All $\mathbf{U}_q^0$-modules in $\boldsymbol{\mathcal{C}}^0$ are clearly injective and the induction functor from (10) takes injective $\mathbf{U}_q^0$-modules to injective $\mathbf{U}_q^0\mathbf{U}_q^-$-modules (see Corollary 2.13 in [3]). Thus, (12) is an injective resolution of $\mathbb{K}_\lambda$ in $\boldsymbol{\mathcal{C}}^-$. Moreover, by the above observation on the weights of $\mathbb{K}[\mathbf{U}_q^-]$, we get

$$I_0(\lambda)_\mu = 0 \quad \text{for all } \mu \ngeq 0$$

and

$$I_k(\lambda)_\mu = 0 \quad \text{for all } \mu \not> 0,\ k > 0.$$

It follows that $\mathrm{Hom}_{\boldsymbol{\mathcal{C}}^-}(\mathbb{K}_0, I_k(\lambda)) = 0$ for $k > 0$ which shows the second line in (11).

Note now that

$$\mathrm{Ext}^i_{\boldsymbol{\mathcal{C}}^-}(\mathbb{K}_\mu, \mathbb{K}_\lambda) \cong \mathrm{Ext}^i_{\boldsymbol{\mathcal{C}}^-}(\mathbb{K}_0, \mathbb{K}_{\lambda-\mu}) \tag{13}$$

for all $i \in \mathbb{N}$ and all $\lambda, \mu \in X$.

Let $M \in \boldsymbol{\mathcal{C}}^-$ be finite-dimensional such that no weight of M is strictly bigger than $\lambda \in X$. Then (11) and (13) imply

$$\mathrm{Ext}^i_{\boldsymbol{\mathcal{C}}^-}(M, \mathbb{K}_\lambda) = 0 \quad \text{for all } k > 0. \tag{14}$$

We are now aiming to prove the Ext-vanishing theorem. Recall that $\nabla_q(\lambda) = \mathrm{Ind}_{\boldsymbol{\mathcal{C}}^-}^{\boldsymbol{\mathcal{C}}}\mathbb{K}_\lambda$. From the q-version of Kempf's vanishing theorem (see Theorem 5.5 in [29]) we get

$$\mathrm{Ext}^i_{\boldsymbol{\mathcal{C}}}(\Delta_q(\lambda), \nabla_q(\mu)) \cong \mathrm{Ext}^i_{\boldsymbol{\mathcal{C}}^-}(\Delta_q(\lambda), \mathbb{K}_\mu). \tag{15}$$

Thus, the Ext-vanishing follows for $\mu \not< \lambda$ from (14). So let $\mu < \lambda$. Recall from Subsection 2.2 of [4] that the character-preserving duality functor $\mathcal{D}(\cdot)$ satisfies $\mathcal{D}(\nabla_q(\lambda)) \cong \Delta_q(\lambda)$ and $\mathcal{D}(\Delta_q(\lambda)) \cong \nabla_q(\lambda)$ for all $\lambda \in X^+$. This gives

$$\mathrm{Ext}^i_{\boldsymbol{\mathcal{C}}}(\Delta_q(\lambda), \nabla_q(\mu)) \cong \mathrm{Ext}^i_{\boldsymbol{\mathcal{C}}}(\Delta_q(\mu), \nabla_q(\lambda)).$$

Thus, we can conclude as before, since now $\lambda \not< \mu$. Finally, if $i = 0$, then we get from (15) that

$$\mathrm{Hom}_{\boldsymbol{\mathcal{C}}}(\Delta_q(\lambda), \nabla_q(\mu)) \cong \mathrm{Hom}_{\boldsymbol{\mathcal{C}}^-}(\Delta_q(\lambda), \mathbb{K}_\mu) = \begin{cases} \mathbb{K}, & \text{if } \lambda = \mu, \\ 0, & \mu \nleq \lambda. \end{cases}$$

If $\mu < \lambda$, then we apply $\mathcal{D}$ as before which finally shows that

$$\mathrm{Hom}_{\boldsymbol{\mathcal{C}}}(\Delta_q(\lambda), \nabla_q(\mu)) \cong \begin{cases} \mathbb{K}c^\lambda, & \lambda = \mu, \\ 0, & \text{else}, \end{cases}$$

for all $\lambda, \mu \in X^+$. This proves the statement since $\mathbf{U}_q$-**Mod** is a full subcategory of $\boldsymbol{\mathcal{C}}$. ∎

Definition 3.2. (**Δ_q- and ∇_q-filtration**) We say that a $\mathbf{U}_q$-module M has a *Δ_q-filtration* if there exists some $k \in \mathbb{N}$ and a descending sequence of $\mathbf{U}_q$-submodules

$$M = M_0 \supset M_1 \supset \cdots \supset M_{k'} \supset \cdots \supset M_{k-1} \supset M_k = 0,$$

such that for all $k' = 0, \ldots, k-1$ we have $M_{k'}/M_{k'+1} \cong \Delta_q(\lambda_{k'})$ for some $\lambda_{k'} \in X^+$.

A *∇_q-filtration* is defined similarly, but using $\nabla_q(\lambda)$ instead of $\Delta_q(\lambda)$ and an ascending sequence of $\mathbf{U}_q$-submodules, that is,

$$0 = M_0 \subset M_1 \subset \cdots \subset M_{k'} \subset \cdots \subset M_{k-1} \subset M_k = M,$$

such that for all $k' = 0, \ldots, k-1$ we have $M_{k'+1}/M_{k'} \cong \nabla_q(\lambda_{k'})$ for some $\lambda_{k'} \in X^+$. ▲

Clearly a $\mathbf{U}_q$-module M has a Δ_q-filtration iff its dual $\mathcal{D}(M)$ has a ∇_q-filtration.

Example 3.3. The simple $\mathbf{U}_q$-module $L_q(\lambda)$ has a Δ_q-filtration iff $L_q(\lambda) \cong \Delta_q(\lambda)$. In that case we have also $L_q(\lambda) \cong \nabla_q(\lambda)$ and thus, $L_q(\lambda)$ has a ∇_q-filtration as well. ▲

A corollary of the Ext-vanishing theorem is the following.

Corollary 3.4. Let $M, N \in \mathbf{U}_q$-**Mod** and $\lambda \in X^+$. Assume that M has a Δ_q-filtration and N has a ∇_q-filtration.

(a) We have $\dim(\mathrm{Hom}_{\mathbf{U}_q}(M, \nabla_q(\lambda))) = (M : \Delta_q(\lambda)) = |\{k' \mid \lambda_{k'} = \lambda\}|$.
(b) We have $\dim(\mathrm{Hom}_{\mathbf{U}_q}(\Delta_q(\lambda), N)) = (N : \nabla_q(\lambda)) = |\{k' \mid \lambda_{k'} = \lambda\}|$.

Here the $\mathbf{U}_q$-weights $\lambda_{k'}$ are as in Definition 3.2. In particular, the multiplicities $(M : \Delta_q(\lambda))$ and $(N : \nabla_q(\lambda))$ are independent of the choice of filtration. □

Note that the proof of Corollary 3.4 below gives a method to find and construct bases of $\mathrm{Hom}_{\mathbf{U}_q}(M, \nabla_q(\lambda))$ and $\mathrm{Hom}_{\mathbf{U}_q}(\Delta_q(\lambda), N)$ respectively.

Proof. Let k be the length of the Δ_q-filtration of M. If $k = 1$, then

$$\dim(\mathrm{Hom}_{\mathbf{U}_q}(M, \nabla_q(\lambda))) = (M : \Delta_q(\lambda)) \tag{16}$$

follows from the uniqueness of c^λ from (9). Otherwise, we take the short exact sequence

$$0 \longrightarrow M' \hookrightarrow M \twoheadrightarrow \Delta_q(\mu) \longrightarrow 0$$

for some $\mu \in X^+$. Since both sides of (16) are additive with respect to short exact sequences by Theorem 3.1, the claim in (a) follows by induction. Similarly for (b) by duality. ■

In fact, following Donkin [10] who obtained the result below in the modular case, we can state two useful consequences of the Ext-vanishing theorem. These are very useful criteria to determine if given $\mathbf{U}_q$-modules M or N have a Δ_q- or ∇_q-filtration respectively.

Proposition 3.5. (**Ext-criteria**) Let $M, N \in \mathbf{U}_q$-**Mod**. Then the following are equivalent.

(a) The $\mathbf{U}_q$-module M has a Δ_q-filtration (respectively N has a ∇_q-filtration).
(b) We have $\mathrm{Ext}^i_{\mathbf{U}_q}(M, \nabla_q(\lambda)) = 0$ (respectively $\mathrm{Ext}^i_{\mathbf{U}_q}(\Delta_q(\lambda), N) = 0$) for all $\lambda \in X^+$ and all $i > 0$.
(c) We have $\mathrm{Ext}^1_{\mathbf{U}_q}(M, \nabla_q(\lambda)) = 0$ (respectively $\mathrm{Ext}^1_{\mathbf{U}_q}(\Delta_q(\lambda), N) = 0$) for all $\lambda \in X^+$. □

Proof. As usual: we are lazy and only show the statement about the Δ_q-filtrations and leave the other to the reader.

Suppose the $\mathbf{U}_q$-module M has a Δ_q-filtration. Then, by the results from Theorem 3.1, $\mathrm{Ext}^i_{\mathbf{U}_q}(M, \nabla_q(\lambda)) = 0$ for all $\lambda \in X^+$ and all $i > 0$ (which shows that (a) implies (b)).

Since (b) clearly implies (c), we only need to show that (c) implies (a).

To this end, suppose the $\mathbf{U}_q$-module M satisfies $\mathrm{Ext}^1_{\mathbf{U}_q}(M, \nabla_q(\lambda)) = 0$ for all $\lambda \in X^+$. We inductively, with respect to the filtration (by simples $L_q(\lambda)$) length $\ell(M)$ of M, construct the Δ_q-filtration for M.

So, by Proposition 2.11 in [4], we can assume that $M = L_q(\lambda)$ for some $\lambda \in X^+$.

Consider the short exact sequence

$$0 \longrightarrow \ker(\mathrm{pro}^\lambda) \hookrightarrow \Delta_q(\lambda) \xrightarrow{\mathrm{pro}^\lambda} L_q(\lambda) \longrightarrow 0. \tag{17}$$

By Theorem 3.1 we get from (17) a short exact sequence

$$0 \longleftarrow \mathrm{Hom}_{\mathbf{U}_q}(\ker(\mathrm{pro}^\lambda), \nabla_q(\mu)) \twoheadleftarrow \mathrm{Hom}_{\mathbf{U}_q}(\Delta_q(\lambda), \nabla_q(\mu)) \hookleftarrow \mathrm{Hom}_{\mathbf{U}_q}(L_q(\lambda), \nabla_q(\mu)) \longleftarrow 0$$

for all $\mu \in X^+$. Note that, by Theorem 3.1 again, $\mathrm{Hom}_{\mathbf{U}_q}(\Delta_q(\lambda), \nabla_q(\mu))$ is zero if $\mu \neq \lambda$ and 1-dimensional if $\mu = \lambda$. Moreover, by construction, $\mathrm{Hom}_{\mathbf{U}_q}(L_q(\lambda), \nabla_q(\lambda))$ is 1-dimensional. Thus, $\mathrm{Hom}_{\mathbf{U}_q}(\ker(\mathrm{pro}^\lambda), \nabla_q(\mu)) = 0$ for all $\mu \in X^+$ showing that $\ker(\mathrm{pro}^\lambda) = 0$. This, by (17), implies $\Delta_q(\lambda) \cong L_q(\lambda)$.

Now assume that $\ell(M) > 1$. Choose $\lambda \in X^+$ minimal such that $\mathrm{Hom}_{\mathbf{U}_q}(M, L_q(\lambda)) \neq 0$. As before in (17), we consider the canonical projection $\mathrm{pro}^\lambda \colon \Delta_q(\lambda) \twoheadrightarrow L_q(\lambda)$ and its kernel $\ker(\mathrm{pro}^\lambda)$.

Note now that $\mathrm{Ext}^1_{\mathbf{U}_q}(M, \nabla_q(\lambda)) = 0$ implies $\mathrm{Ext}^1_{\mathbf{U}_q}(M, \ker(\mathrm{pro}^\lambda)) = 0$:

Assume the contradiction. Then we can find a composition factor $L_q(\mu)$ for $\mu < \lambda$ of $\ker(\mathrm{pro}^\lambda)$ such that $\mathrm{Ext}^1_{\mathbf{U}_q}(M, L_q(\mu)) \neq 0$. Then the exact sequence

$$\mathrm{Hom}_{\mathbf{U}_q}(M, \nabla_q(\mu)/L_q(\mu)) \longrightarrow \mathrm{Ext}^1_{\mathbf{U}_q}(M, L_q(\mu)) \neq 0 \longrightarrow \mathrm{Ext}^1_{\mathbf{U}_q}(M, \nabla_q(\mu)) = 0$$

implies that $\mathrm{Hom}_{\mathbf{U}_q}(M, \nabla_q(\mu)/L_q(\mu)) \neq 0$. Since $\mu < \lambda$, this gives a contradiction to the minimality of λ.

Hence, any non-zero $\mathbf{U}_q$-homomorphism $\mathrm{pro} \in \mathrm{Hom}_{\mathbf{U}_q}(M, L_q(\lambda))$ lifts to a surjection

$$\overline{\mathrm{pro}} \colon M \twoheadrightarrow \Delta_q(\lambda).$$

By assumption and Theorem 3.1 we have $\mathrm{Ext}^1_{\mathbf{U}_q}(M, \nabla_q(\mu)) = 0 = \mathrm{Ext}^1_{\mathbf{U}_q}(\Delta_q(\lambda), \nabla_q(\mu))$ for all $\mu \in X^+$. Thus, we have $\mathrm{Ext}^1_{\mathbf{U}_q}(\ker(\overline{\mathrm{pro}}), \nabla_q(\mu)) = 0$ for all $\mu \in X^+$ and we can proceed by induction (since $\ell(\ker(\overline{\mathrm{pro}})) < \ell(M)$, by construction). ■

Example 3.6. Let us come back to our favorite example, i.e. q being a complex, primitive third root of unity for $\mathbf{U}_q(\mathfrak{sl}_2)$. The simple $\mathbf{U}_q$-module $L_q(3)$ does neither have a Δ_q- nor a ∇_q-filtration (compare Example 2.12 with Example 3.3). This can also be seen with Proposition 3.5, because $\mathrm{Ext}^1_{\mathbf{U}_q}(L_q(3), L_q(1))$ is not trivial: by Example 2.12 from above we have $\Delta_q(1) \cong L_q(1) \cong \nabla_q(1)$, but

$$0 \longrightarrow L_q(1) \hookrightarrow \Delta_q(3) \twoheadrightarrow L_q(3) \longrightarrow 0$$

does not split. Analogously, $\mathrm{Ext}^1_{\mathbf{U}_q}(L_q(1), L_q(3)) \neq 0$ by duality. ▲

3.2. **$\mathbf{U}_q$-tilting modules.** Following Donkin [10], we are now ready to define the *category of $\mathbf{U}_q$-tilting modules* that we denote by $\boldsymbol{\mathcal{T}}$. This category is our main object of study.

Definition 3.7. (**Category of $\mathbf{U}_q$-tilting modules**) The category $\boldsymbol{\mathcal{T}}$ is the full subcategory of $\mathbf{U}_q$-**Mod** whose objects are all $\mathbf{U}_q$-*tilting modules*, that is, $\mathbf{U}_q$-modules T which have both, a Δ_q- and a ∇_q-filtration. ▲

From Proposition 3.5 we obtain directly an important statement.

Corollary 3.8. Let $T \in \mathbf{U}_q$-**Mod**. Then

$$T \in \boldsymbol{\mathcal{T}} \quad \text{iff} \quad \mathrm{Ext}^1_{\mathbf{U}_q}(T, \nabla_q(\lambda)) = 0 = \mathrm{Ext}^1_{\mathbf{U}_q}(\Delta_q(\lambda), T) \quad \text{for all } \lambda \in X^+.$$

Moreover, the corresponding higher Ext-groups vanish as well. ■

Recall the contravariant, character preserving functor $\mathcal{D}\colon \mathbf{U}_q\text{-}\mathbf{Mod} \to \mathbf{U}_q\text{-}\mathbf{Mod}$ from (5). Clearly, by Corollary 3.8, $T \in \boldsymbol{\mathcal{T}}$ iff $\mathcal{D}(T) \in \boldsymbol{\mathcal{T}}$. Thus, $\mathcal{D}(\cdot)$ restricts to a functor $\mathcal{D}\colon \boldsymbol{\mathcal{T}} \to \boldsymbol{\mathcal{T}}$. In fact, we show below in Corollary 3.13, that the functor $\mathcal{D}(\cdot)$ restricts to (a functor isomorphic to) the identity functor on objects of $\boldsymbol{\mathcal{T}}$.

Example 3.9. The $L_q(\lambda)$ are $\mathbf{U}_q$-tilting modules iff $\Delta_q(\lambda) \cong L_q(\lambda) \cong \nabla_q(\lambda)$.

Coming back to our favourite example, that is the case $\mathfrak{g} = \mathfrak{sl}_2$ and q is a complex, primitive third root of unity: a direct computation using similar reasoning as in Example 2.12 (that is, the appearance of some actions equals zero as in (8)) shows that $L_q(i)$ is a $\mathbf{U}_q$-tilting module iff $i = 0, 1$ or $i \equiv -1 \bmod 3$. More general: if q is a complex, primitive l-th root of unity, then $L_q(i)$ is a $\mathbf{U}_q$-tilting module iff $i = 0, \dots, l-1$ or $i \equiv -1 \bmod l$. ▲

Proposition 3.10. $\boldsymbol{\mathcal{T}}$ is a Krull-Schmidt category, closed under duality $\mathcal{D}(\cdot)$ and under finite direct sums. Furthermore, $\boldsymbol{\mathcal{T}}$ is closed under finite tensor products. □

Proof. That $\boldsymbol{\mathcal{T}}$ is Krull-Schmidt is immediate. By Corollary 3.8 in [4] we see that $\boldsymbol{\mathcal{T}}$ is closed under duality $\mathcal{D}(\cdot)$ and under finite direct sums.

Only that $\boldsymbol{\mathcal{T}}$ is closed under finite tensor products remains to be proven. By duality, this reduces to show the statement that, given $M, N \in \mathbf{U}_q$-**Mod** where both have a ∇_q-filtration, then $M \otimes N$ has a ∇_q-filtration. In addition, this reduces further to the statement

$$\nabla_q(\lambda) \otimes \nabla_q(\mu) \quad \text{has a } \nabla_q\text{-filtration for all } \lambda, \mu \in X^+. \tag{18}$$

In this note we give a proof of (18) in type A where it is true that ω_i is minuscule for all $i = 1, \dots, n$. The idea of the proof goes back to [34]. As we point out, this case (and the arguments used here) are enough for most of the examples considered in [4]. For the general case the only known proofs of (18) rely on crystal bases, see Theorem 3.3 in [25] or alternatively Corollary 1.9 in [18].

Proof of (18) in types A: we claim that is suffices[2] to show

$$\nabla_q(\lambda) \otimes \nabla_q(\omega_i) \quad \text{has a } \nabla_q\text{-filtration for all } \lambda \in X^+ \text{ and all } i = 1, \dots, n. \tag{19}$$

To see that (19) implies (18) we shall work with the the $\mathbb{Q}_{\geq 0}$-version of the partial ordering $\leq$ on X given by $\mu \leq_{\mathbb{Q}} \lambda$ iff $\lambda - \mu$ is a $\mathbb{Q}_{\geq 0}$-valued linear combination of the simple roots, that

[2] Note that our proof of the fact that (19) implies (18) works in all types.

is, $\lambda-\mu=\sum_{i=1}^{n}a_i\alpha_i$ with $a_i\in\mathbb{Q}_{\geq 0}$. Clearly $\mu\leq_{\mathbb{Q}}\lambda$ implies $\mu\leq\lambda$. Note that $0\leq_{\mathbb{Q}}\omega_i$ for all $i=1,\dots,n$ which means that 0 is the unique minimal $\mathbf{U}_q$-weight in X^+ with respect to $\leq_{\mathbb{Q}}$.

Assume now that (19) holds. We shall prove (18) by induction with respect to $\leq_{\mathbb{Q}}$. For $\lambda=0$ we have $\nabla_q(\lambda)\cong\mathbb{K}$ and there is nothing to prove.

So let $\lambda\in X^+-\{0\}$ and assume that (18) holds for all $\mu<_{\mathbb{Q}}\lambda$. Note that there exists a fundamental $\mathbf{U}_q$-weight ω such that $\mu=\lambda-\omega$. This means that, by (19), we have a short exact sequence of the form

$$0\longrightarrow M\hookrightarrow\nabla_q(\mu)\otimes\nabla_q(\omega)\twoheadrightarrow\nabla_q(\lambda)\longrightarrow 0. \tag{20}$$

Here the $\mathbf{U}_q$-module M has a ∇_q-filtration. By induction, $\nabla_q(\lambda')\otimes\nabla_q(\mu)$ has a ∇_q-filtration for all $\lambda'\in X^+$ and so, by (19), has $\nabla_q(\lambda')\otimes\nabla_q(\mu)\otimes\nabla_q(\omega)$. Moreover, the ∇_q-factors of M have the form $\nabla_q(\nu)$ for $\nu<_{\mathbb{Q}}\lambda$. Hence, by the induction hypothesis, we have that $\nabla_q(\lambda')\otimes M$ has a ∇_q-filtration for all $\lambda'\in X^+$. Thus, tensoring (20) with $\nabla_q(\lambda')$ from the left gives a ∇_q-filtration for the two leftmost terms. Therefore, also the third has a ∇_q-filtration (by Proposition 3.5). This shows that (19) implies (18).

From now we have to assume that the fundamental $\mathbf{U}_q$-weights are minuscule. By the above, it remains to show (19). For this purpose, recall that

$$\nabla_v(\lambda)=\mathrm{Ind}_{\mathbf{U}_v^-\mathbf{U}_v^0}^{\mathbf{U}_v}\mathbb{K}_\lambda.$$

By the tensor identity (see Proposition 2.16 in [3]) this implies

$$\nabla_q(\lambda)\otimes\nabla_q(\omega_i)\cong\mathrm{Ind}_{\mathbf{U}_v^-\mathbf{U}_v^0}^{\mathbf{U}_v}(\mathbb{K}_\lambda\otimes\nabla_q(\omega_i))$$

for all $i=1,\dots,n$. Now take a filtration of $\mathbb{K}_\lambda\otimes\nabla_q(\omega_i)$ of the form

$$0=M_0\subset M_1\subset\cdots\subset M_{k'}\subset\cdots\subset M_{k-1}\subset M_k=\mathbb{K}_\lambda\otimes\nabla_q(\omega_i), \tag{21}$$

such that for all $k'=0,\dots,k-1$ we have $M_{k'+1}/M_{k'}\cong\mathbb{K}_{\lambda_{k'+1}}$ for some $\lambda_{k'}\in X^+$. Thus, the set $\{\lambda_{k'}\mid k'=1,\dots,k\}$ is the set of $\mathbf{U}_q$-weights of $\mathbb{K}_\lambda\otimes\nabla_q(\omega_i)$. But the $\mathbf{U}_q$-weights of $\nabla_q(\omega_i)$ are of the form $\{w(\omega_i)\mid w\in W\}$ where W is the Weyl group associated to $\mathbf{U}_q$. Hence, $\lambda_{k'}=\lambda+w_{k'}(\omega_i)$ for some $w_{k'}\in W$. We get[3]

$$\langle\lambda_{k'},\alpha_j^\vee\rangle=\langle\lambda,\alpha_j^\vee\rangle+\langle\omega_i,w_{k'}^{-1}(\alpha_j^\vee)\rangle\geq 0+(-1)=-1$$

for all $j=1,\dots,n$. Said otherwise, $\lambda_{k'}+\rho\in X^+$. Hence, the q-version of Kempf's vanishing theorem (see Theorem 5.5 in [29]) shows that we can apply the functor $\mathrm{Ind}_{\mathbf{U}_v^-\mathbf{U}_v^0}^{\mathbf{U}_v}(\cdot)$ to (21) to obtain a ∇_q-filtration of $\nabla_q(\lambda)\otimes\nabla_q(\omega_i)$. Thus, we obtain (19). ■

In particular, for $\mathfrak{g}$ of type A, the proof of Proposition 3.10 gives us the special case that $T=\Delta_q(\omega_{i_1})\otimes\cdots\otimes\Delta_q(\omega_{i_d})$ is a $\mathbf{U}_q$-tilting module for any $i_k\in\{1,\dots,n\}$. Moreover, the proof of Proposition 3.10 generalizes: using similar arguments, one can prove that, given the vector representation $V=\Delta_q(\omega_1)$ and $\mathfrak{g}$ of type A, C or D, then $T=V^{\otimes d}$ is a $\mathbf{U}_q$-tilting module. Even more general, the arguments also generalize to show that, given the $\mathbf{U}_q$-module $V=\Delta_q(\lambda)$ with $\lambda\in X^+$ minuscule, then $T=V^{\otimes d}$ is a $\mathbf{U}_q$-tilting module.

The indecomposable $\mathbf{U}_q$-modules in $\boldsymbol{\mathcal{T}}$, that we denote by $T_q(\lambda)$, are indexed by the dominant (integral) $\mathbf{U}_q$-weights $\lambda\in X^+$ (see Proposition 3.11 below). The $\mathbf{U}_q$-tilting module $T_q(\lambda)$

[3] Here we need that the ω_i are minuscule because we need that $\langle\omega_i,w_{k'}^{-1}(\alpha_j^\vee)\rangle\geq -1$.

is determined by the property that it is indecomposable with λ as its unique maximal weight. Then λ appears in fact with multiplicity 1.

The following classification is, in the modular case, due to Ringel [28] and Donkin [10].

Proposition 3.11. (**Classification of the indecomposable $\mathbf{U}_q$-tilting modules**) For each $\lambda \in X^+$ there exists an indecomposable $\mathbf{U}_q$-tilting module, that we denote by $T_q(\lambda)$, with $\mathbf{U}_q$-weight spaces $T_q(\lambda)_\mu = 0$ unless $\mu \leq \lambda$. Moreover, $T_q(\lambda)_\lambda \cong \mathbb{K}$. In addition, given any indecomposable $\mathbf{U}_q$-tilting module $T \in \boldsymbol{\mathcal{T}}$, then there exists $\lambda \in X^+$ such that $T \cong T_q(\lambda)$.

Thus, the $T_q(\lambda)$'s form a complete set of non-isomorphic indecomposables of $\boldsymbol{\mathcal{T}}$ and all indecomposable $\mathbf{U}_q$-tilting modules $T_q(\lambda)$ are uniquely determined by their maximal weight $\lambda \in X^+$, that is,

$$\{\text{indecomposable } \mathbf{U}_q\text{-tilting modules}\} \xleftrightarrow{1:1} X^+. \qquad \square$$

Proof. We start by constructing $T_q(\lambda)$ for a given, fixed $\lambda \in X^+$.

If the Weyl $\mathbf{U}_q$-module $\Delta_q(\lambda)$ is a $\mathbf{U}_q$-tilting module, then we simply define $T_q(\lambda) = \Delta_q(\lambda)$.

Otherwise, by Theorem 3.1, we can choose a $\mathbf{U}_q$-weight $\mu_2 \in X^+$ minimal such that $\dim(\mathrm{Ext}^1_{\mathbf{U}_q}(\Delta_q(\mu_2), \Delta_q(\lambda))) = m_2 \neq 0$ (note that all appearing Ext's are finite-dimensional). Then there is a non-splitting extension

$$0 \longrightarrow \Delta_q(\lambda) = M_1 \hookrightarrow M_2 \twoheadrightarrow \Delta_q(\mu_2)^{\oplus m_2} \longrightarrow 0.$$

Note the important fact that necessarily $\mu_2 < \lambda$. This follows from the universal property of $\Delta_q(\lambda)$ saying that

$$\mathrm{Hom}_{\mathbf{U}_q}(\Delta_q(\lambda), M) = \{m \in M_\lambda \mid E_i^{(r)} m = 0 \text{ for all } i = 1, \ldots, n,\ r \in \mathbb{N}\}$$

for any $\mathbf{U}_q$-module M (here M_λ again denotes the λ-weight space of M). This is the dual of the (q-version of the) Frobenius reciprocity, i.e. the dual of equation (4) in [4].

If M_2 is a $\mathbf{U}_q$-tilting module, then we set $T_q(\lambda) = M_2$. Otherwise, by Theorem 3.1 again, we can choose $\mu_3 \in X^+$ minimal with $\dim(\mathrm{Ext}^1_{\mathbf{U}_q}(\Delta_q(\mu_3), M_2)) = m_3 \neq 0$ and we get a non-split extension

$$0 \longrightarrow M_2 \hookrightarrow M_3 \twoheadrightarrow \Delta_q(\mu_3)^{\oplus m_3} \longrightarrow 0.$$

Again $\mu_3 < \lambda$ and also $\mu_3 < \mu_2$.

And hence, we can continue as above and obtain a filtration of the form

$$\cdots \supset M_3 \supset M_2 \supset M_1 \supset M_0 = 0 \tag{22}$$

which is a "Δ_q-filtration" by construction, since we have $M_{k'+1}/M_{k'} \cong \Delta_q(\mu_{k'+1})^{\oplus m_{k'+1}}$ for all $k' = 0, 1, 2, \ldots$, where we use $\mu_1 = \lambda$ and $m_1 = 1$.

Thus, because there are only finitely many $\mu < \lambda$ (with $\mu \in X^+$), this process stops at some point giving a $\mathbf{U}_q$-module M_k. The $\mathbf{U}_q$-module M_k has a ∇_q-filtration, since otherwise there would, by Proposition 3.5, exist a $\mu_{k+1} \in X^+$ with $\mathrm{Ext}^1_{\mathbf{U}_q}(\Delta_q(\mu_{k+1}), M_k) \neq 0$. Moreover, we have constructed a "Δ_q-filtration" for M_k in (22) which shows that M_k is a $\mathbf{U}_q$-tilting module.

By construction, the $\mathbf{U}_q$-tilting module M_k has a unique indecomposable summand $T_q(\lambda)$ with $T_q(\lambda)_\lambda = (M_k)_\lambda \cong \mathbb{K}$. This is the indecomposable $\mathbf{U}_q$-tilting module we were looking for, since by the Krull-Schmidt property, $T_q(\lambda)$ is a $\mathbf{U}_q$-tilting module.

Now let us suppose that $T \in \boldsymbol{\mathcal{T}}$ is indecomposable. Choose any maximal $\mathbf{U}_q$-weight λ of T. Then we have $\mathrm{Hom}_{\mathbf{U}_v^- \mathbf{U}_v^0}(T, \mathbb{K}_\lambda) \neq 0$. By the Frobenius reciprocity (or, to be more precise, the

q-version of it) from equation (4) in [4], we get a non-zero $\mathbf{U}_q$-homomorphism $f\colon T \to \nabla_q(\lambda)$. By duality, we also get a non-zero $\mathbf{U}_q$-homomorphism $g\colon \Delta_q(\lambda) \to T$ with $f \circ g \neq 0$. Consider now the diagram

$$\begin{array}{ccccc} \Delta_q(\lambda) & \xhookrightarrow{\iota^\lambda} & T_q(\lambda) & \xrightarrow{\pi^\lambda} & \nabla_q(\lambda) \\ & {\scriptstyle g}\searrow & & \nearrow{\scriptstyle f} & \\ & & T & & \end{array} \tag{23}$$

where ι^λ is the inclusion of the first $\mathbf{U}_q$-submodule in a Δ_q-filtration of $T_q(\lambda)$ and π^λ is the surjection onto the last quotient of in a ∇_q-filtration of $T_q(\lambda)$. Since both path in the diagram (23) are non-zero, we can scale everything by some non-zero scalars in $\mathbb{K}$ such that (23) commutes (which we assume in the following)[4].

As in the proof of Proposition 3.5 (see also (12) in [4]), we see that

$$\mathrm{Ext}^1_{\mathbf{U}_q}(\Delta_q(\lambda), T) = 0 = \mathrm{Ext}^1_{\mathbf{U}_q}(T, \nabla_q(\lambda)) \Rightarrow \mathrm{Ext}^1_{\mathbf{U}_q}(\mathrm{coker}(\iota^\lambda), T) = 0 = \mathrm{Ext}^1_{\mathbf{U}_q}(T, \ker(\pi^\lambda)) \tag{24}$$

holds. Here $\ker(\pi^\lambda)$ and $\mathrm{coker}(\iota^\lambda)$ are the corresponding kernel and co-kernel respectively.

Thus, we see that the $\mathbf{U}_q$-homomorphism g extends to an $\mathbf{U}_q$-homomorphism $\overline{g}\colon T_q(\lambda) \to T$ whereas f factors through T via $\overline{f}\colon T \to T_q(\lambda)$. Then the composition $\overline{f} \circ \overline{g}$ is an isomorphism since it is so on $T_q(\lambda)_\lambda$. Hence, $T_q(\lambda)$ is a summand of T which shows $T \cong T_q(\lambda)$ since we have assumed that T is indecomposable.

The other statements are direct consequences of the first two which finishes the proof. ■

Remark 3.12. For a fixed $\lambda \in X^+$ we have $\mathbf{U}_q$-homomorphisms

$$\Delta_q(\lambda) \xhookrightarrow{\iota^\lambda} T_q(\lambda) \xrightarrow{\pi^\lambda} \nabla_q(\lambda)$$

where ι^λ is the inclusion of the first $\mathbf{U}_q$-submodule in a Δ_q-filtration of $T_q(\lambda)$ and π^λ is the surjection onto the last quotient in a ∇_q-filtration of $T_q(\lambda)$. Note that these are only defined up to scalars and we fix scalars in the following such that $\pi^\lambda \circ \iota^\lambda = c^\lambda$ (where c^λ is again the $\mathbf{U}_q$-homomorphism from (9)).

Take any $\mathbf{U}_q$-tilting module $T \in \boldsymbol{\mathcal{T}}$. An easy argument shows (see also the proof of Proposition 3.5) the following crucial fact:

$$\mathrm{Ext}^1_{\mathbf{U}_q}(\Delta_q(\lambda), T) = 0 = \mathrm{Ext}^1_{\mathbf{U}_q}(T, \nabla_q(\lambda)) \Rightarrow \mathrm{Ext}^1_{\mathbf{U}_q}(\mathrm{coker}(\iota^\lambda), T) = 0 = \mathrm{Ext}^1_{\mathbf{U}_q}(T, \ker(\pi^\lambda))$$

for all $\lambda \in X^+$. Hence, we see that any $\mathbf{U}_q$-homomorphism $g\colon \Delta_q(\lambda) \to T$ extends to an $\mathbf{U}_q$-homomorphism $\overline{g}\colon T_q(\lambda) \to T$ whereas any $\mathbf{U}_q$-homomorphism $f\colon T \to \nabla_q(\lambda)$ factors through $T_q(\lambda)$ via $\overline{f}\colon T \to T_q(\lambda)$. ▲

Corollary 3.13. We have $\mathcal{D}(T) \cong T$ for $T \in \boldsymbol{\mathcal{T}}$, that is, all $\mathbf{U}_q$-tilting modules T are self-dual. In particular, we have for all $\lambda \in X^+$ that

$$(T : \Delta_q(\lambda)) = \dim(\mathrm{Hom}_{\mathbf{U}_q}(T, \nabla_q(\lambda))) = \dim(\mathrm{Hom}_{\mathbf{U}_q}(\Delta_q(\lambda), T)) = (T : \nabla_q(\lambda)). \qquad \square$$

[4] To see this, recall that there is an (up to scalars) unique $\mathbf{U}_q$-homomorphism $c^\lambda\colon \Delta_q(\lambda) \to \nabla_q(\lambda)$.

Proof. By the Krull-Schmidt property it suffices to show the statement for the indecomposable $\mathbf{U}_q$-tilting modules $T_q(\lambda)$. Since $\mathcal{D}$ preserves characters, we see that $\mathcal{D}(T_q(\lambda))$ has λ as unique maximal weight, therefore $\mathcal{D}(T_q(\lambda)) \cong T_q(\lambda)$ by Proposition 3.11. Moreover, the leftmost and the rightmost equalities follow directly from Corollary 3.4. Finally

$$(T_q(\lambda) : \Delta_q(\lambda)) = (\mathcal{D}(T_q(\lambda)) : \mathcal{D}(\Delta_q(\lambda))) = (\mathcal{D}(T_q(\lambda)) : \nabla_q(\lambda)) = (T_q(\lambda) : \nabla_q(\lambda))$$

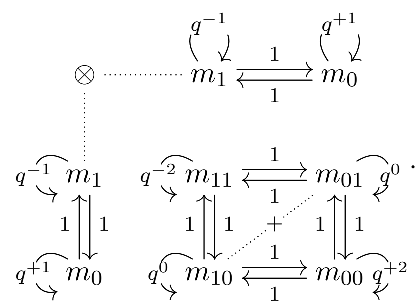
by definition and $\mathcal{D}(T_q(\lambda)) \cong T_q(\lambda)$ from above, which settles also the middle equality. ■

Example 3.14. Let us go back to the $\mathfrak{sl}_2$ case again. Then we obtain the family $(T_q(i))_{i\in\mathbb{N}}$ of indecomposable $\mathbf{U}_q$-tilting modules as follows.

Start by setting $T_q(0) \cong \Delta_q(0) \cong L_q(0) \cong \nabla_q(0)$ and $T_q(1) \cong \Delta_q(1) \cong L_q(1) \cong \nabla_q(1)$. Then we denote by $m_0 \in T_q(1)$ any eigenvector for K with eigenvalue q. For each $i > 1$ we define $T_q(i)$ to be the indecomposable summand of $T_q(1)^{\otimes i}$ which contains the vector $m_0 \otimes \cdots \otimes m_0 \in T_q(1)^{\otimes i}$. The $\mathbf{U}_q(\mathfrak{sl}_2)$-tilting module $T_q(1)^{\otimes i}$ is not indecomposable if $i > 1$: by Proposition 3.11 we have $(T_q(1)^{\otimes i} : \Delta_q(i)) = 1$ and

$$T_q(1)^{\otimes i} \cong T_q(i) \oplus \bigoplus_{k<i} T_q(k)^{\oplus \mathrm{mult}_k} \quad \text{for some } \mathrm{mult}_k \in \mathbb{N}.$$

In the case $l = 3$, we have for instance $T_q(1)^{\otimes 2} \cong T_q(2) \oplus T_q(0)$ since the tensor product $T_q(1) \otimes T_q(1)$ looks as follows (abbreviation $m_{ij} = m_i \otimes m_j$):



By construction, the indecomposable $\mathbf{U}_q(\mathfrak{sl}_2)$-module $T_q(2)$ contains m_{00} and therefore has to be the $\mathbb{C}$-span of $\{m_{00}, q^{-1}m_{10} + m_{01}, m_{11}\}$ as indicated above. The remaining summand is the 1-dimensional $\mathbf{U}_q$-tilting module $T_q(0) \cong L_q(0)$ from before. ▲

The following is interesting in its own right.

Corollary 3.15. Let $\mu \in X^+$ be a minuscule $\mathbf{U}_q$-weight. Then $T = \Delta_q(\mu)^{\otimes d}$ is a $\mathbf{U}_q$-tilting module for any $d \in \mathbb{N}$ and $\dim(\mathrm{End}_{\mathbf{U}_q}(T))$ is independent of the field $\mathbb{K}$ and of $q \in \mathbb{K}^*$ and are given by

$$\dim(\mathrm{Hom}_{\mathbf{U}_q}(M, N)) = \sum_{\lambda \in X^+} (M : \Delta_q(\lambda))(N : \nabla_q(\lambda)).$$

In particular, this holds for $\Delta_q(\omega_1)$ being the vector representation of $\mathbf{U}_q = \mathbf{U}_q(\mathfrak{g})$ for $\mathfrak{g}$ of type A, C or D. □

Proof. Since $\mu \in X^+$ is minuscule: $\Delta_q(\mu) \cong L_q(\mu)$ is a simple $\mathbf{U}_q$-tilting module for any field $\mathbb{K}$ and any $q \in \mathbb{K}^*$. Thus, by Proposition 3.10 we see that T is a $\mathbf{U}_q$-tilting module for any $d \in \mathbb{N}$. Hence, by Corollaries 3.4 and 3.13, we have $\dim(\mathrm{End}_{\mathbf{U}_q}(T)) = \sum_{\lambda \in X^+} (T : \Delta_q(\lambda))^2$. Now use the fact that $\chi(\Delta_q(\mu))$ is as in the classical case which implies the statement. ■

3.3. **The characters of indecomposable $\mathbf{U}_q$-tilting modules.** In this subsection we describe how to compute $(T_q(\lambda) : \Delta_q(\mu))$ for all $\lambda, \mu \in X^+$ (which can be done algorithmically in the case where q is a complex, primitive l-th root of unity). As an application, we illustrate how to decompose tensor products of $\mathbf{U}_q$-tilting modules. This shows that, in principle, our cellular basis for endomorphism rings $\mathrm{End}_{\mathbf{U}_q}(T)$ of $\mathbf{U}_q$-tilting modules T (that we introduce in [4, Section 4], arXiv version) can be made completely explicit.

Given an abelian category $\mathcal{AB}$, we denote its *Grothendieck group* by $K_0(\mathcal{AB})$ and its *split Grothendieck group* by $K_0^{\oplus}(\mathcal{AB})$. We point out that the notation of the split Grothendieck group also makes sense for a given additive category $\mathcal{AD}$ that satisfies the Krull-Schmidt property where we use the same notation (we refer the reader unfamiliar with these and the notation we use to [24, Section 1.2]).

By Propositions 2.9 and 2.11, a $\mathbb{Z}$-basis of the Grothendieck group $K_0(\mathbf{U}_q\text{-}\mathbf{Mod})$ is given by isomorphism classes $\{[\Delta_q(\lambda)] \mid \lambda \in X^+\}$.

On the other hand, $\boldsymbol{\mathcal{T}}$ is not abelian (see Example 3.9), but additive and satisfies the Krull-Schmidt property. A $\mathbb{Z}$-basis of $K_0^{\oplus}(\boldsymbol{\mathcal{T}})$ is, by Proposition 3.11, spanned by isomorphism classes $\{[T_q(\lambda)]_{\oplus} \mid \lambda \in X^+\}$.

Since both $\mathbf{U}_q$-**Mod** and $\boldsymbol{\mathcal{T}}$ are closed under tensor products, $K_0(\mathbf{U}_q\text{-}\mathbf{Mod})$ and $K_0^{\oplus}(\boldsymbol{\mathcal{T}})$ get an (in fact isomorphic) induced ring structure.

Corollary 3.16. The inclusion of categories $\iota \colon \boldsymbol{\mathcal{T}} \to \mathbf{U}_q\text{-}\mathbf{Mod}$ induces an isomorphism

$$[\iota] \colon K_0^{\oplus}(\boldsymbol{\mathcal{T}}) \to K_0(\mathbf{U}_q\text{-}\mathbf{Mod}), \quad [T_q(\lambda)]_{\oplus} \mapsto [T_q(\lambda)], \quad \lambda \in X^+$$

of rings. □

Proof. The set $B = \{[T_q(\lambda)] \mid \lambda \in X^+\}$ forms a $\mathbb{Z}$-basis of $K_0^{\oplus}(\boldsymbol{\mathcal{T}})$ by Proposition 3.11 and it is clear that $[\iota]$ is a well-defined ring homomorphism.

Moreover, we have

$$[T_q(\lambda)] = [\Delta_q(\lambda)] + \sum_{\mu < \lambda \in X^+} (T_q(\mu) : \Delta_q(\mu))[\Delta_q(\mu)] \in K_0(\mathbf{U}_q\text{-}\mathbf{Mod}) \tag{25}$$

with $T_q(0) \cong \Delta_q(0)$ by Proposition 3.11. Hence, $[\iota](B)$ is also a $\mathbb{Z}$-basis of $K_0(\mathbf{U}_q\text{-}\mathbf{Mod})$ since the $\Delta_q(\lambda)$'s form a $\mathbb{Z}$-basis and the claim follows. ■

Recall that $\mathbb{Z}[X]$ carries an action of the Weyl group W associated to the Cartan datum (see below). Thus, we can look at the invariant part of this action, denoted by $\mathbb{Z}[X]^W$, which is known as *Weyl's character ring*.

We obtain the following (known) categorification result.

Corollary 3.17. The tilting category $\boldsymbol{\mathcal{T}}$ (naively) categorifies Weyl's character ring, that is,

$$K_0^{\oplus}(\boldsymbol{\mathcal{T}}) \cong \mathbb{Z}[X]^W \quad \text{as rings.}$$

□

Proof. It is known that there is an isomorphism $K_0(\mathfrak{g}\text{-}\mathbf{Mod}) \xrightarrow{\cong} \mathbb{Z}[X]^W$ given by sending finite-dimensional $\mathfrak{g}$-modules to their characters (which can be regarded as elements in $\mathbb{Z}[X]^W$).

Now the characters $\chi(\Delta_q(\lambda))$ of the $\Delta_q(\lambda)$'s are (as mentioned below Example 2.12) the same as in the classical case. Thus, we can adopt the isomorphism from $K_0(\mathfrak{g}\text{-}\mathbf{Mod})$ to $\mathbb{Z}[X]^W$ from above (non-quantized!). Details can, for example, be found in [6, Chapter VIII, §7.7].

Then the statement follows from Corollary 3.16. ■

For each simple root $\alpha_i \in \Pi$ let s_i be the reflection

$$s_i(\lambda) = \lambda - \langle \lambda, \alpha_i^\vee \rangle \alpha_i, \quad \text{for } \lambda \in E,$$

in the hyperplane $H_{\alpha_i^\vee}$ orthogonal to α_i. These reflections s_i generate a group W, called *Weyl group*, associated to our Cartan datum.

For any fixed $l \in \mathbb{N}$, the *affine Weyl group* $W_l \cong W \ltimes l\mathbb{Z}\Pi$ is the group generated by the reflections $s_{\beta,r}$ in the *affine hyperplanes* $H_{\beta^\vee,r} = \{x \in E \mid \langle x, \beta^\vee \rangle = lr\}$ for $\beta \in \Phi$ and $r \in \mathbb{Z}$. Note that, if $l = 0$, then $W_0 \cong W$.

For $\beta \in \Phi$ there exists $w \in W$ such that $\beta = w(\alpha_i)$ for some $i = 1, \dots, n$. We set $l_\beta = l_i$ where $l_i = \frac{l}{\gcd(l,d_i)}$. Using this, we have the *dot-action* of W_l on the $\mathbf{U}_q$-weight lattice X via

$$s_{\beta,r}.\lambda = s_\beta(\lambda + \rho) - \rho + l_\beta r \beta.$$

Note that the case $l = 1$ recovers the usual action of the affine Weyl group W_1 on X.

Definition 3.18. (**Alcove combinatorics**) The *fundamental alcove* $\mathcal{A}_0$ is

$$\mathcal{A}_0 = \{\lambda \in X \mid 0 < \langle \lambda + \rho, \alpha^\vee \rangle < l, \text{ for all } \alpha \in \Phi^+\} \subset X^+. \tag{26}$$

Its *closure* $\overline{\mathcal{A}}_0$ is given by

$$\overline{\mathcal{A}}_0 = \{\lambda \in X \mid 0 \leq \langle \lambda + \rho, \alpha^\vee \rangle \leq l, \text{ for all } \alpha \in \Phi^+\} \subset X^+ - \rho. \tag{27}$$

The *non-affine walls* of $\mathcal{A}_0$ are

$$\check{\partial}\mathcal{A}_0^i = \overline{\mathcal{A}}_0 \cap (H_{\alpha_i^\vee,0} - \rho), i = 1, \dots, n, \quad \check{\partial}\mathcal{A}_0 = \bigcup_{i=1}^{n} \check{\partial}\mathcal{A}_0^i.$$

The set

$$\hat{\partial}\mathcal{A}_0 = \overline{\mathcal{A}}_0 \cap (H_{\alpha_0^\vee,1} - \rho)$$

is called the *affine* wall of $\mathcal{A}_0$. Here α_0 is the maximal short root. We call the union of all these walls the *boundary* $\partial\mathcal{A}_0$ of $\mathcal{A}_0$. More generally, an *alcove* $\mathcal{A}$ is a connected component of

$$E - \bigcup_{r \in \mathbb{Z}, \beta \in \Phi} (H_{\beta^\vee,r} - \rho).$$

We denote the set of alcoves by $\mathcal{AL}$. ▲

Note that the affine Weyl group W_l acts simply transitively on $\mathcal{AL}$. Thus, we can associate $1 \in W_l \mapsto \mathcal{A}(1) = \mathcal{A}_0 \in \mathcal{AL}$ and in general $w \in W_l \mapsto \mathcal{A}(w) \in \mathcal{AL}$.

Example 3.19. In the case $\mathfrak{g} = \mathfrak{sl}_2$ we have $\rho = \omega_1 = 1$. Consider for instance again $l = 3$. Then $k \in \mathbb{N} = X^+$ is contained in the fundamental alcove $\mathcal{A}_0$ iff $0 < k + 1 < 3$.

Moreover, $-\rho \in \check{\partial}\mathcal{A}_0$ and $2 \in \hat{\partial}\mathcal{A}_0$ are on the walls. Thus, $\overline{\mathcal{A}}_0$ can be visualized as

where the affine wall on the right is indicated in red and the non-affine wall on the left is indicated in green. ▲

Example 3.20. Let us leave our running $\mathfrak{sl}_2$ example for a second and do another example which is graphically more interesting.

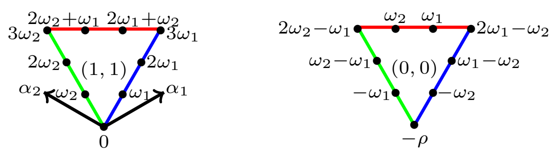
In the case $\mathfrak{g} = \mathfrak{sl}_3$ we have $\rho = \alpha_1 + \alpha_2 = \omega_1 + \omega_2 \in X^+$ and $\alpha_0 = \alpha_1 + \alpha_2$. Now consider again $l = 3$. The condition (26) means that $\mathcal{A}_0$ consists of those $\lambda = \lambda_1\omega_1 + \lambda_2\omega_2$ for which

$$0 < \langle \lambda_1\omega_1 + \lambda_2\omega_2 + \omega_1 + \omega_2, \alpha_i^\vee \rangle < 3 \quad \text{for } i = 1, 2, 0.$$

Thus, $0 < \lambda_1 + 1 < 3$, $0 < \lambda_2 + 1 < 3$ and $0 < \lambda_1 + \lambda_2 + 2 < 3$. Hence, only the $\mathbf{U}_q(\mathfrak{sl}_3)$-weight $\lambda = (0, 0) \in X^+$ is in $\mathcal{A}_0$. In addition, we have by condition (27) that

$$\check{\partial}\mathcal{A}_0 = \{-\rho, -\omega_1, -\omega_2, \omega_1 - \omega_2, \omega_2 - \omega_1\}, \qquad \hat{\partial}\mathcal{A}_0 = \{\omega_1, \omega_2, 2\omega_1 - \omega_2, 2\omega_2 - \omega_1\}.$$

Hence, $\overline{\mathcal{A}}_0$ can be visualized as (displayed without the $-\rho$ shift on the left)



where, as before, the affine wall at the top is indicated in red, the hyperplane orthogonal to α_1 on the left in green and the hyperplane orthogonal to α_2 on the right in blue. ▲

We say $\lambda \in X^+ - \rho$ is *linked* to $\mu \in X^+$ if there exists $w \in W_l$ such that $w.\lambda = \mu$. We note the following theorem, called *the linkage principle*, where we, by convention, set $T_q(\lambda) = \Delta_q(\lambda) = \nabla_q(\lambda) = L_q(\lambda) = 0$ for $\lambda \in \check{\partial}\mathcal{A}_0$.

Theorem 3.21. (**The linkage principle**) All composition factors of $T_q(\lambda)$ have maximal weights μ linked to λ. Moreover, $T_q(\lambda)$ is a simple $\mathbf{U}_q$-module if $\lambda \in \overline{\mathcal{A}}_0$.

If λ is linked to an element of $\mathcal{A}_0$, then $T_q(\lambda)$ is a simple $\mathbf{U}_q$-module iff $\lambda \in \mathcal{A}_0$. □

Proof. This is a slight reformulation of [1, Corollaries 4.4 and 4.6]. ■

The linkage principle gives us now a decomposition into a direct sum of categories

$$\boldsymbol{\mathcal{T}} \cong \bigoplus_{\lambda \in \mathcal{A}_0} \boldsymbol{\mathcal{T}}_\lambda \oplus \bigoplus_{\lambda \in \partial\mathcal{A}_0} \boldsymbol{\mathcal{T}}_\lambda,$$

where each $\boldsymbol{\mathcal{T}}_\lambda$ consists of all $T \in \boldsymbol{\mathcal{T}}$ whose indecomposable summands are all of the form $T_q(\mu)$ for $\mu \in X^+$ lying in the W_l-dot-orbit of $\lambda \in \mathcal{A}_0$ (or of $\lambda \in \partial\mathcal{A}_0$). We call these categories *blocks* to stress that they are homologically unconnected (although they might be decomposable). Moreover, if $\lambda \in \mathcal{A}_0$, then we call $\boldsymbol{\mathcal{T}}_\lambda$ an *l-regular* block, while the $\boldsymbol{\mathcal{T}}_\lambda$'s with $\lambda \in \partial\mathcal{A}_0$ are called *l-singular* blocks (we say for short just regular and singular blocks in what follows).

In fact, by Proposition 3.11, the $\mathbf{U}_q$-weights labelling the indecomposable $\mathbf{U}_q$-tilting modules are only the dominant (integral) weights $\lambda \in X^+$. Let $\mathcal{DC} = \{x \in E \mid \langle x, \beta^\vee \rangle \geq 0, \beta \in \Phi\}$. Then these $\mathbf{U}_q$-weights correspond blockwise precisely to the alcoves

$$\mathcal{AL}^+ = \mathcal{AL} \cap \mathcal{DC},$$

contained in the dominant chamber $\mathcal{DC}$. That is, they correspond to the set of coset representatives of minimal length in $\{wW_0 \mid w \in W_1\}$. In formulas,

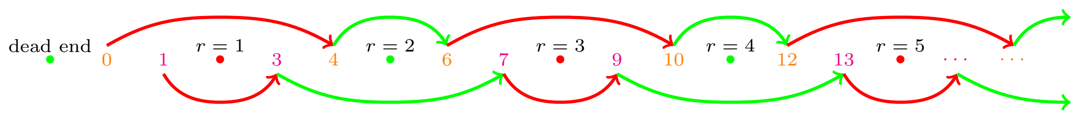
$$T_q(w.\lambda) \in \boldsymbol{\mathcal{T}}_\lambda \leftrightsquigarrow \mathcal{A}(w) \in \mathcal{AL}^+ \leftrightsquigarrow wW_0 \subset W_1, \tag{28}$$

for all $\lambda \in \mathcal{A}_0$.

Example 3.22. In our pet example with $\mathfrak{g} = \mathfrak{sl}_2$ and $l = 3$ we have, by Theorem 3.21 and Example 3.19 a block decomposition

$$\boldsymbol{\mathcal{T}} \cong \boldsymbol{\mathcal{T}}_{-1} \oplus \boldsymbol{\mathcal{T}}_0 \oplus \boldsymbol{\mathcal{T}}_1 \oplus \boldsymbol{\mathcal{T}}_2.$$

The W_l-dot-orbit of $0 \in \mathcal{A}_0$ respectively $1 \in \mathcal{A}_0$ can be visualized as



Compare also to [5, (2.4.1)].

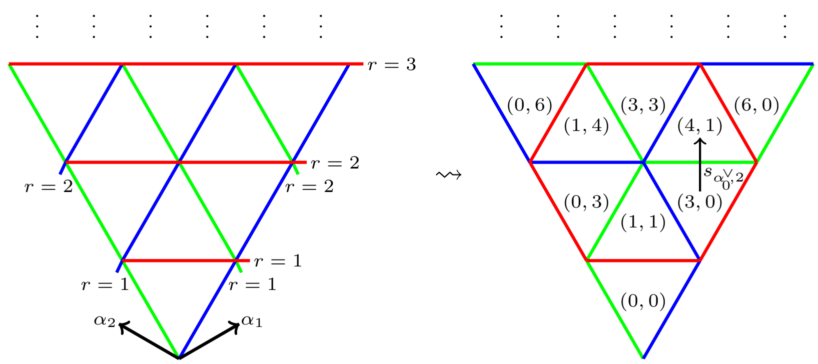
It turns out that, for $\mathbb{K} = \mathbb{C}$, both singular blocks $\boldsymbol{\mathcal{T}}_{-1}$ and $\boldsymbol{\mathcal{T}}_2$ are semisimple (in particular, these blocks decompose further), see Example 3.27 or [5, Lemma 2.25]. ▲

Example 3.23. In the $\mathfrak{sl}_3$ case with $l = 3$ we have the block decomposition

$$\boldsymbol{\mathcal{T}} \cong \boldsymbol{\mathcal{T}}_{-\rho} \oplus \boldsymbol{\mathcal{T}}_{-\omega_2} \oplus \boldsymbol{\mathcal{T}}_{-\omega_1} \oplus \boldsymbol{\mathcal{T}}_{\omega_1-\omega_2} \oplus \boldsymbol{\mathcal{T}}_{\omega_2-\omega_1} \oplus \boldsymbol{\mathcal{T}}_{(0,0)} \oplus \boldsymbol{\mathcal{T}}_{2\omega_1-\omega_2} \oplus \boldsymbol{\mathcal{T}}_{\omega_1} \oplus \boldsymbol{\mathcal{T}}_{\omega_2} \oplus \boldsymbol{\mathcal{T}}_{2\omega_2-\omega_1}$$

(note that the singular blocks are not necessarily semisimple anymore, even when $\mathbb{K} = \mathbb{C}$).

The W_l-dot-orbit in $\mathcal{AC}^+$ of the regular block $\boldsymbol{\mathcal{T}}_{(0,0)}$ looks as follows.



Here we reflect either in a red (that is, $\alpha_0 = (1,1)$), green (that is, $\alpha_1 = (2,-1)$) or blue (that is, $\alpha_2 = (-1,2)$) hyperplane, and the r measures the hyperplane-distance from the origin (both indicated in the left picture above). In the right picture we have indicated the linkage (we have also displayed one of the dot-reflections).

Theorem 3.21 means now that $T_q((1,1))$ satisfies

$$(T_q((1,1)) : \Delta_q(\mu)) \neq 0 \quad \Rightarrow \quad \mu \in \{(0,0),(1,1)\}$$

and $T_q((3,3))$ satisfies

$$(T_q((3,3)) : \Delta_q(\mu)) \neq 0 \quad \Rightarrow \quad \mu \in \{(0,0),(1,1),(3,0),(0,3),(4,1),(1,4),(3,3)\}.$$

We calculate the precise values later in Example 3.25. ▲

In order to get our hands on the multiplicities, we need Soergel's version of the *(affine) parabolic Kazhdan-Lusztig polynomials*, which we denote by

$$n_{\mu\lambda}(t) \in \mathbb{Z}[v,v^{-1}], \quad \lambda,\mu \in X^+ - \rho. \tag{29}$$

For brevity, we do not recall the definition of these polynomials (which can be computed algorithmically) here, but refer to [31, Section 3] where the relevant polynomial is denoted

$n_{y,x}$ for $x, y \in W_l$ (which translates by (28) to our notation). The main point for us is the following theorem due to Soergel.

Theorem 3.24. (**Multiplicity formula**) Suppose $\mathbb{K} = \mathbb{C}$ and q is a complex, primitive l-th root of unity. For each pair $\lambda, \mu \in X^+$ with λ being an l-regular $\mathbf{U}_q$-weight (that is, $T_q(\lambda)$ belongs to a regular block of $\boldsymbol{\mathcal{T}}$) we have

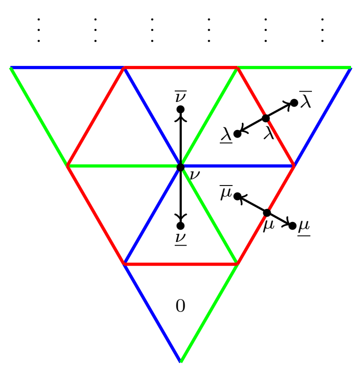
$$(T_q(\lambda) : \Delta_q(\mu)) = (T_q(\lambda) : \nabla_q(\mu)) = n_{\mu\lambda}(1).$$

In particular, if $\lambda, \mu \in X^+$ are not linked, then $n_{\mu\lambda}(v) = 0$. □

Proof. This follows from [30, Theorem 5.12] (see also [31, Conjecture 7.1]). ■

In addition to Theorem 3.24, we are going to describe now an algorithmic way to compute $(T_q(\lambda) : \Delta_q(\mu))$ for all $T_q(\lambda)$ lying in a singular blocks of $\boldsymbol{\mathcal{T}}$. We point out that Theorem 3.26 below is valid for $q \in \mathbb{K}$ being a primitive l-th root of unity (where $\mathbb{K}$ is, in contrast to Theorem 3.24, an arbitrary field).

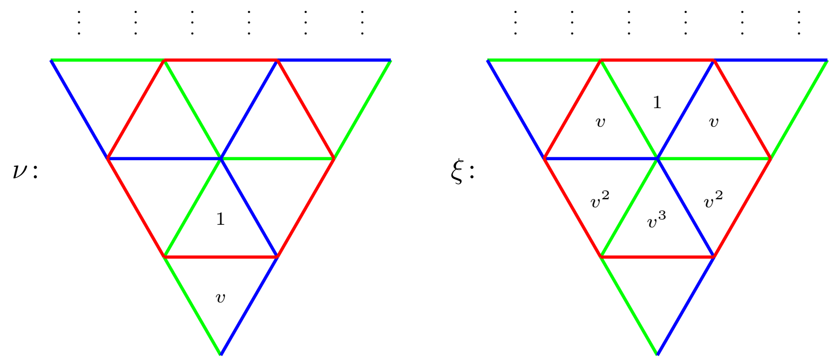
Assume in the following that $\lambda \in X^+$ is not l-regular. Set $W_\lambda = \{w \in W_l \mid w.\lambda = \lambda\}$. Then we can find a unique l-regular $\mathbf{U}_q$-weight $\overline{\lambda} \in W_l.0$ such that λ is in the closure of the alcove containing $\overline{\lambda}$ and $\overline{\lambda}$ is maximal in $W_\lambda.\overline{\lambda}$. Similarly, we find a can find a unique l-regular $\mathbf{U}_q$-weight $\underline{\lambda} \in W_l.0$ such that λ is in the closure of the alcove containing $\underline{\lambda}$ and $\underline{\lambda}$ is minimal in $W_\lambda.\overline{\lambda}$. Some examples in the $\mathfrak{g} = \mathfrak{sl}_3$ case are



We stress that, in the μ case above, Theorem 3.26 is not valid: recall that in those cases $T_q(\mu) = \Delta_q(\mu) = L_q(\mu) = \nabla_q(\mu) = 0$ and thus, we do not have to worry about these in the following.

Example 3.25. Back to Example 3.23: for $\nu = \omega_1 + \omega_2 = (1,1)$ we have $n_{\nu\nu}(v) = 1$ and $n_{\nu(0,0)}(v) = v$ as shown in the left picture below. Similarly, for $\xi = 3\omega_1 + 3\omega_2 = (3,3)$ the only non-zero parabolic Kazhdan-Lusztig polynomials are $n_{\xi\xi}(v) = 1$, $n_{\xi(1,4)}(v) = v = n_{\xi(4,1)}(v)$,

$n_{\xi(0,3)}(v) = v^2 = n_{\xi(3,0)}(v)$ and $n_{\xi\nu}(v) = v^3$ as illustrated on the right below.



Therefore, we have, by Theorem 3.24, that $(T_q(\nu) : \Delta_q(\mu)) = 1$ if $\mu \in \{(0,0),(1,1)\}$ and $(T_q(\nu) : \Delta_q(\mu)) = 0$ if $\mu \notin \{(0,0),(1,1)\}$. We encourage the reader to work out $(T_q(\xi) : \Delta_q(\mu))$ by using the above patterns and Example 3.23. For all patterns in rank 2 see [32]. ▲

We are aiming to show the following Theorem (compare to Remark 3.26 in [4]).

Theorem 3.26. (**Multiplicity formula - singular case**) We have

$$(T_q(\lambda) : \Delta_q(\mu)) = (T_q(\overline{\lambda}) : \Delta_q(\overline{\mu}))$$

for all $\mu \in W_l.\lambda \cap X^+$.

We consider the translation functors $\mathcal{T}_{\xi}^{\xi'} : \boldsymbol{\mathcal{T}}_{\xi} \to \boldsymbol{\mathcal{T}}_{\xi'}$ for various $\xi, \xi' \in X^+$ in the proof. The reader unfamiliar with these and their basic properties can for example consider Part II, Chapter 7 in [16]. We only stress here that $\mathcal{T}_{\xi}^{\xi'} : \boldsymbol{\mathcal{T}}_{\xi} \to \boldsymbol{\mathcal{T}}_{\xi'}$ is the biadjoint of $\mathcal{T}_{\xi'}^{\xi} : \boldsymbol{\mathcal{T}}_{\xi'} \to \boldsymbol{\mathcal{T}}_{\xi}$.

Proof. In order to prove Theorem 3.26, we have to show some intermediate steps. We start with the following two claims.

$$\text{We have} \quad [\Delta_q(\lambda') : L_q(\underline{\lambda})] = 1 \quad \text{for all} \quad \lambda' \in W_{\lambda}.\overline{\lambda}. \tag{30}$$

Moreover, for all $\lambda' \in W_{\lambda}.\overline{\lambda}$:

$$\text{there is a unique} \quad \varphi \in \mathrm{Hom}_{\mathbf{U}_q}(\Delta_q(\lambda'), \Delta_q(\overline{\lambda})) \quad \text{with} \quad [\mathrm{Im}(\varphi) : L_q(\underline{\lambda})] = 1. \tag{31}$$

Here uniqueness is meant up to scalars.

Proof of (30): we have $\mathcal{T}_{\overline{\lambda}}^{\lambda}(\Delta_q(\lambda')) \cong \Delta_q(\lambda)$. In addition, for any $\lambda'' \in W_l.\overline{\lambda} \cap X^+$, we have $\mathcal{T}_{\overline{\lambda}}^{\lambda}(L_q(\lambda'')) \cong L_q(\lambda)$ iff $\lambda'' = \underline{\lambda} \in X^+$. This proves (30).

Proof of (31): we use descending induction. If $\lambda' = \overline{\lambda}$, then (31) is clear. So assume $\lambda' < \overline{\lambda}$ and denote by $\mathcal{A}'$ the alcove containing λ'. Choose an upper wall H of $\mathcal{A}'$ such that the corresponding reflection s_H belongs to W_{λ}. Then $\lambda'' = s_H.\lambda' > \lambda'$. Thus, by induction, there exists an (up to scalars) unique non-zero $\mathbf{U}_q$-homomorphism $\psi \colon \Delta_q(\lambda'') \to \Delta_q(\overline{\lambda})$ with $[\mathrm{Im}(\psi) : L_q(\underline{\lambda})] = 1$. We claim now that for all $\lambda' \in W_{\lambda}.\overline{\lambda}$:

$$\text{there exists a unique} \quad \tilde{\varphi} \in \mathrm{Hom}_{\mathbf{U}_q}(\Delta_q(\lambda'), \Delta_q(\lambda'')) \quad \text{with} \quad [\mathrm{Im}(\tilde{\varphi}) : L_q(\underline{\lambda})] = 1. \tag{32}$$

Again uniqueness is meant up to scalars.

Because (32) implies that $\varphi = \psi \circ \tilde{\varphi}$ is the (up to scalars) unique non-zero $\mathbf{U}_q$-homomorphism we are looking for, it remains to show (32). To this end, choose $\nu \in H$. Then we have a short exact sequence

$$0 \longrightarrow \Delta_q(\lambda'') \hookrightarrow \mathcal{T}_\nu^{\overline{\lambda}} \Delta_q(\nu) \twoheadrightarrow \Delta_q(\lambda') \longrightarrow 0.$$

This sequence does not split since $\mathcal{T}_\nu^{\overline{\lambda}} \Delta_q(\nu)$ has simple head equal to $L_q(\lambda')$. Therefore, the inclusion

$$\begin{aligned} \mathrm{Hom}_{\mathbf{U}_q}(\Delta_q(\lambda'), \Delta_q(\lambda'')) &\hookrightarrow \mathrm{Hom}_{\mathbf{U}_q}(\Delta_q(\lambda'), \mathcal{T}_\nu^{\overline{\lambda}} \Delta_q(\nu)) \\ &\cong \mathrm{Hom}_{\mathbf{U}_q}(\mathcal{T}_{\overline{\lambda}}^{\nu} \Delta_q(\lambda'), \Delta_q(\nu)) \\ &\cong \mathrm{End}_{\mathbf{U}_q}(\Delta_q(\nu)) \cong \mathbb{K} \end{aligned}$$

is an equality. So we can pick any non-zero $\mathbf{U}_q$-homomorphism $\tilde{\varphi} \in \mathrm{Hom}_{\mathbf{U}_q}(\Delta_q(\lambda'), \Delta_q(\lambda''))$ which will be unique up to scalars. Then $L_q(\lambda')$ is a composition factor of $\mathrm{Im}(\tilde{\varphi})$. This implies that $\mathcal{T}_{\overline{\lambda}}^{\nu} \tilde{\varphi} \in \mathrm{End}_{\mathbf{U}_q}(\Delta_q(\nu))$ is non-zero and thus, an isomorphism. In particular, $L_q(\underline{\lambda})$ is a composition factor of $\mathrm{Im}(\tilde{\varphi})$, because $\mathcal{T}_{\overline{\lambda}}^{\nu} L_q(\lambda') \neq 0$. Hence, (32) follows and thus, (31) holds.

We keep the notation from before.

(33) $$\text{We have} \quad (T_q(\overline{\lambda}) : \Delta_q(\lambda')) = 1 \quad \text{for all} \quad \lambda' \in W_\lambda . \overline{\lambda}.$$

Proof of (33): by (31) we have $\mathrm{Hom}_{\mathbf{U}_q}(\Delta_q(\lambda'), \Delta_q(\overline{\lambda})) \cong \mathbb{K}$. This together with

$$\mathrm{Hom}_{\mathbf{U}_q}(\Delta_q(\lambda'), T_q(\overline{\lambda})) \supset \mathrm{Hom}_{\mathbf{U}_q}(\Delta_q(\lambda'), \Delta_q(\overline{\lambda})) \cong \mathbb{K}$$

implies (33).

(34) $$\text{We have} \quad \mathcal{T}_\lambda^{\overline{\lambda}} T_q(\lambda) = T_q(\overline{\lambda}).$$

Proof of (34): we have $\mathcal{T}_\lambda^{\overline{\lambda}} T_q(\lambda) = T_q(\overline{\lambda}) \oplus \mathrm{rest}$ where rest is some $\mathbf{U}_q$-tilting module with $\mathbf{U}_q$-weights $< \overline{\lambda}$. However, applying $\mathcal{T}_{\overline{\lambda}}^{\lambda}(\cdot)$, we get

$$T_q(\lambda)^{\oplus |W_\lambda|} \cong \mathcal{T}_{\overline{\lambda}}^{\lambda} T_q(\lambda) \oplus \mathcal{T}_{\overline{\lambda}}^{\lambda}(\mathrm{rest}).$$

However, by (33), we also have

$$\mathcal{T}_{\overline{\lambda}}^{\lambda} T_q(\overline{\lambda}) \cong T_q(\lambda)^{\oplus |W_\lambda|}.$$

Hence, $\mathcal{T}_{\overline{\lambda}}^{\lambda}(\mathrm{rest}) = 0$ which implies rest $= 0$: suppose otherwise. Then there exists $\tilde{\lambda} \in X^+$ with

$$0 \neq \mathrm{Hom}_{\mathbf{U}_q}(L_q(\tilde{\lambda}), \mathrm{rest}) \subset \mathrm{Hom}_{\mathbf{U}_q}(L_q(\tilde{\lambda}), \mathcal{T}_\lambda^{\overline{\lambda}} T_q(\lambda)) \cong \mathrm{Hom}_{\mathbf{U}_q}(\mathcal{T}_{\overline{\lambda}}^{\lambda} L_q(\tilde{\lambda}), T_q(\lambda)).$$

But then $0 \neq \mathcal{T}_{\overline{\lambda}}^{\lambda} L_q(\tilde{\lambda}) \subset \mathcal{T}_{\overline{\lambda}}^{\lambda}(\mathrm{rest})$. This is a contradiction, hence, (34) follows.

We are now ready to prove the theorem itself. For this purpose, note that we get

$$(T_q(\lambda) : \Delta_q(w.\lambda)) = (T_q(\overline{\lambda}) : \Delta_q(w.\overline{\lambda}))) \quad \text{for all } w \in W_l \text{ with } w.\lambda \in X^+.$$

from (34). This in turn implies the statement of the theorem by the linkage principle, see Theorem 3.21 in [4]. ■

Since the polynomials from (29) can be computed inductively, we can use Theorems 3.24 and 3.26 in the case $\mathbb{K} = \mathbb{C}$ to *explicitly* calculate the decomposition of a tensor product of $\mathbf{U}_q$-tilting modules $T = T_q(\lambda_1) \otimes \cdots \otimes T_q(\lambda_d)$ into its indecomposable summands:

- Calculate, by using Theorems 3.24 and 3.26, $(T_q(\lambda_i) : \Delta_q(\mu))$ for $i = 1, \ldots, d$.
- This gives the multiplicities of T, by the Corollary 3.16 and the fact that $\chi(\Delta_q(\lambda))$ are as in the classical case.
- Use (25) to recursively compute the decomposition of T (starting with any maximal $\mathbf{U}_q$-weight of T).

Example 3.27. Let us come back to our favourite case, that is, $\mathfrak{g} = \mathfrak{sl}_2$, $\mathbb{K} = \mathbb{C}$ and $l = 3$. In the regular cases we have $T_q(k) \cong \Delta_q(k)$ for $k = 0, 1$ and the parabolic Kazhdan-Lusztig polynomials are

$$n_{jk}(v) = \begin{cases} 1, & \text{if } j = k, \\ v, & \text{if } j < k \text{ are separated by precisely one wall,} \\ 0, & \text{else,} \end{cases}$$

for $k > 1$. By the above we obtain $T_q(k) \cong \Delta_q(k)$ for $k \in \mathbb{N}$ singular, hence, the two singular blocks $\boldsymbol{\mathcal{T}}_{-1}$ and $\boldsymbol{\mathcal{T}}_2$ are semisimple.

In Example 3.14 we have already calculated $T_q(1) \otimes T_q(1) \cong T_q(2) \oplus T_q(0)$. Let us go one step further now: $T_q(1) \otimes T_q(1) \otimes T_q(1)$ has only $(T_q(1)^{\otimes 3} : \Delta_q(3)) = 1$ and $(T_q(1)^{\otimes 3} : \Delta_q(1)) = 2$ as non-zero multiplicities. This means that $T_q(3)$ is a summand of $T_q(1) \otimes T_q(1) \otimes T_q(1)$. Since $T_q(3)$ has only $(T_q(3) : \Delta_q(3)) = 1$ and $(T_q(3) : \Delta_q(1)) = 1$ as non-zero multiplicities (by the calculation of the periodic Kazhdan-Lusztig polynomials), we have

$$T_q(1) \otimes T_q(1) \otimes T_q(1) \cong T_q(3) \oplus T_q(1) \in \boldsymbol{\mathcal{T}}_1. \tag{35}$$

Moreover, we have (as we, as usual, encourage the reader to work out)

$$T_q(1) \otimes T_q(1) \otimes T_q(1) \otimes T_q(1) \cong (T_q(4) \oplus T_q(0)) \oplus (T_q(2) \oplus T_q(2) \oplus T_q(2)) \in \boldsymbol{\mathcal{T}}_0 \oplus \boldsymbol{\mathcal{T}}_2.$$

To illustrate how this decomposition depends on l: assume now that $l > 3$. Then, which can be verified similarly as in Example 3.19, the $\mathbf{U}_q$-tilting module $T_q(3)$ is in the fundamental alcove $\mathcal{A}_0$. Thus, by Theorem 3.21, $T_q(3)$ is simple as in the "classical" case. Said otherwise, we have $T_q(3) \cong \Delta_q(3)$. Hence, in the same spirit as above, we obtain (as in the generic case)

$$T_q(1) \otimes T_q(1) \otimes T_q(1) \cong T_q(3) \oplus (T_q(1) \oplus T_q(1)) \in \boldsymbol{\mathcal{T}}_3 \oplus \boldsymbol{\mathcal{T}}_1 \tag{36}$$

in contrast to the decomposition in (35). ▲

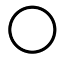
5. Cellular structures: examples and applications

5.2. **(Graded) cellular structures and the Temperley-Lieb algebras: a comparison.** Finally we want to present one explicit example, the Temperley-Lieb algebras, which is of particular interest in low-dimensional topology and categorification. Our main goal is to construct new (graded) cellular bases, and use our approach to establish semisimplicity conditions, and construct and compute the dimensions of its simple modules in new ways.

Fix $\delta = q + q^{-1}$ for $q \in \mathbb{K}^*$.[5] Recall that the *Temperley-Lieb algebra* $\mathcal{TL}_d(\delta)$ in d strands with parameter δ is the free diagram algebra over $\mathbb{K}$ with basis consisting of all possible

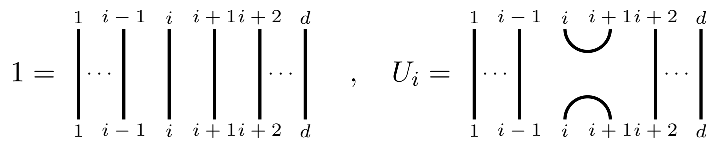
[5]The $\mathfrak{sl}_2$ case works with any $q \in \mathbb{K}^*$, including even roots of unity, see e.g. [5, Definition 2.3].

non-intersecting tangle diagrams with d bottom and top boundary points modulo boundary preserving isotopy and the local relation for evaluating circles given by the parameter[6] δ:



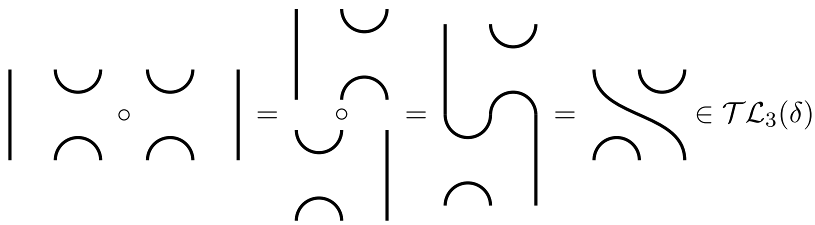
$$= \delta = q + q^{-1} \in \mathbb{K}.$$

The algebra $\mathcal{TL}_d(\delta)$ is locally generated by



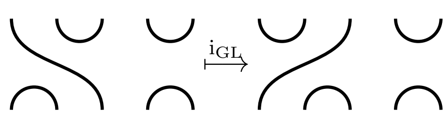
for $i = 1, \ldots, d-1$ called *identity* 1 and *cap-cup* U_i (which takes place between the strand i and $i+1$). For simplicity, we suppress the boundary labels in the following.

The multiplication $y \circ x$ is giving by stacking diagram y on top of diagram x. For example



Recall from [4, 5.1.3] (whose notation we use now) that, by quantum Schur-Weyl duality, we can use [4, Theorem 3.9] to obtain a cellular basis of $\mathcal{TL}_d(\delta)$. The aim now is to compare our cellular bases to the one given by Graham and Lehrer in [12, Theorem 6.7], where we point out that we do not obtain their cellular basis: our cellular basis depends for instance on whether $\mathcal{TL}_d(\delta)$ is semisimple or not. In the non-semisimple case, at least for $\mathbb{K} = \mathbb{C}$, we obtain a non-trivially $\mathbb{Z}$-graded cellular basis in the sense of [13, Definition 2.1], see Proposition 5.17.

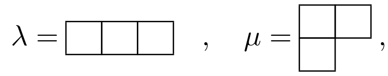
We want to compare our cell datum $(\mathcal{P}, \mathcal{I}, \mathcal{C}, \mathrm{i})$ to the one of Graham and Lehrer (indicated by a subscript GL) from [12, Section 6]. To this end, let us recall Graham and Lehrer's cell datum $(\mathcal{P}_{\mathrm{GL}}, \mathcal{I}_{\mathrm{GL}}, \mathcal{C}_{\mathrm{GL}}, \mathrm{i}_{\mathrm{GL}})$. The $\mathbb{K}$-linear anti-involution i_{GL} is given by "turning pictures upside down". For example



For the insistent reader: more formally, the $\mathbb{K}$-linear anti-involution i_{GL} is the unique $\mathbb{K}$-linear anti-involution which fixes all U_i's for $i = 1, \ldots, d-1$.

The data $\mathcal{P}_{\mathrm{GL}}$ and $\mathcal{I}_{\mathrm{GL}}$ are given combinatorially: $\mathcal{P}_{\mathrm{GL}}$ is the set $\Lambda^+(2,d)$ of all Young diagrams with d nodes and at most two rows. For example, the elements of $\Lambda^+(2,3)$ are

(37)



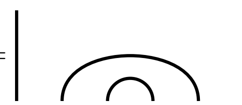
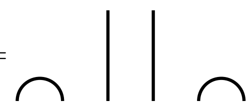
[6]We point out that there are two different conventions about circle evaluations in the literature: evaluating to δ or to $-\delta$. We use the first convention because we want to stay close to the cited literature.

where we point out that we use the English notation for Young diagrams. Now $\mathcal{I}_{\mathrm{GL}}^{\lambda}$ is the set of all standard tableaux of shape λ, denoted by $\mathrm{Std}(\lambda)$, that is, all fillings of λ with non-repeating numbers $1,\dots,d$ such that the entries strictly increase along rows and columns. For example, the elements of $\mathrm{Std}(\mu)$ for μ as in (37) are

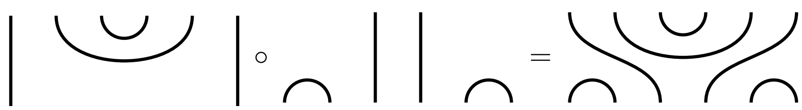
$$t_1 = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \\ \cline{1-1}\end{array}\;,\quad t_2 = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \\ \cline{1-1}\end{array}\,. \tag{38}$$

The set $\mathcal{P}_{\mathrm{GL}}$ is a poset where the order $\leq$ is the so-called dominance order on Young diagrams. In the "at most two rows case" this is $\mu \leq \lambda$ iff μ has fewer columns (an example is (37) with the same notation).

The only thing missing is thus the parametrization of the cellular basis. This works as follows: fix $\lambda \in \Lambda^{+}(2,d)$ and assign to each $t \in \mathrm{Std}(\lambda)$ a "half diagram" x_t via the rule that one "caps off" the strands whose numbers appear in the second row with the biggest possible candidate to the left of the corresponding number (going from left to right in the second row). Note that this is well-defined due to planarity. For example,

(39) $s = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 6 \\ \hline 4 & 5 \\ \cline{1-2}\end{array} \rightsquigarrow x_s =$  , $t = \begin{array}{|c|c|c|c|}\hline 1 & 3 & 4 & 5 \\ \hline 2 & 6 \\ \cline{1-2}\end{array} \rightsquigarrow x_t =$ 

Then one obtains c_{st}^{λ} by "turning x_s upside down and stacking it on top of x_t". For example,

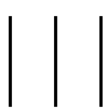
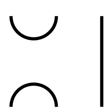
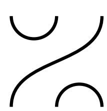
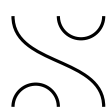
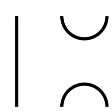
$c_{st}^{\lambda} = \mathrm{i}_{\mathrm{GL}}(x_s) \circ x_t =$ 

for $\lambda \in \Lambda^{+}(2,6)$ and $s,t \in \mathrm{Std}(\lambda)$ as in (39). The map $\mathcal{C}_{\mathrm{GL}}$ sends $(s,t) \in \mathcal{I}_{\mathrm{GL}}^{\lambda} \times \mathcal{I}_{\mathrm{GL}}^{\lambda}$ to c_{st}^{λ}.

Theorem 5.1. (**Cellular basis for $\mathcal{TL}_d(\delta)$ - the first**) The quadruple $(\mathcal{P}_{\mathrm{GL}}, \mathcal{I}_{\mathrm{GL}}, \mathcal{C}_{\mathrm{GL}}, \mathrm{i}_{\mathrm{GL}})$ is a cell datum for $\mathcal{TL}_d(\delta)$. □

Proof. This is [12, Theorem 6.7]. ■

Example 5.2. For $\mathcal{TL}_3(\delta)$ we have five basis elements, namely

$c_{cc}^{\lambda} =$  , $c_{t_1t_1}^{\mu} =$  , $c_{t_1t_2}^{\mu} =$  , $c_{t_2t_1}^{\mu} =$  , $c_{t_2t_2}^{\mu} =$ 

where we use the notation from (37) and (38) (and the "canonical" filling c for λ). ▲

Before stating our cellular basis, we provide a criterion which tells precisely whether $\mathcal{TL}_d(\delta)$ is semisimple or not. Recall that there is a known criteria for which Weyl modules $\Delta_q(i)$ are simple, see e.g. [5, Proposition 2.7].

Proposition 5.3. (**Semisimplicity criterion for $\mathcal{TL}_d(\delta)$**) We have the following.

(a) Let $\delta \neq 0$. Then $\mathcal{TL}_d(\delta)$ is semisimple iff $[i] = q^{1-i} + \cdots + q^{i-1} \neq 0$ for all $i = 1,\dots,d$ iff q is not a root of unity with $d < l = \mathrm{ord}(q^2)$, or $q = 1$ and $\mathrm{char}(\mathbb{K}) > d$.
(b) Let $\mathrm{char}(\mathbb{K}) = 0$. Then $\mathcal{TL}_d(0)$ is semisimple iff d is odd (or $d = 0$).
(c) Let $\mathrm{char}(\mathbb{K}) = p > 0$. Then $\mathcal{TL}_d(0)$ is semisimple iff $d \in \{0,1,3,5,\dots,2p-1\}$. □

Proof. (a): We want to show that $T = V^{\otimes d}$ decomposes into simple $\mathbf{U}_q$-modules iff $d < l$, or $q = 1$ and $\mathrm{char}(\mathbb{K}) > d$, which is clearly equivalent to the non-vanishing of the $[i]$'s.

Assume that $d < l$. Since the maximal $\mathbf{U}_q$-weight of $V^{\otimes d}$ is d and since all Weyl $\mathbf{U}_q$-modules $\Delta_q(i)$ for $i < l$ are simple, we see that all indecomposable summands of $V^{\otimes d}$ are simple.

Otherwise, if $l \leq d$, then $T_q(d)$ (or $T_q(d-2)$ in the case $d \equiv -1 \bmod l$) is a non-simple, indecomposable summand of $V^{\otimes d}$ (note that this arguments fails if $l = 2$, i.e. $\delta = 0$).

The case $q = 1$ works similar, and we can now use [4, Theorem 4.13] to finish the proof of (a).

(b): Since $\delta = 0$ iff $q = \pm\sqrt[2]{-1}$, we can use the linkage from e.g. [5, Theorem 2.23] in the case $l = 2$ to see that $T = V^{\otimes d}$ decomposes into a direct sum of simple $\mathbf{U}_q$-modules iff d is odd (or $d = 0$). This implies that $\mathcal{TL}_d(0)$ is semisimple iff d is odd (or $d = 0$) by [4, Theorem 4.13].

(c): If $\mathrm{char}(\mathbb{K}) = p > 0$ and $\delta = 0$ (for $p = 2$ this is equivalent to $q = 1$), then we have $\Delta_q(i) \cong L_q(i)$ iff $i = 0$ or $i \in \{2ap^n - 1 \mid n \in \mathbb{Z}_{\geq 0}, 1 \leq a < p\}$. In particular, this means that for $d \geq 2$ we have that either $T_q(d)$ or $T_q(d-2)$ is a simple $\mathbf{U}_q$-module iff $d \in \{3, 5, \ldots, 2p-1\}$. Hence, using the same reasoning as above, we see that $T = V^{\otimes d}$ is semisimple iff $d \in \{0, 1, 3, 5, \ldots, 2p-1\}$. By [4, Theorem 4.13] we see that $\mathcal{TL}_d(0)$ is semisimple iff $d \in \{0, 1, 3, 5, \ldots, 2p-1\}$. ■

Example 5.4. We have that $[k] \neq 0$ for all $k = 1, 2, 3$ is satisfied iff q is not a forth or a sixth root of unity. By Proposition 5.3 we see that $\mathcal{TL}_3(\delta)$ is semisimple as long as q is not one of these values from above. The other way around is only true for q being a sixth root of unity (the conclusion from semisimplicity to non-vanishing of the quantum numbers above does not work in the case $q = \pm\sqrt[2]{-1}$). ▲

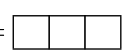
Remark 5.5. The semisimplicity criterion for $\mathcal{TL}_d(\delta)$ was already already found, using quite different methods, in [36, Section 5] in the case $\delta \neq 0$, and in the case $\delta = 0$ in [23, Chapter 7] or [27, above Proposition 4.9]. For us it is an easy application of [4, Theorem 4.13]. ▲

A direct consequence of Proposition 5.3 is that the Temperley-Lieb algebra $\mathcal{TL}_d(\delta)$ for $q \in \mathbb{K}^* - \{1\}$ not a root of unity is semisimple (or $q = \pm 1$ and $\mathrm{char}(\mathbb{K}) = 0$), regardless of d.

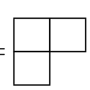
5.2.1. *Temperley-Lieb algebra: the semisimple case.* Assume that $q \in \mathbb{K}^* - \{1\}$ is not a root of unity (or $q = \pm 1$ and $\mathrm{char}(\mathbb{K}) = 0$). Thus, we are in the semisimple case.

Let us first compare the cell datum of Graham and Lehrer with our cell datum. We have the poset $\mathcal{P}_{\mathrm{GL}}$ consisting of all $\lambda \in \Lambda^+(2, d)$ in Graham and Lehrer's case and the poset $\mathcal{P}$ consisting of all $\lambda \in X^+$ such that $\Delta_q(\lambda)$ is a factor of T in our case.

The two sets are the same: an element $\lambda = (\lambda_1, \lambda_2) \in \mathcal{P}_{\mathrm{GL}}$ corresponds to $\lambda_1 - \lambda_2 \in \mathcal{P}$. This is clearly an injection of sets. Moreover, $\Delta_q(i) \otimes \Delta_q(1) \cong \Delta_q(i+1) \oplus \Delta_q(i-1)$ for $i > 0$ shows surjectivity. Two easy examples are

$\lambda = (\lambda_1, \lambda_2) = (3, 0) =$  $\in \mathcal{P}_{\mathrm{GL}} \rightsquigarrow \lambda_1 - \lambda_2 = 3 \in \mathcal{P},$

and

$\mu = (\mu_1, \mu_2) = (2, 1) =$  $\in \mathcal{P}_{\mathrm{GL}} \rightsquigarrow \mu_1 - \mu_2 = 1 \in \mathcal{P}$,

which fits to the decomposition as in (36).

Similarly, an inductive reasoning shows that there is a factor $\Delta_q(i)$ of T for any standard filling for the Young diagram that gives rise to i under the identification from above. Thus, $\mathcal{I}_{\mathrm{GL}}$ is also the same as our $\mathcal{I}$.

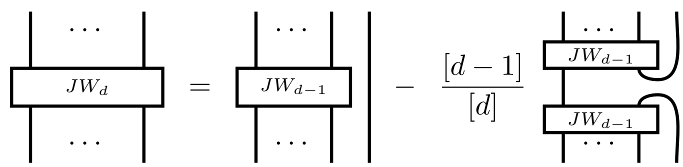
As an example, we encourage the reader to compare (37) and (38) with (36).

To see that the $\mathbb{K}$-linear anti-involution i_{GL} is also our involution i, we note that we build our basis from a "top" part g_i^λ and a "bottom" part f_j^λ and i switches top and bottom similarly as the $\mathbb{K}$-linear anti-involution i_{GL}.

Thus, except for $\mathcal{C}$ and $\mathcal{C}_{\mathrm{GL}}$, the cell data agree.

In order to state how our cellular basis for $\mathcal{TL}_d(\delta)$ looks like, recall the following definition of the *(generalized) Jones-Wenzl projectors*.

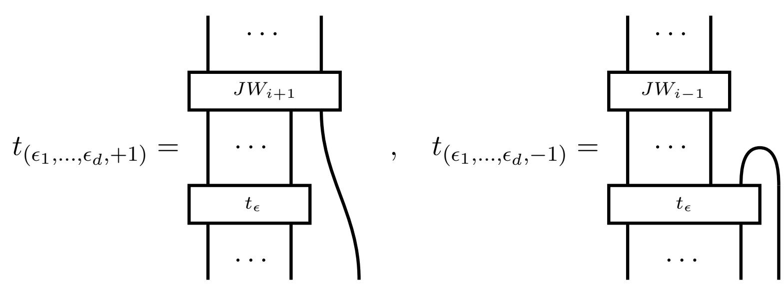
Definition 5.6. (**Jones-Wenzl projectors**) The d-th *Jones-Wenzl projectors*, which we denote by $JW_d \in \mathcal{TL}_d(\delta)$, is recursively defined via the recursion rule



where we assume that JW_1 is the identity diagram in one strand. ▲

Note that the projector JW_d can be identified with the projection of $T = V^{\otimes d}$ onto its maximal weight summand. These projectors were introduced by Jones in [17] and then further studied by Wenzl in [35]. In fact, they can be generalized as follows.

Definition 5.7. (**Generalized Jones-Wenzl projectors**) Given any d-tuple (with $d > 0$) of the form $\vec{\epsilon} = (\epsilon_1, \dots, \epsilon_d) \in \{\pm 1\}^d$ such that $\sum_{j=1}^k \epsilon_j \geq 0$ for all $k = 1, \dots, d$. Set $i = \sum_{j=1}^d \epsilon_j$. We define recursively two certain "half-diagrams" $t_{(\epsilon_1,\dots,\epsilon_d,\pm 1)}$ via



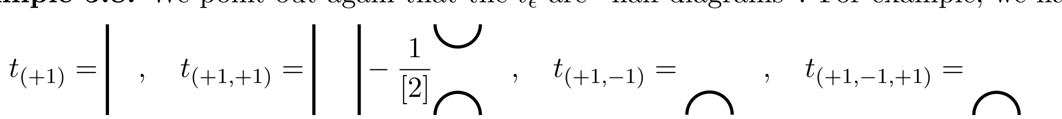
where $t_{(+1)} \in \mathcal{TL}_1(\delta)$ is defined to be the identity element. By convention, $t_{(\epsilon_1,\dots,\epsilon_d,-1)} = 0$ if $i - 1 < 0$. Note that $t_{(\epsilon_1,\dots,\epsilon_d,\pm 1)}$ has always $d+1$ bottom boundary points, but $i \pm 1$ top boundary points.

Then we assign to any such $\vec{\epsilon}$ a *generalized Jones-Wenzl "projector"* $JW_{\vec{\epsilon}} \in \mathcal{TL}_d(\delta)$ via

$$JW_{\vec{\epsilon}} = \mathrm{i}(t_{\vec{\epsilon}}) \circ t_{\vec{\epsilon}},$$

where i is, as above, the $\mathbb{K}$-linear anti-involution that "turns pictures upside down". ▲

Example 5.8. We point out again that the $t_{\vec{\epsilon}}$ are "half-diagrams". For example, we have



where we can read off the top boundary points by summing the ϵ_i's. ▲

Note that Jones-Wenzl projectors are special cases of the construction in Definition 5.7, i.e. $JW_d = JW_{(1,\dots,1)}$. Moreover, while we think about the Jones-Wenzl projectors as projecting to the maximal weight summand of $T = V^{\otimes d}$, the generalized Jones-Wenzl projectors $JW_{\vec{\epsilon}}$ project to the summands of $T = V^{\otimes d}$ of the form $\Delta_q(i)$ where i is as above $i = \sum_{j=1}^{d} \epsilon_j$. To be more precise, we have the following.

Proposition 5.9. (**Diagrammatic projectors**) There exists non-zero scalars $a_{\vec{\epsilon}} \in \mathbb{K}$ such that $JW'_{\vec{\epsilon}} = a_{\vec{\epsilon}} JW_{\vec{\epsilon}}$ are well-defined idempotents forming a complete set of mutually orthogonal, commuting, primitive idempotents in $\mathcal{TL}_d(\delta)$. □

Proof. That they are well-defined follows from the fact that no quantum numbers vanish if $q \in \mathbb{K}^* - \{1\}$ is not a root of unity.

The other statements can be proven as in [9, Proposition 2.19 and Theorem 2.20] (beware that they call these projectors higher Jones-Wenzl projectors). ■

Example 5.10. Recall from Example 3.27 that we have the following decompositions.

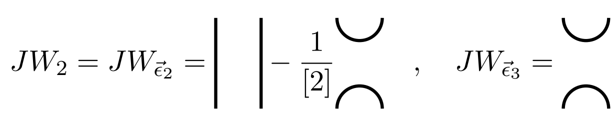
$$V^{\otimes 1} = \Delta_q(1), \quad V^{\otimes 2} \cong \Delta_q(2) \oplus \Delta_q(0), \quad V^{\otimes 3} \cong \Delta_q(3) \oplus \Delta_q(1) \oplus \Delta_q(1). \tag{40}$$

Moreover, there are the following $\vec{\epsilon}$ vectors. We have $\vec{\epsilon}_1 = (+1)$ and

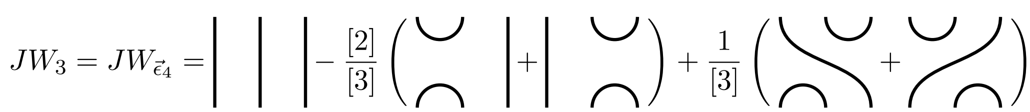
$$\vec{\epsilon}_2 = (+1,+1),\ \vec{\epsilon}_3 = (+1,-1),\ \vec{\epsilon}_4 = (+1,+1,+1),\ \vec{\epsilon}_5 = (+1,+1,-1),\ \vec{\epsilon}_6 = (+1,-1,+1).$$

We point out that $(+1,-1,-1)$ does not satisfy the sum property from Definition 5.7 and thus, does not count.

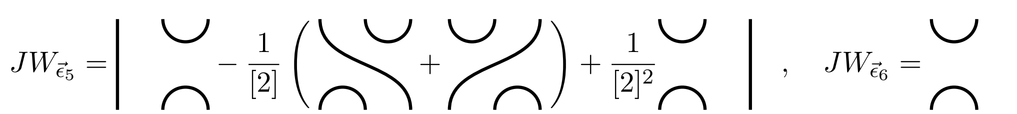
By construction, $JW_{\vec{\epsilon}_1}$ is the identity strand in one variable and hence, is the projector on the unique factor in (40). Moreover, we have



where $JW_{\vec{\epsilon}_2}$ and $JW_{\vec{\epsilon}_3}$ are the (up to scalars) projectors onto the $\Delta_q(2)$ and the $\Delta_q(0)$ summand in (40) respectively. Furthermore, we have



is the projection to the $\Delta_q(3)$ summand in (40). The other two projectors are (up to scalars)



as we invite the reader to check. ▲

Proposition 5.11. (**(New) cellular bases**) The datum given by the quadruple $(\mathcal{P}, \mathcal{I}, \mathcal{C}, \mathrm{i})$ for $\mathcal{TL}_d(\delta) \cong \mathrm{End}_{\mathbf{U}_q}(T)$ is a cell datum for $\mathcal{TL}_d(\delta)$. Moreover, $\mathcal{C} \neq \mathcal{C}_{\mathrm{GL}}$ for all $d > 1$ and all choices involved in the definition of $\mathrm{im}(\mathcal{C})$. In particular, there is a choice such that all generalized Jones-Wenzl projectors are part of $\mathrm{im}(\mathcal{C})$. □

Proof. That we get a cell datum as stated follows from [4, Theorem 4.13] and the discussion above.

That our cellular basis $\mathcal{C}$ will never be $\mathcal{C}_{\mathrm{GL}}$ for $d > 1$ is due to the fact that Graham and Lehrer's cellular basis always contains the identity (which corresponds to the unique standard filling of the Young diagram associated to $\lambda = (d, 0)$).

In contrast, let $\lambda_k = (d-k, k)$ for $0 \leq k \leq \lfloor \frac{d}{2} \rfloor$. Then

$$T = V^{\otimes d} \cong \Delta_q(d) \oplus \bigoplus_{0 < k \leq \lfloor \frac{d}{2} \rfloor} \Delta_q(d-2k)^{\oplus m_{\lambda_k}} \tag{41}$$

for some multiplicities $m_{\lambda_k} \in \mathbb{Z}_{>0}$, we see that for $d > 1$ the identity is never part of any of our bases: all the $\Delta_q(i)$'s are simple $\mathbf{U}_q$-modules and each c^k_{ij} factors only through $\Delta_q(k)$. In particular, the basis element c^λ_{11} for $\lambda = \lambda_d$ has to be (a scalar multiple) of $JW_{(1,\dots,1)}$.

As in [4, 5.1.1] we can choose for $\mathcal{C}$ an Artin-Wedderburn basis of $\mathrm{End}_{\mathbf{U}_q}(T) \cong \mathcal{TL}_d(\delta)$.

By our construction, all basis elements c^k_{ij} are block matrices of the form

$$\begin{pmatrix} \mathbf{M}_d & 0 & \cdots & 0 \\ 0 & \mathbf{M}_{d-2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{M}_\varepsilon \end{pmatrix}$$

with $\varepsilon = 0$ if d is even and $\varepsilon = 1$ if d is odd (where we regard V as decomposed as in (41), the indices should indicate the summands and $\mathbf{M}_{d-2k}$ is of size $m_k \times m_k$).

Clearly, the block matrices of the form $\mathbf{E}^k_{ii}$ for $i = 1, \dots, m_k$ with only non-zero entry in the i-th column and row of $\mathbf{M}_k$ form a set of mutually orthogonal, commuting, primitive idempotents. Hence, by Proposition 5.9, these have to be the generalized Jones-Wenzl projectors JW_ϵ for $k = \sum_{j=1}^k \epsilon_j$. ■

Example 5.12. Let us consider $\mathcal{TL}_3(\delta)$ as in Example 5.2 for any $q \in \mathbb{K}^* - \{1, \pm \sqrt[2]{-1}\}$ that is not a critical value as in Example 5.4. Then $\mathcal{TL}_3(\delta)$ is semisimple by Proposition 5.3.

In particular, we have a decomposition of $V^{\otimes 3}$ as in (40). Fix the same order as in (40). Then we can choose five basis elements as

$$c^\lambda_{cc} = \mathbf{E}_{11}, \quad c^\mu_{t_1t_1} = \mathbf{E}_{22}, \quad c^\mu_{t_1t_2} = \mathbf{E}_{23}, \quad c^\mu_{t_2t_1} = \mathbf{E}_{32}, \quad c^\mu_{t_2t_2} = \mathbf{E}_{33},$$

where we use the notation from (37) and (38) (and the "canonical" filling c for λ) again.

Note that c^λ_{cc} corresponds to the Jones-Wenzl projector $JW_3 = JW_{(+1+1+1)}$, $c^\mu_{t_1t_1}$ corresponds to $JW_{(+1+1-1)}$ and $c^\mu_{t_2t_2}$ corresponds to $JW_{(+1-1+1)}$. Compare to Example 5.10. ▲

Note the following classification result (see for example [27, Corollary 5.2] for $\mathbb{K} = \mathbb{C}$).

Corollary 5.13. We have a complete set of pairwise non-isomorphic, simple $\mathcal{TL}_d(\delta)$-modules $L(\lambda)$ for λ being a length-two partition of d with $\dim(L(\lambda)) = |\mathrm{Std}(\lambda)|$, where $\mathrm{Std}(\lambda)$ is the set of all standard tableaux of shape λ. □

Proof. Directly from Proposition 5.11 and [4, Theorems 4.11 and 4.12] because $m_\lambda = |\mathrm{Std}(\lambda)|$ (with the notation from [4, Theorem 4.12]). ■

Example 5.14. The Temperley-Lieb algebra $\mathcal{TL}_3(\delta)$ in the semisimple case has

$$\dim\left(L\left(\ydiagram{3}\right)\right)=1,\qquad \dim\left(L\left(\ydiagram{2,1}\right)\right)=2.$$

Compare to (38). ▲

5.2.2. *Temperley-Lieb algebra: the non-semisimple case.* Let us assume that we have fixed $q\in\mathbb{K}^*-\{1,\pm\sqrt[2]{-1}\}$ to be a critical value such that $[k]=0$ for some $k=1,\dots,d$. Then, by Proposition 5.3, the algebra $\mathcal{TL}_d(\delta)$ is no longer semisimple. In particular, to the best of our knowledge, there is no diagrammatic analogon of the Jones-Wenzl projectors in general.

Proposition 5.15. (**(New) cellular basis - the second**) The datum $(\mathcal{P},\mathcal{I},\mathcal{C},\mathrm{i})$ with $\mathcal{C}$ as in [4, Theorem 3.9] for $\mathcal{TL}_d(\delta)\cong\mathrm{End}_{\mathbf{U}_q}(T)$ is a cell datum for $\mathcal{TL}_d(\delta)$. Moreover, $\mathcal{C}\neq\mathcal{C}_{\mathrm{GL}}$ for all $d>1$ and all choices involved in the definition of our basis. Thus, there is a choice such that all generalized, non-semisimple Jones-Wenzl projectors are part of $\mathrm{im}(\mathcal{C})$. □

Proof. As in the proof of Proposition 5.11 and left to the reader. ■

Note that we can do better: as in Example 3.22 one gets a decomposition

$$\boldsymbol{\mathcal{T}}\cong\boldsymbol{\mathcal{T}}_{-1}\oplus\boldsymbol{\mathcal{T}}_0\oplus\boldsymbol{\mathcal{T}}_1\oplus\cdots\oplus\boldsymbol{\mathcal{T}}_{l-3}\oplus\boldsymbol{\mathcal{T}}_{l-2}\oplus\boldsymbol{\mathcal{T}}_{l-1}, \tag{42}$$

where the blocks $\boldsymbol{\mathcal{T}}_{-1}$ and $\boldsymbol{\mathcal{T}}_{l-1}$ are semisimple if $\mathbb{K}=\mathbb{C}$. Compare also to [5, Lemma 2.25].

If we fix $\mathbb{K}=\mathbb{C}$: as explained in [5, Section 3.5] each block in the decomposition (42) can be equipped with a non-trivial $\mathbb{Z}$-grading coming from Khovanov and Seidel's quiver algebra from [19]. Hence, we have the following.

Lemma 5.16. The $\mathbb{C}$-algebra $\mathrm{End}_{\mathbf{U}_q}(T)$ can be equipped with a non-trivial $\mathbb{Z}$-grading. Thus, $\mathcal{TL}_d(\delta)$ over $\mathbb{C}$ can be equipped with a non-trivial $\mathbb{Z}$-grading. □

Proof. The second statement follows directly from the first using quantum Schur-Weyl duality. Hence, we only need to show the first.

Note that $T=V^{\otimes d}$ decomposes as in (41), we can order this decomposition by blocks. Each block carries a $\mathbb{Z}$-grading coming from Khovanov and Seidel's quiver algebra (as explained in details in [5, Section 3]). In particular, we can choose the basis elements c_{ij}^{λ} in such a way that we get the $\mathbb{Z}$-graded basis obtained in [5, Corollary 4.23]. Since there is no interaction between different blocks, the statement follows. ■

Recall from [13, Definition 2.1] that a $\mathbb{Z}$-graded cell datum of a $\mathbb{Z}$-graded algebra is a cell datum for the algebra together with an additional *degree function* $\deg\colon\coprod_{\lambda\in\mathcal{P}}\mathcal{I}^{\lambda}\to\mathbb{Z}$, such that $\deg(c_{ij}^{\lambda})=\deg(i)+\deg(j)$. For us the choice of $\deg(\cdot)$ is as follows.

If $\lambda\in\mathcal{P}$ is in one of the semisimple blocks, then we simply set $\deg(i)=0$ for all $i\in\mathcal{I}^{\lambda}$.

Assume that $\lambda\in\mathcal{P}$ is not in the semisimple blocks. It is known that every $T_q(\lambda)$ has precisely two Weyl factors. The g_i^{λ} that map $\Delta_q(\lambda)$ into a higher $T_q(\mu)$ should be indexed by a 1-colored i whereas the g_i^{λ} mapping $\Delta_q(\lambda)$ into $T_q(\lambda)$ should have 0-colored i. Similarly for the f_j^{λ}'s. Then the degree of the elements $i\in\mathcal{I}^{\lambda}$ should be the corresponding color. We get the following. (Here $\mathcal{C}$ is as in [4, Theorem 3.9].)

Proposition 5.17. (**Graded cellular basis**) The datum $(\mathcal{P},\mathcal{I},\mathcal{C},\mathrm{i})$ supplemented with the function $\deg(\cdot)$ from above is a $\mathbb{Z}$-graded cell datum for the $\mathbb{C}$-algebra $\mathcal{TL}_d(\delta)\cong\mathrm{End}_{\mathbf{U}_q}(T)$. □

Proof. The hardest part is cellularity which directly follows from [4, Theorem 3.9]. That the quintuple $(\mathcal{P}, \mathcal{I}, \mathcal{C}, \mathrm{i}, \deg)$ gives a $\mathbb{Z}$-graded cell datum follows from the construction. ■

Example 5.18. Let us consider $\mathcal{TL}_3(\delta)$ as in Example 5.12, namely q being a complex, primitive third root of unity. Then $\mathcal{TL}_3(\delta)$ is non-semisimple by Proposition 5.3. In particular, we have a decomposition of $V^{\otimes 3}$ different from (40), namely as in (35). In this case $\mathcal{P} = \{1, 3\}$, $\mathcal{I}^3 = \{1, 3\}$ and $\mathcal{I}^1 = \{1\}$. By our choice from above

$$\deg(i) = \begin{cases} 0, & \text{if } i = 1 \in \mathcal{I}^1 \text{ or } i = 3 \in \mathcal{I}^3, \\ 1, & \text{if } i = 1 \in \mathcal{I}^3. \end{cases}$$

As in Example 5.12 (if we use the ordering induced by the decomposition from (35)), we can choose basis elements as

$$c^{\lambda}_{cc} = \mathbf{E}_{11}, \quad c^{\mu}_{t_1t_1} = \mathbf{E}_{22}, \quad c^{\mu}_{t_1t_2} = \mathbf{E}_{21}, \quad c^{\mu}_{t_2t_1} = \mathbf{E}_{12}, \quad c^{\mu}_{t_2t_2} = \mathbf{E}_{33},$$

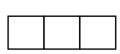
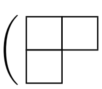
where we use the notation from (37) and (38) (and the "canonical" filling c for λ) again. These are of degrees $0, 1, 1, 2$ and 0 respectively. We also note the difference to the basis in the semisimple case from Example 5.12. ▲

Remark 5.19. Our grading and the one found by Plaza and Ryom-Hansen in [26] agree (up to a shift of the indecomposable summands). To see this, note that our algebra is isomorphic to the algebra $K_{1,n}$ studied in [7] which is by [7, (4.8)] and [8, Theorem 6.3] a quotient of some particular cyclotomic KL-R algebra (the compatibility of the grading follows for example from [14, Corollary B.6]). The same holds, by construction, for the grading in [26]. ▲

Corollary 5.20. Let $\mathbb{K} = \mathbb{C}$. We have a complete set of pairwise non-isomorphic, simple $\mathcal{TL}_d(\delta)$-modules $L(\lambda)$ for $\lambda \in \Lambda^+(2, d)$ such that $T_q(\lambda)$ is a summand of $T = V^{\otimes d}$ with $\dim(L(\lambda)) = m_\lambda$, where m_λ is the multiplicity of $T_q(\lambda)$ as a summand of $T = V^{\otimes d}$. □

Proof. As in Corollary 5.13. ■

Example 5.21. If q is a complex, primitive third root of unity, then $\mathcal{TL}_3(\delta)$ has

$\dim\left(L\left(\right.\right.$  $\left.\left.\right)\right) = 1, \qquad \dim\left(L\left(\right.\right.$  $\left.\left.\right)\right) = 1.$

Note the contrast to the semisimple case from Example 5.14. ▲

Remark 5.22. In the case $\mathbb{K} = \mathbb{C}$ we can give a dimension formula, namely

$$\dim(L(\lambda)) = m_\lambda = \begin{cases} |\mathrm{Std}(\lambda)|, & \text{if } \lambda_1 - \lambda_2 \equiv -1 \bmod l, \\ \sum_{\mu = w.\lambda, \mu \geq \lambda \in \Lambda^+(2,d)} (-1)^{\ell(w)} |\mathrm{Std}(\mu)|, & \text{if } \lambda_1 - \lambda_2 \not\equiv -1 \bmod l, \end{cases}$$

where $w \in W_l$ is the affine Weyl group and $\ell(w)$ is the length of a reduced word $w \in W_l$. This matches the formulas from, for example, [2, Proposition 6.7] or [27, Corollary 5.2]. In the case where $\mathrm{char}(\mathbb{K}) > 0$ one can in principle also obtain a formula. But this time we do not encourage the reader to work out the (rather complicated) formula. ▲

References

[1] H.H. Andersen. The strong linkage principle for quantum groups at roots of 1. *J. Algebra*, 260(1):2–15, 2003. doi:10.1016/S0021-8693(02)00618-X.

[2] H.H. Andersen, G. Lehrer, and R. Zhang. Cellularity of certain quantum endomorphism algebras. To appear in Pacific J. Math. URL: http://arxiv.org/abs/1303.0984.

[3] H.H. Andersen, P. Polo, and K.X. Wen. Representations of quantum algebras. *Invent. Math.*, 104(1):1–59, 1991. doi:10.1007/BF01245066.

[4] H.H. Andersen, C. Stroppel, and D. Tubbenhauer. Semisimplicity of hecke and (walled) brauer algebras. URL: http://arxiv.org/abs/1507.07676.

[5] H.H. Andersen and D. Tubbenhauer. Diagram categories for $\mathbf{U}_q$-tilting modules at roots of unity. To appear in Transform. Groups. URL: http://arxiv.org/abs/1503.00224.

[6] N. Bourbaki. *Lie groups and Lie algebras. Chapters 7–9*. Elements of Mathematics (Berlin). Springer-Verlag, Berlin, 2005. Translated from the 1975 and 1982 French originals by Andrew Pressley.

[7] J. Brundan and C. Stroppel. Highest weight categories arising from Khovanov's diagram algebra I: cellularity. *Mosc. Math. J.*, 11(4):685–722, 821–822, 2011. URL: http://arxiv.org/abs/0806.1532.

[8] J. Brundan and C. Stroppel. Highest weight categories arising from Khovanov's diagram algebra III: category $\mathcal{O}$. *Represent. Theory*, 15:170–243, 2011. URL: http://arxiv.org/abs/0812.1090, doi:10.1090/S1088-4165-2011-00389-7.

[9] B. Cooper and M. Hogancamp. An exceptional collection for Khovanov homology. URL: http://arxiv.org/abs/1209.1002.

[10] S. Donkin. On tilting modules for algebraic groups. *Math. Z.*, 212(1):39–60, 1993. doi:10.1007/BF02571640.

[11] S. Donkin. *The q-Schur algebra*, volume 253 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1998. doi:10.1017/CBO9780511600708.

[12] J.J. Graham and G. Lehrer. Cellular algebras. *Invent. Math.*, 123(1):1–34, 1996. doi:10.1007/BF01232365.

[13] J. Hu and A. Mathas. Graded cellular bases for the cyclotomic Khovanov-Lauda-Rouquier algebras of type *A*. *Adv. Math.*, 225(2):598–642, 2010. URL: http://arxiv.org/abs/0907.2985, doi:10.1016/j.aim.2010.03.002.

[14] J. Hu and A. Mathas. Quiver Schur algebras for linear quivers. *Proc. Lond. Math. Soc. (3)*, 110(6):1315–1386, 2015. URL: http://arxiv.org/abs/1110.1699, doi:10.1112/plms/pdv007.

[15] J.C. Jantzen. *Lectures on quantum groups*, volume 6 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 1996.

[16] J.C. Jantzen. *Representations of algebraic groups*, volume 107 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, second edition, 2003.

[17] V.F.R. Jones. Index for subfactors. *Invent. Math.*, 72(1):1–25, 1983. doi:10.1007/BF01389127.

[18] M. Kaneda. Based modules and good filtrations in algebraic groups. *Hiroshima Math. J.*, 28(2):337–344, 1998.

[19] M. Khovanov and P. Seidel. Quivers, Floer cohomology, and braid group actions. *J. Amer. Math. Soc.*, 15(1):203–271, 2002. URL: http://arxiv.org/abs/math/0006056, doi:10.1090/S0894-0347-01-00374-5.

[20] G. Lusztig. Finite-dimensional Hopf algebras arising from quantized universal enveloping algebra. *J. Amer. Math. Soc.*, 3(1):257–296, 1990. doi:10.2307/1990988.

[21] G. Lusztig. *Introduction to quantum groups*. Modern Birkhäuser Classics. Birkhäuser/Springer, New York, 2010. Reprint of the 1994 edition. doi:10.1007/978-0-8176-4717-9.

[22] G. Lusztig. Quantum groups at roots of 1. *Geom. Dedicata*, 35(1-3):89–113, 1990. doi:10.1007/BF00147341.

[23] P. Martin. *Potts models and related problems in statistical mechanics*, volume 5 of *Series on Advances in Statistical Mechanics*. World Scientific Publishing Co., Inc., Teaneck, NJ, 1991. doi:10.1142/0983.

[24] V. Mazorchuk. *Lectures on algebraic categorification*. QGM Master Class Series. European Mathematical Society (EMS), Zürich, 2012. URL: http://arxiv.org/abs/1011.0144, doi:10.4171/108.

[25] J. Paradowski. *Filtrations of modules over the quantum algebra*, volume 56 of *Proc. Sympos. Pure Math.* Amer. Math. Soc., Providence, RI, 1994.

[26] D. Plaza and S. Ryom-Hansen. Graded cellular bases for Temperley-Lieb algebras of type A and B. *J. Algebraic Combin.*, 40(1):137–177, 2014. URL: http://arxiv.org/abs/1203.2592, doi:10.1007/s10801-013-0481-6.

[27] D. Ridout and Y. Saint-Aubin. Standard modules, induction and the structure of the Temperley-Lieb algebra. *Adv. Theor. Math. Phys.*, 18(5):957–1041, 2014. URL: http://arxiv.org/abs/1204.4505.

[28] C.M. Ringel. The category of modules with good filtrations over a quasi-hereditary algebra has almost split sequences. *Math. Z.*, 208(2):209–223, 1991. doi:10.1007/BF02571521.

[29] S. Ryom-Hansen. A q-analogue of Kempf's vanishing theorem. *Mosc. Math. J.*, 3(1):173–187, 260, 2003. URL: http://arxiv.org/abs/0905.0236.

[30] W. Soergel. Character formulas for tilting modules over Kac-Moody algebras. *Represent. Theory*, 2:432–448 (electronic), 1998. doi:10.1090/S1088-4165-98-00057-0.

[31] W. Soergel. Kazhdan-Lusztig polynomials and a combinatoric[s] for tilting modules. *Represent. Theory*, 1:83–114 (electronic), 1997. doi:10.1090/S1088-4165-97-00021-6.

[32] C. Stroppel. Untersuchungen zu den parabolischen Kazhdan-Lusztig-Polynomen für affine Weyl-Gruppen. Diploma Thesis (1997), 74 pages (German). URL: http://www.math.uni-bonn.de/ag/stroppel/arbeit_Stroppel.pdf.

[33] T. Tanisaki. *Character formulas of Kazhdan-Lusztig type*, volume 40 of *Fields Inst. Commun.* Amer. Math. Soc., Providence, RI, 2004.

[34] J.P. Wang. Sheaf cohomology on G/B and tensor products of Weyl modules. *J. Algebra*, 77(1):162–185, 1982. doi:10.1016/0021-8693(82)90284-8.

[35] H. Wenzl. On sequences of projections. *C. R. Math. Rep. Acad. Sci. Canada*, 9(1):5–9, 1987.

[36] B.W. Westbury. The representation theory of the Temperley-Lieb algebras. *Math. Z.*, 219(4):539–565, 1995. doi:10.1007/BF02572380.

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