GRADINGS ON WALLED BRAUER ALGEBRAS AND KHOVANOV'S ARC ALGEBRA

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ABSTRACT. We introduce some $\mathbb{Z}$-graded versions of the walled Brauer algebra $B_{r,s}(\delta)$, working over a field of characteristic zero. This allows us to prove that $B_{r,s}(\delta)$ is Morita equivalent to an idempotent truncation of a certain infinite dimensional version of Khovanov's arc algebra. We deduce that the walled Brauer algebra is Koszul whenever $\delta \neq 0$.

CONTENTS

1. INTRODUCTION AND WEIGHT DICTIONARY

Let $\delta \in \mathbb{C}$ be a fixed parameter. The *walled Brauer algebra* $B_{r,s}(\delta)$ is a subalgebra of the classical Brauer algebra $B_{r+s}(\delta)$; see §2 for the precise definition. It was introduced independently by Turaev [T] and Koike [Ko] in the late 1980s motivated in part by a Schur-Weyl duality between $B_{r,s}(m)$ and the general linear group $GL_m(\mathbb{C})$ arising from mutually commuting actions on the "mixed" tensor space $V^{\otimes r} \otimes W^{\otimes s}$, where V is the natural representation of $GL_m(\mathbb{C})$ and $W := V^*$; see also [BCHLLS].

If $\delta \notin \mathbb{Z}$ then the algebra $B_{r,s}(\delta)$ is semisimple, and its representation theory can be described using character-theoretic methods analogous to ones used in the study of the complex representation theory of the symmetric group; see e.g. [H], [N]. In this article we are mainly interested in the non-semisimple cases, when the representation theory is still surprisingly tractable. For example, it is known that $B_{r,s}(\delta)$ is a quasi-hereditary algebra if $\delta \neq 0$, and in these cases explicit closed formulae for the composition multiplicities of standard modules

and the corresponding Kazhdan-Lusztig polynomials have been worked out by Cox and De Visscher; see [CD, Theorem 4.10] and [CD, Corollary 9.3].

Assuming $\delta \in \mathbb{Z}$, the work just mentioned reveals some striking combinatorial coincidences between the representation theory of $B_{r,s}(\delta)$ and a certain infinite dimensional generalisation $K(\delta)$ of Khovanov's arc algebra. The latter algebra was introduced originally in a special case in [Kh] where it plays a role in Khovanov's categorification of the Jones polynomial. Khovanov's definition was extended in [BS1] to include various infinite dimensional algebras like $K(\delta)$.

The main goal of this article is to explain these combinatorial coincidences by proving that $B_{r,s}(\delta)$ is Morita equivalent to an idempotent truncation of $K(\delta)$. This adds a new twist to the results of [BS3], [BS4] which construct equivalences between other generalisations of Khovanov's arc algebra from [BS1] and various naturally occuring algebras and categories including level two cyclotomic Hecke algebras and Khovanov-Lauda-Rouquier algebras, perverse sheaves on Grassmannians, and rational representations of the supergroup $GL_{m|n}(\mathbb{C})$.

To formulate our results in more detail, and to make the definition of the algebra $K(\delta)$ precise, we must recall the key combinatorial observation from [CD, §4]. This gives a "weight dictionary" between the bipartitions which appear when studying the representation theory of $B_{r,s}(\delta)$ and the weight diagrams of [BS1] which appear in the definition of the algebra $K(\delta)$. Recall a *bipartition* means a pair $\lambda = (\lambda^{\mathrm{L}}, \lambda^{\mathrm{R}})$ of partitions. Let Λ be the set of all bipartitions and

$$\Lambda_{r,s} := \left\{ \lambda \in \Lambda \mid |\lambda^{\mathrm{L}}| = r - t \text{ and } |\lambda^{\mathrm{R}}| = s - t \text{ for } 0 \le t \le \min(r,s) \right\}, \quad (1.1)$$

$$\dot{\Lambda}_{r,s} := \begin{cases} \Lambda_{r,s} & \text{if } \delta \ne 0 \text{ or } r \ne s \text{ or } r = s = 0, \\ \Lambda_{r,s} \setminus \{(\varnothing, \varnothing)\} & \text{if } \delta = 0 \text{ and } r = s > 0. \end{cases} \quad (1.2)$$

For each $\lambda \in \Lambda_{r,s}$, there is a *cell module* $C_{r,s}(\lambda)$ for $B_{r,s}(\delta)$; see §2 below. Moreover $C_{r,s}(\lambda)$ has irreducible head denoted $D_{r,s}(\lambda)$ whenever $\lambda \in \dot{\Lambda}_{r,s}$, and the modules $\{D_{r,s}(\lambda) \mid \lambda \in \dot{\Lambda}_{r,s}\}$ give a complete set of pairwise non-isomorphic irreducible $B_{r,s}(\delta)$-modules.

On the other hand a *weight diagram* in the sense of [BS1] is a horizontal "number line" with vertices at all integers labelled by one of the symbols $\circ, \wedge, \vee$ and $\times$; we require moreover that it is impossible to find a vertex labelled $\vee$ to the left of a vertex labelled $\wedge$ outside of some finite subset of the vertices. To identify bipartitions with certain weight diagrams, take $\lambda \in \Lambda$ and let

$$I_{\wedge}(\lambda) := \{\lambda_1^{\mathrm{L}}, \lambda_2^{\mathrm{L}} - 1, \lambda_3^{\mathrm{L}} - 2, \dots \}, \quad (1.3)$$

$$I_{\vee}(\lambda) := \{1 - \delta - \lambda_1^{\mathrm{R}}, 2 - \delta - \lambda_2^{\mathrm{R}}, 3 - \delta - \lambda_3^{\mathrm{R}}, \dots \}, \quad (1.4)$$

where $\lambda_1^{\mathrm{L}} \ge \lambda_2^{\mathrm{L}} \ge \cdots$ are the parts of λ^{L} and $\lambda_1^{\mathrm{R}} \ge \lambda_2^{\mathrm{R}} \ge \cdots$ are the parts of λ^{R}. Then we identify λ with the weight diagram whose ith vertex is labelled

$$\begin{cases} \circ & \text{if } i \text{ does not belong to either } I_{\vee}(\lambda) \text{ or } I_{\wedge}(\lambda), \\ \wedge & \text{if } i \text{ belongs to } I_{\wedge}(\lambda) \text{ but not to } I_{\vee}(\lambda), \\ \vee & \text{if } i \text{ belongs to } I_{\vee}(\lambda) \text{ but not to } I_{\wedge}(\lambda), \\ \times & \text{if } i \text{ belongs to both } I_{\vee}(\lambda) \text{ and } I_{\wedge}(\lambda), \end{cases} \quad (1.5)$$

for each $i \in \mathbb{Z}$. Of course this depends implicitly on the fixed parameter $\delta \in \mathbb{Z}$. For example, if $\delta = -2$ then

$$(\varnothing, \varnothing) = \cdots \overset{0 \qquad\qquad 1-\delta}{\wedge\ \ \wedge\ \ \wedge\ \ \wedge\ \ \circ\ \ \circ\ \ \vee\ \ \vee\ \ \vee\ \ \vee} \cdots$$

$$((2^2, 1), (3, 2)) = \cdots \overline{\wedge\ \ \circ\ \ \wedge\ \ \vee\ \ \wedge\ \ \times\ \ \circ\ \ \circ\ \ \vee\ \ \vee} \cdots$$

(where all the omitted vertices on the left are labelled $\wedge$ and the ones on the right are labelled $\vee$). In this way, the set Λ of bipartitions is identified with a set of weight diagrams in the sense of [BS1]. Now the general construction from [BS1, §4] attaches a positively graded, locally unital algebra $K(\delta)$ to this set Λ of weight diagrams; see §§3–4 below. Finally for each $\lambda \in \Lambda$ there are some *standard modules* $V(\lambda)$ for $K(\delta)$, each of which has one-dimensional head denoted $L(\lambda)$. The modules $\{L(\lambda) \mid \lambda \in \Lambda\}$ give a complete set of pairwise non-isomorphic irreducible $K(\delta)$-modules.

The main result of the article can now be formulated as follows.

Theorem 1.1. *There exists a Morita equivalence between the walled Brauer algebra $B_{r,s}(\delta)$ and the finite dimensional algebra $K_{r,s}(\delta) := e_{r,s}K(\delta)e_{r,s}$, where*

$$e_{r,s} := \sum_{\lambda \in \dot\Lambda_{r,s}} e_\lambda \in K(\delta). \tag{1.6}$$

The equivalence can be chosen so that the cell module $C_{r,s}(\lambda)$ corresponds to the $K_{r,s}(\delta)$-module $V_{r,s}(\lambda) := e_{r,s}V(\lambda)$ for each $\lambda \in \Lambda_{r,s}$. Hence the irreducible module $D_{r,s}(\lambda)$ corresponds to $L_{r,s}(\lambda) := e_{r,s}L(\lambda)$ for each $\lambda \in \dot\Lambda_{r,s}$.

The set $\Lambda_{r,s}$ is an upper set in Λ with respect to the natural Bruhat ordering. Combined with known properties of $K(\delta)$ from [BS2], it follows that the idempotent truncation $K_{r,s}(\delta)$ is a standard Koszul algebra when $\Lambda_{r,s} = \dot\Lambda_{r,s}$, i.e. it is both graded quasi-hereditary and Koszul; see Theorem 4.4. So we obtain the following corollary.

Corollary 1.2. *If $\Lambda_{r,s} = \dot\Lambda_{r,s}$ then $B_{r,s}(\delta)$ admits a Koszul grading. In particular, $B_{r,s}(\delta)$ is standard Koszul when $\delta \neq 0$.*

The Koszul grading on $B_{r,s}(\delta)$ is only unique up to automorphism (see [BGS, §2.5]), and we don't expect that there is any distinguished way to realise it explicitly. Instead, we construct some graded algebras $B_R(\delta)$, which we call *graded walled Brauer algebras*, indexed by elements R of the set $\mathscr{R}_{r,s}$ of sequences consisting of r E's and s F's. We show that $B_R(\delta)$ is graded Morita equivalent to $K_{r,s}(\delta)$ in a canonical way; see Theorem 6.2. Moreover we show that $B_R(\delta)$ is isomorphic (though not canonically) to $B_{r,s}(\delta)$ as an ungraded algebra; see Corollary 8.2. Putting these two things together, we obtain a family of equivalences of categories

$$\mathbf{g}_R : K_{r,s}(\delta)\text{-mod} \to B_{r,s}(\delta)\text{-mod}, \tag{1.7}$$

one for each $R \in \mathscr{R}_{r,s}$, each of which satisfies the hypotheses of Theorem 1.1; see Theorem 8.10.

To give a little more idea about the definition of $B_R(\delta)$, recall for the walled Brauer algebra that there are two basic branching rules, one from $B_{r,s}(\delta)$

to $B_{r-1,s}(\delta)$ ("E-restriction") and the other from $B_{r,s}(\delta)$ to $B_{r,s-1}(\delta)$ ("F-restriction"). The sequences $R \in \mathscr{R}_{r,s}$ also parametrise the $\binom{r+s}{r}$ different ways to restrict from $B_{r,s}(\delta)$ all the way down to the trivial subalgebra $B_{0,0}(\delta)$ by iterating these basic branching rules, e.g. $E^r F^s$ and $F^s E^r$ encode

$$B_{r,s}(\delta) \downarrow B_{r,s-1}(\delta) \downarrow \cdots \downarrow B_{r,0}(\delta) \downarrow B_{r-1,0}(\delta) \downarrow \cdots \downarrow B_{0,0}(\delta),$$

$$B_{r,s}(\delta) \downarrow B_{r-1,s}(\delta) \downarrow \cdots \downarrow B_{0,s}(\delta) \downarrow B_{0,s-1}(\delta) \downarrow \cdots \downarrow B_{0,0}(\delta),$$

respectively. The graded walled Brauer algebra $B_R(\delta)$ has a distinguished basis indexed by certain paths in the Bratelli diagram which describes the branching of cell modules with respect to the sequence R; see §6 for some examples. Multiplication is defined by representing the basis elements diagrammatically then applying an explicit combinatorial procedure similar in spirit to the multiplication in Khovanov's arc algebra; this diagrammatics is quite different from the sort that appears in the definition of $B_{r,s}(\delta)$. We also construct a graded cellular basis for each $B_R(\delta)$, hence defining *graded cell modules* $C_R(\lambda)$; see Theorem 6.4 and Corollary 8.11. Moreover we compute the graded dimensions of the irreducible representations; see Theorem 6.7.

Once the algebras $B_R(\delta)$ have been constructed, the main problem is to prove that $B_{r,s}(\delta) \cong B_R(\delta)$. We do this by exploiting the super analogue of the mixed Schur-Weyl duality mentioned at the beginning of the introduction. Assume for this that $\delta = m - n$ for integers $m, n \geq 0$. Let G denote the general linear supergroup $GL_{m|n}(\mathbb{C})$ with natural module V and dual natural module W. There is a natural right action of $B_{r,s}(\delta)$ on $V^{\otimes r} \otimes W^{\otimes s}$ by G-module endomorphisms. We first show that the induced homomorphism

$$\Psi^{m,n}_{r,s} : B_{r,s}(\delta) \to \operatorname{End}_G(V^{\otimes r} \otimes W^{\otimes s})^{\mathrm{op}} \tag{1.8}$$

is surjective; see Theorem 7.8. Moreover it is an isomorphism if and only if

$$r + s < (m+1)(n+1). \tag{1.9}$$

Then we apply the results of [BS4] to describe $\operatorname{End}_G(V^{\otimes r} \otimes W^{\otimes s})^{\mathrm{op}}$ in purely diagrammatic terms. Assuming (1.9) holds, we get exactly the algebra $B_R(\delta)$, hence obtain the desired isomorphism $B_{r,s}(\delta) \cong B_R(\delta)$. We expect that this isomorphism is independent of the particular choice of m and n, but are unable to prove this at present; see Remark 8.3 for further discussion of this issue.

Our approach has several other consequences. First, if either $m = 0$ or $n = 0$, i.e. G is just the general linear group not the general linear supergroup, we are able to give an explicit description of $\ker \Psi^{m,n}_{r,s}$, showing it is generated by an idempotent; see Theorem 8.15. This result is an analogue for the walled Brauer algebra of a result about Brauer algebras obtained recently by Lehrer and Zhang [LZ]. For $m, n > 0$, the ideal $\ker \Psi^{m,n}_{r,s}$ need not be generated by an idempotent; see Remark 6.17.

We also determine the irreducible representations of the quotient algebra $B_{r,s}(\delta)/\ker \Psi^{m,n}_{r,s}$, showing up to isomorphism that they are modules $D_{r,s}(\lambda)$ for $\lambda \in \dot{\Lambda}_{r,s}$ such that λ is an (m,n)-cross bipartition in the sense of Comes and Wilson [CW]; see Theorem 8.17. This allows us to recover the classification of indecomposable summands of $V^{\otimes r} \otimes W^{\otimes s}$ obtained originally in [CW] by a different approach; see Theorem 8.19. Our approach gives a little more

information about these interesting modules, e.g. we compute their irreducible socles and heads explicitly.

The work of Comes and Wilson is motivated by the study of Deligne's tensor category $\underline{\text{Rep}}(GL_\delta)$; see also [D], [Kn]. For $\delta \in \mathbb{Z}$, we conjecture that Deligne's category is equivalent (on forgetting the tensor structure) to the category $\text{Proj}(K(\delta))$ of (locally unital) finitely generated projective $K(\delta)$-modules. The main obstacle preventing us from proving this conjecture using the techniques of the article is related to the question of independence of m and n mentioned earlier.

Acknowledgements. The results in this article were obtained during the program "On the Interaction of Representation Theory with Geometry and Combinatorics" at the Hausdorff Institute for Mathematics in Bonn in January, 2011, and written up in part during a visit by the first author to the University of Sydney in March–May, 2011. We thank J. Comes for some helpful discussions about $\underline{\text{Rep}}(GL_\delta)$ and for the proof of Lemma 8.16.

## 2. Refined branching rules for the walled Brauer algebra

In this preliminary section, we fix $r, s \geq 0$ and $\delta \in \mathbb{C}$. We are going to recall the definition of the walled Brauer algebra $B_{r,s}(\delta)$ and establish some basic facts about its representation theory. In particular, we will define some refined induction and restriction functors and describe their effect on cell modules.

The walled Brauer algebra. As a $\mathbb{C}$-vector space, $B_{r,s}(\delta)$ has a basis consisting of isotopy classes of *walled Brauer diagrams*. These are certain diagrams drawn in a rectangle with $(r+s)$ vertices on its top and bottom edges, numbered $1, \ldots, r + s$ in order from left to right. Each vertex must be connected to exactly one other vertex by a smooth curve drawn in the interior of the rectangle; curves can cross transversally, no triple intersections. We refer to the curves whose endpoints are on different edges of the rectangle as *vertical strands*, and the curves whose endpoints are on the same edge of the rectangle are *horizontal strands*. In addition, there is a vertical wall separating vertices $1, \ldots, r$ from vertices $r + 1, \ldots, r + s$. We require that the endpoints of each vertical strand are on the same side of the wall and the endpoints of each horizontal strand are on opposite sides of the wall. For example here are two basis vectors in $B_{2,2}(\delta)$:
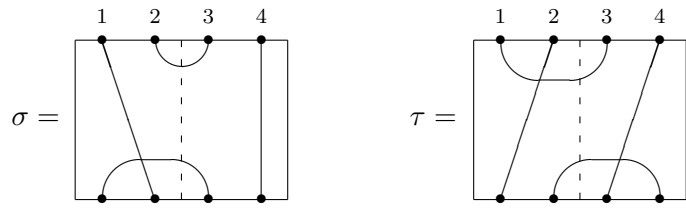
Multiplication is by the following concatenation procedure: $\sigma\tau$ is obtained by putting σ *under* τ, removing internal circles, then multiplying the resulting basis vector by δ^t where t is the number of internal circles removed. For example,

for σ and τ as above, we have:

$$\sigma\tau = \qquad\qquad\qquad \tau\sigma = \delta\cdot$$
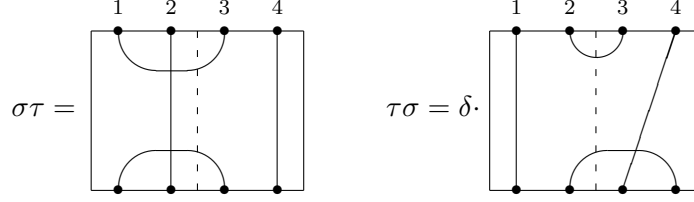
It is useful to compare $B_{r,s}(\delta)$ with the group algebra $\mathbb{C}\Sigma_{r+s}$ of the symmetric group. Of course the latter can also be viewed diagrammatically if we identify $\sigma \in \Sigma_{r+s}$ with its *permutation diagram*, the diagram drawn in the same rectangle as before with a vertical strand connecting the ath vertex on the top to the $\sigma(a)$th vertex on the bottom for each $1 \leq a \leq r + s$. It is clear from the definitions that there is vector space isomorphism

$$\operatorname{flip}_{r,s} : \mathbb{C}\Sigma_{r+s} \overset{\sim}{\to} B_{r,s}(\delta) \qquad\qquad (2.1)$$

mapping a permutation diagram to the walled Brauer diagram obtained by adding a wall between the rth and $(r + 1)$th vertices, then flipping the part of the diagram that is to the right of the wall in its horizontal axis without disconnecting any strands. For example the walled Brauer diagrams σ and τ above arise by applying $\operatorname{flip}_{2,2}$ to the following permutation diagrams, respectively:

$$(1\,2\,3) = \qquad\qquad\qquad (1\,3\,4\,2) =$$
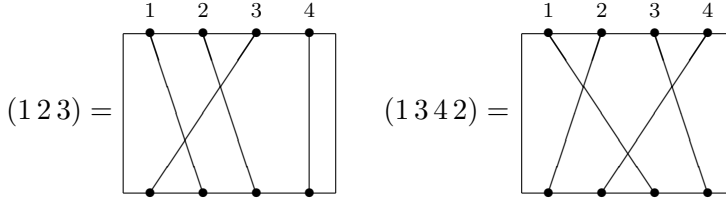
It follows in particular that

$$\dim B_{r,s}(\delta) = \dim \mathbb{C}\Sigma_{r+s} = (r + s)!. \qquad\qquad (2.2)$$

Generators, relations, and the central element $z_{r,s}$. The algebra $\mathbb{C}\Sigma_{r+s}$ is generated by the basic transpositions $\sigma_a := (a\,a+1)$ for $a = 1, \ldots, r + s - 1$, subject only to the usual Coxeter relations. Let $\tau_a := \operatorname{flip}_{r,s}(\sigma_a) \in B_{r,s}(\delta)$. So for $a \neq r$ the diagram τ_a is just the same as σ_a (with the addition of the wall), while τ_r is the diagram

$$\tau_r =$$
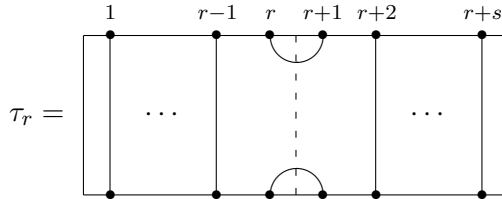
It is an exercise to see that $B_{r,s}(\delta)$ is generated by the elements $\tau_1, \ldots, \tau_{r+s-1}$ and that the following relations hold whenever they make sense:

$$\tau_a^2 = 1 \quad (a \neq r), \qquad\qquad \tau_a \tau_b = \tau_b \tau_a \qquad (|a-b| > 1), \qquad (2.3)$$

$$\tau_r^2 = \delta \tau_r, \qquad\qquad \tau_a \tau_b \tau_a = \tau_b \tau_a \tau_b \quad (|a-b| = 1, a \neq r \neq b), \qquad (2.4)$$

$$\tau_r = \tau_r \tau_{r-1} \tau_r, \qquad \tau_r \tau_{r-1} \tau_{r+1} \tau_r = \tau_{r-1} \tau_{r+1} \tau_r \tau_{r-1} \tau_{r+1} \tau_r, \qquad (2.5)$$

$$\tau_r = \tau_r \tau_{r+1} \tau_r, \qquad \tau_r \tau_{r-1} \tau_{r+1} \tau_r = \tau_r \tau_{r-1} \tau_{r+1} \tau_r \tau_{r-1} \tau_{r+1}. \qquad (2.6)$$

Although never needed in the present article, it is known moreover that this is a full set of relations for the algebra $B_{r,s}(\delta)$; see [H, Corollary 4.5] for a special case or [N, Theorem 4.1] in general.

More generally, for $1 \leq a, b \leq r+s$ with $a \neq b$, we introduce the following "transpositions" in $B_{r,s}(\delta)$:

$$\overline{(a\ b)} := \begin{cases} \mathrm{flip}_{r,s}((a\ b)) & \text{if } a \text{ and } b \text{ are on the same side of the wall,} \\ -\mathrm{flip}_{r,s}((a\ b)) & \text{if } a \text{ and } b \text{ are on opposite sides of the wall.} \end{cases}$$

We stress the presence of the sign in the second case; it means that $\tau_a = \overline{(a\ a+1)}$ for $a \neq r$ but $\tau_r = -\overline{(r\ r+1)}$. Then let

$$z_{r,s} := \sum_{1 \leq a < b \leq r+s} \overline{(a\ b)}. \qquad (2.7)$$

Lemma 2.1. *The element $z_{r,s}$ belongs to the center of $B_{r,s}(\delta)$.*

Proof. It suffices to show that $z_{r,s}$ commutes with the generators of $B_{r,s}(\delta)$, which is a routine exercise. □

Cell modules. Recall the set $\Lambda_{r,s}$ of bipartitions from (1.1). For each $\lambda \in \Lambda_{r,s}$, there is a corresponding $B_{r,s}(\delta)$-module $C_{r,s}(\lambda)$, which is a *cell module* in the sense of [GL] arising from a certain cellular algebra structure on $B_{r,s}(\delta)$ discussed in [CDDM, §2]. We won't recall the definition of this cellular algebra structure here, as it is enough for our purposes to work with the explicit construction of $C_{r,s}(\lambda)$ given in [CDDM, §3].

Let $J_{r,s}^t$ be the two-sided ideal of $B_{r,s}(\delta)$ spanned by the walled Brauer diagrams with at least t horizontal strands at the top and bottom, giving a chain

$$B_{r,s}(\delta) = J_{r,s}^0 \supset J_{r,s}^1 \supset \cdots \supset J_{r,s}^{\min(r,s)} \supset \{0\}.$$

The images of the walled Brauer diagrams with exactly t horizontal strands at the top and bottom give a basis for the quotient $J_{r,s}^t / J_{r,s}^{t+1}$.

Now take $\lambda \in \Lambda_{r,s}$ and let $t := r - |\lambda^{\mathrm{L}}| = s - |\lambda^{\mathrm{R}}|$. Let $I_{r,s}^t$ be the subspace of $J_{r,s}^t / J_{r,s}^{t+1}$ spanned by the images of the diagrams with exactly t horizontal strands at the top connecting the $(r+1-k)$th vertex to the $(r+k)$th vertex for each $k = 1, \ldots, t$. Identify $\mathbb{C}(\Sigma_{r-t} \times \Sigma_{s-t})$ with the subalgebra of $B_{r,s}(\delta)$ generated by $\tau_1, \ldots, \tau_{r-t-1}, \tau_{r+t+1}, \ldots, \tau_{r+s-1}$ in the obvious way. Then $I_{r,s}^t$ is invariant under left multiplication by elements of $B_{r,s}(\delta)$ and right multiplication by elements of $\mathbb{C}(\Sigma_{r-t} \times \Sigma_{s-t})$, hence it is a $(B_{r,s}(\delta), \mathbb{C}(\Sigma_{r-t} \times \Sigma_{s-t}))$-bimodule. Let $S(\lambda^{\mathrm{L}})$ be the irreducible Σ_{r-t}-module parametrised by λ^{L} and $S(\lambda^{\mathrm{R}})$ be the irreducible Σ_{s-t}-module parametrised by λ^{R}. Let $S(\lambda)$ denote their outer

tensor product $S(\lambda^{\mathrm{L}}) \boxtimes S(\lambda^{\mathrm{R}})$, which is an irreducible $\Sigma_{r-t} \times \Sigma_{s-t}$-module. Then we have by definition that

$$C_{r,s}(\lambda) := I_{r,s}^t \otimes_{\mathbb{C}(\Sigma_{r-t} \times \Sigma_{s-t})} S(\lambda). \tag{2.8}$$

The bimodule $I_{r,s}^t$ is free as a right $\mathbb{C}(\Sigma_{r-t} \times \Sigma_{s-t})$-module, with basis $X_{r,s}^t$ given by images of the walled Brauer diagrams with exactly t horizontal strands at the top connecting the $(r + 1 - k)$th vertex to the $(r + k)$th vertex for $k = 1, \dots, t$ in which no two vertical strands cross. So as a vector space we have that

$$C_{r,s}(\lambda) = \bigoplus_{\tau \in X_{r,s}^t} \tau \otimes S(\lambda). \tag{2.9}$$

In particular, this implies that

$$\dim C_{r,s}(\lambda) = \frac{r!s!}{(r-t)!t!(s-t)!} \dim S(\lambda^{\mathrm{L}}) \dim S(\lambda^{\mathrm{R}}). \tag{2.10}$$

To compute the action of a walled Brauer diagram $\sigma \in B_{r,s}(\delta)$ on a vector $\tau \otimes v$ for $\tau \in X_{r,s}^t$ and $v \in S(\lambda)$ we proceed as follows. If σ has more than t horizontal strands then $\sigma(\tau \otimes v) = 0$; otherwise $\sigma(\tau \otimes v) = \tau' \otimes (c\sigma'v)$ where $\sigma\tau = c\tau'\sigma'$ for $c \in \mathbb{C}, \tau' \in X_{r,s}^t$ and $\sigma' \in \Sigma_{r-t} \times \Sigma_{s-t}$. It follows easily from this that $C_{r,s}(\lambda)$ is a cyclic $B_{r,s}(\delta)$-module generated by any vector of the form $\tau \otimes v$ for $\tau \in X_{r,s}^t$ and $0 \neq v \in S(\lambda)$.

Lemma 2.2. *For any $\lambda \in \Lambda_{r,s}$ we have that $\mathrm{End}_{B_{r,s}(\delta)}(C_{r,s}(\lambda)) \cong \mathbb{C}$.*

Proof. Set $t := r - |\lambda^{\mathrm{L}}| = s - |\lambda^{\mathrm{R}}|$. Let τ be the unique walled Brauer diagram in $X_{r,s}^t$ with horizontal strands connecting the $(r+1-k)$th vertex to the $(r+k)$th vertex for $k = 1, \dots, t$ at both the top and the bottom.

We first treat the case that $t < r$. Let $\sigma := \tau\tau_{r-t} \cdots \tau_{r-2}\tau_{r-1}$. On drawing the diagrams, it is easy to check that $\sigma\tau = \tau$, and that $\sigma C_{r,s}(\lambda) = \tau \otimes S(\lambda)$ in the decomposition (2.9). It follows for any $f \in \mathrm{End}_{B_{r,s}(\delta)}(C_{r,s}(\lambda))$ and $v \in S(\lambda)$ that $f(\tau \otimes v) = f(\sigma\tau \otimes v) = \sigma f(\tau \otimes v) = \tau \otimes \bar{f}(v)$ for a unique $\bar{f}(v) \in S(\lambda)$. In other words, f induces a linear map $\bar{f} : S(\lambda) \to S(\lambda)$, which is easily seen to be a $\Sigma_{r-t} \times \Sigma_{s-t}$-module homomorphism. Hence $\bar{f}(v) = cv$ for a unique scalar $c \in \mathbb{C}$ by Schur's lemma, i.e. $f(\tau \otimes v) = c(\tau \otimes v)$. Since $\tau \otimes v$ generates $C_{r,s}(\lambda)$, we deduce that $f = c \cdot \mathrm{id}$.

The case that $t < s$ is similar.

It remains to treat the case that $t = r = s$. This is trivial if $t = 0$, so assume moreover that $t > 0$. Let $\sigma := \tau'\tau_1 \cdots \tau_{r-1}$ where τ' is the unique walled Brauer diagram in $X_{r,s}^{t-1}$ with horizontal strands connecting the $(r+1-k)$th vertex to the $(r+k)$th vertex for $k = 1, \dots, t-1$ at both the top and the bottom. Then again we have that $\sigma\tau = \tau$ and $\sigma C_{r,s}(\lambda) = \tau \otimes S(\lambda)$. Now the proof can be completed as before. $\square$

Action of $z_{r,s}$ on cell modules. We draw the Young diagram of a partition λ in the usual English way, and let $[\lambda]$ denote its set of boxes. For $A \in [\lambda]$, its *content* $\mathrm{cont}(A)$ is the integer $(j-i)$ if A is in row i and column j. For example,

here is the Young diagram of the partition $\lambda = (5, 4, 4)$ with boxes labelled by their contents:

0	1	2	3	4
-1	0	1	2	
-2	-1	0	1	

.

For $\lambda \in \Lambda_{r,s}$, let

$$z_{r,s}(\lambda) := \sum_{A \in [\lambda^{\mathrm{L}}]} \mathrm{cont}(A) + \sum_{A \in [\lambda^{\mathrm{R}}]} \mathrm{cont}(A) - t\delta \in \mathbb{C},$$

where $t := r - |\lambda^{\mathrm{L}}| = s - |\lambda^{\mathrm{R}}|$.

Lemma 2.3. *For $\lambda \in \Lambda_{r,s}$, the central element $z_{r,s} \in B_{r,s}(\delta)$ acts on the cell module $C_{r,s}(\lambda)$ by multiplication by the scalar $z_{r,s}(\lambda)$.*

Proof. Let $t := r - |\lambda^{\mathrm{L}}| = s - |\lambda^{\mathrm{R}}|$, $I := \{1, \ldots, r-t\}$, $J := \{r+t+1, \ldots, r+s\}$, and $K := \{r-t+1, \ldots, r\}$. For $a \in K$, let $\tilde{a} := 2r+1-a$. Recalling (2.8), let τ be the image in $I_{r,s}^t$ of the walled Brauer diagram with a vertical strand connecting the ath vertex at the top to the ath vertex at the bottom for each $a \in I \cup J$, and a horizontal strand connecting a to $\tilde{a}$ both at the top and the bottom for each $a \in K$. Also pick any non-zero vector $v \in S(\lambda)$. As $C_{r,s}(\lambda)$ is generated by $\tau \otimes v$ and $z_{r,s}$ is central, it suffices to show that $z_{r,s}(\tau \otimes v) = z_{r,s}(\lambda)(\tau \otimes v)$. We have that $z_{r,s} = z_I + z_J + z_K + z_K' + z_K'' + z_{IJ} + z_{IK} + z_{JK}$ where

$$z_I := \sum_{\substack{a,b \in I \\ a<b}} \overline{(a\ b)}, \qquad z_{IJ} := \sum_{\substack{a \in I \\ b \in J}} \overline{(a\ b)},$$

$$z_J := \sum_{\substack{a,b \in J \\ a<b}} \overline{(a\ b)}, \qquad z_{IK} := \sum_{\substack{a \in I \\ b \in K}} \left(\overline{(a\ b)} + \overline{(a\ \tilde{b})} \right),$$

$$z_K := \sum_{a \in K} \overline{(a\ \tilde{a})}, \qquad z_{JK} := \sum_{\substack{a \in J \\ b \in K}} \left(\overline{(a\ b)} + \overline{(a\ \tilde{b})} \right),$$

$$z_K' := \sum_{\substack{a,b \in K \\ a<b}} \left(\overline{(a\ b)} + \overline{(a\ \tilde{b})} \right), \qquad z_K'' := \sum_{\substack{a,b \in K \\ a<b}} \left(\overline{(\tilde{a}\ b)} + \overline{(\tilde{a}\ \tilde{b})} \right).$$

Now we observe that

$$z_{IJ}(\tau \otimes v) = z_{IK}(\tau \otimes v) = z_{JK}(\tau \otimes v) = z_K'(\tau \otimes v) = z_K''(\tau \otimes v) = 0$$

because each of the terms in the summations above already act as zero. Moreover $z_I(\tau \otimes v) = \sum_{A \in [\lambda^{\mathrm{L}}]} \mathrm{cont}(A)(\tau \otimes v)$, $z_J(\tau \otimes v) = \sum_{A \in [\lambda^{\mathrm{R}}]} \mathrm{cont}(A)(\tau \otimes v)$, and $z_K(\tau \otimes v) = -t\delta(\tau \otimes v)$. The first two equalities follow because z_I and z_J are the sums of all transpositions in Σ_{r-t} and Σ_{s-t}, respectively, and it is well known (e.g. as a consequence of Young's orthogonal form or by the Murnaghan-Nakayama rule) that these act on $S(\lambda^{\mathrm{L}})$ and $S(\lambda^{\mathrm{R}})$ by the scalars indicated. $\square$

Refined induction and restriction functors. There are embeddings

$$\iota_{r,s}^{r+1,s} : B_{r,s}(\delta) \hookrightarrow B_{r+1,s}(\delta), \qquad \iota_{r,s}^{r,s+1} : B_{r,s}(\delta) \hookrightarrow B_{r,s+1}(\delta) \qquad (2.11)$$

defined by inserting a vertical strand immediately to the left or right of the wall, respectively. Thus $\iota_{r,s}^{r+1,s}(\tau_r) = \tau_r\tau_{r+1}\tau_r$, $\iota_{r,s}^{r,s+1}(\tau_r) = \tau_{r+1}\tau_r\tau_{r+1}$, and both $\iota_{r,s}^{r+1,s}$ and $\iota_{r,s}^{r,s+1}$ map $\tau_a \mapsto \tau_a$ for $1 \leq a \leq r-1$ and $\tau_b \mapsto \tau_{b+1}$ for $r+1 \leq b \leq r+s-1$. Let $\left(\mathrm{ind}_{r,s}^{r+1,s}, \mathrm{res}_{r,s}^{r+1,s}\right)$ and $\left(\mathrm{ind}_{r,s}^{r,s+1}, \mathrm{res}_{r,s}^{r,s+1}\right)$ be the adjoint pairs of induction and restriction functors attached to the embeddings $\iota_{r,s}^{r+1,s}$ and $\iota_{r,s}^{r,s+1}$, respectively.

We want to define refined versions of these functors. Fix $i \in \mathbb{C}$. Let

$$x_{r,s}^{r+1,s} := z_{r+1,s} - \iota_{r,s}^{r+1,s}(z_{r,s}) = \sum_{\substack{1 \leq a \leq r+s+1 \\ a \neq r+1}} \overline{(a\ r+1)} \quad \in B_{r+1,s}(\delta), \qquad (2.12)$$

$$x_{r,s}^{r,s+1} := z_{r,s+1} - \iota_{r,s}^{r,s+1}(z_{r,s}) = \sum_{\substack{1 \leq a \leq r+s+1 \\ a \neq r+1}} \overline{(a\ r+1)} \quad \in B_{r,s+1}(\delta). \qquad (2.13)$$

The elements $x_{r,s}^{r+1,s}$ and $x_{r,s}^{r,s+1}$ centralise $\iota_{r,s}^{r+1,s}(B_{r,s}(\delta))$ and $\iota_{r,s}^{r,s+1}(B_{r,s}(\delta))$, respectively. In the finite dimensional commutative subalgebra of $B_{r+1,s}(\delta)$ generated by $x_{r,s}^{r+1,s}$, there is a unique idempotent $1_{r,s;i}^{r+1,s}$ that acts on any module as the projection onto the generalised i-eigenspace of $x_{r,s}^{r+1,s}$. Similarly in the subalgebra of $B_{r,s+1}(\delta)$ generated by $x_{r,s}^{r,s+1}$, there is a unique idempotent $1_{r,s;i}^{r,s+1}$ that acts as the projection onto the generalised $(-i-\delta)$-eigenspace of $x_{r,s}^{r,s+1}$. Let

$$\iota_{r,s;i}^{r+1,s} : B_{r,s}(\delta) \to B_{r+1,s}(\delta), \qquad \iota_{r,s;i}^{r,s+1} : B_{r,s}(\delta) \to B_{r,s+1}(\delta) \qquad (2.14)$$

be the non-unital algebra homomorphisms defined by first applying the embedding $\iota_{r,s}^{r+1,s}$ then multiplying by the centralising idempotent $1_{r,s;i}^{r+1,s}$, or by applying $\iota_{r,s}^{r,s+1}$ then multiplying by $1_{r,s;i}^{r,s+1}$, respectively. Let $(i\text{-}\mathrm{ind}_{r,s}^{r+1,s}, i\text{-}\mathrm{res}_{r,s}^{r+1,s})$ and $(i\text{-}\mathrm{ind}_{r,s}^{r,s+1}, i\text{-}\mathrm{res}_{r,s}^{r,s+1})$ be the adjoint pairs of i-induction and i-restriction functors attached to the homomorphisms $\iota_{r,s;i}^{r+1,s}$ and $\iota_{r,s;i}^{r,s+1}$, respectively. Thus,

$$i\text{-}\mathrm{res}_{r,s}^{r+1,s} : B_{r+1,s}(\delta)\text{-mod} \to B_{r,s}(\delta)\text{-mod}, \qquad (2.15)$$

$$i\text{-}\mathrm{res}_{r,s}^{r,s+1} : B_{r,s+1}(\delta)\text{-mod} \to B_{r,s}(\delta)\text{-mod} \qquad (2.16)$$

are the exact functors defined by multiplication by the idempotents $1_{r,s;i}^{r+1,s}$ and $1_{r,s;i}^{r,s+1}$, respectively, viewing the result as a left $B_{r,s}(\delta)$-module via the homomorphisms (2.14). Also

$$i\text{-}\mathrm{ind}_{r,s}^{r+1,s} : B_{r,s}(\delta)\text{-mod} \to B_{r+1,s}(\delta)\text{-mod}, \qquad (2.17)$$

$$i\text{-}\mathrm{ind}_{r,s}^{r,s+1} : B_{r,s}(\delta)\text{-mod} \to B_{r,s+1}(\delta)\text{-mod} \qquad (2.18)$$

are the functors $B_{r+1,s}(\delta)1_{r,s;i}^{r+1,s} \otimes_{B_{r,s}(\delta)} ?$ and $B_{r,s+1}(\delta)1_{r,s;i}^{r,s+1} \otimes_{B_{r,s}(\delta)} ?$, respectively, where the right $B_{r,s}(\delta)$-module structure is again defined using (2.14).

Since $\sum_{i \in \mathbb{C}} 1^{r+1,s}_{r,s;i} = 1 \in B_{r+1,s}(\delta)$ and $\sum_{i \in \mathbb{C}} 1^{r,s+1}_{r,s;i} \in B_{r,s+1}(\delta)$, we have that

$$\text{res}^{r+1,s}_{r,s} = \bigoplus_{i \in \mathbb{C}} i\text{-res}^{r+1,s}_{r,s}, \qquad\qquad \text{ind}^{r+1,s}_{r,s} = \bigoplus_{i \in \mathbb{C}} i\text{-ind}^{r+1,s}_{r,s},$$

$$\text{res}^{r,s+1}_{r,s} = \bigoplus_{i \in \mathbb{C}} i\text{-res}^{r,s+1}_{r,s}, \qquad\qquad \text{ind}^{r,s+1}_{r,s} = \bigoplus_{i \in \mathbb{C}} i\text{-ind}^{r,s+1}_{r,s}.$$

The following lemma realises the i-restriction and i-induction functors in terms of taking certain generalised eigenspaces.

Lemma 2.4. *Let M be a finite dimensional indecomposable $B_{r,s}(\delta)$-module and $\lambda_M \in \mathbb{C}$ be the unique eigenvalue of the (not necessarily diagonalisable) endomorphism of M defined by multiplication by $z_{r,s}$.*

(1) *The module $i\text{-res}^{r,s}_{r-1,s}M$ is the generalised $(\lambda_M - i)$-eigenspace of $z_{r-1,s}$ on $\text{res}^{r,s}_{r-1,s}M$.*

(2) *The module $i\text{-res}^{r,s}_{r,s-1}M$ is the generalised $(\lambda_M + i + \delta)$-eigenspace of $z_{r,s-1}$ on $\text{res}^{r,s}_{r,s-1}M$.*

(3) *The module $i\text{-ind}^{r+1,s}_{r,s}M$ is the generalised $(\lambda_M + i)$-eigenspace of $z_{r+1,s}$ on $\text{ind}^{r+1,s}_{r,s}M$.*

(4) *The module $i\text{-ind}^{r,s+1}_{r,s}M$ is the generalised $(\lambda_M - i - \delta)$-eigenspace of $z_{r,s+1}$ on $\text{ind}^{r+1,s}_{r,s}M$.*

Proof. This follows from the definitions. $\qquad\qquad\qquad\qquad\qquad\qquad \square$

We give the set Λ of all bipartitions the structure of a labelled directed graph, with a directed edge $\lambda \xrightarrow{i} \mu$ labelled by $i \in \mathbb{C}$ if μ is obtained from λ either by adding a box of content i to the Young diagram of λ^{L} or by removing a box of content $-i - \delta$ from the Young diagram of λ^{R}. Equivalently, $\lambda \xleftarrow{i} \mu$ if μ is obtained from λ by removing a box of content i from the Young diagram of λ^{L} or by adding a box of content $-i - \delta$ to the Young diagram of λ^{R}. The following theorem is a refinement of [CDDM, Theorem 3.3].

Theorem 2.5. *The following hold for $\lambda \in \Lambda_{r,s}$ and $i \in \mathbb{C}$.*

(1) *The module $i\text{-res}^{r,s}_{r-1,s}C_{r,s}(\lambda)$ has a filtration with sections isomorphic to $\{C_{r-1,s}(\mu) \,|\, \text{for all } \mu \in \Lambda_{r-1,s} \text{ with } \mu \xrightarrow{i} \lambda\}$.*

(2) *The module $i\text{-res}^{r,s}_{r,s-1}C_{r,s}(\lambda)$ has a filtration with sections isomorphic to $\{C_{r,s-1}(\mu) \,|\, \text{for all } \mu \in \Lambda_{r,s-1} \text{ with } \mu \xleftarrow{i} \lambda\}$.*

(3) *The module $i\text{-ind}^{r+1,s}_{r,s}C_{r,s}(\lambda)$ has a filtration with sections isomorphic to $\{C_{r+1,s}(\mu) \,|\, \text{for all } \mu \in \Lambda_{r+1,s} \text{ with } \lambda \xrightarrow{i} \mu\}$*

(4) *The module $i\text{-ind}^{r,s+1}_{r,s}C_{r,s}(\lambda)$ has a filtration with sections isomorphic to $\{C_{r,s+1}(\mu) \,|\, \text{for all } \mu \in \Lambda_{r,s+1} \text{ with } \lambda \xleftarrow{i} \mu\}$.*

In all cases, the filtration is of length at most two. When it is of length exactly two, the sections are indexed by bipartitions μ' and μ'' such that $|(\mu')^{\text{L}}| < |(\mu'')^{\text{L}}|$ and $|(\mu')^{\text{R}}| < |(\mu'')^{\text{R}}|$, and the following hold:

(a) *for $M = i\text{-res}^{r,s}_{r-1,s}C_{r,s}(\lambda)$ or $M = i\text{-res}^{r,s}_{r,s-1}C_{r,s}(\lambda)$, there is a unique submodule M' isomorphic to the cell module indexed by μ'; the quotient M/M' is isomorphic to the cell module indexed by μ'';*

(b) *for $M = i\text{-ind}^{r+1,s}_{r,s}C_{r,s}(\lambda)$ or $M = i\text{-ind}^{r,s+1}_{r,s}C_{r,s}(\lambda)$, there is a unique submodule M' such that M/M' is isomorphic to the cell module indexed by μ''; the submodule M' is isomorphic to the cell module indexed by μ'.*

Proof. The existence of the filtrations follows from [CDDM, Theorem 3.3] and Lemmas 2.3–2.4. The fact that the filtrations are of length at most two follows because λ^{L} has at most one addable or removable box of content i and λ^{R} has at most one addable or removable box of content $-i - \delta$.

For the final uniqueness statements (a)–(b), we just explain the proof of (a) for $M = i\text{-res}^{r,s}_{r-1,s}C_{r,s}(\lambda)$, since the other cases are similar. So assume both that λ^{L} has a removable box of content i (hence $r > 0$) and λ^{R} has an addable box of content $-i - \delta$. Let μ' and μ'' be obtained from λ by removing the former box from λ^{L} or by adding the latter box to λ^{R}, respectively. The existence argument indicated in the previous paragraph actually shows that M has a submodule $M' \cong C_{r-1,s}(\mu')$ such that $M/M' \cong C_{r-1,s}(\mu'')$. To get the uniqueness, it remains to observe that

$$\dim \operatorname{Hom}_{B_{r-1,s}(\delta)}(C_{r-1,s}(\mu'), i\text{-res}^{r,s}_{r-1,s}C_{r,s}(\lambda)) = 1,$$

as follows from Lemma 2.2 using the fact that $i\text{-ind}^{r,s}_{r-1,s}C_{r-1,s}(\mu') \cong C_{r,s}(\lambda)$ and adjointness. $\qquad\square$

Jucys-Murphy elements. The elements (2.12)–(2.13) are examples of *Jucys-Murphy elements* in the walled Brauer algebra. The definition of these elements in general depends also on a fixed choice of a sequence $R = R^{(1)}R^{(2)} \cdots R^{(r+s)}$ consisting of r E's and s F's, i.e. an element of the set $\mathscr{R}_{r,s}$ from the introduction. The corresponding Jucys-Murphy elements are denoted $x_1^R, \ldots, x_{r+s}^R$ and are defined recursively as follows. The base case $r + s = 0$ is vacuous. For the induction step, let RE and RF be the sequences obtained from R by adding an E or an F to the end. Assuming $x_1^R, \ldots, x_{r+s}^R \in B_{r,s}(\delta)$ have been defined already, let $x_1^{RE}, \ldots, x_{r+s+1}^{RE} \in B_{r+1,s}(\delta)$ and $x_1^{RF}, \ldots, x_{r+s+1}^{RF} \in B_{r,s+1}(\delta)$ be defined by setting

$$x_a^{RE} := \iota_{r,s}^{r+1,s}(x_a^R), \qquad x_{r+s+1}^{RE} := x_{r,s}^{r+1,s}, \tag{2.19}$$

$$x_a^{RF} := \iota_{r,s}^{r,s+1}(x_a^R), \qquad x_{r+s+1}^{RF} := x_{r,s}^{r,s+1}, \tag{2.20}$$

for $a = 1, \ldots, r + s$. When it is not ambiguous, we drop the superscript R, denoting the Jucys-Murphy elements in $B_{r,s}(\delta)$ simply by $x_1, \ldots, x_{r+s}$. Note always that $x_1 = 0$.

It is clear from the definition that $x_1, \ldots, x_{r+s}$ generate a finite dimensional commutative subalgebra of $B_{r,s}(\delta)$. For $\boldsymbol{i} \in \mathbb{C}^{r+s}$, define the *weight idempotent* $e(\boldsymbol{i})$ to be the unique idempotent in this commutative subalgebra with the property that $e(\boldsymbol{i})M = M_{\boldsymbol{i}}$ for any finite dimensional module M, where

$$M_{\boldsymbol{i}} = \left\{ v \in M \ \middle| \ \begin{array}{l} (x_a - i_a)^N v = 0 \text{ for } N \gg 0 \text{ if } R^{(a)} = E, \\ (x_a + i_a + \delta)^N v = 0 \text{ for } N \gg 0 \text{ if } R^{(a)} = F \end{array} \right\}. \tag{2.21}$$

The idempotents $\{e(\boldsymbol{i})|\boldsymbol{i} \in \mathbb{C}^{r+s}\}$ are mutually orthogonal, all but finitely many of them are zero, and they sum to 1.

Remark 2.6. One can check that any symmetric polynomial in the Jucys-Murphy elements $x_1, \ldots, x_{r+s}$ belongs to the center of $B_{r,s}(\delta)$. For example, the central element $z_{r,s}$ from (2.7) is $x_1 + \cdots + x_{r+s}$. We conjecture by analogy with a well-known result about the symmetric group that every element of the center of $B_{r,s}(\delta)$ can be expressed as a symmetric polynomial in $x_1, \ldots, x_{r+s}$.

3. Arc algebras and special projective functors

Next we review the construction and basic properties of Khovanov's arc algebras and their generalisations from [Kh], [BS1]. Actually for notational convenience we work only with the universal arc algebra K, which is the (infinite) direct sum of all the indecomposable arc algebras considered in [BS1]. We also recall the definition of projective functors following [BS2], and introduce certain special projective functors which play a key role in all the subsequent applications.

The universal arc algebra. Recall the definition of a *weight diagram* from the introduction. There is a partial order $\leq$ on the set of weight diagrams called the *Bruhat order*, which is generated by the basic operation of switching a pair of (not necessarily neighbouring) labels $\vee \wedge$; getting bigger in the Bruhat order means $\vee$'s move to the right. Also let $\sim$ be the equivalence relation defined by declaring that $\lambda \sim \mu$ if μ can be obtained from λ by permuting $\vee$'s and $\wedge$'s (and doing nothing to $\circ$'s and $\times$'s). We refer to $\sim$-equivalence classes of weight diagrams as *blocks*.

Given any weight diagram λ, there is a general procedure to close λ below to obtain its *cup diagram* $\underline{\lambda}$; see [BS1, §2]. Briefly, to construct $\underline{\lambda}$, one connects as many $\vee \wedge$ pairs of vertices as possible together by anti-clockwise cups then adds vertical rays down to infinity at all other vertices labelled $\wedge$ or $\vee$, without allowing any crossings of cups and/or rays. For example, if

$$\lambda = \cdots \ \overset{\vee}{\underset{\wedge}{\rule{0pt}{0pt}}} \ \overset{}{\underset{\wedge}{\rule{0pt}{0pt}}} \ \overset{\vee}{\rule{0pt}{0pt}} \ \overset{\vee}{\rule{0pt}{0pt}} \ \times \ \overset{}{\underset{\wedge}{\rule{0pt}{0pt}}} \ \circ \ \overset{}{\underset{\wedge}{\rule{0pt}{0pt}}} \ \overset{\vee}{\rule{0pt}{0pt}} \cdots$$

(where all the omitted vertices to the left are labelled $\wedge$ and the ones to the right are labelled $\vee$) then

$$\underline{\lambda} = \cdots \ \text{⌣} \ | \ \text{⌣} \ | \cdots$$

(with a ray at all other vertices). There is a similar procedure to close a weight diagram μ above to obtain its *cap diagram* $\overline{\mu}$; it is just the mirror image of $\underline{\mu}$ in the horizontal axis. We stress that the vertices in $\underline{\lambda}$ and $\overline{\mu}$ are not labelled.

Given weight diagrams $\lambda, \mu \sim \alpha$, we can concatenate to obtain the labelled diagrams $\underline{\lambda}\alpha$ and $\alpha\overline{\mu}$. We say that these diagrams are *consistently oriented* if each cup or cap has exactly one endpoint labelled $\vee$ and one labelled $\wedge$, and moreover all the rays labelled $\wedge$ are to the left of all the rays labelled $\vee$. Introduce the shorthand $\lambda \subset \alpha$ to indicate that $\lambda \sim \alpha$ and $\underline{\lambda}\alpha$ is consistently oriented. By [BS1, Lemma 2.3], this implies automatically that $\lambda \leq \alpha$ in the Bruhat order, indeed, $\leq$ is the transitive closure of the relation $\subset$. Similarly

we write $\alpha \supset \mu$ if $\alpha \sim \mu$ and $\alpha\overline{\mu}$ is consistently oriented, which implies that $\alpha \geq \mu$. Finally if $\lambda \subset \alpha \supset \mu$ we can draw the oriented cup diagram $\underline{\lambda}\alpha$ under the oriented cap diagram $\alpha\overline{\mu}$ to obtain the *oriented circle diagram* $\underline{\lambda}\alpha\overline{\mu}$. Here is an example of such a diagram:
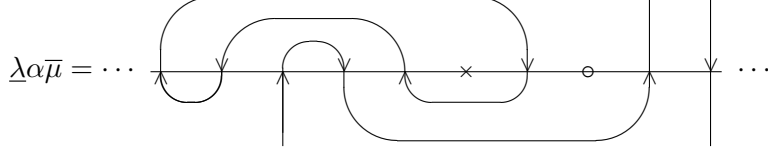
We define the *degree* of any consistently oriented diagram to be the total number of clockwise cups and caps that it contains; the example above is of degree 4.

Now we follow the construction in [BS1] to define a positively graded algebra K, which we call the *universal arc algebra*. As a graded vector space, K has the homogeneous basis

$$\left\{(\underline{\lambda}\alpha\overline{\mu}) \,\big|\, \text{for all weight diagrams } \alpha \text{ and } \lambda \subset \alpha \supset \mu\right\}, \tag{3.1}$$

and the degree of the vector $(\underline{\lambda}\alpha\overline{\mu})$ is the degree of the underlying diagram. For $\lambda \subset \alpha \supset \mu$ and $\nu \subset \beta \supset \omega$, the product $(\underline{\lambda}\alpha\overline{\mu})(\underline{\nu}\beta\overline{\omega})$ is zero whenever $\mu \neq \nu$. If $\mu = \nu$ the product is computed by drawing $\underline{\lambda}\alpha\overline{\mu}$ under $\underline{\nu}\beta\overline{\omega}$, joining up corresponding pairs of rays, then iterating a certain (generalised) *surgery procedure* to contract the symmetric $\overline{\mu}\underline{\nu}$-section of the diagram to obtain a (possibly zero) sum of new basis vectors of the form $(\underline{\lambda}\gamma\overline{\omega})$ with $\alpha \leq \gamma \geq \beta$. This surgery procedure is explained in detail in [BS1, §6]; see especially the example [BS1, (6.3)] and also [BS1, Corollary 4.5]. The fact that the multiplication in K is well defined independent of the order chosen for the surgery procedures, and that it is associative, is established in [BS1, §4] by comparing with Khovanov's topological definition of the multiplication in terms of the two-dimensional TQFT associated to the commutative Frobenius algebra $H^*(\mathbb{P}^1, \mathbb{C}) \cong \mathbb{C}[x]/(x^2)$.

The degree zero subalgebra K_0 has a basis consisting of mutually orthogonal primitive idempotents

$$\{e_\lambda \mid \text{for all weight diagrams } \lambda\} \tag{3.2}$$

defined from $e_\lambda := (\underline{\lambda}\lambda\overline{\lambda})$. For $\mu \subset \alpha \supset \nu$, we have that

$$e_\lambda(\underline{\mu}\alpha\overline{\nu}) = \begin{cases} (\underline{\mu}\alpha\overline{\nu}) & \text{if } \lambda = \mu, \\ 0 & \text{otherwise,} \end{cases} \qquad (\underline{\mu}\alpha\overline{\nu})e_\lambda = \begin{cases} (\underline{\mu}\alpha\overline{\nu}) & \text{if } \lambda = \nu, \\ 0 & \text{otherwise.} \end{cases} \tag{3.3}$$

The algebra K is not unital. Instead the idempotents (3.2) form a system of local units, i.e. we have that $K = \bigoplus_{\lambda,\mu} e_\lambda K e_\mu$ summing over all weight diagrams λ and μ. This is clear from (3.3).

For any block Γ, we let $K_\Gamma := \bigoplus_{\lambda,\mu \in \Gamma} e_\lambda K e_\mu$, which is a subalgebra of K. We then have that

$$K = \bigoplus_{\text{all blocks } \Gamma} K_\Gamma \tag{3.4}$$

as an algebra. Each of the subalgebras K_Γ is indecomposable; this follows by [BS1, (5.9)] as $\subset$ generates the partial order $\sim$. So (3.4) is the decomposition of the algebra K into blocks. If Γ is finite then K_Γ is a finite dimensional unital

algebra with $1 = \sum_{\lambda \in \Gamma} e_\lambda$. If Γ is infinite then K_Γ is only locally unital, and the idempotents $\{e_\lambda \mid \lambda \in \Gamma\}$ provide a system of local units.

Graded modules. Let $K\text{-mod}_{lf}$ be the category of graded left K-modules M such that

(1) M is locally finite dimensional, i.e. $\dim M_i < \infty$ for each $i \in \mathbb{Z}$;
(2) M is bounded below, i.e. $M_i = \{0\}$ for $i \ll 0$;
(3) M is locally unital in the sense that $M = \bigoplus_\lambda e_\lambda M$ summing over all weight diagrams λ.

We write $M\langle j \rangle$ for the module M shifted up in degree by j, so $M\langle j \rangle_i = M_{i-j}$. Also for $M, N \in K\text{-mod}_{lf}$ we write $\mathrm{Hom}_K(M, N)_j$ for the space of all homogeneous K-module homomorphisms of degree j, that is, the ones that map M_i to N_{i+j} for each $i \in \mathbb{Z}$, then define the graded vector space

$$\mathrm{Hom}_K(M, N) = \bigoplus_{j \in \mathbb{Z}} \mathrm{Hom}_K(M, N)_j.$$

For a weight diagram μ, we have the *projective indecomposable module* $P(\mu) := K e_\mu$, which has basis

$$\{(\underline{\lambda}\alpha\overline{\mu}) \mid \text{for all } \lambda \subset \alpha \text{ such that } \alpha \supset \mu\}. \tag{3.5}$$

The vectors $(\underline{\lambda}\alpha\overline{\mu})$ from this basis with $\alpha \neq \mu$ span a proper submodule of $P(\mu)$, and the corresponding quotient is the *standard module* $V(\mu)$; see [BS1, Theorem 5.1]. Finally we have the one-dimensional module $L(\mu) := P(\mu)/\mathrm{rad}\, P(\mu) = V(\mu)/\mathrm{rad}\, V(\mu)$. The modules $\{L(\mu)\}$ give a full set of pairwise non-isomorphic irreducible K-modules (up to grading shift). For later use, let

$$p_\mu : P(\mu) \twoheadrightarrow L(\mu) \tag{3.6}$$

denote the canonical quotient map.

Theorem 3.1 ([BS1],[BS2])**.** *The following hold for weight diagrams λ and μ.*

(1) *The graded composition multiplicity $d_{\lambda,\mu}(q) = \sum_{i \in \mathbb{Z}} d^{(i)}_{\lambda,\mu} q^i$ defined from $d^{(i)}_{\lambda,\mu} := [V(\mu) : L(\lambda)\langle i \rangle]$ is given explicitly by*

$$d_{\lambda,\mu}(q) = \begin{cases} q^{\deg(\underline{\lambda}\mu)} & \text{if } \lambda \subset \mu, \\ 0 & \text{otherwise.} \end{cases}$$

(2) *The projective indecomposable module $P(\lambda)$ has a finite length filtration whose sections are isomorphic to degree-shifted standard modules, with $V(\lambda)$ at the top and $V(\mu)\langle j \rangle$ appearing with multiplicity $d^{(j)}_{\lambda,\mu}$.*

(3) *Let $p_{\lambda,\mu}(q) = \sum_{i \geq 0} p^{(i)}_{\lambda,\mu} q^i \in \mathbb{N}[q]$ be the polynomial from [BS2, (5.4)], which is a certain Kazhdan-Lusztig polynomial for a Grassmannian (see [BS2, Remark 5.1]) with $p_{\lambda,\mu}(q) = 0$ if $\lambda \not\subseteq \mu$. Then the matrices $(d_{\lambda,\mu}(q))_{\lambda,\mu}$ and $(p_{\lambda,\mu}(-q))_{\lambda,\mu}$ are inverse to each other.*

(4) *The standard module $V(\lambda)$ has a canonical linear projective resolution*

$$\cdots \to P_1(\lambda) \to P_0(\lambda) \to V(\lambda) \to 0$$

in which $P_0(\lambda) = P(\lambda)$ *and* $P_i(\lambda) = \bigoplus_{\mu \geq \lambda} p_{\lambda,\mu}^{(i)} P(\mu)\langle i \rangle$. *Moreover the irreducible modules* $L(\lambda)$ *also have linear projective resolutions, hence the algebra* K *is Koszul.*

Proof. See [BS1, Theorem 5.2], [BS1, Theorem 5.1], [BS2, Corollary 5.4] and [BS2, Theorem 5.3, Corollary 5.13], respectively. □

Geometric bimodules and projective functors. Recall from [BS2, §2] that a *crossingless matching* is a diagram obtained by drawing a cap diagram underneath a cup diagram, then joining pairs of rays according to some order-preserving bijection between the vertices at their endpoints. This means that t is a union of finitely many cups and caps and (possibly infinitely many) line segments. Let $\text{cups}(t)$ and $\text{caps}(t)$ denote the number of cups and caps in t, respectively, and let t^* be the mirror image of t in its horizontal axis. Given blocks Δ and Γ, a $\Delta\Gamma$-*matching* means a crossingless matching t such that the vertices at the bottom (resp. top) of t that are not at the ends of cups, caps or line segments are in exactly the same positions as the vertices labelled ∘ or × in the weight diagrams from Δ (resp. Γ).

Assume we are given a $\Delta\Gamma$-matching t. For $\alpha \in \Delta$ and $\beta \in \Gamma$, we can form the labelled diagram $\alpha t \beta$ (so α labels the bottom number line and β labels the top). We say $\alpha t \beta$ is *consistently oriented* if each cup and cap has exactly one endpoint labelled ∨ and one labelled ∧, and the endpoints of each line segment are labelled by the same symbol. We write $\alpha \overset{t}{-} \beta$ to indicate that $\alpha \in \Delta, \beta \in \Gamma$ and $\alpha t \beta$ is consistently oriented. This is a slight abuse of notation: we are assuming that t remembers the blocks Δ and Γ. For $\lambda \subset \alpha \overset{t}{-} \beta \supset \mu$, we can concatenate (in order from bottom to top as usual) to get the composite diagram $\underline{\lambda}\alpha t \beta \overline{\mu}$.

The *geometric bimodule* $K_{\Delta\Gamma}^t$ is a certain graded (K, K)-bimodule constructed explicitly in [BS2, §3] (extending Khovanov's definition in a special case from [Kh]). As a graded vector space, it has homogeneous basis

$$\left\{ (\underline{\lambda}\alpha t \beta \overline{\mu}) \,\middle|\, \text{for all } \lambda \subset \alpha \overset{t}{-} \beta \supset \mu \right\}$$

where the degree of a basis vector is the degree of the diagram as above. The bimodule structure is defined as follows. Given basis vectors $(\underline{\nu}\alpha\overline{\omega}) \in K$ and $(\underline{\lambda}\gamma t \beta \overline{\mu}) \in K_{\Delta\Gamma}^t$, the product $(\underline{\nu}\alpha\overline{\omega})(\underline{\lambda}\alpha t \beta \overline{\mu})$ is zero unless $\omega = \lambda$, in which case it is computed by drawing $\underline{\nu}\gamma\overline{\omega}$ underneath $\underline{\lambda}\alpha t \beta \overline{\mu}$ then applying the surgery procedure to contract the symmetric $\overline{\omega}\underline{\lambda}$-section of the diagram. The product $(\underline{\lambda}\alpha t \beta \overline{\mu})(\underline{\nu}\alpha\overline{\omega})$ is defined similarly, this time putting $\underline{\nu}\alpha\overline{\omega}$ on top. Note in particular that all blocks from (3.4) other than K_Δ act on the left by zero; similarly all blocks other than K_Γ act on the right by zero.

Lemma 3.2 ([BS2, Corollary 4.3]). *The geometric bimodule* $K_{\Delta\Gamma}^t$ *is projective both as a left* K_Δ-*module and as a right* K_Γ-*module.*

Suppose in addition that we are given a block Π and a $\Pi\Delta$-matching s. We can glue s under t to obtain the *composite matching* st, which has three number lines corresponding in order from bottom to top to the blocks Π, Δ and Γ. Its *reduction* $\text{red}(st)$ is the $\Pi\Gamma$-matching obtained by removing the middle number

line and all internal circles; see [BS2, §2] for an example. Let $K^{st}_{\Pi\Delta\Gamma}$ be the graded vector space with homogeneous basis

$$\left\{ (\underline{\lambda}\alpha s\beta t\gamma\overline{\mu}) \,\middle|\, \text{for all } \lambda \subset \alpha \overset{s}{-} \beta \overset{t}{-} \gamma \supset \mu \right\}.$$

Mimicking the construction just before Lemma 3.2, we make $K^{st}_{\Pi\Delta\Gamma}$ into a graded (K, K)-bimodule.

Lemma 3.3 ([BS2, Theorem 3.5(iii), Theorem 3.6])**.** *Let s be a $\Pi\Delta$-matching, t be a $\Delta\Gamma$-matching and $u := \mathrm{red}(st)$. There are canonical graded bimodule isomorphisms*

$$K^s_{\Pi\Delta} \otimes_K K^t_{\Delta\Gamma} \cong K^{st}_{\Pi\Delta\Gamma} \cong \bigoplus_{\beta \text{ with } \alpha \overset{s}{-} \beta \overset{t}{-} \gamma} K^u_{\Pi\Gamma}\langle \deg(\alpha s\beta t\gamma) - \deg(\alpha u\gamma)\rangle$$

where in the direct sum α and γ are fixed weight diagrams satisfying $\alpha \overset{u}{-} \gamma$; the sum should be interpreted as zero if no such α and γ exist.

Lemma 3.4 ([BS2, Theorem 4.7])**.** *There is a canonical graded bimodule isomorphism $K^t_{\Delta\Gamma}\langle -\mathrm{caps}(t)\rangle \cong \mathrm{Hom}_K(K^{t^*}_{\Gamma\Delta}\langle -\mathrm{caps}(t)\rangle, K)$.*

Now we introduce the *indecomposable projective functors*

$$G^t_{\Delta\Gamma} := K^t_{\Delta\Gamma}\langle -\mathrm{caps}(t)\rangle \otimes_K ? : K\text{-mod}_{lf} \to K\text{-mod}_{lf}, \qquad (3.7)$$

$${}^*G^t_{\Delta\Gamma} := K^{t^*}_{\Gamma\Delta}\langle -\mathrm{caps}(t)\rangle \otimes_K ? : K\text{-mod}_{lf} \to K\text{-mod}_{lf}. \qquad (3.8)$$

By Lemmas 3.2 and 3.4 (or see [BS2, Corollary 4.8]), the indecomposable projective functor $G^t_{\Delta\Gamma}$ is isomorphic to the functor $\mathrm{Hom}_K(K^{t^*}_{\Gamma\Delta}\langle -\mathrm{caps}(t)\rangle, ?)$, which is right adjoint to ${}^*G^t_{\Delta\Gamma}$ by adjointness of tensor and hom. Hence there is a canonical adjunction making $({}^*G^t_{\Delta\Gamma}, G^t_{\Delta\Gamma})$ into an adjoint pair of functors. Note also from the definitions that

$${}^*G^t_{\Delta\Gamma} = G^{t^*}_{\Gamma\Delta}\langle \mathrm{cups}(t) - \mathrm{caps}(t)\rangle. \qquad (3.9)$$

A *projective functor* means any endofunctor of $K\text{-mod}_{lf}$ that is isomorphic to a direct sum of indecomposable projective functors (possibly shifted in degree). Projective functors are exact thanks to Lemma 3.2. The composition of two projective functors is again a projective functor thanks to Lemma 3.3. Finally projective functors map finite dimensional modules to finite dimensional modules by [BS2, Corollary 4.12].

Projective functors on irreducible modules. Continue with t being a $\Delta\Gamma$-matching. For $\lambda \in \Delta$ and $\mu \in \Gamma$, we can concatenate in order from bottom to top as usual to obtain the composite diagrams $\underline{\lambda}t$ and $t\overline{\mu}$. The *lower reduction* $\mathrm{red}(\underline{\lambda}t)$ is the ordinary cup diagram obtained from $\underline{\lambda}t$ by removing the bottom number line and all connected components that do not extend up to the top number line. Similarly the *upper reduction* $\mathrm{red}(t\overline{\mu})$ is the ordinary cap diagram obtained from $t\overline{\mu}$ by removing the top number line.

For $\mu \in \Gamma$, the K-module $G^t_{\Delta\Gamma}P(\mu)$ is identified with $K^t_{\Delta\Gamma}e_\mu\langle -\mathrm{caps}(t)\rangle$, hence it has the diagram basis

$$\left\{ (\underline{\lambda}\alpha t\beta\overline{\mu}) \,\middle|\, \text{for all } \lambda \subset \alpha \overset{t}{-} \beta \text{ such that } \beta \supset \mu \right\}. \qquad (3.10)$$

Applying $G_{\Delta\Gamma}^t$ to (3.6), we get a surjection $G_{\Delta\Gamma}^t(p_\mu) : G_{\Delta\Gamma}^t P(\mu) \twoheadrightarrow G_{\Delta\Gamma}^t L(\mu)$ which identifies $G_{\Delta\Gamma}^t L(\mu)$ with a quotient of $G_{\Delta\Gamma}^t P(\mu)$. The following theorem describes this quotient explicitly.

Theorem 3.5. *Let t be a $\Delta\Gamma$-matching and $\mu \in \Gamma$. Then $G_{\Delta\Gamma}^t L(\mu)$ is the quotient of $G_{\Delta\Gamma}^t P(\mu)$ by the submodule spanned by all the vectors from (3.10) in which either $\mathrm{red}(\underline{\lambda}t) \neq \underline{\mu}$ or $\beta \neq \mu$. Hence $G_{\Delta\Gamma}^t L(\mu)$ is finite dimensional with basis given by the image of the vectors*

$$\left\{ (\underline{\lambda}\alpha t\mu\overline{\mu}) \,\big|\, \text{for all } \lambda \subset \alpha \overset{t}{-} \mu \text{ such that } \mathrm{red}(\underline{\lambda}t) = \underline{\mu} \right\}. \tag{3.11}$$

Moreover, $G_{\Delta\Gamma}^t L(\mu)$ is non-zero if and only if there exists a (necessarily unique) λ such that $\lambda \overset{t}{-} \mu$ and $\deg(\lambda t\mu) = 0$, in which case $G_{\Delta\Gamma}^t L(\mu)$ is a self-dual indecomposable module with head $L(\lambda)\langle-\mathrm{caps}(t)\rangle$ and socle $L(\lambda)\langle\mathrm{caps}(t)\rangle$.

Proof. The final statement is [BS2, Theorem 4.11(ii)–(iii)]. For the other parts, [BS2, Theorem 4.11(i)] and [BS2, Corollary 4.12] show that $G_{\Delta\Gamma}^t L(\mu)$ is finite dimensional with

$$[G_{\Delta\Gamma}^t L(\mu)] = \sum_{\substack{\lambda \subset \alpha \overset{t}{-} \mu \\ \mathrm{red}(\underline{\lambda}t)=\underline{\mu}}} [L(\lambda)\langle\deg(\underline{\lambda}\alpha t\mu) - \mathrm{caps}(t)\rangle], \tag{3.12}$$

equality written in the Grothendieck group of K-mod_{lf}. In particular the dimension of $G_{\Delta\Gamma}^t L(\mu)$ is exactly the number of vectors listed in (3.11). It remains to show that $(\underline{\lambda}\alpha t\beta\overline{\mu}) \in G_{\Delta\Gamma}^t P(\mu)$ belongs to the kernel of the canonical map $G_{\Delta\Gamma}^t(p_\mu) : G_{\Delta\Gamma}^t P(\mu) \twoheadrightarrow G_{\Delta\Gamma}^t L(\mu)$ either if $\beta \neq \mu$ or if $\mathrm{red}(\underline{\lambda}t) \neq \underline{\mu}$. The first paragraph of the proof of [BS2, Theorem 4.5] shows that $G_{\Delta\Gamma}^t V(\mu)$ is the quotient of $G_{\Delta\Gamma}^t P(\mu)$ by the submodule spanned by all the vectors $(\underline{\lambda}\alpha t\beta\overline{\mu})$ with $\beta \neq \mu$. Hence all of these vectors lie in $\ker G_{\Delta\Gamma}^t(p_\mu)$. Moreover (3.12) implies $e_\lambda G_{\Delta\Gamma}^t L(\mu) = \{0\}$ for all $\lambda \in \Delta$ with $\mathrm{red}(\underline{\lambda}t) \neq \underline{\mu}$, hence all the vectors $(\underline{\lambda}\alpha t\beta\overline{\mu})$ with $\mathrm{red}(\underline{\lambda}t) \neq \underline{\mu}$ lie in the kernel too. $\qquad\square$

Corollary 3.6. *Let t be a $\Delta\Gamma$-matching and $\mu \in \Gamma$ such that $G_{\Delta\Gamma}^t L(\mu)$ is non-zero. Let λ be the unique weight diagram with $\lambda \overset{t}{-} \mu$ and $\deg(\lambda t\mu) = 0$, and let λ' be obtained from λ by reversing the orientation of all the caps in $\lambda t\mu$. Then $G_{\Delta\Gamma}^t L(\mu)$ is a cyclic module generated by the image of the vector $(\underline{\lambda}\lambda t\mu\overline{\mu})$, and its irreducible socle is generated by the image of the vector $(\underline{\lambda}\lambda' t\mu\overline{\mu})$.*

Proof. This follows from the descriptions of the head and socle of $G_{\Delta\Gamma}^t L(\mu)$ in Theorem 3.5, on noting that the vectors $(\underline{\lambda}\lambda t\mu\overline{\mu})$ and $(\underline{\lambda}\lambda' t\mu\overline{\mu})$ in the basis (3.11) are of degrees $-\mathrm{caps}(t)$ and $\mathrm{caps}(t)$, respectively. $\qquad\square$

Special projective functors. For $i \in \mathbb{Z}$, we introduce the following *special projective functors*:

$$E_i := \bigoplus_{\Gamma,\Delta,s} G_{\Gamma\Delta}^s, \qquad\qquad F_i := \bigoplus_{\Delta,\Gamma,t} G_{\Delta\Gamma}^t, \tag{3.13}$$

$$^*E_i := \bigoplus_{\Gamma,\Delta,s} {}^*G_{\Gamma\Delta}^s, \qquad\qquad {}^*F_i := \bigoplus_{\Delta,\Gamma,t} {}^*G_{\Delta\Gamma}^t, \tag{3.14}$$

where the direct sums are over all blocks Γ, Δ and crossingless matchings s, t such that

(1) the labels $\circ$ and $\times$ on the ith and $(i+1)$th vertices of weights in Γ and Δ match one of the configurations displayed in (3.15);

(2) the labels $\circ$ and $\times$ on all other vertices are in the exactly same positions in weights from Γ and from Δ;

(3) s is the $\Gamma\Delta$-matching and t is the $\Delta\Gamma$-matching displayed in the strip between the ith and $(i+1)$th vertices in (3.15), and s and t have vertical line segments connecting all other vertices not labelled $\circ$ or $\times$ in weights from Γ and Δ.

$$(3.15)$$

So $E_i, F_i, {}^*E_i$ and ${}^*F_i$ are the endofunctors of $K_\Lambda\text{-mod}_{lf}$ defined by tensoring with the graded (K_Λ, K_Λ)-bimodules

$$\widetilde{E}_i := \bigoplus_{\Gamma,\Delta,s} K^s_{\Gamma\Delta}\langle -\mathrm{caps}(s)\rangle, \qquad \widetilde{F}_i := \bigoplus_{\Delta,\Gamma,t} K^t_{\Delta\Gamma}\langle -\mathrm{caps}(t)\rangle, \qquad (3.16)$$

$${}^*\widetilde{E}_i := \bigoplus_{\Gamma,\Delta,s} K^{s*}_{\Delta\Gamma}\langle -\mathrm{caps}(s)\rangle, \qquad {}^*\widetilde{F}_i := \bigoplus_{\Delta,\Gamma,t} K^{t*}_{\Gamma\Delta}\langle -\mathrm{caps}(t)\rangle, \qquad (3.17)$$

respectively, summing over the same sets as before. By the general theory of projective functors reviewed earlier, there are canonical adjunctions making $({}^*E_i, E_i)$ and $({}^*F_i, F_i)$ into adjoint pairs. All four functors are exact, they map projectives to projectives, and they map finite dimensional modules to finite dimensional modules.

Special projective functors on standard modules. Given weight diagrams λ and μ and $i \in \mathbb{Z}$, we write $\lambda \xrightarrow{i} \mu$ if λ and μ have the same labels on all but the ith and $(i+1)$th vertices, and the ith and $(i+1)$th vertices of λ and μ are labelled as in one of the following cases:

$$\lambda \xrightarrow{i} \mu:$$

λ	$\vee\circ$	$\wedge\circ$	$\times\vee$	$\times\wedge$	$\times\circ$	$\times\circ$	$\vee\wedge$	$\wedge\vee$
μ	$\circ\vee$	$\circ\wedge$	$\vee\times$	$\wedge\times$	$\vee\wedge$	$\wedge\vee$	$\circ\times$	$\circ\times$
degree	0	0	0	0	0	1	-1	0

$$(3.18)$$

Similarly, we write $\lambda \xleftarrow{i} \mu$ if λ and μ have the same labels on all but the ith and $(i+1)$th vertices, and the ith and $(i+1)$th vertices of λ and μ are labelled as in one of the following cases:

$$\lambda \xleftarrow{i} \mu:$$

λ	$\circ\vee$	$\circ\wedge$	$\vee\times$	$\wedge\times$	$\vee\wedge$	$\wedge\vee$	$\circ\times$	$\circ\times$
μ	$\vee\circ$	$\wedge\circ$	$\times\vee$	$\times\wedge$	$\times\circ$	$\times\circ$	$\vee\wedge$	$\wedge\vee$
degree	0	0	0	0	-1	0	0	1

$$(3.19)$$

Then define the *edge degrees*

$$\deg(\lambda \xrightarrow{i} \mu) := \#\{\mu' < \mu \mid \lambda \xrightarrow{i} \mu'\} - \#\{\lambda' > \lambda \mid \lambda' \xrightarrow{i} \mu\}, \qquad (3.20)$$

$$\deg(\lambda \xleftarrow{i} \mu) := \#\{\mu' < \mu \mid \lambda \xleftarrow{i} \mu'\} - \#\{\lambda' > \lambda \mid \lambda' \xleftarrow{i} \mu\}. \qquad (3.21)$$

The explicit values of these degrees are the numbers listed in the third rows of (3.18) and (3.19), respectively. Note that $\lambda \xrightarrow{i} \mu$ if and only if $\mu \xleftarrow{i} \lambda$; however the degrees $\deg(\lambda \xrightarrow{i} \mu)$ and $\deg(\mu \xleftarrow{i} \lambda)$ are in general different.

Theorem 3.7. *The following hold for any weight diagram λ and $i \in \mathbb{Z}$.*
  (1) *The module $E_i V(\lambda)$ has a filtration with sections isomorphic to the modules $\{V(\mu)\langle\deg(\mu \xrightarrow{i} \lambda)\rangle \mid$ for all μ such that $\mu \xrightarrow{i} \lambda\}$.*
  (2) *The module $F_i V(\lambda)$ has a filtration with sections isomorphic to the modules $\{V(\mu)\langle\deg(\mu \xleftarrow{i} \lambda)\rangle \mid$ for all μ such that $\mu \xleftarrow{i} \lambda\}$.*
  (3) *The module $^*E_i V(\lambda)$ has a filtration with sections isomorphic to the modules $\{V(\mu)\langle\deg(\lambda \xrightarrow{i} \mu)\rangle \mid$ for all μ such that $\lambda \xrightarrow{i} \mu\}$.*
  (4) *The module $^*F_i V(\lambda)$ has a filtration with sections isomorphic to the modules $\{V(\mu)\langle\deg(\lambda \xleftarrow{i} \mu)\rangle \mid$ for all μ such that $\lambda \xleftarrow{i} \mu\}$.*

In all cases, the filtration is of length at most two; when it is of length exactly two the standard module indexed by the biggest weight in the Bruhat ordering $\geq$ appears as a submodule and the other is a quotient.

Proof. Parts (1) and (2) follow from [BS2, Theorem 4.5]. Then (3) and (4) follow from (2) and (1), respectively, using (3.9). $\qquad\square$

Remark 3.8. To make the connection with Theorem 2.5 in the previous section, fix $\delta \in \mathbb{Z}$, and suppose that Λ is the set of all bipartitions identified with a set of weight diagrams according to the weight dictionary (1.3)–(1.5). Then Λ is a connected component of the labelled directed graph just defined, and the labelled directed edges between elements of Λ are exactly the same as in the labelled directed graph introduced just before Theorem 2.5.

Special projective functors on projective modules. For a weight diagram $\lambda, i \in \mathbb{Z}$ and a symbol $x \in \{\circ, \wedge, \vee, \times\}$ we write λ^i_x for the weight diagram obtained from λ by relabelling the ith vertex by x. Then for $i < j$ and symbols x, y we set $\lambda^{ij}_{xy} := (\lambda^i_x)^j_y$, and so on.

Theorem 3.9. *The following hold for any weight diagram λ and $i \in \mathbb{Z}$.*
  (1) *If $\lambda = \lambda^{i(i+1)}_{\vee\circ}$ then $^*E_i P(\lambda) \cong P(\lambda^{i(i+1)}_{\circ\vee})$.*
  (2) *If $\lambda = \lambda^{i(i+1)}_{\wedge\circ}$ then $^*E_i P(\lambda) \cong P(\lambda^{i(i+1)}_{\circ\wedge})$.*
  (3) *If $\lambda = \lambda^{i(i+1)}_{\times\vee}$ then $^*E_i P(\lambda) \cong P(\lambda^{i(i+1)}_{\vee\times})$.*
  (4) *If $\lambda = \lambda^{i(i+1)}_{\times\wedge}$ then $^*E_i P(\lambda) \cong P(\lambda^{i(i+1)}_{\wedge\times})$.*
  (5) *If $\lambda = \lambda^{i(i+1)}_{\times\circ}$ then $^*E_i P(\lambda) \cong P(\lambda^{i(i+1)}_{\vee\wedge})$.*
  (6) *If $\lambda = \lambda^{i(i+1)}_{\vee\wedge}$ then $^*E_i P(\lambda) \cong P(\lambda^{i(i+1)}_{\circ\times})\langle 1\rangle \oplus P(\lambda^{i(i+1)}_{\circ\times})\langle -1\rangle$.*
  (7) *If $\lambda = \lambda^{i(i+1)}_{\wedge\vee}$ then $^*E_i P(\lambda) \cong P(\lambda^{i(i+1)}_{\circ\times})$.*

(8) *If $\lambda = \lambda_{\vee\vee}^{i(i+1)}$ and there is a cap connecting vertex $i+1$ to vertex $j > i+1$ in $\overline{\lambda}$ then ${}^*E_iP(\lambda) \cong P(\lambda_{\circ\times\vee}^{i(i+1)j})$.*

(9) *If $\lambda = \lambda_{\wedge\wedge}^{i(i+1)}$ and there is a cap connecting vertex i to vertex $j < i$ in $\overline{\lambda}$ then ${}^*E_iP(\lambda) \cong P(\lambda_{\wedge\circ\times}^{ji(i+1)})$.*

(10) *For all other λ we have that ${}^*E_iP(\lambda) = \{0\}$.*

For the analogous results with ${}^*E_i$ replaced by ${}^*F_i$, interchange all occurrences of $\circ$ and $\times$.

Proof. This can be deduced using [BS2, Theorem 4.2]. $\qquad\square$

4. Idempotent truncations of $K(\delta)$

In this section, we fix $\delta \in \mathbb{Z}$. We are going to establish some basic properties of the algebra $K_{r,s}(\delta)$ from Theorem 1.1, i.e. the algebra which will eventually be shown to be Morita equivalent to the walled Brauer algebra $B_{r,s}(\delta)$. In particular we introduce counterparts for $K_{r,s}(\delta)$ of the walled Brauer algebras's i-induction and i-restriction functors, culminating in a result which is parallel to Theorem 2.5.

Graded cellular algebras. We begin the section by recalling briefly the definition of a *graded cellular algebra* from [GL], [HM]. This means an associative unital algebra H together with a *cell datum* $(X, I, C, \deg)$ such that

(1) X is a finite partially ordered set;

(2) $I(\lambda)$ is a finite set for each $\lambda \in X$;

(3) $C : \dot{\bigcup}_{\lambda \in X} I(\lambda) \times I(\lambda) \to H, (i,j) \mapsto C_{i,j}^\lambda$ is an injective map whose image is a basis for H;

(4) the map $H \to H, C_{i,j}^\lambda \mapsto C_{j,i}^\lambda$ is an algebra anti-automorphism;

(5) if $\lambda \in X$ and $i, j \in I(\lambda)$ then for any $x \in H$ we have that

$$x\, C_{i,j}^\lambda \equiv \sum_{i' \in I(\lambda)} r_x(i', i) C_{i',j}^\lambda \pmod{H_{>\lambda}}$$

where the scalar $r_x(i', i)$ is independent of j and $H_{>\lambda}$ denotes the subspace of H spanned by $\{C_{k,l}^\mu \mid \mu > \lambda, k, l \in I(\mu)\}$;

(6) $\deg : \dot{\bigcup}_{\lambda \in X} I(\lambda) \to \mathbb{Z}, i \mapsto \deg_i^\lambda$ is a function such that the $\mathbb{Z}$-grading on H defined by declaring each $C_{i,j}^\lambda$ is of degree $\deg_i^\lambda + \deg_j^\lambda$ makes H into a graded algebra.

Assuming H is a graded cellular algebra, we get for $\lambda \in X$ an associated *graded cell module* $V(\lambda)$ with basis $\{v_i \mid i \in I(\lambda)\}$. It is a graded left H-module with $\deg(v_i) := \deg_i^\lambda$ and the action of $x \in H$ defined from $x\, v_i := \sum_{i' \in I(\lambda)} r_x(i', i)v_{i'}$. Moreover there is a canonically defined homogeneous symmetric bilinear form $(.,.)$ on $V(\lambda)$ with radical $\mathrm{rad}\, V(\lambda)$. The quotient $L(\lambda) := V(\lambda)/\mathrm{rad}\, V(\lambda)$ is either zero or an irreducible graded H-module, and the non-zero $L(\lambda)$'s give a complete set of pairwise inequivalent irreducible graded H-modules (up to degree shift). See [GL], [HM] for details.

Graded cellular structure on $K_{r,s}(\delta)$. Let Λ be the set of all bipartitions identified with a set of weight diagrams using the weight dictionary (1.3)–(1.5).

Let $K(\delta)$ be the subalgebra $\bigoplus_{\lambda,\mu\in\Lambda} e_\lambda K e_\mu$ of the universal arc algebra K from the previous section. Note that Λ is a union of blocks. So the algebra $K(\delta)$ is a sum of some of the blocks of K from (3.4):

$$K(\delta) = \bigoplus_{\Gamma\in\Lambda/\sim} K_\Gamma. \qquad (4.1)$$

Let $K(\delta)\text{-mod}_{lf}$ be the full subcategory of $K\text{-mod}_{lf}$ consisting of all modules M such that $M = \bigoplus_{\lambda\in\Lambda} e_\lambda M$. For $\lambda \in \Lambda$, the K-modules $P(\lambda)$, $V(\lambda)$ and $L(\lambda)$ from the previous section belong to $K(\delta)\text{-mod}_{lf}$. Moreover Theorem 3.1 specialises in an obvious way.

Let $e_{r,s} = \sum_{\lambda\in\dot{\Lambda}_{r,s}} e_\lambda \in K(\delta)$ be the idempotent from (1.6) and set

$$K_{r,s}(\delta) := e_{r,s} K(\delta) e_{r,s}. \qquad (4.2)$$

It is immediate from (3.3) that $K_{r,s}(\delta)$ has a basis consisting of diagrams $(\underline{\lambda}\alpha\overline{\mu})$ for all $\lambda \subset \alpha \supset \mu$ with $\lambda, \mu \in \dot{\Lambda}_{r,s}$ and $\alpha \in \Lambda$. The weight α here automatically belongs to $\Lambda_{r,s}$; this follows because $\Lambda_{r,s}$ is an upper set in the poset $(\Lambda, \leq)$, i.e. $\lambda \in \Lambda_{r,s}, \mu \in \Lambda, \lambda \leq \mu \Rightarrow \mu \in \Lambda_{r,s}$. So the basis for $K_{r,s}(\delta)$ can be denoted

$$\{(\underline{\lambda}\alpha\overline{\mu}) \mid \text{for all } \lambda \subset \alpha \supset \mu \text{ with } \lambda, \mu \in \dot{\Lambda}_{r,s} \text{ and } \alpha \in \Lambda_{r,s}\}.$$

In particular $K_{r,s}(\delta)$ is a positively graded, finite dimensional algebra with identity $e_{r,s}$.

Multiplication by the idempotent $e_{r,s}$ defines an exact functor

$$e_{r,s} : K(\delta)\text{-mod}_{lf} \to K_{r,s}(\delta)\text{-mod}_{lf},$$

where $K_{r,s}(\delta)\text{-mod}_{lf}$ is the category of graded left $K_{r,s}(\delta)$-modules that are locally finite dimensional and bounded below.

Lemma 4.1. *Let M and N be left $K(\delta)$-modules. If M is a finite direct sum of summands of $K(\delta)e_{r,s}$ then the natural map*

$$\mathrm{Hom}_{K(\delta)}(M, N) \to \mathrm{Hom}_{K_{r,s}(\delta)}(e_{r,s}M, e_{r,s}N)$$

defined by the functor $e_{r,s}$ is an isomorphism.

Proof. It is obviously an isomorphism for $M = K(\delta)e_{r,s}$. $\qquad\square$

Lemma 4.2. *Let M be a right $K(\delta)$-module and N be a left $K(\delta)$-module. If M is a direct sum of summands of $e_{r,s}K(\delta)$ or N is a direct sum of summands of $K(\delta)e_{r,s}$, then the natural map*

$$Me_{r,s} \otimes_{K_{r,s}(\delta)} e_{r,s}N \to M \otimes_{K(\delta)} N, \qquad m \otimes n \mapsto m \otimes n$$

is an isomorphism.

Proof. It is obviously true if $M = e_{r,s}K(\delta)$ or $N = K(\delta)e_{r,s}$. $\qquad\square$

For $\lambda \in \Lambda$, let

$$P_{r,s}(\lambda) := e_{r,s}P(\lambda), \qquad (4.3)$$
$$V_{r,s}(\lambda) := e_{r,s}V(\lambda), \qquad (4.4)$$
$$L_{r,s}(\mu) := e_{r,s}L(\mu). \qquad (4.5)$$

Lemma 4.3. *The following statements hold for any $\lambda \in \Lambda$.*

(1) $L_{r,s}(\lambda) \neq \{0\}$ *if and only if* $\lambda \in \dot{\Lambda}_{r,s}$.
(2) $V_{r,s}(\lambda) \neq \{0\}$ *if and only if* $\lambda \in \Lambda_{r,s}$.

Proof. (1) This is immediate from the definition of the idempotent $e_{r,s}$.
(2) Suppose first that $\lambda \in \Lambda \setminus \Lambda_{r,s}$. All composition factors of $V(\lambda)$ are of the form $L(\mu)\langle j\rangle$ for $\mu \leq \lambda$, hence $\mu \notin \Lambda_{r,s}$, so we get that $e_{r,s}V(\lambda) = \{0\}$ by (1). Conversely if $\lambda \in \dot{\Lambda}_{r,s}$ then $V(\lambda)$ has head $L(\lambda)$, so $e_{r,s}V(\lambda) \neq \{0\}$ by (1). Finally if $\lambda = (\varnothing, \varnothing) \in \Lambda_{r,s} \setminus \dot{\Lambda}_{r,s}$, then $V(\lambda)$ has a composition factor isomorphic to $L(\mu)\langle 1\rangle$ where $\mu = ((1),(1)) \in \dot{\Lambda}_{r,s}$ by Theorem 3.1(1), hence again $e_{r,s}V(\lambda) \neq \{0\}$. $\qquad\square$

Theorem 4.4. *The algebra $K_{r,s}(\delta)$ is a graded cellular algebra with cell datum $(X, I, C, \deg)$ defined by*

$$X := \Lambda_{r,s}, \qquad\qquad I(\alpha) := \{\lambda \in \dot{\Lambda}_{r,s} \mid \lambda \subset \alpha\},$$
$$C^{\alpha}_{\lambda,\mu} := (\underline{\lambda}\alpha\overline{\mu}), \qquad\qquad \deg^{\alpha}_{\lambda} := \deg(\underline{\lambda}\alpha)$$

for $\alpha \in \Lambda_{r,s}$ and $\lambda, \mu \in I(\alpha)$. Moreover the following hold.
  (1) *The graded cell modules are the modules $\{V_{r,s}(\lambda) \mid \lambda \in \Lambda_{r,s}\}$.*
  (2) *For $\lambda \in \dot{\Lambda}_{r,s}$ the cell module $V_{r,s}(\lambda)$ has one-dimensional head $L_{r,s}(\lambda)$, and the modules $\{L_{r,s}(\lambda) \mid \lambda \in \dot{\Lambda}_{r,s}\}$ give a complete set of pairwise non-isomorphic irreducible $K_{r,s}(\delta)$-modules (up to degree shift).*
  (3) *The projective cover of $L_{r,s}(\lambda)$ is $P_{r,s}(\lambda) = K_{r,s}(\delta)e_{\lambda}$ for each $\lambda \in \dot{\Lambda}_{r,s}$.*
  (4) *For $\lambda \in \dot{\Lambda}_{r,s}$ and $\mu \in \Lambda_{r,s}$, the graded composition multiplicity*

$$d_{\lambda,\mu}(q) = \sum_{i \in \mathbb{Z}} [V_{r,s}(\mu) : L_{r,s}(\lambda)\langle i\rangle]q^i$$

   is given explicitly by the formula from Theorem 3.1(1).
  (5) *Irreducible modules $L_{r,s}(\lambda)$ and $L_{r,s}(\mu)$ for $\lambda, \mu \in \dot{\Lambda}_{r,s}$ belong to the same block if and only if $\lambda \sim \mu$.*
  (6) *For each $\lambda \in \Lambda_{r,s}$, there is a resolution*

$$\cdots \to e_{r,s}P_1(\lambda) \to e_{r,s}P_0(\lambda) \to V_{r,s}(\lambda) \to 0$$

   in which $e_{r,s}P_i(\lambda) = \bigoplus_{\mu \in \Lambda_{r,s}} p^{(i)}_{\lambda,\mu}P_{r,s}(\mu)\langle i\rangle$, and $p_{\lambda,\mu}(q) = \sum_{i \geq 0} p^{(i)}_{\lambda,\mu}q^i$ is as in Theorem 3.1(3).
  (7) *If $\Lambda_{r,s} = \dot{\Lambda}_{r,s}$, then $K_{r,s}(\delta)$ is a graded quasi-hereditary algebra in the sense of [CPS], and the $V_{r,s}(\lambda)$'s are its graded standard modules. Moreover the resolution in (6) is a linear projective resolution, hence $K_{r,s}(\delta)$ is standard Koszul in the sense of [ADL].*

Proof. The cellularity is deduced from [BS1, Theorem 4.4] in exactly the same way as [BS1, Corollary 4.6].
(1) By definition, the graded cell module indexed by $\lambda \in \Lambda_{r,s}$ has a distinguished homogeneous basis

$$\{v_{\mu} \mid \text{for all } \mu \subset \lambda \text{ with } \mu \in \dot{\Lambda}_{r,s}\}.$$

Moreover the action of a basis vector $(\underline{\nu}\alpha\overline{\omega}) \in K_{r,s}(\delta)$ on v_{μ} is computed as follows. First compute the product $(\underline{\nu}\alpha\overline{\omega})(\underline{\mu}\lambda\overline{\lambda})$ in the algebra $K_{r,s}(\delta)$ to get a

(possibly zero) sum of basis vectors of the form $(\underline{\nu}\beta\overline{\lambda})$. Then $(\underline{\nu}\alpha\overline{\omega})v_\mu$ is equal to v_ν if $(\underline{\nu}\lambda\overline{\lambda})$ appears in this sum, and $(\underline{\nu}\alpha\overline{\omega})v_\mu = 0$ otherwise. On the other hand, using the original definition of $V(\lambda)$ as a quotient of $P(\lambda)$ given just after (3.5), we see that $V_{r,s}(\lambda)$ has a basis given by the canonical images of the vectors

$$\{(\underline{\mu}\lambda\overline{\lambda}) \mid \text{for all } \mu \subset \lambda \text{ with } \mu \in \dot{\Lambda}_{r,s}\},$$

and the action is defined by the algorithm just explained. Hence $V_{r,s}(\lambda)$ is isomorphic to the graded cell module parametrised by λ.

(2), (3) These are standard consequences of Lemma 4.3(1), since $L(\lambda)$ is the irreducible head of both $V(\lambda)$ and $P(\lambda)$.

(4) This follows from Theorem 3.1(1).

(5) This is a combinatorial exercise using (4) and the usual description of the blocks of a cellular algebra in terms of its decomposition matrix.

(6) This follows by applying $e_{r,s}$ to the resolution in Theorem 3.1(4), using also Lemma 4.3(2).

(7) The fact that $K_{r,s}(\delta)$ is quasi-hereditary follows from the general theory of cellular algebras since the number of isomorphism classes of irreducible modules is equal to the number of cell modules when $\Lambda_{r,s} = \dot{\Lambda}_{r,s}$. The fact that the resolution is a linear projective resolution follows from (3). □

Idempotent truncations of special projective functors. In view of Remark 3.8, the special projective functors from (3.13)–(3.14) restrict to well-defined functors

$$E_i, F_i, {}^*E_i, {}^*F_i : K(\delta)\text{-mod}_{lf} \to K(\delta)\text{-mod}_{lf}.$$

By definition, these functors are defined by tensoring with the graded bimodules from (3.16)–(3.17). Tensoring with various truncations of these bimodules gives us the following right exact functors:

$$i\text{-res}_{r,s}^{r+1,s} := e_{r,s}\widetilde{E}_i e_{r+1,s} \otimes_{K_{r+1,s}(\delta)} ? : K_{r+1,s}(\delta)\text{-mod}_{lf} \to K_{r,s}(\delta)\text{-mod}_{lf}, \quad (4.6)$$

$$i\text{-res}_{r,s}^{r,s+1} := e_{r,s}\widetilde{F}_i e_{r,s+1} \otimes_{K_{r,s+1}(\delta)} ? : K_{r,s+1}(\delta)\text{-mod}_{lf} \to K_{r,s}(\delta)\text{-mod}_{lf}, \quad (4.7)$$

$$i\text{-ind}_{r,s}^{r+1,s} := e_{r+1,s}{}^*\widetilde{E}_i e_{r,s} \otimes_{K_{r,s}(\delta)} ? : K_{r,s}(\delta)\text{-mod}_{lf} \to K_{r+1,s}(\delta)\text{-mod}_{lf}, \quad (4.8)$$

$$i\text{-ind}_{r,s}^{r,s+1} := e_{r,s+1}{}^*\widetilde{F}_i e_{r,s} \otimes_{K_{r,s}(\delta)} ? : K_{r,s}(\delta)\text{-mod}_{lf} \to K_{r,s+1}(\delta)\text{-mod}_{lf}. \quad (4.9)$$

Lemma 4.5. *For $\lambda \in \dot{\Lambda}_{r,s}$ and $i \in \mathbb{Z}$, ${}^*E_i P(\lambda)$ is isomorphic to a direct sum of degree-shifted copies of $P(\mu)$ for some $\mu \in \dot{\Lambda}_{r+1,s}$. Conversely, given $\mu \in \dot{\Lambda}_{r+1,s}$, it is possible to choose $\lambda \in \dot{\Lambda}_{r,s}$ and $i \in \mathbb{Z}$ such that ${}^*E_i P(\lambda)$ has a summand isomorphic (up to a degree shift) to $P(\mu)$. The same two statements hold with ${}^*E_i$ replaced by ${}^*F_i$ and $\dot{\Lambda}_{r+1,s}$ replaced by $\dot{\Lambda}_{r,s+1}$.*

Proof. We just prove the lemma for ${}^*E_i$. First take $\lambda \in \dot{\Lambda}_{r,s}$. The fact that ${}^*E_i P(\lambda)$ is a sum of copies of degree-shifted copies $P(\mu)$ for some $\mu \in \dot{\Lambda}_{r+1,s}$ follows from Theorem 3.9. Conversely, take $\mu \in \dot{\Lambda}_{r+1,s}$. We want to show there exists $\lambda \in \dot{\Lambda}_{r,s}$ and $i \in \mathbb{Z}$ such that $P(\mu)$ is isomorphic (up to a degree shift) to a summand of ${}^*E_i P(\lambda)$. Some observations:

(1) If we can find $i \in \mathbb{Z}$ such that the ith and $(i+1)$th vertices of the weight diagram of μ are labelled $\circ\wedge$, $\vee\times$ or $\vee\wedge$, then we are done by Theorem 3.9(2), (3) or (5), respectively.

(2) If we can find $i \in \mathbb{Z}$ such that the ith and $(i+1)$th vertices of μ are labelled $\circ\vee$, $\wedge\times$ or $\circ\times$ then we are done by Theorem 3.9(1), (4) or (6), respectively, unless $s = |\mu^{\mathrm{R}}|$.

(3) If we can find $i \in \mathbb{Z}$ such that the ith and $(i+1)$th vertices of μ are labelled $\circ\times$, we are done by Theorem 3.9(7) unless $\mu = ((1), \varnothing)$, $\delta = 0$ and $s > 0$; moreover we are done in the exceptional case by (2).

Using (1) and (3) we are reduced to the situation that every $\times$ or $\wedge$ in μ has either $\times$ or $\wedge$ immediately to its left. This implies that $\mu^{\mathrm{L}} = \varnothing$, hence $s > |\mu^{\mathrm{R}}|$. Hence by (2) every $\times$ must have actually $\times$ to its left, i.e. since the number of $\times$'s is finite there are actually no $\times$'s at all. Then (2) once again gives that $\mu = (\varnothing, \varnothing)$ and $\delta = 0$, which contradicts the assumption that $\mu \in \dot{\Lambda}_{r+1,s}$. $\qquad\square$

Lemma 4.6. *The following hold for all $r, s \geq 0$ and $i \in \mathbb{Z}$.*

(1) *The graded left $K(\delta)$-module $^{*}\widetilde{E}_i e_{r,s}$ is isomorphic to a finite direct sum of degree-shifted summands of $K(\delta)e_{r+1,s}$.*

(2) *The graded left $K(\delta)$-module $^{*}\widetilde{F}_i e_{r,s}$ is isomorphic to a finite direct sum of degree-shifted summands of $K(\delta)e_{r,s+1}$.*

(3) *The graded right $K(\delta)$-module $e_{r,s}\widetilde{E}_i$ is isomorphic to a finite direct sum of degree-shifted summands of $e_{r+1,s}K(\delta)$.*

(4) *The graded right $K(\delta)$-module $e_{r,s}\widetilde{F}_i$ is isomorphic to a finite direct sum of degree-shifted summands of $e_{r,s+1}K(\delta)$.*

Proof. Parts (1) and (2) follow from Lemma 4.5. Parts (3) and (4) follow from (1) and (2), respectively, on considering the mirror images of the diagrams in a horizontal axis. $\qquad\square$

Lemma 4.7. *For $i \in \mathbb{Z}$ and $r, s \geq 0$, there are canonical isomorphisms of functors*

$$i\text{-res}_{r,s}^{r+1,s} \circ e_{r+1,s} \cong e_{r,s} \circ E_i : K(\delta)\text{-mod}_{lf} \to K_{r,s}(\delta)\text{-mod}_{lf},$$

$$i\text{-res}_{r,s}^{r,s+1} \circ e_{r,s+1} \cong e_{r,s} \circ F_i : K(\delta)\text{-mod}_{lf} \to K_{r,s}(\delta)\text{-mod}_{lf} .$$

Proof. We just construct the first isomorphism, since the second is similar. By Lemma 4.6(3) and Lemma 4.2, the natural multiplication map is a bimodule isomorphism $e_{r,s}\widetilde{E}_i e_{r+1,s} \otimes_{K_{r+1,s}(\delta)} e_{r+1,s}K(\delta) \cong e_{r,s}\widetilde{E}_i$. It remains to observe that $i\text{-res}_{r,s}^{r+1,s} \circ e_{r+1,s}$ and $e_{r,s} \circ E_i$ are the functors defined by tensoring with these bimodules. $\qquad\square$

Lemma 4.8. *There are canonical adjunctions making $(i\text{-ind}_{r,s}^{r+1,s}, i\text{-res}_{r,s}^{r+1,s})$ and $(i\text{-ind}_{r,s}^{r,s+1}, i\text{-res}_{r,s}^{r,s+1})$ into adjoint pairs of functors. In particular, the functors $i\text{-res}_{r,s}^{r+1,s}$ and $i\text{-res}_{r,s}^{r,s+1}$ are exact.*

Proof. We just construct the adjunction for $(i\text{-ind}_{r,s}^{r,s+1}, i\text{-res}_{r,s}^{r,s+1})$, since the argument for $(i\text{-ind}_{r,s}^{r+1,s}, i\text{-res}_{r,s}^{r+1,s})$ is similar. The restriction of the isomorphism from Lemma 3.4 gives a graded $(K_{r,s}(\delta), K_{r,s+1}(\delta))$-bimodule isomorphism

$$e_{r,s}K_{\Delta\Gamma}^{t}e_{r,s+1}\langle -\mathrm{caps}(t)\rangle \cong e_{r,s}\mathrm{Hom}_{K(\delta)}(K_{\Gamma\Delta}^{t*}\langle -\mathrm{caps}(t)\rangle, K(\delta))e_{r,s+1}$$

for each Δ, Γ and t as in (3.17). The bimodule on the right is identified with $\mathrm{Hom}_{K(\delta)}(K_{\Gamma\Delta}^{t^*}\langle -\mathrm{caps}(t)\rangle e_{r,s}, K(\delta)e_{r,s+1})$. Summing over all Δ, Γ and t, we get a graded bimodule isomorphism

$$e_{r,s}\widetilde{F}_i e_{r,s+1} \cong \mathrm{Hom}_{K(\delta)}(^*\widetilde{F}_i e_{r,s}, K(\delta)e_{r,s+1}).$$

By Lemma 4.6(2) and Lemma 4.1, the functor $e_{r,s+1}$ defines an isomorphism

$$\mathrm{Hom}_{K(\delta)}(^*\widetilde{F}_i e_{r,s}, K(\delta)e_{r,s+1}) \overset{\sim}{\to} \mathrm{Hom}_{K_{r,s+1}(\delta)}(e_{r,s+1}{}^*\widetilde{F}_i e_{r,s}, e_{r,s+1}K(\delta)e_{r,s+1}).$$

We have now constructed a graded $(K_{r,s}(\delta), K_{r,s+1}(\delta))$-bimodule isomorphism

$$e_{r,s}\widetilde{F}_i e_{r,s+1} \cong \mathrm{Hom}_{K_{r,s+1}(\delta)}(e_{r,s+1}{}^*\widetilde{F}_i e_{r,s}, K_{r,s+1}(\delta)).$$

Applying Lemma 4.6(2) once more, $e_{r,s+1}{}^*\widetilde{F}_i e_{r,s}$ is projective as a left $K_{r,s+1}(\delta)$-module, so we deduce that the functor $i\text{-res}_{r,s}^{r,s+1} = e_{r,s}\widetilde{F}_i e_{r,s+1}\otimes_{K_{r,s+1}(\delta)}?$ is isomorphic to the functor $\mathrm{Hom}_{K_{r,s+1}(\delta)}(e_{r,s+1}{}^*\widetilde{F}_i e_{r,s}, ?)$. The latter is canonically right adjoint to $i\text{-ind}_{r,s}^{r,s+1} = e_{r,s+1}{}^*\widetilde{F}_i e_{r,s}\otimes_{K_{r,s}(\delta)}?$. $\qquad \square$

Lemma 4.9. *For $r,s \geq 0$ such that $(r,s,\delta) \neq (0,0,0)$, the right exact functor*

$$G_{r,s} := K_{r+1,s+1}(\delta)e_{r,s}\otimes_{K_{r,s}(\delta)}? : K_{r,s}(\delta)\text{-mod}_{lf} \to K_{r+1,s+1}(\delta)\text{-mod}_{lf}$$

has the following properties:

   (1) $G_{r,s}P_{r,s}(\lambda) \cong P_{r+1,s+1}(\lambda)$ *for each $\lambda \in \dot{\Lambda}_{r,s}$;*
   (2) $G_{r,s}V_{r,s}(\lambda) \cong V_{r+1,s+1}(\lambda)$ *for each $\lambda \in \Lambda_{r,s}$.*

Proof. (1) Take $\lambda \in \dot{\Lambda}_{r,s}$ and recall from Theorem 4.4(3) that $P_{r,s}(\lambda) = K_{r,s}(\delta)e_\lambda$. Hence $G_{r,s}P_{r,s}(\lambda) = K_{r+1,s+1}(\delta)e_{r,s}\otimes_{K_{r,s}}P_{r,s}(\lambda) \cong K_{r+1,s+1}(\delta)e_\lambda = P_{r+1,s+1}(\lambda)$.
(2) We first show that $G_{r,s}V_{r,s}(\lambda) \cong V_{r+1,s+1}(\lambda)$ in the exceptional case $\delta = 0$, $r = s \neq 0$ and $\lambda = (\varnothing, \varnothing)$. Observe that $P(\lambda) = V(\lambda)$ by Theorem 3.1(2). Let $\mu = ((1),(1))$, $\nu' = ((2),(1^2))$ and $\nu'' = ((1^2),(2))$. Right multiplication by $(\underline{\mu}\lambda\overline{\lambda})$ defines a graded $K(\delta)$-module homomorphism $P(\mu)\langle 1\rangle \twoheadrightarrow \mathrm{rad}\, V(\lambda)$. It is easy to see that the kernel of this is generated by the vectors $(\underline{\nu'}\mu\overline{\mu})$ and $(\underline{\nu''}\mu\overline{\mu})$. Hence there is an exact sequence

$$(P(\nu') \oplus P(\nu''))\langle 2\rangle \to P(\mu)\langle 1\rangle \to \mathrm{rad}\, V(\lambda) \to 0. \qquad (4.10)$$

Multiplication defines a bimodule homorphism

$$e_{r+1,s+1}K(\delta)e_{r,s} \otimes_{K_{r,s}} e_{r,s}K(\delta) \to e_{r+1,s+1}K(\delta). \qquad (4.11)$$

Tensoring (4.11) and (4.10) together, noting $e_{r,s}(\mathrm{rad}\, V(\lambda)) = e_{r,s}V(\lambda)$ because $e_{r,s}L(\lambda) = \{0\}$, we get the following commutative diagram with exact rows:

$$(G_{r,s}P_{r,s}(\nu') \oplus G_{r,s}P_{r,s}(\nu''))\langle 2\rangle \longrightarrow G_{r,s}P_{r,s}(\mu)\langle 1\rangle \longrightarrow G_{r,s}V_{r,s}(\lambda) \to 0$$

$$\downarrow \qquad\qquad\qquad\qquad\qquad \downarrow \qquad\qquad\qquad \downarrow$$

$$(P_{r+1,s+1}(\nu') \oplus P_{r+1,s+1}(\nu''))\langle 2\rangle \longrightarrow P_{r+1,s+1}(\mu)\langle 1\rangle \longrightarrow V_{r+1,s+1}(\lambda) \to 0.$$

The two vertical arrows on the left are isomorphisms by the previous paragraph. Hence the vertical arrow on the right is an isomorphism too by the five lemma.

Finally for the general case, we may assume by the previous paragraph that $\lambda \in \dot{\Lambda}_{r,s}$. By Theorem 3.1(3)–(4), there is an exact sequence

$$\bigoplus_{\mu \in \Lambda_{r,s}} p^{(1)}_{\lambda,\mu} P(\mu)\langle 1 \rangle \to P(\lambda) \to V(\lambda) \to 0. \qquad (4.12)$$

Tensoring (4.11) and (4.12) together, we get a commutative diagram with exact rows:

$$
\begin{array}{ccccc}
\displaystyle\bigoplus_{\mu \in \Lambda_{r,s}} p^{(1)}_{\lambda,\mu} G_{r,s} P_{r,s}(\mu)\langle 1 \rangle & \longrightarrow & G_{r,s} P_{r,s}(\lambda) & \longrightarrow & G_{r,s} V_{r,s}(\lambda) \to 0 \\[2mm]
\downarrow & & \downarrow & & \downarrow \\[2mm]
\displaystyle\bigoplus_{\mu \in \Lambda_{r,s}} p^{(1)}_{\lambda,\mu} P_{r+1,s+1}(\mu)\langle 1 \rangle & \longrightarrow & P_{r+1,s+1}(\lambda) & \longrightarrow & V_{r+1,s+1}(\lambda) \to 0.
\end{array}
$$

The two vertical arrows on the left are isomorphisms either by (1) or by the previous paragraph in case $\mu = (\varnothing, \varnothing)$. Hence the arrow on the right is an isomorphism too. $\qquad\square$

The following theorem should be compared with Theorem 2.5.

Theorem 4.10. *The following hold for $\lambda \in \Lambda_{r,s}$ and $i \in \mathbb{Z}$.*
  (1) *The module $i\text{-res}^{r,s}_{r-1,s} V_{r,s}(\lambda)$ has a filtration with sections isomorphic to*
      $\{V_{r-1,s}(\mu)\langle \deg(\mu \xrightarrow{i} \lambda \rangle \mid \text{for all } \mu \in \Lambda_{r-1,s} \text{ such that } \mu \xrightarrow{i} \lambda\}.$
  (2) *The module $i\text{-res}^{r,s}_{r,s-1} V_{r,s}(\lambda)$ has a filtration with sections isomorphic to*
      $\{V_{r,s-1}(\mu)\langle \deg(\mu \xleftarrow{i} \lambda)\rangle \mid \text{for all } \mu \in \Lambda_{r,s-1} \text{ such that } \mu \xleftarrow{i} \lambda\}.$
  (3) *The module $i\text{-ind}^{r+1,s}_{r,s} V_{r,s}(\lambda)$ has a filtration with sections isomorphic to*
      $\{V_{r+1,s}(\mu)\langle \deg(\lambda \xrightarrow{i} \mu)\rangle \mid \text{for all } \mu \in \Lambda_{r+1,s} \text{ such that } \lambda \xrightarrow{i} \mu\}.$
  (4) *The module $i\text{-ind}^{r,s+1}_{r,s} V_{r,s}(\lambda)$ has a filtration with sections isomorphic to*
      $\{V_{r,s+1}(\mu)\langle \deg(\lambda \xleftarrow{i} \mu)\rangle \mid \text{for all } \mu \in \Lambda_{r,s+1} \text{ such that } \lambda \xleftarrow{i} \mu\}.$
In all cases, the filtration is of length at most two; when it is of length exactly two the module indexed by the biggest weight in the Bruhat ordering $\geq$ appears as a submodule and the other is a quotient.

Proof. We just explain in the first and last cases. For the first case, we have by Lemma 4.7 that $i\text{-res}^{r,s}_{r-1,s} V_{r,s}(\lambda) = i\text{-res}^{r,s}_{r-1,s}(e_{r,s} V(\lambda)) \cong e_{r-1,s}(E_i V(\lambda))$. Now apply Theorem 3.7(1). Instead consider $i\text{-ind}^{r,s+1}_{r,s} V_{r,s}(\lambda)$. The case $(r,s,\delta) = (0,0,0)$ can be checked directly using the observation that $i\text{-ind}^{r,s+1}_{r,s} V_{r,s}(\lambda) = e_{0,1} {}^{*}\widetilde{F}_i e_{0,0}$. Assuming $(r,s,\delta) \neq (0,0,0)$, we can use Lemma 4.9 to get that

$$i\text{-ind}^{r,s+1}_{r,s} V_{r,s}(\lambda) = e_{r,s+1} {}^{*}\widetilde{F}_i e_{r,s} \otimes_{K_{r,s}(\delta)} V_{r,s}(\lambda)$$
$$\cong e_{r,s+1} {}^{*}\widetilde{F}_i e_{r+1,s+1} \otimes_{K_{r+1,s+1}(\delta)} K_{r+1,s+1}(\delta) e_{r,s} \otimes_{K_{r,s}(\delta)} V_{r,s}(\lambda)$$
$$\cong e_{r,s+1} {}^{*}\widetilde{F}_i e_{r+1,s+1} \otimes_{K_{r+1,s+1}(\delta)} V_{r+1,s+1}(\lambda).$$

Up to a grading shift, this is the same as $i\text{-res}^{r+1,s+1}_{r,s+1} V_{r+1,s+1}(\lambda)$, which has the appropriate filtration by the first case. $\qquad\square$

5. Diagram bases for special endomorphism algebras

We return once again to the general setting of §3, so K is the universal arc algebra. We are going to construct diagram bases for the endomorphism algebras of the modules obtained by applying sequences of special projective functors either to projective indecomposable modules or to irreducible modules.

Good homomorphisms. Suppose we are given blocks Δ, Π and Γ. Let s be a $\Delta\Pi$-matching and t be a $\Delta\Gamma$-matching. Also fix $\lambda \in \Pi$ and $\mu \in \Gamma$. In this subsection, we construct bases for $\operatorname{Hom}_K(G^s_{\Delta\Pi}P(\lambda), G^t_{\Delta\Gamma}P(\mu))$ and $\operatorname{Hom}_K(G^s_{\Delta\Pi}L(\lambda), G^t_{\Delta\Gamma}L(\mu))$.

Given $\alpha \overset{s^*}{-} \beta \overset{t}{-} \gamma$ such that $\lambda \subset \alpha$ and $\gamma \supset \mu$, define a homomorphism

$$f_{\alpha\beta\gamma} : G^s_{\Delta\Pi}P(\lambda) \to G^t_{\Delta\Gamma}P(\mu) \tag{5.1}$$

by declaring for $(\underline{\nu}\omega s \kappa \overline{\lambda}) \in G^s_{\Delta\Pi}P(\lambda)$ that $f_{\alpha\beta\gamma}(\underline{\nu}\omega s \kappa \overline{\lambda})$ is the (possibly zero) sum of basis vectors of $G^t_{\Delta\Gamma}$ obtained by drawing the diagram $\underline{\lambda}\alpha s^*\beta t\gamma\overline{\mu}$ on top of $\underline{\nu}\omega s \kappa \overline{\lambda}$, then applying the following *extended surgery procedure* to contract the $\omega s \kappa \overline{\lambda}|\underline{\lambda}\alpha s^*\beta$-part of the diagram.

(1) First consider all the mirror image pairs of closed circles from the diagrams $\omega s \kappa \overline{\lambda}$ and $\underline{\lambda}\alpha s^*\beta$. If the circles from at least one of these pairs are oriented in the same way (both anti-clockwise or both clockwise), the extended surgery procedure gives 0.

(2) Assuming all the mirror image pairs of circles are oppositely oriented, we replace $s\kappa\overline{\lambda}$ by $\operatorname{red}(s\overline{\lambda})$ and $\underline{\lambda}\alpha s^*$ by $\operatorname{red}(\underline{\lambda}s^*)$, join corresponding pairs of rays, then apply the usual surgery procedure to contract the symmetric $\operatorname{red}(s\overline{\lambda})\operatorname{red}(\underline{\lambda}s^*)$-section of the resulting diagram.

Theorem 5.1. *The homomorphisms*

$$\left\{ f_{\alpha\beta\gamma} \mid \text{for all } \alpha \overset{s^*}{-} \beta \overset{t}{-} \gamma \text{ such that } \lambda \subset \alpha \text{ and } \gamma \supset \mu \right\} \tag{5.2}$$

give a basis for $\operatorname{Hom}_K(G^s_{\Delta\Pi}P(\lambda), G^t_{\Delta\Gamma}P(\mu))$. *Moreover* $f_{\alpha\beta\gamma}$ *is of degree*

$$\deg(\underline{\lambda}\alpha s^*\beta t\gamma\overline{\mu}) + \operatorname{caps}(s) - 2\operatorname{cups}(s) - \operatorname{caps}(t).$$

Proof. Using (3.9), Lemma 3.3 and the canonical adjunction, we get graded vector space isomorphisms

$$\operatorname{Hom}_K(G^s_{\Delta\Pi}P(\lambda), G^t_{\Delta\Gamma}P(\mu))$$
$$\cong \operatorname{Hom}_K({}^*G^{s^*}_{\Pi\Delta}P(\lambda), G^t_{\Delta\Gamma}P(\mu))\langle\operatorname{caps}(s) - \operatorname{cups}(s)\rangle$$
$$\cong \operatorname{Hom}_K(P(\lambda), G^{s^*}_{\Pi\Delta}G^t_{\Delta\Gamma}P(\mu))\langle\operatorname{caps}(s) - \operatorname{cups}(s)\rangle$$
$$\cong \operatorname{Hom}_K(Ke_\lambda, K^{s^*t}_{\Pi\Delta\Gamma}e_\mu)\langle\operatorname{caps}(s) - \operatorname{cups}(s) - \operatorname{caps}(s^*) - \operatorname{caps}(t)\rangle$$
$$\cong e_\lambda K^{s^*t}_{\Pi\Delta\Gamma}e_\mu\langle\operatorname{caps}(s) - 2\operatorname{cups}(s) - \operatorname{caps}(t)\rangle.$$

The space $e_\lambda K^{s^*t}_{\Pi\Delta\Gamma}e_\mu$ has basis given by all $(\underline{\lambda}\alpha s^*\beta t\gamma\overline{\mu})$ for $\alpha \overset{s^*}{-} \beta \overset{t}{-} \gamma$ with $\lambda \subset \alpha$ and $\gamma \supset \mu$. It remains to trace these basis vectors through the explicit definitions of above isomorphisms to see that they correspond exactly to the homomorphisms $f_{\alpha\beta\gamma}$. □

Remark 5.2. One can also work with the adjoint projective functors from (3.8). In those terms, Theorem 5.1 shows that the homomorphisms (5.2) give a basis for $\mathrm{Hom}_K({}^*G^{s^*}_{\Pi\Delta}P(\lambda), {}^*G^{t^*}_{\Gamma\Delta}P(\mu))$. Moreover now $f_{\alpha\beta\gamma}$ is of degree $\deg(\underline{\lambda}\alpha s^*\beta t\gamma\overline{\mu}) - \mathrm{caps}(s^*) - \mathrm{cups}(t)$; this follows from (3.9).

A *good homomorphism* is a map $f : G^s_{\Delta\Pi}P(\lambda) \to G^t_{\Delta\Gamma}P(\mu)$ that factors through the canonical quotient maps to induce $\bar{f} : G^s_{\Delta\Pi}L(\lambda) \to G^t_{\Delta\Gamma}L(\mu)$ making the following diagram commute:

$$
\begin{array}{ccc}
G^s_{\Delta\Pi}P(\lambda) & \xrightarrow{\ f\ } & G^t_{\Delta\Gamma}P(\mu) \\
{\scriptstyle G^s_{\Delta\Pi}(p_\lambda)}\big\downarrow & & \big\downarrow{\scriptstyle G^t_{\Delta\Gamma}(p_\mu)} \\
G^s_{\Delta\Pi}L(\lambda) & \xrightarrow{\ \bar{f}\ } & G^t_{\Delta\Gamma}L(\mu).
\end{array}
$$

Theorem 5.3. *Let* $u := \mathrm{red}(s^*t)$. *The space* $\mathrm{Hom}_K(G^s_{\Delta\Pi}L(\lambda), G^t_{\Delta\Gamma}L(\mu))$ *is non-zero if and only if* $\lambda \overset{u}{-} \mu$ *and* $\deg(\lambda u\mu) = 0$. *Assuming this condition holds, let* λ' *be obtained from* λ *by reversing the orientation of all the caps in* $\lambda u\mu$.

(1) *The homomorphisms*

$$
\left\{ f_{\alpha\beta\gamma} \;\middle|\; \begin{array}{l} \text{for all } \alpha \overset{s^*}{-} \beta \overset{t}{-} \gamma \text{ such that } \lambda \subset \alpha, \; \gamma \supset \mu, \\ \text{and either } \alpha = \lambda' \text{ or } \gamma \neq \mu \end{array} \right\}
$$

give a basis for the subspace of $\mathrm{Hom}_K(G^s_{\Delta\Pi}P(\lambda), G^t_{\Delta\Gamma}P(\mu))$ *consisting of all good homomorphisms.*

(2) *The induced homomorphisms* $\bar{f}_{\alpha\beta\gamma} : G^s_{\Delta\Pi}L(\lambda) \to G^t_{\Delta\Gamma}L(\mu)$ *are zero for all* $\alpha \overset{s^*}{-} \beta \overset{t}{-} \gamma$ *such that* $\lambda \subset \alpha, \gamma \supset \mu$ *and* $\gamma \neq \mu$.

(3) *The induced homomorphisms* $\bar{f}_{\lambda'\beta\mu} : G^s_{\Delta\Pi}L(\lambda) \to G^t_{\Delta\Gamma}L(\mu)$ *for all* β *such that* $\lambda' \overset{s^*}{-} \beta \overset{t}{-} \mu$ *give a basis for* $\mathrm{Hom}_K(G^s_{\Delta\Pi}L(\lambda), G^t_{\Delta\Gamma}L(\mu))$.

Proof. We will ignore grading shifts throughout the proof. Also we'll often use Lemma 3.3 without explicit reference to identify the composite functor $G^{s^*}_{\Pi\Delta} \circ G^t_{\Delta\Gamma}$ with the functor $K^{s^*t}_{\Pi\Delta\Gamma}\otimes_K ?$ hence with a direct sum of copies of $G^u_{\Pi\Gamma}$. This allows us to apply Theorem 3.5 and Corollary 3.6 to $G^{s^*}_{\Pi\Delta}G^t_{\Delta\Gamma}L(\mu)$.

For the opening statement, $\mathrm{Hom}_K(G^s_{\Delta\Pi}L(\lambda), G^t_{\Delta\Gamma}L(\mu)) \neq \{0\}$ if and only if $L(\lambda)$ appears in the socle of $G^{s^*}_{\Pi\Delta}G^t_{\Delta\Gamma}L(\mu)$. Now write $G^{s^*}_{\Pi\Delta}G^t_{\Delta\Gamma}L(\mu)$ as a direct sum of copies of $G^u_{\Pi\Gamma}L(\mu)$ and apply the last part of Theorem 3.5.

For the rest of the proof, we assume that $\lambda \overset{u}{-} \mu$ and $\deg(\lambda u\mu) = 0$. Let λ' be defined as in the statement of the theorem. Note the assumption on λ implies automatically that $\lambda \subset \lambda'$ and $\mathrm{red}(\underline{\lambda}u) = \underline{\mu}$. Let $Q := e_\lambda K^{s^*t}_{\Pi\Delta\Gamma}e_\mu$ and R, S be the subspaces spanned by the vectors

$$
\left\{ (\underline{\lambda}\lambda' s^*\beta t\mu\overline{\mu}) \;\middle|\; \text{for all } \beta \text{ such that } \lambda' \overset{s^*}{-} \beta \overset{t}{-} \mu \right\}, \tag{5.3}
$$

$$
\left\{ (\underline{\lambda}\alpha s^*\beta t\gamma\overline{\mu}) \;\middle|\; \text{for all } \alpha \overset{s^*}{-} \beta \overset{t}{-} \gamma \text{ such that } \lambda \subset \alpha, \; \gamma \supset \mu \text{ and } \gamma \neq \mu \right\}, \tag{5.4}
$$

respectively. Let $p : Q \twoheadrightarrow Q/S$ be the quotient and $i : (R+S)/S \hookrightarrow Q/S$ be the inclusion. Adopting the shorthand $\mathrm{H} := \mathrm{Hom}_K$, let q, r, j and k be the other

vertical maps indicated in the following diagram:

$$\begin{array}{ccccc}
\mathrm{H}(G^s_{\Delta\Pi}P(\lambda), G^t_{\Delta\Gamma}P(\mu)) & \xrightarrow[\mathrm{adj}]{\sim} & \mathrm{H}(P(\lambda), G^{s^*}_{\Pi\Delta}G^t_{\Delta\Gamma}P(\mu)) & \xrightarrow[a]{\sim} & Q \\
\downarrow{\scriptstyle r:f\mapsto G^t_{\Delta\Gamma}(p_\mu)\circ f} & & \downarrow{\scriptstyle q:f\mapsto G^{s^*}_{\Pi\Delta}G^t_{\Delta\Gamma}(p_\mu)\circ f} & & \downarrow{\scriptstyle p} \\
\mathrm{H}(G^s_{\Delta\Pi}P(\lambda), G^t_{\Delta\Gamma}L(\mu)) & \xrightarrow[\mathrm{adj}]{\sim} & \mathrm{H}(P(\lambda), G^{s^*}_{\Pi\Delta}G^t_{\Delta\Gamma}L(\mu)) & \xrightarrow[b]{\sim} & Q/S \\
\uparrow{\scriptstyle k:f\mapsto f\circ G^s_{\Delta\Pi}(p_\lambda)} & & \uparrow{\scriptstyle j:f\mapsto f\circ p_\lambda} & & \uparrow{\scriptstyle i} \\
\mathrm{H}(G^s_{\Delta\Pi}L(\lambda), G^t_{\Delta\Gamma}L(\mu)) & \xrightarrow[\mathrm{adj}]{\sim} & \mathrm{H}(L(\lambda), G^{s^*}_{\Pi\Delta}G^t_{\Delta\Gamma}L(\mu)) & \xrightarrow[c]{\sim} & (R+S)/S.
\end{array}$$

Horizontal isomorphisms making the diagram commute are defined as follows.

adj: The three maps adj are the isomorphisms coming from the canonical adjunction. The two squares on the left commute by naturality.

a: We can identify $G^{s^*}_{\Pi\Delta}G^t_{\Delta\Gamma}P(\mu)$ with $K^{s^*t}_{\Pi\Delta\Gamma}e_\mu$. Then evaluation at e_λ gives the isomorphism a.

b: The vertical map q in the diagram is surjective by projectivity. Its kernel is the set of all homomorphisms $P(\lambda) \to G^{s^*}_{\Pi\Delta}G^t_{\Delta\Gamma}P(\mu)$ which map e_λ to $\ker G^{s^*}_{\Pi\Delta}G^t_{\Delta\Gamma}(p_\mu)$. The first statement of Theorem 3.5 explains how to identify $\ker G^{s^*}_{\Pi\Delta}G^t_{\Delta\Gamma}(p_\mu)$ with an explicit subspace of $K^{s^*t}_{\Pi\Delta\Gamma}e_\mu$. It follows that $a(\ker q) = S$. Hence the isomorphism a factors uniquely to induce the isomorphism b making the top right square commute.

c: The vertical map j in the diagram is obviously injective. Its image is the set of all homomorphisms $P(\lambda) \to G^{s^*}_{\Pi\Delta}G^t_{\Delta\Gamma}L(\mu)$ which map e_λ to $\mathrm{soc}\, G^{s^*}_{\Pi\Delta}G^t_{\Delta\Gamma}L(\mu)$. Using the explicit description of this socle from Corollary 3.6, it follows that $b(\mathrm{im}\, j) = (R + S)/S$. Hence the isomorphism b restricts to an isomorphism c making the bottom right square commute.

Recall also from the proof of Theorem 5.1 that the composite $a \circ \mathrm{adj}$ of the maps at the top of our commuting diagram sends $f_{\alpha\beta\gamma}$ to $(\underline{\lambda}\alpha s^*\beta t\gamma\overline{\mu})$.

To complete the proof of the theorem, we have that $p^{-1}(\mathrm{im}\, i) = R+S$. By definition, the space of good homomorphisms is $r^{-1}(\mathrm{im}\, k)$. Hence by the commutativity of the diagram, the space of good homomorphisms is $(a \circ \mathrm{adj})^{-1}(R + S)$. The union of the vectors (5.3)–(5.4) gives a basis for $R + S$, and applying $(a \circ \mathrm{adj})^{-1}$ gives the basis in (1). To deduce (2), we need to show that $r(f_{\alpha\beta\gamma}) = 0$ for all $\alpha \overset{s^*}{-} \beta \overset{t}{-} \gamma$ such that $\lambda \subset \alpha, \gamma \supset \mu$ and $\gamma \neq \mu$. This follows because

$$(b \circ \mathrm{adj} \circ r)(f_{\alpha\beta\gamma}) = (p \circ a \circ \mathrm{adj})(f_{\alpha\beta\gamma}) = (\underline{\lambda}\alpha s^*\beta t\gamma\overline{\mu}) + S = 0$$

by the definition of S. Finally for (3), the image of the vectors (5.3) gives a basis for $(R + S)/S$. Now apply $(c \circ \mathrm{adj})^{-1}$ and use the commutativity of the diagram once again. $\qquad\square$

Remark 5.4. Suppose we are given $\alpha \overset{s^*}{-} \beta \overset{t}{-} \gamma$ such that $\lambda \subset \alpha$ and $\gamma \supset \mu$. Then $f_{\alpha\beta\gamma}$ is a good homomorphism inducing a non-zero $\bar{f}_{\alpha\beta\gamma}$ as in Theorem 5.3(3) if and only if $\underline{\lambda}\alpha s^*\beta t\gamma\overline{\mu}$ satisfies the following properties.

(1) Each connected component in the diagram crosses the top number line at most twice. If it crosses the top number line exactly twice then it is an anti-clockwise circle.

(2) Each connected component crosses the bottom number line at most twice. If it crosses the bottom number line exactly twice and does not cross the top number line then it is a clockwise circle.

This follows because the properties (1) and (2) are equivalent to the assertions $\lambda \overset{u}{-} \mu$, $\deg(\lambda u \mu) = 0$, $\gamma = \mu$ and $\alpha = \lambda'$, where u and λ' are as in Theorem 5.3.

Assume finally that we are given another block Ω, a $\Delta\Omega$-matching u, and $\nu \in \Omega$. We want to explain an algorithm to compose the homomorphisms

$$f_{\alpha\beta\gamma} : G^s_{\Delta\Pi}P(\lambda) \to G^t_{\Delta\Gamma}P(\mu) \qquad f_{\delta\epsilon\varphi} : G^t_{\Delta\Gamma}P(\mu) \to G^u_{\Delta\Omega}P(\nu) \qquad (5.5)$$

for $\lambda \subset \alpha \overset{s^*}{-} \beta \overset{t}{-} \gamma \supset \mu \subset \delta \overset{t^*}{-} \epsilon \overset{u}{-} \varphi \supset \nu$. To do this, draw the diagram $\underline{\lambda}\alpha s^*\beta t\gamma\overline{\mu}$ underneath the diagram $\underline{\mu}\delta t^*\epsilon u\varphi\overline{\nu}$, then apply the extended surgery procedure as above to contract the $\overline{\beta t\gamma\overline{\mu}|\underline{\mu}\delta t^*\epsilon}$-part of the diagram. This produces a (possibly zero) sum of diagrams of the form $\underline{\lambda}\rho s^*\sigma u\tau\overline{\nu}$. The composition $f_{\delta\epsilon\varphi} \circ f_{\alpha\beta\gamma}$ is the sum of the corresponding $f_{\rho\sigma\tau} \in \operatorname{Hom}_K(G^s_{\Delta\Pi}P(\lambda), G^u_{\Delta\Omega}P(\nu))$.

If $f_{\alpha\beta\gamma}$ and $f_{\delta\epsilon\varphi}$ are both good homomorphisms with $\bar{f}_{\alpha\beta\gamma} \neq 0 \neq \bar{f}_{\delta\epsilon\varphi}$, so that the induced maps $\bar{f}_{\alpha\beta\gamma}$ and $\bar{f}_{\delta\epsilon\varphi}$ are basis vectors like in Theorem 5.3(3), then we can modify the algorithm in the previous paragraph to compute $\bar{f}_{\delta\epsilon\varphi} \circ \bar{f}_{\alpha\beta\gamma}$ too. We may as well assume for this that $\operatorname{Hom}_K(G^s_{\Delta\Pi}L(\lambda), G^u_{\Delta\Omega}L(\nu)) \neq \{0\}$. First write $f_{\delta\epsilon\varphi} \circ f_{\alpha\beta\gamma}$ as a sum of maps $f_{\rho\sigma\tau} \in \operatorname{Hom}_K(G^s_{\Delta\Pi}P(\lambda), G^u_{\Delta\Omega}P(\nu))$. Since the composition of two good homomorphisms is automatically good, Theorem 5.3(1) implies that the resulting sum only involves $f_{\rho\sigma\tau}$'s that are themselves good homomorphisms. Then descend to $\operatorname{Hom}_K(G^s_{\Delta\Pi}L(\lambda), G^u_{\Delta\Omega}L(\nu))$ by replacing each $f_{\rho\sigma\tau}$ in the sum in which $\tau = \nu$ by $\bar{f}_{\rho\sigma\tau}$, and discarding all other $f_{\rho\sigma\tau}$ since these induce zero by Theorem 5.3(2). This produces the required linear combination of basis vectors.

Special endomorphism algebras. Recall from the introduction that $\mathscr{R}_{r,s}$ denotes the set of all sequences consisting of r E's and s F's shuffled together in some way. For $R = R^{(1)}R^{(2)} \cdots R^{(r+s)} \in \mathscr{R}_{r,s}$, let ${}^*R := {}^*R^{(r+s)} \cdots {}^*R^{(2)*}R^{(1)}$, which is a sequence consisting of r ${}^*E$'s and s ${}^*F$'s. We interpret R and ${}^*R$ as compositions of the functors

$$E := \bigoplus_{i\in\mathbb{Z}} E_i, \qquad F := \bigoplus_{i\in\mathbb{Z}} F_i, \qquad {}^*E := \bigoplus_{i\in\mathbb{Z}} {}^*E_i, \qquad {}^*F := \bigoplus_{i\in\mathbb{Z}} {}^*F_i. \qquad (5.6)$$

So R and ${}^*R$ are both endofunctors of $K\text{-mod}_{lf}$, and there is a canonical adjunction making $({}^*R, R)$ into an adjoint pair. Moreover we have the natural decompositions

$$R = \bigoplus_{\boldsymbol{i}\in\mathbb{Z}^{r+s}} R_{\boldsymbol{i}}, \qquad {}^*R = \bigoplus_{\boldsymbol{i}\in\mathbb{Z}^{r+s}} {}^*R_{\boldsymbol{i}} \qquad (5.7)$$

where $R_{\boldsymbol{i}} := R^{(1)}_{i_1} R^{(2)}_{i_2} \cdots R^{(r+s)}_{i_{r+s}}$ and ${}^*R_{\boldsymbol{i}} := {}^*R^{(r+s)}_{i_{r+s}} \cdots {}^*R^{(2)}_{i_2}{}^*R^{(1)}_{i_1}$.

Given $M \in K\text{-mod}_{lf}$ and $R \in \mathscr{R}_{r,s}$, we can consider the endomorphism algebra $\operatorname{End}_K({}^*R\,M)^{\mathrm{op}}$. However this might not have finite (or even countable)

dimension in general, so instead we work with the subalgebra

$$\operatorname{End}_K^{fin}(^*R\,M)^{\mathrm{op}} := \bigoplus_{\boldsymbol{i},\boldsymbol{j}\in\mathbb{Z}^{r+s}} \operatorname{Hom}_K(^*R_{\boldsymbol{i}}M, {}^*R_{\boldsymbol{j}}M) \qquad (5.8)$$

of $\operatorname{End}_K(^*R\,M)^{\mathrm{op}}$ consisting of *locally finite endomorphisms* of $^*R\,M$. In this subalgebra, we have an obvious system $\{e(\boldsymbol{i})\,|\,\boldsymbol{i}\in\mathbb{Z}^{r+s}\}$ of mutually orthogonal idempotents such that

$$e(\boldsymbol{i})\operatorname{End}_K^{fin}(^*R\,M)^{\mathrm{op}}e(\boldsymbol{j}) = \operatorname{Hom}_K(^*R_{\boldsymbol{i}}M, {}^*R_{\boldsymbol{j}}M). \qquad (5.9)$$

In all the applications we will meet later in the article, $\operatorname{End}_K^{fin}(^*R\,M)^{\mathrm{op}}$ will actually be finite dimensional. In that case, we have simply that $\operatorname{End}_K^{fin}(^*R\,M)^{\mathrm{op}} = \operatorname{End}_K(^*R\,M)^{\mathrm{op}}$, all but finitely many of the $e(\boldsymbol{i})$'s are zero, and they sum to 1.

We have that $^*(RE) = {}^*E^*R$ and $^*(RF) = {}^*F^*R$. The functors $^*E_i$ and $^*F_i$ define algebra homomorphisms

$$\iota_{R;i}^{RE} : \operatorname{End}_K^{fin}(^*R\,M)^{\mathrm{op}} \to \operatorname{End}_K^{fin}(^*(RE)M)^{\mathrm{op}}, \qquad f \mapsto {}^*E_i(f), \qquad (5.10)$$

$$\iota_{R;i}^{RF} : \operatorname{End}_K^{fin}(^*R\,M)^{\mathrm{op}} \to \operatorname{End}_K^{fin}(^*(RF)M)^{\mathrm{op}}, \qquad f \mapsto {}^*F_i(f). \qquad (5.11)$$

These maps send the idempotent $e(\boldsymbol{i})$ to $e(\boldsymbol{i}i)$ where $\boldsymbol{i}i$ denotes $(i_1,\dots,i_{r+s},i)$. In particular if $\operatorname{End}_K^{fin}(^*R\,M)$ is finite dimensional, then $\iota_{R;i}^{RE}$ and $\iota_{R;i}^{RF}$ map 1 to the idempotents

$$1_{R;i}^{RE} := \sum_{\substack{\boldsymbol{i}\in\mathbb{Z}^{r+s+1}\\ i_{r+s+1}=i}} e(\boldsymbol{i}) \in \operatorname{End}_K^{fin}(^*(RE)M)^{\mathrm{op}}, \qquad (5.12)$$

$$1_{R;i}^{RF} := \sum_{\substack{\boldsymbol{i}\in\mathbb{Z}^{r+s+1}\\ i_{r+s+1}=i}} e(\boldsymbol{i}) \in \operatorname{End}_K^{fin}(^*(RF)M)^{\mathrm{op}}, \qquad (5.13)$$

respectively.

Generalised tableaux. Fix $R = R^{(1)}R^{(2)}\cdots R^{(r+s)} \in \mathscr{R}_{r,s}$ and a weight diagram η. In the rest of the section, we are going to construct bases for the algebras $\operatorname{End}_K^{fin}(^*R\,P(\eta))^{\mathrm{op}}$ and $\operatorname{End}_K^{fin}(^*R\,L(\eta))^{\mathrm{op}}$. We must first develop some combinatorics of paths in the labelled directed graph from (3.18)–(3.19).

An *R-tableau* of *type* η is a chain $\mathtt{T} = (\lambda^{(0)}, \lambda^{(1)}, \dots, \lambda^{(r+s)})$ of weight diagrams such that $\eta \subset \lambda^{(0)}$ and the following hold for each $a = 1,\dots,r+s$.

   (1) If $R^{(a)} = E$ then $\lambda^{(a-1)} \xrightarrow{i_a} \lambda^{(a)}$ for some $i_a \in \mathbb{Z}$.

   (2) If $R^{(a)} = F$ then $\lambda^{(a-1)} \xleftarrow{j_a} \lambda^{(a)}$ for some $i_a \in \mathbb{Z}$.

Let $\mathscr{T}_R(\eta)$ denote the set of all R-tableaux of type η. The *shape* $\mathrm{sh}(\mathtt{T})$ of $\mathtt{T}$ is the final weight diagram $\lambda^{(r+s)}$ in the chain, and the *content* is the tuple $\boldsymbol{i}^{\mathtt{T}} = (i_1,\dots,i_{r+s}) \in \mathbb{Z}^{r+s}$ arising from the labels on the edges in (1)–(2). For

example if

$$\eta =$$

$$\lambda^{(0)} =$$

$$\lambda^{(1)} =$$

$$\lambda^{(2)} =$$

$$\lambda^{(3)} =$$

$$\lambda^{(4)} =$$
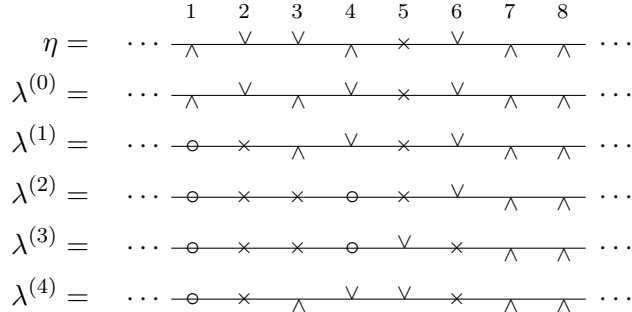
(and all other vertices are labelled $\circ$) then $\mathtt{T} = (\lambda^{(0)}, \lambda^{(1)}, \lambda^{(2)}, \lambda^{(3)}, \lambda^{(4)})$ is an EFE^2-tableau of type η and content $(1, 3, 5, 3)$.

We need several other combinatorial notions related to R-tableaux, all of which depend implicitly on η. Suppose $\mathtt{T} = (\lambda^{(0)}, \dots, \lambda^{(r+s)}) \in \mathscr{T}_R(\eta)$. First we associate two more diagrams denoted $\underline{\mathtt{T}}$ and $\overline{\mathtt{T}}$. The former is defined as follows.

(1) Draw the weight diagrams of the bipartitions $\lambda^{(0)}, \dots, \lambda^{(r+s)}$ in order *from bottom to top*, leaving some vertical space between them.
(2) For each $a = 1, \dots, r + s$, connect the weight diagrams $\lambda^{(a-1)}$ and $\lambda^{(a)}$ together by inserting into the space between them the unique crossingless matching from (3.15) that is consistent with their labels.
(3) Finally glue $\underline{\eta}$ onto the bottom of the diagram.

The definition of $\overline{\mathtt{T}}$ is similar: draw the bipartitions $\lambda^{(0)}, \dots, \lambda^{(r+s)}$ in order *from top to bottom*, connect pairs of weight diagrams using the unique crossingless matching from (3.15) that fits, then glue $\overline{\eta}$ onto the top of the diagram. For the above example, these diagrams are as follows:

$$\underline{\mathtt{T}} = \qquad\qquad \overline{\mathtt{T}} =$$
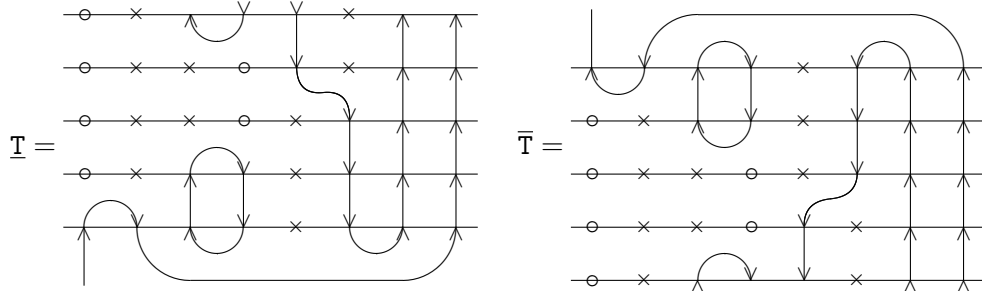
We let $\mathrm{red}(\underline{\mathtt{T}})$ be the labelled cup diagram obtained from $\underline{\mathtt{T}}$ by removing all of the number lines except for the top one, together with all connected components that do not cross the top number line. In the running example, we have that

$$\mathrm{red}(\underline{\mathtt{T}}) =$$

Similarly we define $\mathrm{red}(\overline{\mathtt{T}})$, which is a labelled cap diagram. Finally the *degree* of $\mathtt{T}$ is

$$\deg(\mathtt{T}) := \deg(\underline{\mathtt{T}}) - \mathrm{caps}(\underline{\mathtt{T}}) = \deg(\overline{\mathtt{T}}) - \mathrm{cups}(\overline{\mathtt{T}}), \qquad (5.14)$$

where $\mathrm{caps}(\underline{\mathtt{T}})$ and $\mathrm{cups}(\overline{\mathtt{T}})$ are the total numbers of caps and cups in the diagrams, respectively. For the running example, $\deg(\mathtt{T}) = 2$.

Remark 5.5. Using (3.18)–(3.19), one can check that

$$\deg(\mathtt{T}) = \deg(\underline{\eta}\lambda^{(0)}) + \sum_{\substack{1 \leq a \leq r+s \\ R^{(a)} = E}} \deg(\lambda^{(a-1)} \overset{i_a}{\rightarrow} \lambda^{(a)}) + \sum_{\substack{1 \leq a \leq r+s \\ R^{(a)} = F}} \deg(\lambda^{(a-1)} \overset{i_a}{\leftarrow} \lambda^{(a)})$$

for $\mathtt{T} \in \mathscr{T}_R(\eta)$ with $\boldsymbol{i}^{\mathtt{T}} = (i_1, \dots, i_{r+s})$.

Diagram bases for $\mathbf{End}_K^{fin}(^*{}_R P(\eta))^{\mathrm{op}}$ ***and*** $\mathbf{End}_K^{fin}(^*{}_R L(\eta))^{\mathrm{op}}$***.*** Continue with R and η fixed as in the previous subsection.

Lemma 5.6. *The left K-module $^*{}_R P(\eta)$ has a distinguished basis*

$$\{(\underline{\nu}\overline{\mathtt{U}}) \mid \text{for all } \mathtt{U} \in \mathscr{T}_R(\eta) \text{ and } \nu \subset \mathrm{sh}(\mathtt{U})\}. \tag{5.15}$$

The action of $(\underline{\lambda}\alpha\overline{\mu}) \in K$ on $(\underline{\nu}\overline{\mathtt{U}}) \in {}^*{}_R P(\eta)$ is as follows. It acts as zero unless $\mu = \nu$. Assuming $\mu = \nu$, draw $\underline{\lambda}\alpha\overline{\mu}$ underneath the diagram $\underline{\nu}\overline{\mathtt{U}}$, connect corresponding pairs of rays, then apply the surgery procedure to contract the symmetric $\overline{\mu}\underline{\nu}$-section of the diagram.

Proof. This is constructed in the same way as the basis (3.10), using the iterated version of the first isomorphism in Lemma 3.3 (see [BS2, Theorem 3.5(iii)]). □

Given $\mathtt{S}, \mathtt{T} \in \mathscr{T}_R(\eta)$ with $\mathrm{sh}(\mathtt{S}) = \mathrm{sh}(\mathtt{T})$, we can concatenate to obtain the composite diagram $\underline{\mathtt{S}}\overline{\mathtt{T}}$. We associate the homomorphism

$$f_{\mathtt{ST}} : {}^*{}_R P(\eta) \to {}^*{}_R P(\eta) \tag{5.16}$$

defined on a basis vector $(\underline{\nu}\overline{\mathtt{U}})$ from (5.15) as follows. If $\boldsymbol{i}^{\mathtt{U}} \neq \boldsymbol{i}^{\mathtt{S}}$ then we set $f_{\mathtt{ST}}(\underline{\nu}\overline{\mathtt{U}}) := 0$. If $\boldsymbol{i}^{\mathtt{U}} = \boldsymbol{i}^{\mathtt{S}}$, the diagrams $\overline{\mathtt{U}}$ and $\underline{\mathtt{S}}$ are mirror images of each other (but the labels on their number lines can be different). We compute $f_{\mathtt{ST}}(\underline{\nu}\overline{\mathtt{U}})$ by drawing $\underline{\mathtt{S}}\overline{\mathtt{T}}$ on top of $\underline{\nu}\overline{\mathtt{U}}$, then applying the following *extended surgery procedure* to contract the $\overline{\mathtt{U}}|\underline{\mathtt{S}}$-part of the diagram.

    (1) First consider all the mirror image pairs of closed circles from the diagrams $\overline{\mathtt{U}}$ and $\underline{\mathtt{S}}$. If the circles from at least one of these pairs are oriented in the same way (both anti-clockwise or both clockwise), the extended surgery procedure gives 0.

    (2) Assuming all the mirror image pairs of circles are oppositely oriented, we replace $\overline{\mathtt{U}}$ by $\mathrm{red}(\overline{\mathtt{U}})$ and $\underline{\mathtt{S}}$ by $\mathrm{red}(\underline{\mathtt{S}})$, join corresponding pairs of rays, then apply the usual surgery procedure to contract the symmetric section of the resulting diagram.

This should be compared with the definition of $f_{\alpha\beta\gamma}$ from (5.1).

Note that the vector $(\underline{\nu}\overline{\mathtt{U}})$ from (5.15) lies in the summand $^*{}_{R_{\boldsymbol{i}^{\mathtt{U}}}} P(\eta)$. So $f_{\mathtt{ST}}$ belongs to $\mathrm{Hom}_K(^*{}_{R_{\boldsymbol{i}^{\mathtt{S}}}} P(\eta), {}^*{}_{R_{\boldsymbol{i}^{\mathtt{T}}}} P(\eta))$. We deduce that

$$e(\boldsymbol{i})f_{\mathtt{ST}} = \begin{cases} f_{\mathtt{ST}} & \text{if } \boldsymbol{i} = \boldsymbol{i}^{\mathtt{S}}, \\ 0 & \text{otherwise,} \end{cases} \qquad f_{\mathtt{ST}}e(\boldsymbol{i}) = \begin{cases} f_{\mathtt{ST}} & \text{if } \boldsymbol{i} = \boldsymbol{i}^{\mathtt{T}}, \\ 0 & \text{otherwise.} \end{cases} \tag{5.17}$$

Theorem 5.7. *The homomorphisms*

$$\{f_{\mathtt{ST}} \mid \text{for all } \mathtt{S}, \mathtt{T} \in \mathscr{T}_R(\eta) \text{ with } \mathrm{sh}(\mathtt{S}) = \mathrm{sh}(\mathtt{T})\} \tag{5.18}$$

give a basis for $\mathrm{End}_K^{fin}(^*{}_R P(\eta))^{\mathrm{op}}$. Moreover $f_{\mathtt{ST}}$ is of degree $\deg(\mathtt{S}) + \deg(\mathtt{T})$.

Proof. Using the iterated version of Lemma 3.3 (see [BS2, Theorem 3.6] for the precise formulation), we decompose the functor $^*R$ into a direct sum of indecomposable projective functors, then apply Theorem 5.1. Actually Remark 5.2 is slightly more convenient since it gives directly that $f_{\mathtt{ST}}$ is of degree $\deg(\underline{\mathtt{S}}\overline{\mathtt{T}}) - \mathrm{caps}(\underline{\mathtt{S}}) - \mathrm{cups}(\overline{\mathtt{T}}) = \deg(\mathtt{S}) + \deg(\mathtt{T})$. $\square$

Now we pass from $\mathrm{End}_K^{fin}(^*R\,P(\eta))^{\mathrm{op}}$ to $\mathrm{End}_K^{fin}(^*R\,L(\eta))^{\mathrm{op}}$. A *good pair* $(\mathtt{S},\mathtt{T})$ of R-tableaux of type η means a pair of tableaux $\mathtt{S},\mathtt{T} \in \mathscr{T}_R(\eta)$ of the same shape such that the following hold.

   (1) Each connected component in the diagram $\underline{\mathtt{S}}\overline{\mathtt{T}}$ crosses the top number line at most twice. If it crosses the top number line exactly twice then it is an anti-clockwise circle.
   (2) Each connected component in $\underline{\mathtt{S}}\overline{\mathtt{T}}$ crosses the bottom number line at most twice. If it crosses exactly twice and it does not cross the top number line then it is a clockwise circle.

These are the same conditions as in Remark 5.4.

Theorem 5.8. *If* $(\mathtt{S},\mathtt{T})$ *is a good pair of R-tableaux of type η, then the homomorphism $f_{\mathtt{ST}}$ from (5.16) factors through the quotients to induce a non-zero map $\bar{f}_{\mathtt{ST}} : {}^*R\,L(\eta) \to {}^*R\,L(\eta)$. The homomorphisms*

$$\{\bar{f}_{\mathtt{ST}} \mid \textit{for all good pairs } (\mathtt{S},\mathtt{T}) \textit{ of R-tableaux of type } \eta\} \qquad (5.19)$$

give a basis for $\mathrm{End}_K^{fin}(^*R\,L(\eta))^{\mathrm{op}}$. *Moreover $\bar{f}_{\mathtt{ST}}$ is of degree* $\deg(\mathtt{S}) + \deg(\mathtt{T})$.

Proof. This is deduced from Theorem 5.3 (and Remark 5.4) in the same way that Theorem 5.7 was deduced from Theorem 5.1 above. $\square$

Algorithms to compute with the diagram bases. We conclude the section by explaining various algorithms to compute with the bases (5.18) and (5.19). We will give some concrete examples in a special case in the next section.

Products. We first explain how to compute products. For this, we adapt the algorithms for composing homomorphisms explained after Remark 5.4. To compute products in $\mathrm{End}_K^{fin}(^*R\,P(\eta))^{\mathrm{op}}$, take two basis vectors $f_{\mathtt{UV}}$ and $f_{\mathtt{ST}}$ from (5.18). We have that $f_{\mathtt{ST}}f_{\mathtt{UV}} = 0$ unless $\boldsymbol{i}^{\mathtt{T}} = \boldsymbol{i}^{\mathtt{U}}$. Assuming $\boldsymbol{i}^{\mathtt{T}} = \boldsymbol{i}^{\mathtt{U}}$, we draw $\underline{\mathtt{S}}\overline{\mathtt{T}}$ underneath $\underline{\mathtt{U}}\overline{\mathtt{V}}$, then apply the extended surgery procedure from (5.16) to contract the $\overline{\mathtt{T}}|\underline{\mathtt{U}}$-part of the diagram. This produces a (possibly zero) sum of diagrams of the form $(\underline{\mathtt{X}}\overline{\mathtt{Y}})$ such that $\boldsymbol{i}^{\mathtt{X}} = \boldsymbol{i}^{\mathtt{S}}$ and $\boldsymbol{i}^{\mathtt{Y}} = \boldsymbol{i}^{\mathtt{V}}$; the product $f_{\mathtt{ST}}f_{\mathtt{UV}}$ is the corresponding sum of homomorphisms $f_{\mathtt{XY}} \in \mathrm{End}_K^{fin}(^*R\,P(\eta))^{\mathrm{op}}$.

Now suppose that $(\mathtt{U},\mathtt{V})$ and $(\mathtt{S},\mathtt{T})$ are both good pairs, so that $\bar{f}_{\mathtt{UV}}$ and $\bar{f}_{\mathtt{ST}}$ are basis vectors from (5.19). To compute the product $\bar{f}_{\mathtt{UV}}\bar{f}_{\mathtt{ST}}$ in $\mathrm{End}_K^{fin}(^*R\,L(\eta))^{\mathrm{op}}$, we expand $f_{\mathtt{UV}}f_{\mathtt{ST}}$ as a sum of $f_{\mathtt{XY}}$ using the algorithm in the previous paragraph. Then we replace all $f_{\mathtt{XY}}$ in which $(\mathtt{X},\mathtt{Y})$ is not a good pair by zero, and all remaining $f_{\mathtt{XY}}$ by $\bar{f}_{\mathtt{XY}}$. This gives $\bar{f}_{\mathtt{UV}}\bar{f}_{\mathtt{ST}}$.

Homomorphisms. Next we explain how to compute the maps (5.10)–(5.11) in terms of our diagram bases. Take $\mathtt{S},\mathtt{T} \in \mathscr{T}_R(\eta)$ of the same shape. To compute $\iota_{R;i}^{RE}(f_{\mathtt{ST}})$ (resp. $\iota_{R;i}^{RF}(f_{\mathtt{ST}})$), the rough idea is to insert two new levels at the middle number line of the diagram $\underline{\mathtt{S}}\overline{\mathtt{T}}$ containing the unique composite

matching displayed in (5.20) (resp. (5.21)) that is consistent with the labels on
the number line; if none of the configurations match we set $\iota_{R;i}^{RE}(f_{\mathsf{ST}}) := 0$ (resp.
$\iota_{R;i}^{RF}(f_{\mathsf{ST}}) := 0$).

$$\iota_{R;i}^{RE} : \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (5.20)$$

$$\iota_{R;i}^{RF} : \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (5.21)$$

Here, we display only the strip between the ith and $(i+1)$th vertices, and there
are only vertical line segments joining the vertices labelled $\wedge$ or $\vee$ outside of
this strip. For the second, third and fourth of these configurations, it is quite
clear how to do this: insert the new levels into $\underline{\mathsf{S}}\overline{\mathsf{T}}$, complete the labels on the
new middle number line in the only possible way to get a consistently oriented
diagram, and this produces the desired basis vector in $\mathrm{End}_K^{fin}(*(RE)P(\eta))^{\mathrm{op}}$
(resp. $\mathrm{End}_K^{fin}(*(RF)P(\eta))^{\mathrm{op}}$). The first configuration is a little more subtle and
involves applying one iteration of the surgery procedure. We refer to [BS3,
(6.34)] where this is explained in more detail in an analogous situation. The
proof that this is correct follows the same argument as [BS3, Theorem 6.11].

   Finally we assume in addition that (S,T) is a good pair and modify the
algorithm in the previous paragraph to compute $\iota_{R;i}^{RE}(\bar{f}_{\mathsf{ST}})$ (resp. $\iota_{R;i}^{RF}(\bar{f}_{\mathsf{ST}})$). First
one computes $\iota_{R;i}^{RE}(f_{\mathsf{ST}})$ (resp. $\iota_{R;i}^{RF}(f_{\mathsf{ST}})$) exactly as in the previous paragraph to
obtain a sum of basis vectors for the form f_{XY}. The result is automatically a
good homomorphism, so it remains to replace all f_{XY} in which (X,Y) is not a
good pair by zero, and the remaining ones by $\bar{f}_{\mathsf{XY}}$.

6. The graded walled Brauer algebra

   In this section $\delta \in \mathbb{Z}$ and $R = R^{(1)} \cdots R^{(r+s)} \in \mathscr{R}_{r,s}$ are fixed. We are ready
to define the graded version $B_R(\delta)$ of the walled Brauer algebra $B_{r,s}(\delta)$ and to
show that it is graded Morita equivalent to the basic algebra $K_{r,s}(\delta)$ from §4.
The proof that $B_R(\delta)$ is isomorphic to $B_{r,s}(\delta)$ is deferred to §8.

Definition of the algebra $B_R(\delta)$. According to the weight dictionary (1.3)–
(1.5), the empty bipartition $(\varnothing, \varnothing)$ is identified with the weight diagram

$$\eta := \begin{cases} \cdots \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \cdots & \text{if } \delta \geq 0 \\ \cdots \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \cdots & \text{if } \delta \leq 0 \end{cases} \qquad (6.1)$$

(where there are infinitely many vertices labelled $\wedge$ to the left and $\vee$ to the right). In this section we refer to the generalised R-tableaux of type η as defined in the previous section simply as *R-tableaux*, and denote the set of all such R-tableaux by $\mathscr{T}_R(\delta)$. As η is maximal in the Bruhat order, any $\mathtt{T} = (\lambda^{(0)}, \ldots, \lambda^{(r+s)}) \in \mathscr{T}_R(\delta)$ necessarily has $\lambda^{(0)} = \eta$. For $\mathtt{S}, \mathtt{T} \in \mathscr{T}_R(\delta)$ with $\mathrm{sh}(\mathtt{S}) = \mathrm{sh}(\mathtt{T})$, we often talk about *boundary cups* in the diagrams $\underline{\mathtt{S}}$, $\overline{\mathtt{T}}$ or $\underline{\mathtt{S}}\overline{\mathtt{T}}$, meaning connected components that intersect the top number line twice. Similarly we define *boundary caps* to be components that intersect the bottom number line twice.

It is often convenient to represent elements of $\mathscr{T}_R(\delta)$ as chains of bipartitions in the graph defined just before Theorem 2.5 rather than as chains of weight diagrams; this is justified by Remark 3.8. Let

$$r_a := \#\{b = 1, \ldots, a \mid R^{(b)} = E\}, \qquad s_a := \#\{b = 1, \ldots, a \mid R^{(b)} = F\}.$$

Then elements of $\mathscr{T}_R(\delta)$ are paths $\mathtt{T} = (\lambda^{(0)}, \lambda^{(1)}, \ldots, \lambda^{(r+s)})$ in the Bratelli diagram that describes the branching of cell modules of $B_{r,s}(\delta)$ with respect to the chain of subalgebras $B_{0,0}(\delta) < B_{r_1,s_1}(\delta) < B_{r_2,s_2}(\delta) < \cdots < B_{r,s}(\delta)$; in particular $\lambda^{(0)} = (\varnothing, \varnothing)$ and $\lambda^{(r+s)} \in \Lambda_{r,s}$. Note also for $\mathtt{T} \in \mathscr{T}_R(\delta)$ that the alternative formula for $\deg(\mathtt{T})$ from Remark 5.5 can be written as

$$\deg(\mathtt{T}) = \sum_{\substack{1 \le a \le r+s \\ R^{(a)} = E}} \deg(\lambda^{(a-1)} \overset{i_a}{\to} \lambda^{(a)}) + \sum_{\substack{1 \le a \le r+s \\ R^{(a)} = F}} \deg(\lambda^{(a-1)} \overset{i_a}{\leftarrow} \lambda^{(a)}) \qquad (6.2)$$

for $\boldsymbol{i}^{\mathtt{T}} = (i_1, \ldots, i_{r+s})$. Combined with (3.20)–(3.21), this formula allows the degree of an R-tableau to be computed directly from the underlying chain of bipartitions (and the value of δ). We will give some examples in the next subsection.

Since the set $\mathscr{T}_R(\delta)$ is finite for any $R \in \mathscr{R}_{r,s}$, Theorem 5.7 implies that the graded algebra

$$B_R(\delta) := \mathrm{End}^{fin}_{K(\delta)}({}^*R\, P(\eta))^{\mathrm{op}} \qquad (6.3)$$

is finite dimensional. This means we can drop the "*fin*" in our notation here. We call $B_R(\delta)$ the *graded walled Brauer algebra* of type R. This name will be justified later in the article, when we show (on forgetting the grading) that $B_R(\delta)$ is isomorphic to the walled Brauer algebra $B_{r,s}(\delta)$. By Theorem 5.7, $B_R(\delta)$ has a homogeneous basis

$$\{f_{\mathtt{ST}} \mid \text{for all } \mathtt{S}, \mathtt{T} \in \mathscr{T}_R(\delta) \text{ with } \mathrm{sh}(\mathtt{S}) = \mathrm{sh}(\mathtt{T})\} \qquad (6.4)$$

such that $f_{\mathtt{ST}}$ is of degree $\deg(\mathtt{S}) + \deg(\mathtt{T})$. Products can be computed in this basis using the algorithm explained at the end of §5. Recall also that there is system $\{e(\boldsymbol{i}) \mid \boldsymbol{i} \in \mathbb{Z}^{r+s}\}$ of mutually orthogonal idempotents in $B_R(\delta)$ characterised by (5.17). All but finitely many of the $e(\boldsymbol{i})$'s are zero and their sum is 1. In fact $e(\boldsymbol{i}) \ne 0$ if and only if there exists an R-tableau $\mathtt{T}$ with $\boldsymbol{i}^{\mathtt{T}} = \boldsymbol{i}$. Again we refer the reader to the next subsection for some examples.

The non-unital homomorphisms

$$\iota^{RE}_{R;i} : B_R(\delta) \to B_{RE}(\delta), \qquad \iota^{RF}_{R;i} : B_R(\delta) \to B_{RF}(\delta) \qquad (6.5)$$

from (5.10)–(5.11) map 1 to $1^{RE}_{R;i} \in B_{RE}(\delta)$ and $1^{RF}_{R;i} \in B_{RF}(\delta)$, respectively. The effect of these maps on the basis elements (6.4) can be computed explicitly by the algorithm explained at the end of §5. Associated to these maps, we have adjoint pairs $(i\text{-ind}^{RE}_R, i\text{-res}^{RE}_R)$ and $(i\text{-ind}^{RF}_R, i\text{-res}^{RF}_R)$ of induction and restriction functors defined exactly like in (2.15)–(2.18). Thus

$$i\text{-res}^{RE}_R : B_{RE}(\delta)\text{-mod}_{lf} \to B_R(\delta)\text{-mod}_{lf}, \tag{6.6}$$

$$i\text{-res}^{RF}_R : B_{RF}(\delta)\text{-mod}_{lf} \to B_R(\delta)\text{-mod}_{lf} \tag{6.7}$$

are the exact functors defined by multiplication by the idempotents $1^{RE}_{R;i}$ and $1^{RF}_{R;i}$, respectively, and

$$i\text{-ind}^{RE}_R : B_R(\delta)\text{-mod}_{lf} \to B_{RE}(\delta)\text{-mod}_{lf}, \tag{6.8}$$

$$i\text{-ind}^{RF}_R : B_R(\delta)\text{-mod}_{lf} \to B_{RF}(\delta)\text{-mod}_{lf} \tag{6.9}$$

are the right exact functors $B_{RE}(\delta)1^{RE}_{R;i}\otimes_{B_R(\delta)}?$, and $B_{RF}(\delta)1^{RF}_{R;i}\otimes_{B_R(\delta)}?$, respectively. Later in the article we will also relate these *graded i-induction* and *i-restriction functors* to the ungraded ones defined earlier.

Examples. In this subsection we give some examples to illustrate the definition of $B_R(\delta)$, taking $\delta := 0$ throughout.

We first describe $B_{E^2F}(0)$. Here are all the E^2F-tableaux:

$$\mathtt{S} = ((\varnothing, \varnothing) \xrightarrow{0} ((1), \varnothing) \xrightarrow{-1} ((1^2), \varnothing) \xleftarrow{-1} ((1), \varnothing)),$$

$$\mathtt{T} = ((\varnothing, \varnothing) \xrightarrow{0} ((1), \varnothing) \xrightarrow{1} ((2), \varnothing) \xleftarrow{1} ((1), \varnothing)),$$

$$\mathtt{U} = ((\varnothing, \varnothing) \xrightarrow{0} ((1), \varnothing) \xrightarrow{-1} ((1^2), \varnothing) \xleftarrow{0} ((1^2), (1))),$$

$$\mathtt{V} = ((\varnothing, \varnothing) \xrightarrow{0} ((1), \varnothing) \xrightarrow{1} ((2), \varnothing) \xleftarrow{0} ((2), (1))).$$

The algebra $B_{E^2F}(0)$ has basis $\{f_{\mathtt{SS}}, f_{\mathtt{TT}}, f_{\mathtt{ST}}, f_{\mathtt{TS}}\}\cup\{f_{\mathtt{UU}}\}\cup\{f_{\mathtt{VV}}\}$. These elements are represented by the diagrams
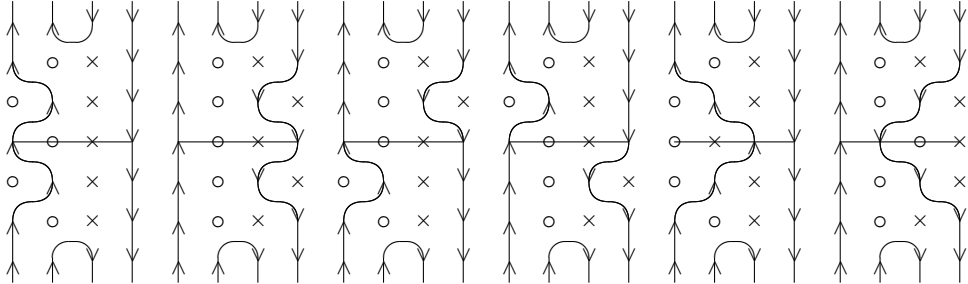
Using the explicit multiplication rule, it follows easily that

$$B_{E^2F}(0) \cong M_2(\mathbb{C}) \oplus \mathbb{C} \oplus \mathbb{C},$$

and the diagram basis consists of matrix units; in particular the idempotents $f_{\mathtt{SS}}$ and $f_{\mathtt{TT}}$ give the diagonal matrix units in $M_2(\mathbb{C})$. Moreover, recalling that the degree of $f_{\mathtt{ST}}$ is the total number of clockwise cups and caps in $\underline{\mathtt{S}}\overline{\mathtt{T}}$ minus the number of caps in $\underline{\mathtt{S}}$ minus the number of cups in $\overline{\mathtt{T}}$, the grading on $B_{E^2F}(0)$ is trivial (concentrated in degree zero).

Consider instead the algebra $B_{FE^2}(0)$. There are also four FE^2-tableaux:

$$\mathtt{S}' = ((\varnothing, \varnothing) \xleftarrow{0} (\varnothing, (1)) \xrightarrow{0} ((1), (1)) \xrightarrow{0} ((1), \varnothing)),$$

$$\mathtt{T}' = ((\varnothing, \varnothing) \xleftarrow{0} (\varnothing, (1)) \xrightarrow{0} (\varnothing, \varnothing) \xrightarrow{0} ((1), \varnothing)),$$

$$\mathtt{U}' = ((\varnothing, \varnothing) \xleftarrow{0} (\varnothing, (1)) \xrightarrow{0} ((1), (1)) \xrightarrow{-1} ((1^2), (1))),$$

$$\mathtt{V}' = ((\varnothing, \varnothing) \xleftarrow{0} (\varnothing, (1)) \xrightarrow{0} ((1), (1)) \xrightarrow{1} ((2), (1))).$$

The algebra $B_{FE^2}(0)$ has basis $\{f_{\mathtt{S}'\mathtt{T}'}, f_{\mathtt{T}'\mathtt{S}'}, f_{\mathtt{S}'\mathtt{S}'}, f_{\mathtt{T}'\mathtt{T}'}\} \cup \{f_{\mathtt{U}'\mathtt{U}'}\} \cup \{f_{\mathtt{V}'\mathtt{V}'}\}$:
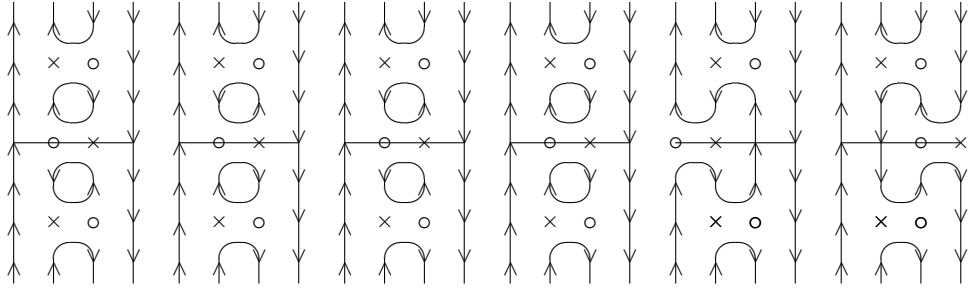
This is also a semisimple algebra isomorphic to $M_2(\mathbb{C}) \oplus \mathbb{C} \oplus \mathbb{C}$, with the idempotents $f_{\mathtt{S}'\mathtt{T}'}$ and $f_{\mathtt{T}'\mathtt{S}'}$ giving the diagonal matrix units in $M_2(\mathbb{C})$. But the grading is no longer trivial as the off-diagonal matrix units $f_{\mathtt{S}'\mathtt{S}'}$ and $f_{\mathtt{T}'\mathtt{T}'}$ in the copy of $M_2(\mathbb{C})$ are of degrees -2 and 2, respectively. Comparing with the preceding example, we see that $B_{E^2F}(0)$ and $B_{FE^2}(0)$ are isomorphic as ungraded algebras but not as graded algebras.

Finally we give a less trivial example which is not semisimple. Consider $R = E^2F^2$. There are ten R-tableaux, namely, the paths in the following Bratelli diagram.
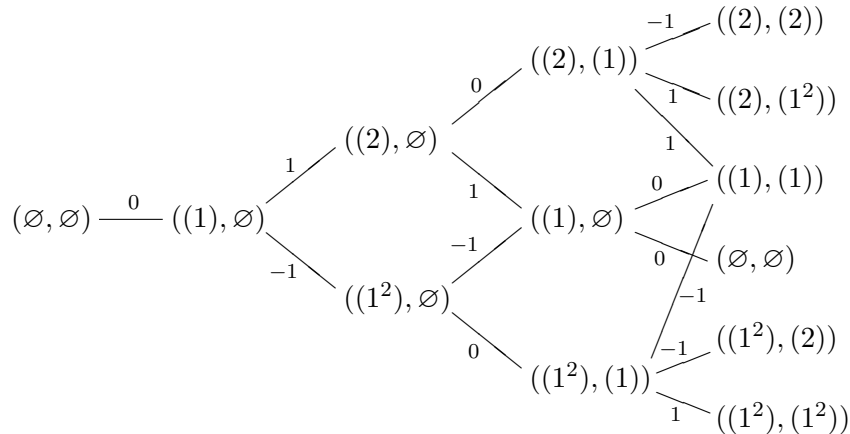
The algebra $B_{E^2F^2}(0)$ is of dimension 24, and the diagrams representing its basis are displayed in Figure 1. Using the multiplication rule, one can check directly in this example that the Jacobson radical is of dimension 16 and is spanned by all the basis vectors $\{f_{\mathtt{S}\mathtt{T}}\}$ such that either $\underline{\mathtt{S}}$ has a clockwise boundary cup or $\overline{\mathtt{T}}$ has a clockwise boundary cap; cf. Corollary 6.8 below. Moreover the quotient

of $B_{E^2F^2}(0)$ by its Jacobson radical is isomorphic to $M_2(\mathbb{C}) \oplus \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}$; cf. Theorem 6.7 below.

Graded Morita equivalence. In this subsection we prove that $B_R(\delta)$ is graded Morita equivalent to $K_{r,s}(\delta)$. Let η be as in (6.1) and recall that $B_R(\delta)$ is the endomorphism algebra of the $K(\delta)$-module ${}^*R\,P(\eta)$, while $K_{r,s}(\delta) = e_{r,s}K(\delta)e_{r,s}$ according to (4.2).

Lemma 6.1. *Every indecomposable summand of ${}^*R\,P(\eta)$ is isomorphic (up to grading shift) to one of the projective indecomposable modules $\{P(\lambda)\,|\,\lambda \in \dot{\Lambda}_{r,s}\}$. Moreover every such $P(\lambda)$ appears as a summand.*

Proof. Proceed by induction on $r + s$, the case $r + s = 0$ being trivial. For the induction step, suppose we have shown already that the indecomposable summands of ${}^*R\,P(\eta)$ are the modules $\{P(\lambda)\,|\,\lambda \in \dot{\Lambda}_{r,s}\}$. We need to show that the indecomposable summands of ${}^*E{}^*R\,P(\eta)$ and ${}^*F{}^*R\,P(\eta)$ are the modules $\{P(\lambda)\,|\,\lambda \in \dot{\Lambda}_{r+1,s}\}$ and $\{P(\lambda)\,|\,\lambda \in \dot{\Lambda}_{r,s+1}\}$, respectively. This follows from Lemma 4.5. $\qquad\square$

Now we can construct the Morita equivalence. Set

$$P_R(\delta) := e_{r,s}{}^*R\,P(\eta), \qquad\qquad (6.10)$$

which is a graded $(K_{r,s}(\delta), B_R(\delta))$-bimodule spanned by the vectors $(\underline{\nu}\,\overline{\mathtt{U}})$ from (5.15) with $\mathtt{U} \in \mathscr{T}_R(\delta)$ and $\nu \in \dot{\Lambda}_{r,s}$ such that $\nu \subset \mathrm{sh}(\mathtt{U})$. Recall also the functors (4.6)–(4.9) and (6.6)–(6.9).

Theorem 6.2. *The functor*

$$\mathbf{f}_R := \mathrm{Hom}_{K_{r,s}(\delta)}(P_R(\delta), ?) : K_{r,s}(\delta)\text{-mod}_{lf} \to B_R(\delta)\text{-mod}_{lf} \qquad (6.11)$$

is an equivalence of categories commuting with i-restriction and i-induction, i.e. there are isomorphisms of functors

$$\mathbf{f}_R \circ i\text{-res}_{r,s}^{r+1,s} \cong i\text{-res}_R^{RE} \circ \mathbf{f}_{RE} : K_{r+1,s}(\delta)\text{-mod}_{lf} \to B_R(\delta)\text{-mod}_{lf},$$

$$\mathbf{f}_R \circ i\text{-res}_{r,s}^{r,s+1} \cong i\text{-res}_R^{RF} \circ \mathbf{f}_{RF} : K_{r,s+1}(\delta)\text{-mod}_{lf} \to B_R(\delta)\text{-mod}_{lf},$$

$$\mathbf{f}_{RE} \circ i\text{-ind}_{r,s}^{r+1,s} \cong i\text{-ind}_R^{RE} \circ \mathbf{f}_R : K_{r,s}(\delta)\text{-mod}_{lf} \to B_{RE}(\delta)\text{-mod}_{lf},$$

$$\mathbf{f}_{RF} \circ i\text{-ind}_{r,s}^{r,s+1} \cong i\text{-ind}_R^{RF} \circ \mathbf{f}_R : K_{r,s}(\delta)\text{-mod}_{lf} \to B_{RF}(\delta)\text{-mod}_{lf} .$$

Proof. By Lemma 6.1 and Theorem 4.4(3), $P_R(\delta) = e_{r,s}{}^*R\,P(\eta)$ is a graded projective generator for the category $K_{r,s}(\delta)\text{-mod}_{lf}$. Moreover the endomorphism algebra $\mathrm{End}_{K_{r,s}(\delta)}(P_R(\delta))^{\mathrm{op}}$ is identified with $B_R(\delta) = \mathrm{End}_{K(\delta)}({}^*R\,P(\eta))^{\mathrm{op}}$ by Lemma 4.1 (and Lemma 6.1 again). Hence $\mathbf{f}_R$ is an equivalence of categories by general principles.

For the four isomorphisms of functors, it is enough to prove the ones involving restriction, since the ones for induction then follow by unicity of adjoints. We just explain the argument for the first isomorphism of functors, since the second
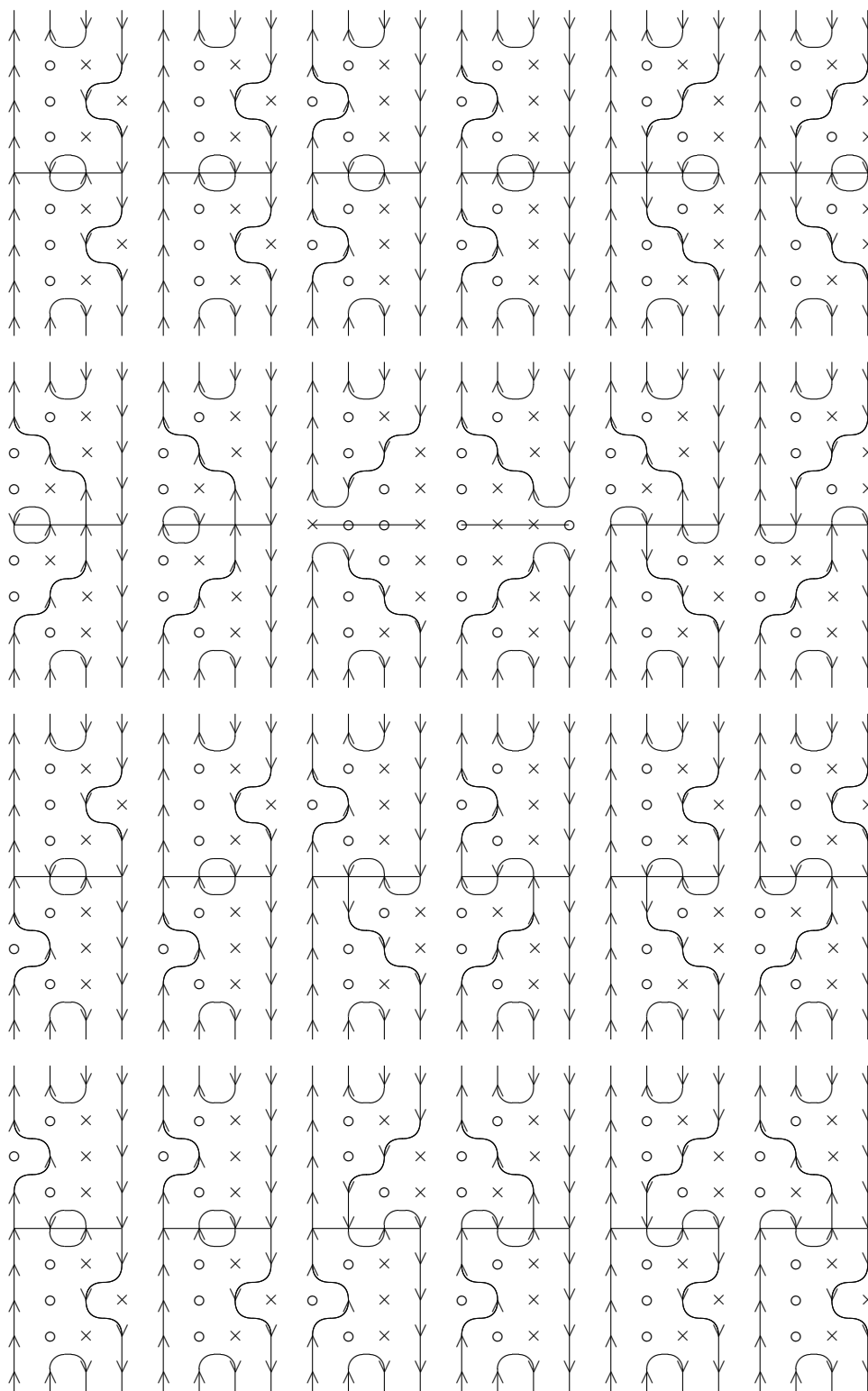
FIGURE 1. The diagram basis for $B_{E^2F^2}(0)$

is similar. For this, we use the following chain of graded $(B_R(\delta), K_{r+1,s}(\delta))$-bimodule isomorphisms.

$$
\begin{aligned}
\mathbf{f}_R(i\text{-res}_{r,s}^{r+1,s} K_{r+1,s}(\delta)) &\cong \mathrm{Hom}_{K_{r,s}(\delta)}(e_{r,s}{}^*R\,P(\eta), e_{r,s}\widetilde{E}_i e_{r+1,s}) \\
&\cong \mathrm{Hom}_{K(\delta)}({}^*R\,P(\eta), \widetilde{E}_i e_{r+1,s}) \\
&\cong \mathrm{Hom}_{K(\delta)}({}^*R\,P(\eta), E_i K(\delta) e_{r+1,s}) \\
&\cong \mathrm{Hom}_{K(\delta)}({}^*E_i{}^*R\,P(\eta), K(\delta) e_{r+1,s}) \\
&\cong i\text{-res}_R^{RE}(\mathrm{Hom}_{K(\delta)}({}^*E^*R\,P(\eta), K(\delta) e_{r+1,s})) \\
&\cong i\text{-res}_R^{RE}(\mathrm{Hom}_{K_{r+1,s}(\delta)}(e_{r+1,s}{}^*(RE)\,P(\eta), K_{r+1,s}(\delta))) \\
&= i\text{-res}_R^{RE}(\mathbf{f}_{RE} K_{r+1,s}(\delta)).
\end{aligned}
$$

The second and the penultimate isomorphisms here follow from Lemmas 4.1 and 6.1.   □

In the rest of the subsection, we give an alternative description of the Morita equivalence $\mathbf{f}_R$ from Theorem 6.2 which is sometimes easier for explicit computations. As well as the left $K(\delta)$-module ${}^*R\,P(\eta) = {}^*R\,K(\delta)e_\eta$, we can consider the right $K(\delta)$-module $e_\eta R\,K(\delta)$. Just like in Lemma 5.6, $e_\eta R\,K(\delta)$ has a distinguished basis

$$
\{(\underline{\mathtt{U}}\nu) \mid \text{for all } \mathtt{U} \in \mathscr{T}_R(\delta) \text{ and } \nu \subset \mathrm{sh}(\mathtt{U})\}. \tag{6.12}
$$

The right action of $K(\delta)$ on elements of this basis can be computed similarly to the left action in Lemma 5.6. We make $e_\eta R\,K(\delta)$ into a left $B_R(\delta)$-module by defining $f_{\mathtt{ST}}(\underline{\mathtt{U}}\nu)$ to be zero if $\boldsymbol{i}^{\mathtt{T}} \neq \boldsymbol{i}^{\mathtt{U}}$, while if $\boldsymbol{i}^{\mathtt{T}} = \boldsymbol{i}^{\mathtt{U}}$ it is computed by drawing $\underline{\mathtt{S}}\overline{\mathtt{T}}$ underneath $\underline{\mathtt{U}}\nu$ and applying the extended surgery procedure as usual. Set

$$
Q_R(\delta) := e_\eta R\,K(\delta)e_{r,s}, \tag{6.13}
$$

which is a graded $(B_R(\delta), K_{r,s}(\delta))$-bimodule spanned by the vectors $(\underline{\mathtt{U}}\nu)$ from (6.12) in which $\nu \in \dot{\Lambda}_{r,s}$. Recalling also the bimodule $P_R(\delta)$ from (6.10), there are bimodule homomorphisms

$$
P_R(\delta) \otimes_{B_R(\delta)} Q_R(\delta) \to K_{r,s}(\delta), \quad Q_R(\delta) \otimes_{K_{r,s}(\delta)} P_R(\delta) \to B_R(\delta), \tag{6.14}
$$

defined by the (by now) obvious multiplication maps; the first involves the extended surgery procedure once again.

Lemma 6.3. *There is a canonical isomorphism of graded $(B_R(\delta), K_{r,s}(\delta))$-bimodules $Q_R(\delta) \cong \mathrm{Hom}_{K_{r,s}(\delta)}(P_R(\delta), K_{r,s}(\delta))$. Hence the functor $\mathbf{f}_R$ from (6.11) is isomorphic to the functor $Q_R(\delta) \otimes_{K_{r,s}(\delta)} ?$. Moreover the bimodule homomorphisms in (6.14) are both isomorphisms.*

Proof. Lemmas 3.3 and 3.4 give us a canonical graded $(B_R(\delta), K(\delta))$-bimodule isomorphism $\mathrm{Hom}_{K(\delta)}({}^*R\,P(\eta), K(\delta)) \cong e_\eta R\,K(\delta)$. It restricts to an isomorphism of $(B_R(\delta), K_{r,s}(\delta))$-bimodules

$$
\mathrm{Hom}_{K(\delta)}({}^*R\,P(\eta), K(\delta)e_{r,s}) \cong e_\eta R\,K(\delta)e_{r,s} = Q_R(\delta).
$$

To establish the first statement, it remains to observe using Lemma 4.1 that

$$
\mathrm{Hom}_{K(\delta)}({}^*R\,P(\eta), K(\delta)e_{r,s}) \cong \mathrm{Hom}_{K_{r,s}(\delta)}(P_R(\delta), K_{r,s}(\delta)).
$$

The rest of the lemma now follows from Theorem 6.2 and standard Morita theory. One needs to use the explicit form of the isomorphism in Lemma 3.4 to check that the first map in (6.14) corresponds to the natural map $P_R(\delta) \otimes \mathrm{Hom}_{K_{r,s}(\delta)}(P_R(\delta), K_{r,s}(\delta)) \to K_{r,s}(\delta)$. $\square$

Graded cell modules. Using Theorem 6.2, one can deduce a great deal of information about the graded representation theory of the algebras $B_R(\delta)$ from Theorems 4.4 and 4.10. Still it is desirable to give intrinsic descriptions of the $B_R(\delta)$-modules $\mathbf{f}_R(V_{r,s}(\lambda))$ and $\mathbf{f}_R(D_{r,s}(\lambda))$. We do this in the next two subsections using the theory of graded cellular algebras once again.

Theorem 6.4. *The algebra $B_R(\delta)$ is a graded cellular algebra with cell datum $(X, I, C, \deg)$ defined by*

$$X := \Lambda_{r,s}, \qquad\qquad I(\lambda) := \{\mathtt{T} \in \mathscr{T}_R(\delta) \mid \mathrm{sh}(\mathtt{T}) = \lambda\},$$

$$C_{\mathtt{S},\mathtt{T}}^\lambda := f_{\mathtt{ST}}, \qquad\qquad \deg_{\mathtt{T}}^\lambda := \deg(\mathtt{T})$$

for $\lambda \in \Lambda_{r,s}$ and $\mathtt{S}, \mathtt{T} \in I(\lambda)$.

Proof. This follows from the algorithm for multiplication by arguments similar to the proof of [BS1, Corollary 3.3]. $\square$

We denote the graded cell modules attached to this graded cellular structure by $\{C_R(\lambda) \mid \lambda \in \Lambda_{r,s}\}$. The general construction gives easily that $C_R(\lambda)$ has a homogeneous basis

$$\big\{ v_{\mathtt{T}} \;\big|\; \text{for all } \mathtt{T} \in \mathscr{T}_R(\delta) \text{ with } \mathrm{sh}(\mathtt{T}) = \lambda \big\} \tag{6.15}$$

with $v_{\mathtt{T}}$ of degree $\deg(\mathtt{T})$. We have that $f_{\mathtt{UV}} v_{\mathtt{T}} = 0$ if $\boldsymbol{i}^{\mathtt{V}} \neq \boldsymbol{i}^{\mathtt{T}}$. Assuming $\boldsymbol{i}^{\mathtt{V}} = \boldsymbol{i}^{\mathtt{T}}$, $f_{\mathtt{UV}} v_{\mathtt{T}}$ is computed as follows. Draw $\underline{\mathtt{U}}\overline{\mathtt{V}}$ underneath $\underline{\mathtt{T}}\overline{\lambda}$. Then use the extended surgery procedure to contract the $\overline{\mathtt{V}}|\underline{\mathtt{T}}$-part of the diagram. This produces a (possibly zero) sum of diagrams $(\underline{\mathtt{S}}\overline{\lambda})$ for various $\mathtt{S}$. Then we discard all the ones in which $\mathtt{S}$ is not of shape λ, and replace the remaining $(\underline{\mathtt{S}}\overline{\lambda})$'s with the corresponding vectors $v_{\mathtt{S}} \in C_R(\lambda)$.

Lemma 6.5. *For any $\lambda \in \Lambda_{r,s}$ we have that $\mathbf{f}_R(V_{r,s}(\lambda)) \cong C_R(\lambda)$ as graded $B_R(\delta)$-modules.*

Proof. By definition, $V_{r,s}(\lambda)$ is the quotient of $K_{r,s}(\delta)e_\lambda$ by the submodule S_1 spanned by all vectors $(\underline{\kappa}\alpha\overline{\lambda})$ for $\dot{\Lambda}_{r,s} \ni \kappa \subset \alpha \supset \lambda$ with $\alpha \neq \lambda$. We claim that S_1 is equal to the submodule S_2 generated by the images of the homomorphisms $K_{r,s}(\delta) \to K_{r,s}(\delta)e_\lambda$ defined by right multiplication by the elements $\{(\underline{\beta}\beta\overline{\lambda}) \mid \beta \supset \lambda, \beta \neq \lambda\}$. To see that $S_2 \subseteq S_1$, it suffices to observe that $(\underline{\kappa}\alpha\overline{\beta})(\underline{\beta}\beta\overline{\lambda})$ is a linear combination of vectors $(\underline{\kappa}\gamma\overline{\lambda})$ for $\gamma \geq \beta$ by [BS1, Theorem 4.4(i)]. Conversely to see that $S_1 \subseteq S_2$, we take $(\underline{\kappa}\alpha\overline{\lambda}) \in S_1$ and compare with the product $(\underline{\kappa}\alpha\overline{\alpha})(\underline{\alpha}\alpha\overline{\lambda}) \in S_2$. By [BS1, Theorem 4.4(iii)], this product is equal to $(\underline{\kappa}\alpha\overline{\lambda})$ plus higher terms of the form $(\underline{\kappa}\beta\overline{\lambda})$ for $\beta > \alpha$. We may assume that these higher terms belong to S_2 by induction. Hence $(\underline{\kappa}\alpha\overline{\lambda}) \in S_2$, as required.

Using the description of the equivalence $\mathbf{f}_R$ as the functor $Q_R(\delta) \otimes_{K_{r,s}(\delta)} ?$ given by Lemma 6.3, we deduce that $\mathbf{f}_R(V_{r,s}(\lambda))$ is isomorphic to the quotient of $Q_R(\delta) \otimes_{K_{r,s}(\delta)} K_{r,s}(\delta)e_\lambda = Q_R(\delta)e_\lambda$ by the submodule

$$S := \sum_{\beta \supset \lambda, \beta \neq \lambda} Q_R(\delta)(\underline{\beta}\beta\overline{\lambda}).$$

In terms of the basis (6.12), $Q_R(\delta)e_\lambda$ has basis

$$\{(\underline{\mathtt{T}}\overline{\lambda}) \,|\, \text{for all } \mathtt{T} \in \mathscr{T}_R(\delta) \text{ with } \mathrm{sh}(\mathtt{T}) \supset \lambda\}.$$

Moreover by similar arguments to the previous paragraph, the submodule S is the span of all the vectors in this basis in which $\mathrm{sh}(\mathtt{T}) \neq \lambda$. Now it is clear the map $Q_R(\delta)e_\lambda/S \to C_R(\lambda), (\underline{\mathtt{T}}\overline{\lambda}) + S \mapsto v_{\mathtt{T}}$ is a graded $B_R(\delta)$-module isomorphism. $\qquad\square$

Characters of graded irreducible modules. We say that $\mathtt{T} \in \mathscr{T}_R(\delta)$ is *restricted* if all the boundary cups in the diagram $\underline{\mathtt{T}}$ are anti-clockwise. Recall by the general theory of graded cellular algebras that the graded cell module $C_R(\lambda)$ is equipped with a homogeneous symmetric bilinear form $(.,.)$.

Lemma 6.6. *Let $\lambda \in \Lambda_{r,s}$. For $\mathtt{S}, \mathtt{T} \in \mathscr{T}_R(\delta)$ with $\mathrm{sh}(\mathtt{S}) = \mathrm{sh}(\mathtt{T}) = \lambda$, the inner product $(v_{\mathtt{S}}, v_{\mathtt{T}})$ is equal to 1 if all of the following hold:*

(1) $\mathtt{S}$ *and* $\mathtt{T}$ *are both restricted;*
(2) $\boldsymbol{i}^{\mathtt{S}} = \boldsymbol{i}^{\mathtt{T}}$, *i.e. the connected components in the diagrams $\underline{\mathtt{S}}$ and $\underline{\mathtt{T}}$ are in exactly the same positions;*
(3) *all matching pairs of circles in $\underline{\mathtt{S}}$ and $\underline{\mathtt{T}}$ are oriented in opposite ways.*

For all other $\mathtt{S}$ and $\mathtt{T}$ we have that $(v_{\mathtt{S}}, v_{\mathtt{T}}) = 0$.

Proof. This follows from the explicit definition of the bilinear form on $C_R(\lambda)$. See [BS3, Lemma 9.7] where this is explained in an analogous situation. $\qquad\square$

Let $D_R(\lambda)$ denote the quotient $C_R(\lambda)/\operatorname{rad} C_R(\lambda)$ by the radical of the bilinear form $(.,.)$. The non-zero $D_R(\lambda)$'s give a complete set of pairwise non-isomorphic irreducible $B_R(\delta)$-modules (up to degree shift). The following theorem gives complete information about the $D_R(\lambda)$'s from a combinatorial viewpoint.

Theorem 6.7. *For $\lambda \in \Lambda_{r,s}$, let $\bar{v}_{\mathtt{T}} \in D_R(\lambda)$ be the image of $v_{\mathtt{T}} \in C_R(\lambda)$. The vectors $\{\bar{v}_{\mathtt{T}} \,|\, \text{for all restricted } \mathtt{T} \in \mathscr{T}_R(\delta) \text{ with } \mathrm{sh}(\mathtt{T}) = \lambda\}$ give a basis for $D_R(\lambda)$. Moreover $D_R(\lambda)$ is non-zero if and only if $\lambda \in \dot{\Lambda}_{r,s}$, in which case we have that $D_R(\lambda) \cong \mathbf{f}_R(L_{r,s}(\lambda))$.*

Proof. The opening statement is a consequence of Lemma 6.6 and the definition of $D_R(\lambda)$. In particular it is clear from this that $D_R(\lambda) = \{0\}$ if $\lambda \notin \dot{\Lambda}_{r,s}$, i.e. when $r = s > 0$, $\delta = 0$ and $\lambda = (\varnothing, \varnothing)$. If $\lambda \in \dot{\Lambda}_{r,s}$ then $D_R(\lambda) \cong \mathbf{f}_R(L_{r,s}(\lambda))$ by Lemma 6.5 and Theorem 4.4(2), in particular, $D_R(\lambda)$ is non-zero in view of Theorem 6.2. $\qquad\square$

Corollary 6.8. *The Jacobson radical of $B_R(\delta)$ is spanned by all the basis vectors $f_{\mathtt{ST}}$ from (6.4) such that either $\mathtt{S}$ or $\mathtt{T}$ is not restricted.*

Proof. Let I be the span of all the basis vectors $f_{\mathtt{ST}}$ in which either $\mathtt{S}$ or $\mathtt{T}$ is not restricted. If $\mathtt{S}$ is not restricted, i.e. $\underline{\mathtt{S}}$ has a clockwise boundary cup, it is clear from Theorem 6.7 and the multiplication rule that $f_{\mathtt{ST}}$ annihilates all irreducible left $B_R(\delta)$-modules, hence it lies in the Jacobson radical $J(B_R(\delta))$. By symmetry this implies also that $f_{\mathtt{ST}} \in J(B_R(\delta))$ if $\mathtt{T}$ is not restricted. Hence $I \subseteq J(B_R(\delta))$. Moreover Theorem 6.7 implies that $\dim B_R(\delta)/J(B_R(\delta))$ is equal to the number of pairs $(\mathtt{S}, \mathtt{T})$ of restricted R-tableaux of the same shape, which is the same as $\dim B_R(\delta)/I$. $\qquad\square$

Primitive idempotents. For a restricted R-tableau $\mathtt{T} \in \mathscr{T}_R(\delta)$, we let

$$e_{\mathtt{T}} := f_{\mathtt{TT}^*} \in B_R(\delta) \qquad\qquad (6.16)$$

where $\mathtt{T}^*$ is the unique R-tableau such that $\underline{\mathtt{T}}^*$ is obtained from $\underline{\mathtt{T}}$ by reversing the orientation of all its circles.

Lemma 6.9. *Suppose $\mathtt{T} \in \mathscr{T}_R(\delta)$ is restricted and set $\lambda := \mathrm{sh}(\mathtt{T})$. Then $e_{\mathtt{T}}$ is a homogeneous primitive idempotent. Moreover the projective indecomposable module $B_R(\delta)e_{\mathtt{T}}$ has irreducible head isomorphic to $D_R(\lambda)\langle -\deg(\mathtt{T})\rangle$.*

Proof. It is clear from the multiplication rule that $e_{\mathtt{T}}$ is an idempotent. For everything else, it suffices to show for $\mu \in \dot{\Lambda}_{r,s}$ that

$$\mathrm{Hom}_{B_R(\delta)}(B_R(\delta)e_{\mathtt{T}}, D_R(\mu)) \cong e_{\mathtt{T}}D_R(\mu)$$

is one-dimensional concentrated in degree $\deg(\mathtt{T})$ if $\mu = \lambda$ and it is zero for all other μ. For this we consider the basis for $D_R(\mu)$ from Theorem 6.7. If $\mu = \lambda$ the only basis vector on which $e_{\mathtt{T}}$ is non-zero is the vector $\bar{v}_{\mathtt{T}}$, which is of degree $\deg(\mathtt{T})$. If $\mu \neq \lambda$ all the basis vectors are annihilated by $e_{\mathtt{T}}$. $\qquad\square$

Now we look again at the idempotents $e(\boldsymbol{i})$ for $\boldsymbol{i} \in \mathbb{Z}^{r+s}$.

Lemma 6.10. *Given $\boldsymbol{i} \in \mathbb{Z}^{r+s}$, the idempotent $e(\boldsymbol{i})$ decomposes into a sum of mutually orthogonal primitive idempotents as $e(\boldsymbol{i}) = \sum_{\mathtt{T}} e_{\mathtt{T}}$ summing over all restricted $\mathtt{T} \in \mathscr{T}_R(\delta)$ with $\boldsymbol{i}^{\mathtt{T}} = \boldsymbol{i}$. Moreover all the tableaux $\mathtt{T}$ appearing in this sum are of the same shape.*

Proof. By the algorithm for multiplication, the given sum $\sum_{\mathtt{T}} e_{\mathtt{T}}$ satisfies the defining property (5.17) of the idempotent $e(\boldsymbol{i})$. Hence $e(\boldsymbol{i}) = \sum_{\mathtt{T}} e_{\mathtt{T}}$. It is clear that the $e_{\mathtt{T}}$'s for different $\mathtt{T}$ are orthogonal, and all the $\mathtt{T}$'s appearing in the sum have the same shape. Finally the $e_{\mathtt{T}}$'s are primitive by Lemma 6.9. $\qquad\square$

Remark 6.11. Lemmas 6.9 and 6.10 together show that the idempotents $e(\boldsymbol{i})$ are *isotypic* in the sense that $B_R(\delta)e(\boldsymbol{i})$ is isomorphic to a direct sum of degree-shifted copies of a single projective indecomposable module, namely, the projective cover of $D_R(\lambda)$ where λ is the shape of any restricted R-tableau $\mathtt{T}$ with $\boldsymbol{i}^{\mathtt{T}} = \boldsymbol{i}$.

The quotient algebra $\boldsymbol{B_R(\delta)/B_R(\delta)_{>k}}$. In this subsection we investigate certain quotients of $B_R(\delta)$. Given $\lambda \in \Lambda_{r,s}$, we let $\mathrm{def}(\lambda)$ be the *defect* of λ, that is, the number of caps in the cap diagram $\overline{\lambda}$. Also let $\mathrm{rk}(\lambda)$ be the number

of $\circ$'s or the number of $\times$'s in λ, whichever is smaller; it is the number of $\circ$'s if $\delta \geq 0$ and the number of $\times$'s if $\delta \leq 0$. Then set

$$k(\lambda) := \operatorname{def}(\lambda) + \operatorname{rk}(\lambda). \tag{6.17}$$

Lemma 6.12. *Suppose we are given* $\mathtt{S}, \mathtt{T} \in \mathscr{T}_R(\delta)$ *of the same shape* λ.

(1) *In the diagram* $\underline{\mathtt{S}}$, *the number of boundary caps is equal to* $\operatorname{rk}(\lambda)$ *plus the number of boundary cups. Similarly in the diagram* $\overline{\mathtt{T}}$, *the number of boundary cups is equal to* $\operatorname{rk}(\lambda)$ *plus the number of boundary caps.*

(2) *In the diagram* $\underline{\mathtt{S}}\overline{\mathtt{T}}$, *the number of boundary cups is equal to the number of boundary caps.*

Proof. (1) In $\underline{\mathtt{S}}$, the number of boundary cups minus the number of boundary caps is equal to the total number of cups minus the total number of caps. Arguing by induction on $r + s$ using (3.15), this is equal to $\operatorname{rk}(\lambda)$.

(2) Suppose there are a boundary caps and b boundary cups in $\underline{\mathtt{S}}$. Suppose there are c boundary caps and d boundary cups in $\overline{\mathtt{T}}$. Finally suppose that e of the boundary cups in $\underline{\mathtt{S}}$ belong to circles in $\underline{\mathtt{S}}\overline{\mathtt{T}}$. This is also the number of boundary caps in $\overline{\mathtt{T}}$ that belong to circles in $\underline{\mathtt{S}}\overline{\mathtt{T}}$. The total number of boundary cups in $\underline{\mathtt{S}}\overline{\mathtt{T}}$ is $b + d - e$. The total number of boundary caps in $\underline{\mathtt{S}}\overline{\mathtt{T}}$ is $a + c - e$. These are equal as $a = b + \operatorname{rk}(\lambda)$ and $d = c + \operatorname{rk}(\lambda)$ by (1). $\qquad\square$

Now we introduce various other "k-values." For $\mathtt{S}, \mathtt{T} \in \mathscr{T}_R(\delta)$ of the same shape, let $k(\mathtt{S}, \mathtt{T})$ be the number of boundary cups in the diagram $\underline{\mathtt{S}}\overline{\mathtt{T}}$; in view of Lemma 6.12(2) this is the same thing as the number of boundary caps in $\underline{\mathtt{S}}\overline{\mathtt{T}}$. We write simply $k(\mathtt{T})$ for $k(\mathtt{T}, \mathtt{T})$; this is just the number of boundary cups in $\overline{\mathtt{T}}$ or the number of boundary caps in $\underline{\mathtt{T}}$. Finally for $\boldsymbol{i} \in \mathbb{Z}^{r+s}$ with $e(\boldsymbol{i}) \neq 0$, let $k(\boldsymbol{i}) := k(\mathtt{T})$ for any $\mathtt{T} \in \mathscr{T}_R(\delta)$ with $\boldsymbol{i}^{\mathtt{T}} = \boldsymbol{i}$; if $e(\boldsymbol{i}) = 0$ we set $k(\boldsymbol{i}) := \infty$.

Let $B_R(\delta)_k$ be the span of all the basis vectors $f_{\mathtt{ST}} \in B_R(\delta)$ in which $k(\mathtt{S}, \mathtt{T}) = k$. Thus we have the vector space decomposition

$$B_R(\delta) = \bigoplus_{k \geq 0} B_R(\delta)_k.$$

For example, in Figure 1, the vector spaces $B_R(\delta)_1$ and $B_R(\delta)_2$ are of dimensions 20 and 4, respectively, and all other $B_R(\delta)_k$ are zero. Note also when non-zero that the idempotent $e(\boldsymbol{i})$ belongs to $B_R(\delta)_{k(\boldsymbol{i})}$.

For $k \geq 0$, let

$$B_R(\delta)_{\leq k} := \bigoplus_{j=0}^{k} B_R(\delta)_j, \qquad B_R(\delta)_{>k} := \bigoplus_{j>k} B_R(\delta)_j. \tag{6.18}$$

Lemma 6.13. *For any* $k, l \geq 0$, *we have that* $B_R(\delta)_k B_R(\delta)_l \subseteq B_R(\delta)_{\max(k,l)}$.

Proof. Take basis vectors $f_{\mathtt{ST}} \in B_R(\delta)_k$ and $f_{\mathtt{UV}} \in B_R(\delta)_l$ such that $\boldsymbol{i}^{\mathtt{T}} = \boldsymbol{i}^{\mathtt{U}}$. By symmetry, we may as well assume that $k \leq l$ and need to show that $f_{\mathtt{ST}} f_{\mathtt{UV}} \in B_R(\delta)_l$. Of the l boundary cups in $\underline{\mathtt{U}}\overline{\mathtt{V}}$, some of them, say l' of them, do not cross the middle number line. These l' boundary cups clearly remain as boundary cups in all the diagrams produced when $f_{\mathtt{ST}} f_{\mathtt{UV}}$ is computed using the extended surgery procedure. The remaining $l'' := l - l'$ boundary cups cross the middle number line. During the extended surgery procedure, these l'' boundary cups

either remain as boundary cups in the product (when they are joined with anti-clockwise circles) or we get zero (when anything else happens). In this way we see there are at least $l = l' + l''$ boundary cups in all the diagrams produced in the product. The only way we could get more than l boundary cups in the product $f_{\mathtt{ST}}f_{\mathtt{UV}}$ is when some boundary cup from $\underline{\mathtt{S}}\overline{\mathtt{T}}$ gets joined to a pair of vertical line segments in $\underline{\mathtt{U}}\overline{\mathtt{V}}$. If this happens, there must be l boundary caps in between these vertical line segments in the diagram $\underline{\mathtt{S}}\overline{\mathtt{T}}$, so that $\underline{\mathtt{S}}\overline{\mathtt{T}}$ has at least $(l + 1)$ boundary cups. This contradicts the initial assumption that $k \leq l$. $\quad\square$

Lemma 6.13 implies that $B_R(\delta)_{>k}$ is a two-sided ideal and $B_R(\delta)_{\leq k}$ is a subalgebra of $B_R(\delta)$. Moreover the canonical quotient map gives an algebra isomorphism

$$B_R(\delta)_{\leq k} \xrightarrow{\sim} B_R(\delta)/B_R(\delta)_{>k}. \tag{6.19}$$

The identity element in $B_R(\delta)_{\leq k}$ is the sum of the idempotents $e(\boldsymbol{i})$ for $\boldsymbol{i} \in \mathbb{Z}^{r+s}$ with $k(\boldsymbol{i}) \leq k$. Lemma 6.10 gives a canonical way to decompose each such $e(\boldsymbol{i})$ into mutually orthogonal primitive idempotents in $B_R(\delta)_{\leq k}$. In particular the identity element of $B_R(\delta)_{\leq k}$ decomposes as $\sum_{\mathtt{T}} e_{\mathtt{T}}$ summing over all restricted $\mathtt{T}$ with $k(\mathtt{T}) \leq k$. Using (6.19), we get an analogous decomposition of the identity $B_R(\delta)/B_R(\delta)_{>k}$.

Theorem 6.14. *For $\lambda \in \dot{\Lambda}_{r,s}$ with $k(\lambda) \leq k$, the ideal $B_R(\delta)_{>k}$ acts as zero on $D_R(\lambda)$, hence $D_R(\lambda)$ can be viewed as an irreducible $B_R(\delta)/B_R(\delta)_{>k}$-module. The modules*

$$\{D_R(\lambda) \mid \lambda \in \dot{\Lambda}_{r,s}, k(\lambda) \leq k\}$$

give a complete set of pairwise non-isomorphic irreducible $B_R(\delta)/B_R(\delta)_{>k}$-modules.

Proof. The remarks just made show that

$$B_R(\delta)/B_R(\delta)_{>k} = \bigoplus_{\mathtt{T}} B_R(\delta)e_{\mathtt{T}}/B_R(\delta)_{>k}e_{\mathtt{T}}$$

summing over all restricted $\mathtt{T} \in \mathscr{T}_R(\delta)$ with $k(\mathtt{T}) \leq k$. Moreover each of the summands on the right hand side is a non-zero projective indecomposable module. Now suppose that $\mathtt{T}$ is a restricted R-tableau with $k(\mathtt{T}) \leq k$, and set $\lambda := \mathrm{sh}(\mathtt{T})$. Lemma 6.9 implies that the head of $B_R(\delta)e_{\mathtt{T}}/B_R(\delta)_{>k}e_{\mathtt{T}}$ is isomorphic (up to degree shift) to $D_R(\lambda)$. Moreover Lemma 6.12(1) implies that $k(\mathtt{T}) = k(\lambda)$, hence $k(\lambda) \leq k$. The theorem follows from these observations. $\quad\square$

We end with a couple more results about $B_R(\delta)/B_R(\delta)_{>k}$ which will be needed later. The following lemma implies that the maps $\iota_{R;i}^{RE}$ and $\iota_{R;i}^{RF}$ from (6.5) factor to induce well-defined maps

$$\iota_{R;i}^{RE} : B_R(\delta)/B_R(\delta)_{>k} \to B_{RE}(\delta)/B_{RE}(\delta)_{>k}, \tag{6.20}$$

$$\iota_{R;i}^{RF} : B_R(\delta)/B_R(\delta)_{>k} \to B_{RF}(\delta)/B_{RF}(\delta)_{>k}. \tag{6.21}$$

Lemma 6.15. *We have that $\iota_{R;i}^{RE}(B_R(\delta)_{>k}) \subseteq B_{RE}(\delta)_{>k}$ and $\iota_{R;i}^{RF}(B_R(\delta)_{>k}) \subseteq B_{RF}(\delta)_{>k}$.*

Proof. We just consider $\iota_{R;i}^{RE}$. Take $f_{\text{ST}} \in B_R(\delta)_{>k}$, i.e. $\underline{\text{S}}\overline{\text{T}}$ has more than k boundary cups. To apply the map $\iota_{R;i}^{RE}$, we use the algorithm explained at the end of the previous section, and need to show that the number of boundary cups does not get smaller. This is immediate for all except the first configuration in (5.20). For the first configuration it follows from the explicit form of the surgery procedure in [BS2, (6.2)]. $\square$

Lemma 6.16. *We have that $B_R(\delta)_{>0} = \langle e(\boldsymbol{i}) \,|\, \text{for all } \boldsymbol{i} \in \mathbb{Z}^{r+s} \text{ with } k(\boldsymbol{i}) > 0\rangle$.*

Proof. All the $e(\boldsymbol{i})$'s with $k(\boldsymbol{i}) > 0$ belong to $B_R(\delta)_{>0}$. Conversely if $f_{\text{ST}} \in B_R(\delta)_{>0}$, then there is at least one boundary cup in the diagram $\underline{\text{S}}\overline{\text{T}}$. This means either that $\overline{\text{T}}$ has a boundary cup or $\underline{\text{S}}$ has a boundary cup. In the former case, $\boldsymbol{i} := \boldsymbol{i}^{\text{T}}$ has $k(\boldsymbol{i}) > 0$ and $f_{\text{ST}} \in B_R(\delta)e(\boldsymbol{i})$. In the latter case we deduce from Lemma 6.12(1) that $\underline{\text{S}}$ also has a boundary cap, so $\boldsymbol{i} := \boldsymbol{i}^{\text{S}}$ has $k(\boldsymbol{i}) > 0$ and $f_{\text{ST}} \in e(\boldsymbol{i})B_R(\delta)$. $\square$

Remark 6.17. For $k > 0$, the ideal $B_R(\delta)_{>k}$ need not be generated by idempotents. See Figure 1 taking $k = 1$ for an example in which it is not.

7. Mixed Schur-Weyl duality for $\mathfrak{gl}_{m|n}(\mathbb{C})$

Fix integers $m, n \geq 0$ and set $\delta := m - n$. In order to identify the walled Brauer algebra $B_{r,s}(\delta)$ with the graded walled Brauer algebra $B_R(\delta)$, we first need to establish a Schur-Weyl duality connecting $B_{r,s}(\delta)$ to the general linear Lie superalgebra $\mathfrak{gl}_{m|n}(\mathbb{C})$. Our approach to this follows the same strategy as the proof of [N, Theorem 3.4] (which treats $\mathfrak{gl}_m(\mathbb{C})$ rather than $\mathfrak{gl}_{m|n}(\mathbb{C})$). The results in this section have also been obtained independently by Comes and Wilson [CW] by a similar trick; actually their exposition is slicker since they exploit fully the tensor category formalism. We also note a partial result along the same lines was obtained earlier in [SM].

The general linear Lie superalgebra. As a general convention, for a homogeneous vector v in a vector superspace, we will write $|v| \in \mathbb{Z}_2$ for its parity. Let V be the vector superspace with homogeneous basis $v_1, \ldots, v_{m+n}$ and $|v_i| := |i| \in \mathbb{Z}_2$, where $|i| := 0$ if $1 \leq i \leq m$ and $|i| := 1$ if $m + 1 \leq i \leq m + n$. The Lie superalgebra $\mathfrak{g} := \mathfrak{gl}_{m|n}(\mathbb{C})$ is the vector superspace $\text{End}(V)$ of all (not necessarily homogeneous) linear endomorphisms of V, with superbracket $[x, y] := xy - (-1)^{|x||y|}yx$ for homogeneous $x, y \in \text{End}(V)$. Let $U(\mathfrak{g})$ be the universal enveloping superalgebra of $\mathfrak{g}$, which is a Hopf superalgebra with comultiplication Δ and antipode S defined on $x \in \mathfrak{g}$ by $\Delta(x) = x \otimes 1 + 1 \otimes x$ and $S(x) = -x$. We recall in particular that the multiplication on $U(\mathfrak{g}) \otimes U(\mathfrak{g})$ is by $(a \otimes b)(c \otimes d) = (-1)^{|b||c|}ac \otimes bd$ for homogeneous $a, b, c, d \in U(\mathfrak{g})$.

Let $W := V^*$ be the dual of the tautological $\mathfrak{g}$-supermodule V, with basis $w_1, \ldots, w_{m+n}$ that is dual to the given basis for V, so $w_i(v_j) = \delta_{i,j}$. It is a $\mathfrak{g}$-supermodule via the usual rule $(xw)(v) := (-1)^{|x||w|}w(S(x)v)$ for homogeneous $x \in \mathfrak{g}, v \in V, w \in W$. In particular, the standard basis elements of $\mathfrak{g}$ consisting of matrix units act on the bases of V and W by the formulae

$$e_{i,j}v_k = \delta_{j,k}v_i, \qquad e_{i,j}w_k = -\delta_{i,k}(-1)^{(|i|+|j|)|i|}w_j \qquad (7.1)$$

for all $1 \leq i, j, k \leq m + n$. If M and N are $\mathfrak{g}$-supermodules, we write $\mathrm{Hom}_{\mathfrak{g}}(M, N)$ for the vector superspace of all $\mathfrak{g}$-supermodule homomorphisms from M to N. By definition, this is the set of all (not necessarily homogeneous) linear maps $f \in \mathrm{Hom}(M, N)$ annihilated by all $x \in \mathfrak{g}$, where the linear action of $\mathfrak{g}$ on $\mathrm{Hom}(M, N)$ is by $(xf)(m) = x(f(m)) - (-1)^{|x||f|}f(xm)$ for homogeneous $x \in \mathfrak{g}, f \in \mathrm{Hom}(M, N)$ and $m \in M$.

An important role is played by the even element

$$\Omega := \sum_{i,j=1}^{m+n} (-1)^{|j|} e_{i,j} \otimes e_{j,i} \in U(\mathfrak{g}) \otimes U(\mathfrak{g}). \tag{7.2}$$

This corresponds to the (invariant) supertrace form on $\mathfrak{g}$, so Ω commutes with $\Delta(x) = x \otimes 1 + 1 \otimes x$ for all $x \in \mathfrak{g}$. Left multiplication by Ω defines a $\mathfrak{g}$-supermodule endomorphism of the tensor product $M \otimes N$ of $\mathfrak{g}$-supermodules M and N. More generally, for $1 \leq a < b \leq r + s$, we let

$$\Omega_{a,b} := \sum_{i,j=1}^{m+n} (-1)^{|j|} 1^{\otimes(a-1)} \otimes e_{i,j} \otimes 1^{\otimes(b-a-1)} \otimes e_{j,i} \otimes 1^{r+s-b} \in U(\mathfrak{g})^{\otimes(r+s)}, \tag{7.3}$$

which defines a $\mathfrak{g}$-supermodule endomorphism of $M_1 \otimes \cdots \otimes M_{r+s}$ given any $\mathfrak{g}$-supermodules $M_1, \ldots, M_{r+s}$.

Lemma 7.1. *For $M, N \in \{V, W\}$, the endomorphism Ω of $M \otimes N$ satisfies*

$$\Omega(v_i \otimes v_j) = (-1)^{|i||j|} v_j \otimes v_i,$$

$$\Omega(w_i \otimes w_j) = (-1)^{|i||j|} w_j \otimes w_i,$$

$$\Omega(v_i \otimes w_j) = -\delta_{i,j}(-1)^{|i|} \sum_{k=1}^{m+n} v_k \otimes w_k,$$

$$\Omega(w_i \otimes v_j) = -\delta_{i,j} \sum_{k=1}^{m+n} (-1)^{|k|} w_k \otimes v_k,$$

for all $1 \leq i, j \leq m + n$.

Proof. Use (7.1), the definition of Ω from (7.2), and recall that the action is by $(a \otimes b)(v \otimes w) = (-1)^{|b||v|} av \otimes bw$. $\square$

Actions on tensor space. The next goal is to define right actions of the algebras $\mathbb{C}\Sigma_{r+s}$ and $B_{r,s}(\delta)$ on the spaces $V^{\otimes(r+s)}$ and $V^{\otimes r} \otimes W^{\otimes s}$, respectively. One way to do this is to write down the actions of generators and then verify relations. For our purposes, it is better to take a different approach which gives explicit closed formulae for the actions of arbitrary basis elements, not just generators. We need some multi-index notation. Let $I := \{1, \ldots, m+n\}$. For any $t \geq 0$, the symmetric group Σ_t acts naturally on the right on the set I^t by place permutation, so $\boldsymbol{i} \cdot \sigma = (i_{\sigma(1)}, \ldots, i_{\sigma(t)})$ for $\boldsymbol{i} = (i_1, \ldots, i_t) \in I^t$ and $\sigma \in \Sigma_t$. For $\boldsymbol{i} \in I^t$, we let

$$|\boldsymbol{i}| := |i_1| + \cdots + |i_t|, \qquad p(\boldsymbol{i}) := \sum_{1 \leq a < b \leq t} |i_a||i_b|. \tag{7.4}$$

Given $\boldsymbol{i} = (i_1, \ldots, i_{r+s}) \in I^{r+s}$, we let $\boldsymbol{i}^{\mathrm{L}} := (i_1, \ldots, i_r) \in I^r$ and $\boldsymbol{i}^{\mathrm{R}} := (i_{r+1}, \ldots, i_{r+s}) \in I^s$, so that $\boldsymbol{i} = \boldsymbol{i}^{\mathrm{L}}\boldsymbol{i}^{\mathrm{R}}$ where the product on the right is by concatenation of tuples. The $\mathfrak{g}$-supermodules $V^{\otimes r}$ and $W^{\otimes s}$ have the obvious bases $\{v_{\boldsymbol{i}} \,|\, \boldsymbol{i} \in I^r\}$ and $\{w_{\boldsymbol{j}} \,|\, \boldsymbol{j} \in I^s\}$, respectively, writing $v_{\boldsymbol{i}} := v_{i_1} \otimes \cdots \otimes v_{i_r}$ and $w_{\boldsymbol{j}} := w_{j_1} \otimes \cdots \otimes w_{j_s}$. Similarly $V^{\otimes(r+s)}$ has basis $\{v_{\boldsymbol{i}} = v_{\boldsymbol{i}^{\mathrm{L}}} \otimes v_{\boldsymbol{i}^{\mathrm{R}}} \,\big|\, \boldsymbol{i} \in I^{r+s}\}$ and $V^{\otimes r} \otimes W^{\otimes s}$ has basis $\{v_{\boldsymbol{i}^{\mathrm{L}}} \otimes w_{\boldsymbol{i}^{\mathrm{R}}} \,\big|\, \boldsymbol{i} \in I^{r+s}\}$.

Now suppose we are given a diagram σ, which could either be a permutation diagram from Σ_{r+s} or a walled Brauer diagram in $B_{r,s}(\delta)$. For $\boldsymbol{i}, \boldsymbol{j} \in I^{r+s}$, let $_{\boldsymbol{i}}\sigma_{\boldsymbol{j}}$ be the labelled diagram obtained by colouring the vertices at the bottom of σ by $i_1, \ldots, i_{r+s}$ and the vertices at the top of σ by $j_1, \ldots, j_{r+s}$ (in order from left to right as usual). We say $_{\boldsymbol{i}}\sigma_{\boldsymbol{j}}$ is *consistently coloured* if the vertices at the ends of each strand are coloured in the same way, in which case we get induced a well-defined colouring of the strands themselves. In a consistently coloured permutation diagram $_{\boldsymbol{i}}\sigma_{\boldsymbol{j}}$ we of course have that $\boldsymbol{j} = \boldsymbol{i} \cdot \sigma$, so the colouring $\boldsymbol{i}$ at the bottom uniquely determines the colouring $\boldsymbol{j}$ at the top; this is not in general true for walled Brauer diagrams.

For a permutation diagram or a walled Brauer diagram σ, and $\boldsymbol{i}, \boldsymbol{j} \in I^{r+s}$, we introduce the *weight*

$$\mathrm{wt}(_{\boldsymbol{i}}\sigma_{\boldsymbol{j}}) := \begin{cases} \prod_c (-1)^{|c|} \cdot \prod_h (-1)^{|h|} & \text{if } _{\boldsymbol{i}}\sigma_{\boldsymbol{j}} \text{ is consistently coloured,} \\ 0 & \text{otherwise,} \end{cases} \qquad (7.5)$$

where

(1) the first product is over all *proper* coloured crossings c in $_{\boldsymbol{i}}\sigma_{\boldsymbol{j}}$, i.e. the crossings that involve two different strands rather than self-intersections;
(2) the second product is over all coloured horizontal strands h in $_{\boldsymbol{i}}\sigma_{\boldsymbol{j}}$ whose endpoints are on the bottom edge (if σ is a permutation diagram this product is empty so can be omitted);
(3) the parity $|c|$ of a coloured crossing $c = \underset{i \quad j}{\times}$ of strands of colours i and j is $|i||j|$;
(4) the parity $|h|$ of a coloured horizontal strand $h = \underset{k \ k}{\cap}$ of colour k is $|k|$.

For example, for $\boldsymbol{i} = (2, 2, 1, 2)$ and $\boldsymbol{j} = (3, 2, 3, 1)$, the following consistently coloured diagram has $\mathrm{wt}(_{\boldsymbol{i}}\sigma_{\boldsymbol{j}}) = 1$ either if $m = 1, n = 2$ or if $m = 2, n = 1$:
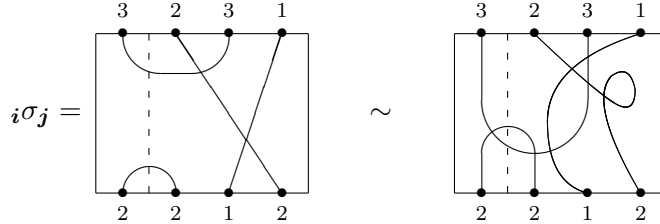
Lemma 7.2. *The weight* $\mathrm{wt}(_{\boldsymbol{i}}\sigma_{\boldsymbol{j}})$ *depends only on the isotopy class of* σ.

Proof. Check that $\mathrm{wt}(_{\boldsymbol{i}}\sigma_{\boldsymbol{j}})$ does not change when any of the three Reidemeister moves are applied to σ. $\qquad\square$

In the following two lemmas, we construct the required actions of $\mathbb{C}\Sigma_{r+s}$ and $B_{r,s}(\delta)$ on $V^{\otimes(r+s)}$ and $V^{\otimes r} \otimes W^{\otimes s}$, respectively. The first of these is well known; see e.g. [BR, Corollary 1.6]. We include the proof only to prepare the reader for the more complicated walled Brauer case in the second lemma.

Lemma 7.3. *There is a well-defined right $\mathbb{C}\Sigma_{r+s}$-module structure on $V^{\otimes(r+s)}$ such that*

$$v_{\boldsymbol{i}} \cdot \sigma = \sum_{\boldsymbol{j} \in I^{r+s}} \mathrm{wt}(_{\boldsymbol{i}}\sigma_{\boldsymbol{j}})\, v_{\boldsymbol{j}} \tag{7.6}$$

for all $\boldsymbol{i} \in I^{r+s}$ and $\sigma \in \Sigma_{r+s}$. Moreover for $1 \leq a < b \leq r+s$ the transposition $(a\,b)$ acts in the same way as the operator $\Omega_{a,b}$ from (7.3). Hence the action of $\mathbb{C}\Sigma_{r+s}$ commutes with the action of $\mathfrak{g}$.

Proof. Once the first statement is proved, the second statement is an easy consequence using also Lemma 7.1. Then the final statement follows because $\Omega_{a,b}$ acts as a $\mathfrak{g}$-supermodule endomorphism and the transpositions generate Σ_{r+s}. To prove the first statement, take $\sigma, \tau \in \Sigma_{r+s}$ and set $\pi := \sigma\tau$. We need to check that (7.6) is consistent with the associativity equation $(v_{\boldsymbol{i}} \cdot \sigma) \cdot \tau = v_{\boldsymbol{i}} \cdot \pi$ for all $\boldsymbol{i} \in I^{r+s}$, i.e.

$$\sum_{\boldsymbol{j},\boldsymbol{k} \in I^{r+s}} \mathrm{wt}(_{\boldsymbol{i}}\sigma_{\boldsymbol{j}})\mathrm{wt}(_{\boldsymbol{j}}\tau_{\boldsymbol{k}})\, v_{\boldsymbol{k}} = \sum_{\boldsymbol{k} \in I^{r+s}} \mathrm{wt}(_{\boldsymbol{i}}\pi_{\boldsymbol{k}})\, v_{\boldsymbol{k}}.$$

Equating coefficients, we therefore need to see that

$$\sum_{\boldsymbol{j} \in I^{r+s}} \mathrm{wt}(_{\boldsymbol{i}}\sigma_{\boldsymbol{j}})\mathrm{wt}(_{\boldsymbol{j}}\sigma_{\boldsymbol{k}}) = \mathrm{wt}(_{\boldsymbol{i}}\pi_{\boldsymbol{k}})$$

for all $\boldsymbol{i}, \boldsymbol{k} \in I^{r+s}$. Both sides are zero unless $_{\boldsymbol{i}}\pi_{\boldsymbol{k}}$ is consistently coloured, so we can assume $\boldsymbol{k} = \boldsymbol{i} \cdot \pi$. Then the sum on the left involves just one non-zero term, so we can also assume that $\boldsymbol{j} := \boldsymbol{i} \cdot \sigma$, and are reduced to checking for this $\boldsymbol{j}$ and $\boldsymbol{k}$ that $\mathrm{wt}(_{\boldsymbol{i}}\sigma_{\boldsymbol{j}})\mathrm{wt}(_{\boldsymbol{j}}\tau_{\boldsymbol{k}}) = \mathrm{wt}(_{\boldsymbol{i}}\pi_{\boldsymbol{k}})$. This is clear from Lemma 7.2. $\square$

Lemma 7.4. *There is a well-defined right $B_{r,s}(\delta)$-module structure on $V^{\otimes r} \otimes W^{\otimes s}$ such that*

$$v_{\boldsymbol{i}^{\mathrm{L}}} \otimes w_{\boldsymbol{i}^{\mathrm{R}}} \cdot \sigma := \sum_{\boldsymbol{j} \in I^{r+s}} \mathrm{wt}(_{\boldsymbol{i}}\sigma_{\boldsymbol{j}})\, v_{\boldsymbol{j}^{\mathrm{L}}} \otimes w_{\boldsymbol{j}^{\mathrm{R}}} \tag{7.7}$$

for all $\boldsymbol{i} \in I^{r+s}$ and all walled Brauer diagrams $\sigma \in B_{r,s}(\delta)$. Moreover for $1 \leq a < b \leq r+s$ the "transposition" $\overline{(a\,b)}$ acts in the same way as the operator $\Omega_{a,b}$ from (7.3). Hence the action of $B_{r,s}(\delta)$ commutes with the action of $\mathfrak{g}$.

Proof. We just check the first statement. Let σ, τ and π be walled Brauer diagrams such that $\sigma\tau = \delta^t\pi$ for $t \geq 0$. Proceeding exactly as in the proof of Lemma 7.3, one reduces to checking the identity

$$\sum_{\boldsymbol{j} \in I^{r+s}} \mathrm{wt}(_{\boldsymbol{i}}\sigma_{\boldsymbol{j}})\mathrm{wt}(_{\boldsymbol{j}}\tau_{\boldsymbol{k}}) = \delta^t \mathrm{wt}(_{\boldsymbol{i}}\pi_{\boldsymbol{k}}) \tag{7.8}$$

for all $\boldsymbol{i}, \boldsymbol{k} \in I^{r+s}$. Both sides are zero unless $_{\boldsymbol{i}}\pi_{\boldsymbol{k}}$ is consistently coloured, so assume this is the case from now on. It is convenient to think in terms of the composite diagram $\sigma|\tau$ that is σ drawn under τ separated by a horizontal boundary line on which the vertices at the top of σ are identified with the vertices at the bottom of τ. Let $_{\boldsymbol{i}}\sigma_{\boldsymbol{j}}\tau_{\boldsymbol{k}}$ be the coloured diagram obtained by colouring the vertices of $\sigma|\tau$ on the bottom edge by $\boldsymbol{i}$, the vertices on the middle boundary line by $\boldsymbol{j}$, and the vertices on the top edge by $\boldsymbol{k}$.

The diagram π is obtained from $\sigma|\tau$ by removing the middle boundary line together with t closed circles from the interior of the diagram. Enumerate these circles by $1, \ldots, t$ in some order. Given $\boldsymbol{j} = (j_1, \ldots, j_t) \in I^t$, there is a unique $\tilde{\boldsymbol{j}} \in I^{r+s}$ such that ${}_i\sigma_{\tilde{\boldsymbol{j}}}$ and ${}_{\tilde{\boldsymbol{j}}}\tau_{\boldsymbol{k}}$ are consistently coloured and the ath circle in ${}_i\sigma_{\tilde{\boldsymbol{j}}}\tau_{\boldsymbol{k}}$ is of colour j_a for each $a = 1, \ldots, t$. In this notation, the identity we are trying to prove is equivalent to

$$\sum_{\boldsymbol{j} \in I^t} \mathrm{wt}({}_i\sigma_{\tilde{\boldsymbol{j}}})\mathrm{wt}({}_{\tilde{\boldsymbol{j}}}\tau_{\boldsymbol{k}}) = \delta^t \mathrm{wt}({}_i\pi_{\boldsymbol{k}}), \qquad (7.9)$$

since all other summands on the left hand side of (7.8) are zero. To prove (7.9), we claim for a fixed $\boldsymbol{j} \in I^t$ that

$$\mathrm{wt}({}_i\sigma_{\tilde{\boldsymbol{j}}})\mathrm{wt}({}_{\tilde{\boldsymbol{j}}}\tau_{\boldsymbol{k}}) = (-1)^{|\boldsymbol{j}|}\mathrm{wt}({}_i\pi_{\boldsymbol{k}}), \qquad (7.10)$$

which implies (7.9) since $\sum_{\boldsymbol{j} \in I^t}(-1)^{|\boldsymbol{j}|} = \delta^t$. To prove (7.10), we compute the contribution to $\mathrm{wt}({}_i\sigma_{\tilde{\boldsymbol{j}}})\mathrm{wt}({}_{\tilde{\boldsymbol{j}}}\tau_{\boldsymbol{k}})\mathrm{wt}({}_i\pi_{\boldsymbol{k}})$ coming from each connected component of $\sigma|\tau$ in turn, showing for $1 \le a \le t$ that the ath circle contributes a factor of $(-1)^{|j_a|}$ and that all other components contribute the factor 1.

To start with, consider the ath circle for some $1 \le a \le t$, which of course is a connected component of $\sigma|\tau$ which does not appear at all in π. This circle is the concatenation $\alpha|\beta$ of two diagrams α and β, the first consisting of z horizontal strands from the top of σ and the second consisting of z horizontal strands from the bottom of τ for some $z \ge 1$. Proceeding by induction on z, one checks that the number of proper crossings in α plus the number of proper crossings in β is equal to $(z - 1)$. Each of these $(z - 1)$ proper crossings contributes a factor of $(-1)^{|j_a|}$ to $\mathrm{wt}({}_i\sigma_{\tilde{\boldsymbol{j}}})\mathrm{wt}({}_{\tilde{\boldsymbol{j}}}\tau_{\boldsymbol{k}})\mathrm{wt}({}_i\pi_{\boldsymbol{k}})$, as does each of the z horizontal strands in β. Since all other connected components of $\sigma|\tau$ cross the circle $\alpha|\beta$ an even number of times, we conclude that the total contribution to $\mathrm{wt}({}_i\sigma_{\tilde{\boldsymbol{j}}})\mathrm{wt}({}_{\tilde{\boldsymbol{j}}}\tau_{\boldsymbol{k}})\mathrm{wt}({}_i\pi_{\boldsymbol{k}})$ coming from *all* crossings or horizontal strands involving at least one strand of α or β is equal to $(-1)^{|j_a|}$, which is what we wanted.

There are two more types of connected component to consider, corresponding to the horizontal and vertical strands in π. The argument for these is quite similar, so let us just explain in the case of a horizontal strand γ whose endpoints lie on the bottom edge of π. The corresponding connected component of $\sigma|\tau$ is of the form $\alpha|\beta$, where α consists of one or more strands from σ and β consists of zero or more strands from τ. Like in the previous paragraph, the number of proper crossings in α plus the number of proper crossings in β is one less than the total number of horizontal strands in β. These proper crossings become self-intersections in γ, and there is exactly one horizontal strand in γ by definition, so all the contributions from these crossings and strands cancel out in the product $\mathrm{wt}({}_i\sigma_{\tilde{\boldsymbol{j}}})\mathrm{wt}({}_{\tilde{\boldsymbol{j}}}\tau_{\boldsymbol{k}})\mathrm{wt}({}_i\pi_{\boldsymbol{k}})$. Combined with Lemma 7.2 this is enough to complete the proof. $\qquad \square$

Mixed Schur-Weyl duality. The right actions constructed in Lemmas 7.3–7.4 induce algebra homomorphisms

$$\Phi_{r+s}^{m,n} : \mathbb{C}\Sigma_{r+s} \to \mathrm{End}_{\mathfrak{g}}(V^{\otimes(r+s)})^{\mathrm{op}}, \qquad (7.11)$$

$$\Psi_{r,s}^{m,n} : B_{r,s}(\delta) \to \mathrm{End}_{\mathfrak{g}}(V^{\otimes r} \otimes W^{\otimes s})^{\mathrm{op}}. \qquad (7.12)$$

The first of these homomorphisms has been extensively studied in the literature starting from the works of Sergeev [S] and Berele-Regev [BR]. In particular, we have the following well-known result.

Theorem 7.5 (Sergeev, Berele-Regev). *The map*

$$\Phi_{r,s}^{m,n} : \mathbb{C}\Sigma_{r+s} \to \mathrm{End}_{\mathfrak{g}}(V^{\otimes(r+s)})^{\mathrm{op}}$$

from (7.11) is surjective. It is injective if and only if $r + s < (m+1)(n+1)$.

Proof. The surjectivity follows from [BR, Remark 4.15]. For the injectivity, [BR, Theorem 3.20] implies that $\Phi_{r,s}^{m,n}$ is injective if and only if all partitions of $(r+s)$ are (m,n)-*hook partitions* in the sense of [BR, Definition 2.3], i.e. their Young diagrams do not contain the box in row $(m+1)$ and column $(n+1)$. This is easily seen to be equivalent to the condition $r + s < (m+1)(n+1)$. (All the results just cited from [BR] were also established independently in [S].) □

We now want to prove that the analogue of Theorem 7.5 holds for the walled Brauer algebra (with $\Phi_{r+s}^{m,n}$ replaced by $\Psi_{r,s}^{m,n}$). We begin by recalling some standard facts about the tensor category of finite dimensional $\mathfrak{g}$-supermodules. Let K, L, M and N be finite dimensional vector superspaces. We identify $M \otimes N = N \otimes M$ so that $m \otimes n = (-1)^{|m||n|} n \otimes m$ for homogeneous $m \in M, n \in N$ (and $\otimes$ always means $\otimes_{\mathbb{C}}$). Similarly, we identify

$$\mathrm{Hom}(M, N) \otimes \mathrm{Hom}(K, L) = \mathrm{Hom}(M \otimes K, N \otimes L) \qquad (7.13)$$

so that $f \otimes g \in \mathrm{Hom}(M, N) \otimes \mathrm{Hom}(K, L)$ is identified with the homomorphism in $\mathrm{Hom}(M \otimes K, N \otimes L)$ defined from $(f \otimes g)(m \otimes k) := (-1)^{|g||m|} f(m) \otimes g(k)$ for homogeneous g and m. Finally we identify M with M^{**}, so a homogeneous vector $m \in M$ is identified with the function $m \in M^{**}$ defined from $m(\mu) = (-1)^{|m||\mu|}\mu(m)$ for all homogeneous $\mu \in M^*$. As usual, for homogeneous $f \in \mathrm{Hom}(M, N)$, we have the dual map $f^* \in \mathrm{Hom}(N^*, M^*)$ defined from $f(\nu)(m) = (-1)^{|f||\nu|}\nu(f(m))$ for homogeneous $\nu \in N^*, m \in M$, and $f^{**} = f$.

The canonical isomorphism $N \otimes L^* \xrightarrow{\sim} \mathrm{Hom}(L, N), f \mapsto \widetilde{f}$ defined from $\widetilde{n \otimes \lambda}(l) = \lambda(l)n$ for $n \in N, \lambda \in L^*, l \in L$ induces the isomorphism

$$\mathrm{adj} : \mathrm{Hom}(M, N \otimes L^*) \xrightarrow{\sim} \mathrm{Hom}(M \otimes L, N), \qquad f \mapsto \widehat{f} \qquad (7.14)$$

such that $\widehat{f}(m \otimes l) = \widetilde{f(m)}(l)$ for $m \in M, l \in L$. If M, N and L are all $\mathfrak{g}$-supermodules then this is a $\mathfrak{g}$-supermodule isomorphism, so it restricts to an isomorphism $\mathrm{Hom}_{\mathfrak{g}}(M, N \otimes L^*) \cong \mathrm{Hom}_{\mathfrak{g}}(M \otimes L, N)$. This is the canonical adjunction making $(? \otimes L, ? \otimes L^*)$ into an adjoint pair of functors on the category of finite dimensional $\mathfrak{g}$-supermodules.

Lemma 7.6. *Let K, L, M and N be finite dimensional $\mathfrak{g}$-supermodules. The map* flip *defined by the following commutative diagram is a $\mathfrak{g}$-supermodule isomorphism:*

$$\begin{array}{ccc} \mathrm{Hom}(M \otimes K, N \otimes L) & \xrightarrow{\text{flip}} & \mathrm{Hom}(M \otimes L^*, N \otimes K^*) \\ \| & & \| \\ \mathrm{Hom}(M, N) \otimes \mathrm{Hom}(K, L) & \xrightarrow{f \otimes g \mapsto f \otimes g^*} & \mathrm{Hom}(M, N) \otimes \mathrm{Hom}(L^*, K^*). \end{array}$$

Proof. Let us instead define flip so that the following diagram commutes:

$$\begin{array}{ccc} \mathrm{Hom}(M \otimes K, N \otimes L) & \xrightarrow{\text{flip}} & \mathrm{Hom}(M \otimes L^*, N \otimes K^*) \\ \| & & \downarrow{\text{adj}} \\ \mathrm{Hom}(M \otimes K, N \otimes L^{**}) & & \mathrm{Hom}(M \otimes L^* \otimes K, N) \\ {\text{adj}}\downarrow & & \| \\ \mathrm{Hom}(M \otimes K \otimes L^*, N) & =\!=\!= & \mathrm{Hom}(M \otimes K \otimes L^*, N). \end{array}$$

Recalling (7.14), all of the other maps in this diagram are $\mathfrak{g}$-supermodule isomorphisms, hence flip is one too. To complete the proof of the lemma, it remains to take $f \in \mathrm{Hom}(M, N)$ and $g \in \mathrm{Hom}(K, L)$ and check that $\mathrm{flip}(f \otimes g) = f \otimes g^*$, which is a routine calculation. $\square$

Like in (7.13), it is natural to identify $W^{\otimes s}$ with $(V^{\otimes s})^*$ so that

$$w_{\boldsymbol{i}}(v_{\boldsymbol{j}}) = (-1)^{p(\boldsymbol{i})}\delta_{\boldsymbol{i},\boldsymbol{j}} \tag{7.15}$$

for all $\boldsymbol{i}, \boldsymbol{j} \in I^s$, recalling (7.4). If we apply Lemma 7.6 to $M = N = V^{\otimes r}$ and $K = L = V^{\otimes s}$, we see that the map $\mathrm{flip}_{r,s}$ defined by the following commutative diagram is a $\mathfrak{g}$-supermodule isomorphism:

$$\begin{array}{ccc} \mathrm{End}(V^{\otimes(r+s)}) & \xrightarrow{\text{flip}_{r,s}} & \mathrm{End}(V^{\otimes r} \otimes W^{\otimes s}) \\ \| & & \| \\ \mathrm{End}(V^{\otimes r}) \otimes \mathrm{End}(V^{\otimes s}) & \xrightarrow{f \otimes g \mapsto f \otimes g^*} & \mathrm{End}(V^{\otimes r}) \otimes \mathrm{End}(W^{\otimes s}). \end{array} \tag{7.16}$$

The main point now is to see that this map is consistent with the linear isomorphism $\mathrm{flip}_{r,s} : \mathbb{C}\Sigma_{r+s} \to B_{r,s}(\delta)$ from (2.1) in the following sense.

Lemma 7.7. *The following diagram commutes:*

$$\begin{array}{ccc} \mathbb{C}\Sigma_{r+s} & \xrightarrow{\text{flip}_{r,s}} & B_{r,s}(\delta) \\ \Phi^{m,n}_{r+s}\downarrow & & \downarrow\Psi^{m,n}_{r,s} \\ \mathrm{End}(V^{\otimes(r+s)}) & \xrightarrow{\text{flip}_{r,s}} & \mathrm{End}(V^{\otimes r} \otimes W^{\otimes s}). \end{array} \tag{7.17}$$

Proof. This is just a direct calculation but the signs are tricky so we go through the details carefully. For $\boldsymbol{i}, \boldsymbol{j} \in I^r$, let $e_{\boldsymbol{i},\boldsymbol{j}} \in \mathrm{End}(V^{\otimes r})$ be the matrix unit defined from $e_{\boldsymbol{i},\boldsymbol{j}}(v_{\boldsymbol{k}}) = \delta_{\boldsymbol{j},\boldsymbol{k}}v_{\boldsymbol{i}}$. Similarly for $\boldsymbol{i}, \boldsymbol{j} \in I^s$, let $f_{\boldsymbol{i},\boldsymbol{j}} \in \mathrm{End}(W^{\otimes s})$ be the

matrix unit defined from $f_{i,j}(w_k) = \delta_{j,k} w_i$. Using (7.15), it is straightforward to check for $i, j \in I^s$ that

$$(e_{i,j})^* = (-1)^{(|i|+|j|)|i|+p(i)+p(j)} f_{j,i}. \tag{7.18}$$

Now take $\sigma \in \Sigma_{r+s}$ and let $\tau := \text{flip}_{r,s}(\sigma) \in B_{r,s}(\delta)$. Using (7.6) and the identification of $\text{End}(V^{\otimes(r+s)})$ with $\text{End}(V^{\otimes r}) \otimes \text{End}(V^{\otimes s})$, we see that

$$\Phi_{r+s}^{m,n}(\sigma) = \sum_{i,j \in I^{r+s}} (-1)^{(|i^{\text{R}}|+|j^{\text{R}}|)|i^{\text{L}}|} \text{wt}(_i\sigma_j) \, e_{j^{\text{L}},i^{\text{L}}} \otimes e_{j^{\text{R}},i^{\text{R}}}. \tag{7.19}$$

Similarly using (7.7) we have that

$$\Psi_{r,s}^{m,n}(\tau) = \sum_{i,j \in I^{r+s}} (-1)^{(|i^{\text{R}}|+|j^{\text{R}}|)|i^{\text{L}}|} \text{wt}(_i\tau_j) \, e_{j^{\text{L}},i^{\text{L}}} \otimes f_{j^{\text{R}},i^{\text{R}}}. \tag{7.20}$$

Reindex the summation on the right hand side of (7.19) to switch the roles of i^{R} and j^{R}, then apply the map $\text{flip}_{r,s}$ from (7.16) and use (7.18) to deduce that

$$\text{flip}_{r,s}(\Phi_{r+s}^{m,n}(\sigma)) = \sum_{i,j \in I^{r+s}} (-1)^{(|i^{\text{R}}|+|j^{\text{R}}|)|i|+p(i^{\text{R}})+p(j^{\text{R}})} \text{wt}(_{i^{\text{L}}j^{\text{R}}}\sigma_{j^{\text{L}}i^{\text{R}}}) \, e_{j^{\text{L}},i^{\text{L}}} \otimes f_{j^{\text{R}},i^{\text{R}}}.$$

Comparing with (7.20) the proof of the lemma is now reduced to verifying the combinatorial identity

$$(-1)^{(|i^{\text{R}}|+|j^{\text{R}}|)|i^{\text{R}}|+p(i^{\text{R}})+p(j^{\text{R}})} \text{wt}(_{i^{\text{L}}j^{\text{R}}}\sigma_{j^{\text{L}}i^{\text{R}}}) = \text{wt}(_i\tau_j) \tag{7.21}$$

for any $i, j \in I^{r+s}$. It is clear that both sides of (7.21) are zero unless $_i\tau_j$ is consistently coloured. Assume this is the case from now on.

Partition the set $\{r+1, \ldots, r+s\}$ as $\{a_1 < \cdots < a_k\} \sqcup \{b_1 < \cdots < b_l\}$ so that the vertices $a_1, \ldots, a_k$ on the bottom edge are at the ends of Horizontal strands of τ and the vertices $b_1, \ldots, b_l$ on the bottom edge are at the ends of Vertical strands of τ. Let $i^{\text{RH}} := (i_{a_1}, \ldots, i_{a_k})$ and $i^{\text{RV}} := (i_{b_1}, \ldots, i_{b_l})$. So the tuple i^{RH} lists the colours of all the horizontal segments in $_i\tau_j$ whose endpoints lie on the bottom edge and we have that

$$|i^{\text{R}}| = |i^{\text{RH}}| + |i^{\text{RV}}|, \qquad p(i^{\text{R}}) = p(i^{\text{RH}}) + |i^{\text{RH}}||i^{\text{RV}}| + p(i^{\text{RV}}). \tag{7.22}$$

Similarly we define $j^{\text{RH}} \in I^k$ and $j^{\text{RV}} \in I^l$ so that j^{RH} lists the colours of all the horizontal line segments in $_i\tau_j$ whose endpoints lie on the top edge and

$$|j^{\text{R}}| = |j^{\text{RH}}| + |j^{\text{RV}}|, \qquad p(j^{\text{R}}) = p(j^{\text{RH}}) + |j^{\text{RH}}||j^{\text{RV}}| + p(j^{\text{RV}}). \tag{7.23}$$

The assumption that $_i\tau_j$ is consistently coloured implies that

$$|i^{\text{RV}}| = |j^{\text{RV}}|. \tag{7.24}$$

Finally let us assume the diagrams representing σ and τ are reduced in the sense that the total number of crossings is as small as possible and each strand crosses the wall at most once. Then by the definition (7.5) we have that

$$\text{wt}(_{i^{\text{L}}j^{\text{R}}}\sigma_{j^{\text{L}}i^{\text{R}}}) = \prod_c (-1)^{|c|} \tag{7.25}$$

where the product is over all coloured crossings c in the diagram $_{i^{\mathrm{L}}j^{\mathrm{R}}}\sigma_{j^{\mathrm{L}}i^{\mathrm{R}}}$. Similarly we have that

$$(-1)^{|i^{\mathrm{RH}}|}\mathrm{wt}(_{i}\tau_{j}) = \prod_{c'}(-1)^{|c'|} \qquad (7.26)$$

where the product is over all coloured crossings c' in the diagram $_{i}\tau_{j}$. When flipping from σ to τ, any strand of σ which does not cross the wall transforms into a vertical strand in τ which has exactly the same crossings with other strands after the flip as it did before. However given two strands in σ which both cross the wall (transforming to two horizontal strands in τ), they cross after the flip if and only if they did *not* cross before the flip, and vice versa. Combined with (7.25)–(7.26), this is enough to see that

$$\mathrm{wt}(_{i^{\mathrm{L}}j^{\mathrm{R}}}\sigma_{j^{\mathrm{L}}i^{\mathrm{R}}}) = (-1)^{p(i^{\mathrm{RH}})+|i^{\mathrm{RH}}||j^{\mathrm{RH}}|+p(j^{\mathrm{RH}})}(-1)^{|i^{\mathrm{RH}}|}\mathrm{wt}(_{i}\tau_{j}).$$

Comparing with (7.21), it remains to observe that

$$(|i^{\mathrm{R}}| + |j^{\mathrm{R}}|)|i^{\mathrm{R}}| + p(i^{\mathrm{R}}) + p(j^{\mathrm{R}}) = p(i^{\mathrm{RH}}) + |i^{\mathrm{RH}}||j^{\mathrm{RH}}| + p(j^{\mathrm{RH}}) + |i^{\mathrm{RH}}|$$

in $\mathbb{Z}_2$, which follows formally from the identities (7.22)–(7.24). $\qquad \square$

Theorem 7.8. *The map*

$$\Psi_{r,s}^{m,n} : B_{r,s}(\delta) \to \mathrm{End}_{\mathfrak{g}}(V^{\otimes r} \otimes W^{\otimes s})^{\mathrm{op}}$$

from (7.12) is surjective. It is injective if and only if $r + s < (m + 1)(n + 1)$.

Proof. We know from Lemma 7.6 that the bottom map in the diagram (7.17) is a $\mathfrak{g}$-supermodule isomorphism, hence it maps the subspace $\mathrm{End}_{\mathfrak{g}}(V^{\otimes(r+s)})$ isomorphically onto the subspace $\mathrm{End}_{\mathfrak{g}}(V^{\otimes r} \otimes W^{\otimes s})$. The theorem follows from this observation combined with Lemma 7.7 and Theorem 7.5. $\qquad \square$

8. Isomorphism theorem and applications

In order to prove the main results of the article, we still need to show that the graded walled Brauer algebra $B_R(\delta)$ is isomorphic to $B_{r,s}(\delta)$. Throughout the section we fix integers $m, n \geq 0$, and set $\delta := m - n$, $k := \min(m, n)$. This is equivalent to fixing $\delta \in \mathbb{Z}$, $k \geq 0$, and letting $m := k, n := k - \delta$ if $\delta \leq 0$ or $m := k + \delta, n := k$ if $\delta \geq 0$.

The main isomorphism theorem. Recall the algebra $B_R(\delta)/B_R(\delta)_{>k}$ from (6.18), and the maps (2.14) and (6.20)–(6.21).

Theorem 8.1. *For each $R \in \mathscr{R}_{r,s}$, there is a surjective algebra homomorphism*

$$\Theta_R^{(k)} : B_{r,s}(\delta) \twoheadrightarrow B_R(\delta)/B_R(\delta)_{>k} \qquad (8.1)$$

with the same kernel as the map $\Psi_{r,s}^{m,n}$ from Theorem 7.8. Moreover the following diagrams commute.

$$
\begin{array}{ccc}
B_{r,s}(\delta) & \xrightarrow{\;\iota_{r,s;i}^{r+1,s}\;} & B_{r+1,s}(\delta) \\[4pt]
\Theta_R^{(k)} \downarrow & & \downarrow \Theta_{RE}^{(k)} \\[4pt]
B_R(\delta)/B_R(\delta)_{>k} & \xrightarrow{\;\iota_{R;i}^{RE}\;} & B_{RE}(\delta)/B_{RE}(\delta)_{>k}
\end{array}
$$

$$
\begin{array}{ccc}
B_{r,s}(\delta) & \xrightarrow{\ \iota^{r,s+1}_{r,s;i}\ } & B_{r,s+1}(\delta) \\[2pt]
\Theta^{(k)}_{R} \downarrow & & \downarrow \Theta^{(k)}_{RF} \\[2pt]
B_{R}(\delta)/B_{R}(\delta)_{>k} & \xrightarrow{\ \iota^{RF}_{R;i}\ } & B_{RF}(\delta)/B_{RF}(\delta)_{>k}
\end{array}
$$

Proof. We assume some notation and results still to be formulated; see the next three subsections. The homomorphism $\Theta^{(k)}_{R}$ is the composite $\Upsilon^{(k)}_{R} \circ \Pi^{(k)}_{R} \circ \Psi^{(k)}_{R}$ of certain maps

$$\Psi^{(k)}_{R} : B_{r,s}(\delta) \twoheadrightarrow \mathrm{End}_{\mathfrak{g}}(^{*}R\,\mathbb{C})^{\mathrm{op}}, \tag{8.2}$$

$$\Pi^{(k)}_{R} : \mathrm{End}_{\mathfrak{g}}(^{*}R\,\mathbb{C})^{\mathrm{op}} \xrightarrow{\sim} \mathrm{End}_{K(m|n)}(^{*}R\,L(\zeta))^{\mathrm{op}}, \tag{8.3}$$

$$\Upsilon^{(k)}_{R} : \mathrm{End}_{K(m|n)}(^{*}R\,L(\zeta))^{\mathrm{op}} \xrightarrow{\sim} B_{r,s}(\delta)/B_{r,s}(\delta)_{>k} \tag{8.4}$$

constructed in Theorems 8.5, 8.7 and 8.9 below. The maps $\Upsilon^{(k)}_{R}$ and $\Pi^{(k)}_{R}$ are isomorphisms and $\Psi^{(k)}_{R}$ is surjective, hence $\Theta^{(k)}_{R}$ is surjective. Moreover it is clear from its explicit definition below that $\Psi^{(k)}_{R}$ has the same kernel as $\Psi^{m,n}_{r,s}$, hence $\Theta^{(k)}_{R}$ has this as its kernel too. Finally the commutativity of the diagrams follows immediately from the commutativity of the diagrams in the statements of Theorems 8.5, 8.7 and 8.9. $\qquad\square$

Corollary 8.2. *For any $R \in \mathscr{R}_{r,s}$ we have that $\dim B_{R}(\delta)_{>k} = \dim \ker \Psi^{m,n}_{r,s}$. So $B_{R}(\delta)_{>k}$ is zero if and only if $r + s < (m+1)(n+1)$, in which case $\Theta^{(k)}_{R}$ is an isomorphism*

$$\Theta^{(k)}_{R} : B_{r,s}(\delta) \xrightarrow{\sim} B_{R}(\delta). \tag{8.5}$$

Proof. It is clear from the definition that $B_{R}(\delta)_{>k} = \{0\}$ if k is sufficiently large. In that case m and n are large too, so $\Psi^{m,n}_{r,s}$ is injective by Theorem 7.8. We deduce that $\Theta^{(k)}_{R}$ is an isomorphism $B_{r,s}(\delta) \xrightarrow{\sim} B_{R}(\delta)$ for $k \gg 0$. This shows that $\dim B_{r,s}(\delta) = \dim B_{R}(\delta)$. Combined with Theorem 8.1 we deduce for any k that $\dim B_{r,s}(\delta)_{>k} = \dim \ker \Psi^{m,n}_{r,s}$. $\qquad\square$

Remark 8.3. There are some implicit choices involved in the definition of $\Theta^{(k)}_{R}$; these come from the isomorphism (8.3) which we will deduce from results in [BS4]. We conjecture that these choices can be made so that the homomorphisms $\Theta^{(k)}_{R}$ stabilise as $k \to \infty$. More precisely, we expect there are isomorphisms

$$\Theta_{R} : B_{r,s}(\delta) \xrightarrow{\sim} B_{R}(\delta)$$

for all R such that $\Theta^{(k)}_{R}$ is the composition of Θ_{R} and the quotient map $B_{R}(\delta) \twoheadrightarrow B_{R}(\delta)/B_{R}(\delta)_{>k}$ for each $k \geq 0$. The arguments in [BS4] are not well-suited to allowing k to vary, so we are not able to see this using the present approach.

The epimorphism $\Psi^{(k)}_{R}$. In this subsection we construct the first map (8.2) needed in the proof of Theorem 8.1. Let $\mathfrak{g} := \mathfrak{gl}_{m|n}(\mathbb{C})$ as in §7. We begin by recalling the definition of the *special projective functors* on a certain category of finite dimensional $\mathfrak{g}$-supermodules. Let $\mathfrak{t}$ be the Cartan subalgebra of $\mathfrak{g}$

consisting of diagonal matrices. Let $\varepsilon_1, \ldots, \varepsilon_{m+n}$ be the usual basis for $\mathfrak{t}^*$, i.e. ε_r picks out the rth diagonal entry of a diagonal matrix. Let $(.,.)$ be the symmetric bilinear form on $\mathfrak{t}^*$ defined by $(\varepsilon_i, \varepsilon_j) = (-1)^{|i|} \delta_{i,j}$, and set $X := \bigoplus_{i=1}^{m+n} \mathbb{Z} \varepsilon_i \subset \mathfrak{t}^*$. As in the introduction of [BS4], we let $\mathscr{F}(m|n)$ be the category of all $\mathfrak{g}$-supermodules M such that

(1) M is finite dimensional;
(2) M decomposes as $M = \bigoplus_{\lambda \in X} M_\lambda$, where M_λ denotes the λ-weight space of M with respect to $\mathfrak{t}$;
(3) the $\mathbb{Z}_2$-grading on M_λ is concentrated in degree $(\lambda, \varepsilon_{m+1} + \cdots + \varepsilon_{m+n})$ (mod 2) for each $\lambda \in X$.

The morphisms in $\mathscr{F}(m|n)$ are all (necessarily even) $\mathfrak{g}$-supermodule homomorphisms. Note further that $\mathscr{F}(m|n)$ is an abelian category closed under tensor product, and it contains both the natural supermodule V and its dual W. We refer to $\mathscr{F}(m|n)$ as the category of *rational $\mathfrak{g}$-supermodules*.

Let $^*E := ? \otimes V$ and $^*F := ? \otimes W$, viewed as endofunctors of $\mathscr{F}(m|n)$. Observe that multiplication by the element Ω from (7.2) defines natural endomorphisms of both $^*EM$ and $^*FM$ for every $M \in \mathscr{F}(m|n)$. Let $^*E_i$ and $^*F_i$ be the subfunctors of $^*E$ and $^*F$, respectively, defined on $M \in \mathscr{F}(m|n)$ by declaring that

(1) $^*E_i M$ is the generalised i-eigenspace of Ω on $^*EM$;
(2) $^*F_i M$ is the generalised $(-i - \delta)$-eigenspace of Ω on $^*FM$.

By [BS4, Corollary 2.9, Lemma 2.10], we have that

$$^*E = \bigoplus_{i \in \mathbb{Z}} {}^*E_i, \qquad\qquad {}^*F = \bigoplus_{i \in \mathbb{Z}} {}^*F_i. \qquad\qquad (8.6)$$

Let

$$b_{r,s}^{r+1,s} : V^{\otimes r} \otimes W^{\otimes s} \otimes V \xrightarrow{\sim} V^{\otimes(r+1)} \otimes W^{\otimes s}, \qquad\qquad (8.7)$$

$$b_{r,s}^{r,s+1} : V^{\otimes r} \otimes W^{\otimes s} \otimes W \xrightarrow{\sim} V^{\otimes r} \otimes W^{\otimes(s+1)} \qquad\qquad (8.8)$$

be the canonical isomorphisms defined by supercommuting $W^{\otimes s}$ past the rightmost copy of V or W, respectively. We stress $b_{r,s}^{r,s+1}$ is *not* the identity (unless $s = 0$). The following lemma connects the special projective functors on $\mathscr{F}(m|n)$ to the i-restriction functors from (2.15)–(2.16).

Lemma 8.4. *For each $i \in \mathbb{Z}$, the restrictions of the maps $b_{r,s}^{r+1,s}$ and $b_{r,s}^{r,s+1}$ just defined give $(U(\mathfrak{g}), B_{r,s}(\delta))$-bimodule isomorphisms*

$$b_{r,s}^{r+1,s} : {}^*E_i(V^{\otimes r} \otimes W^{\otimes s}) \xrightarrow{\sim} (V^{\otimes(r+1)} \otimes W^{\otimes s}) 1_{r,s;i}^{r+1,s},$$

$$b_{r,s}^{r,s+1} : {}^*F_i(V^{\otimes r} \otimes W^{\otimes s}) \xrightarrow{\sim} (V^{\otimes r} \otimes W^{\otimes(s+1)}) 1_{r,s;i}^{r,s+1}.$$

Proof. We just explain for $b_{r,s}^{r+1,s}$; the other case is similar. It is clear that the map (8.7) is a $(U(\mathfrak{g}), B_{r,s}(\delta))$-module homomorphism, where the $B_{r,s}(\delta)$-module structure on the the left hand side is obtained by applying the functor $^*E = ? \otimes V$ to its action on $V^{\otimes r} \otimes W^{\otimes s}$, and on the right hand side it is the restriction of the $B_{r+1,s}(\delta)$-module structure via the homomorphism $\iota_{r,s}^{r+1,s}$ from (2.11). By definition, $^*E_i(V^{\otimes r} \otimes W^{\otimes s})$ is the generalised i-eigenspace of the endomorphism of $V^{\otimes r} \otimes W^{\otimes s} \otimes V$ defined by multiplication by $\sum_{1 \leq a \leq r+s} \Omega_{a, r+s+1}$. This maps

under $b_{r,s}^{r+1,s}$ to the generalised i-eigenspace of the endomorphism of $V^{\otimes(r+1)} \otimes W^{\otimes s}$ defined by multiplication by $\sum_{1 \leq a \leq r} \Omega_{a,r+1} + \sum_{r+2 \leq b \leq r+s+1} \Omega_{r+1,b}$. The latter is the same as the endomorphism defined by $x_{r,s}^{r+\overline{1},s} \in B_{r+1,s}(\delta)$ from (2.12); this follows by the second statement of Lemma 7.4. Hence $b_{r,s}^{r+1,s}$ maps ${}^*E_i(V^{\otimes r} \otimes W^{\otimes s})$ isomorphically onto the generalised i-eigenspace of $x_{r,s}^{r+1,s}$ on $V^{\otimes(r+1)} \otimes W^{\otimes s}$, which is exactly the subspace $(V^{\otimes(r+1)} \otimes W^{\otimes s})1_{r,s;i}^{r+1,s}$. $\qquad\square$

For $R = R^{(1)} \cdots R^{(r+s)} \in \mathscr{R}_{r,s}$, we can interpret ${}^*R = {}^*R^{(r+s)} \cdots {}^*R^{(1)}$ as an endofunctor of $\mathscr{F}(m|n)$. Let $\mathbb{C}$ be the trivial $\mathfrak{g}$-supermodule. Mimicking the definitions (5.10)–(5.11), the functors ${}^*E_i$ and ${}^*F_i$ induce non-unital algebra homomorphisms

$$\iota_{R;i}^{RE} : \operatorname{End}_{\mathfrak{g}}({}^*R\,\mathbb{C})^{\mathrm{op}} \to \operatorname{End}_{\mathfrak{g}}({}^*(RE)\,\mathbb{C})^{\mathrm{op}}, \qquad f \mapsto {}^*E_i(f), \qquad (8.9)$$

$$\iota_{R;i}^{RF} : \operatorname{End}_{\mathfrak{g}}({}^*R\,\mathbb{C})^{\mathrm{op}} \to \operatorname{End}_{\mathfrak{g}}({}^*(RF)\,\mathbb{C})^{\mathrm{op}}, \qquad f \mapsto {}^*F_i(f). \qquad (8.10)$$

We recursively define a $\mathfrak{g}$-supermodule isomorphism

$$\psi_R : {}^*R\,\mathbb{C} \xrightarrow{\sim} V^{\otimes r} \otimes W^{\otimes s}$$

as follows. If $r = s = 0$ then we take ψ_R to be the identity map $\mathbb{C} \to \mathbb{C}$. For the induction step, assume $\psi_R : {}^*R\,\mathbb{C} \xrightarrow{\sim} V^{\otimes r} \otimes W^{\otimes s}$ has been defined already, then set

$$\psi_{RE} := b_{r,s}^{r+1,s} \circ {}^*E(\psi_R) : {}^*(RE)\,\mathbb{C} \to V^{\otimes(r+1)} \otimes W^{\otimes s},$$

$$\psi_{RF} := b_{r,s}^{r,s+1} \circ {}^*F(\psi_R) : {}^*(RF)\,\mathbb{C} \xrightarrow{\sim} V^{\otimes r} \otimes W^{\otimes(s+1)}.$$

Theorem 8.5. *For $R \in \mathscr{R}_{r,s}$, there is a surjective algebra homomorphism*

$$\Psi_R^{(k)} : B_{r,s}(\delta) \twoheadrightarrow \operatorname{End}_{\mathfrak{g}}({}^*R\,\mathbb{C})^{\mathrm{op}}, \qquad \sigma \mapsto \psi_R^{-1} \circ \Psi_{r,s}^{m,n}(\sigma) \circ \psi_R.$$

Moreover the following diagrams commute for each $i \in \mathbb{Z}$.

$$
\begin{array}{ccc}
B_{r,s}(\delta) & \xrightarrow{\ \iota_{r,s;i}^{r+1,s}\ } & B_{r+1,s}(\delta) \\
\Psi_R^{(k)} \downarrow & & \downarrow \Psi_{RE}^{(k)} \\
\operatorname{End}_{\mathfrak{g}}({}^*R\,\mathbb{C})^{\mathrm{op}} & \xrightarrow{\ \iota_{R;i}^{RE}\ } & \operatorname{End}_{\mathfrak{g}}({}^*(RE)\,\mathbb{C})^{\mathrm{op}}
\end{array}
$$

$$
\begin{array}{ccc}
B_{r,s}(\delta) & \xrightarrow{\ \iota_{r,s;i}^{r,s+1}\ } & B_{r,s+1}(\delta) \\
\Psi_R^{(k)} \downarrow & & \downarrow \Psi_{RF}^{(k)} \\
\operatorname{End}_{\mathfrak{g}}({}^*R\,\mathbb{C})^{\mathrm{op}} & \xrightarrow{\ \iota_{R;i}^{RF}\ } & \operatorname{End}_{\mathfrak{g}}({}^*(RF)\,\mathbb{C})^{\mathrm{op}}
\end{array}
$$

Proof. The first part follows from Theorem 7.8. The commutative diagrams follow from Lemma 8.4 and the definitions. $\qquad\square$

The isomorphism $\Pi_R^{(k)}$. To construct the second map (8.3) needed in the proof of Theorem 8.1, we apply the main results of [BS4]. Recall that the

dominance ordering on X is defined by $\lambda \preceq \mu$ if $\mu - \lambda$ is a sum of the positive roots $\{\varepsilon_i - \varepsilon_j \mid 1 \leq i < j \leq m+n\}$. Also let

$$\rho := \sum_{r=1}^{m}(1-r)\varepsilon_r + \sum_{s=1}^{n}(m-s)\varepsilon_{m+s}.$$

It is well known that the isomorphism classes of irreducible supermodules in the category $\mathscr{F}(m|n)$ from the previous subsection are parametrised by their highest weights with respect to the dominance ordering by the set

$$\Xi := \left\{ \lambda \in X \,\middle|\, \begin{array}{c} (\lambda+\rho,\varepsilon_1) > \cdots > (\lambda+\rho,\varepsilon_m), \\ (\lambda+\rho,\varepsilon_{m+1}) < \cdots < (\lambda+\rho,\varepsilon_{m+n}) \end{array} \right\} \tag{8.11}$$

of *dominant integral weights*. We denote the irreducible corresponding to $\lambda \in \Xi$ by $L(\lambda)$.

There is another weight dictionary which allows us to identify Ξ with a set of weight diagrams. Given $\lambda \in \Xi$, let

$$I_\vee(\lambda) := \{(\lambda+\rho,\varepsilon_1),\ldots,(\lambda+\rho,\varepsilon_m)\}, \tag{8.12}$$

$$I_\wedge(\lambda) := \mathbb{Z} \setminus \{(\lambda+\rho,\varepsilon_{m+1}),\ldots,(\lambda+\rho,\varepsilon_{m+n})\}. \tag{8.13}$$

Then identify λ with the weight diagram associated to these sets via (1.5); the finiteness condition for weight diagrams stipulated in the introduction is automatically satisfied as there are only finitely many vertices labelled $\vee$. For example, the zero weight (which parametrises the trivial $\mathfrak{g}$-supermodule) is identified with the diagram

$$\tag{8.14}$$

(where there are infinitely many vertices labelled $\wedge$ to the left and the right).

Let $K(m|n)$ be the subalgebra $\bigoplus_{\lambda,\mu \in \Xi} e_\lambda K e_\mu$ of the universal arc algebra from §3. As Ξ is a union of blocks, the algebra $K(m|n)$ is just the sum of some blocks of K from (3.4). Let $K(m|n)\text{-mod}_{lf}$ be the full subcategory of $K\text{-mod}_{lf}$ consisting of all modules M such that $M = \bigoplus_{\lambda \in \Xi} e_\lambda M$. Up to grading shift, the irreducible objects in this category are the one-dimensional modules $\{L(\lambda) \mid \lambda \in \Xi\}$.

The set Ξ is a connected component in the labelled directed graph defined by (3.18)–(3.19). It follows that the special projective functors ${}^*E_i$ and ${}^*F_i$ from (3.14) restrict to well-defined endofunctors of $K(m|n)\text{-mod}_{lf}$. These functors are defined by tensoring with certain bimodules. Forgetting the gradings on these bimodules, we can view them instead as endofunctors of the category $K(m|n)\text{-mod}$ of locally unital, finite dimensional left $K(m|n)$-modules. The main result of [BS4] can now be formulated as follows.

Theorem 8.6 ([BS4, Theorem 5.11, Theorem 5.12])**.** *There is an equivalence of categories*

$$\mathbf{e} : \mathscr{F}(m|n) \to K(m|n)\text{-mod}.$$

Under the equivalence, the irreducible $\mathfrak{g}$-supermodule $L(\lambda)$ of highest weight $\lambda \in \Xi$ corresponds to the irreducible $K(m|n)$-module with the same name. Moreover for each $i \in \mathbb{Z}$ there are canonical isomorphisms of functors

$$^*E_i \circ \mathbf{e} \cong \mathbf{e} \circ {}^*E_i : \mathscr{F}(m|n) \to K(m|n)\text{-mod}, \qquad (8.15)$$

$$^*F_i \circ \mathbf{e} \cong \mathbf{e} \circ {}^*F_i : \mathscr{F}(m|n) \to K(m|n)\text{-mod}. \qquad (8.16)$$

Now recall for $R \in \mathscr{R}_{r,s}$ that $^*R$ can be viewed either as an endofunctor of $\mathscr{F}(m|n)$ as in the previous section, as an endofunctor of $K(m|n)\text{-mod}_{lf}$ as in (5.6), or even as an endofunctor of $K(m|n)$-mod on forgetting the gradings. Using (8.15)–(8.16) repeatedly, we get an isomorphism of functors $^*R \circ \mathbf{e} \cong \mathbf{e} \circ {}^*R : \mathscr{F}(m|n) \to K(m|n)$-mod. Fixing also a (unique up to scalars) isomorphism $L(\zeta) \cong \mathbf{e}\,\mathbb{C}$, we get from this an isomorphism

$$\pi_R : {}^*R\,L(\zeta) \xrightarrow{\sim} \mathbf{e}^*R\,\mathbb{C}.$$

Note although we made a choice of scalar here, the isomorphism $\Pi_R^{(k)}$ in the following theorem is independent of this choice. For the statement, recall also the maps (8.9)–(8.10) and (5.10)–(5.11).

Theorem 8.7. *For any $R \in \mathscr{R}_{r,s}$, there is an algebra isomorphism*

$$\Pi_R^{(k)} : \operatorname{End}_{\mathfrak{g}}({}^*R\,\mathbb{C})^{\mathrm{op}} \xrightarrow{\sim} \operatorname{End}_{K(m|n)}({}^*R\,L(\zeta))^{\mathrm{op}}, \qquad \theta \mapsto \pi_R^{-1} \circ \mathbf{e}(\theta) \circ \pi_R.$$

Moreover the following diagrams commute.

$$
\begin{array}{ccc}
\operatorname{End}_{\mathfrak{g}}({}^*R\,\mathbb{C})^{\mathrm{op}} & \xrightarrow{\ \iota_{R;i}^{RE}\ } & \operatorname{End}_{\mathfrak{g}}({}^*(RE)\,\mathbb{C})^{\mathrm{op}} \\[2pt]
{\scriptstyle \Pi_R^{(k)}}\downarrow & & \downarrow{\scriptstyle \Pi_{RE}^{(k)}} \\[2pt]
\operatorname{End}_{K(m|n)}({}^*R\,L(\zeta))^{\mathrm{op}} & \xrightarrow{\ \iota_{R;i}^{RE}\ } & \operatorname{End}_{K(m|n)}({}^*(RE)\,L(\zeta))^{\mathrm{op}}
\end{array}
$$

$$
\begin{array}{ccc}
\operatorname{End}_{\mathfrak{g}}({}^*R\,\mathbb{C})^{\mathrm{op}} & \xrightarrow{\ \iota_{R;i}^{RF}\ } & \operatorname{End}_{\mathfrak{g}}({}^*(RF)\,\mathbb{C})^{\mathrm{op}} \\[2pt]
{\scriptstyle \Pi_R^{(k)}}\downarrow & & \downarrow{\scriptstyle \Pi_{RF}^{(k)}} \\[2pt]
\operatorname{End}_{K(m|n)}({}^*R\,L(\zeta))^{\mathrm{op}} & \xrightarrow{\ \iota_{R;i}^{RF}\ } & \operatorname{End}_{K(m|n)}({}^*(RF)\,L(\zeta))^{\mathrm{op}}
\end{array}
$$

Proof. This follows from Theorem 8.6. $\qquad\qquad\square$

The isomorphism $\Upsilon_R^{(k)}$. Let $R \in \mathscr{R}_{r,s}$ be fixed, and η and ζ be the weights from (6.1) and (8.14), respectively. We can now complete the proof of Theorem 8.1 by constructing the third map (8.4). Here there is an interesting diagrammatic trick. Recall Theorem 5.8. It gives us a basis $\{\bar{f}_{\mathtt{ST}}\}$ for the special endomorphism algebra $\operatorname{End}_{K(m|n)}({}^*R\,L(\zeta))^{\mathrm{op}}$ indexed by good pairs $(\mathtt{S}, \mathtt{T})$ of R-tableaux of type ζ. We are going to compare this with the basis $\{f_{\mathtt{ST}}\}$ for the graded walled Brauer algebra $B_R(\delta)$ from (6.4), which is indexed by pairs $(\mathtt{S}, \mathtt{T})$ of R-tableaux of type η with $\operatorname{sh}(\mathtt{S}) = \operatorname{sh}(\mathtt{T})$.
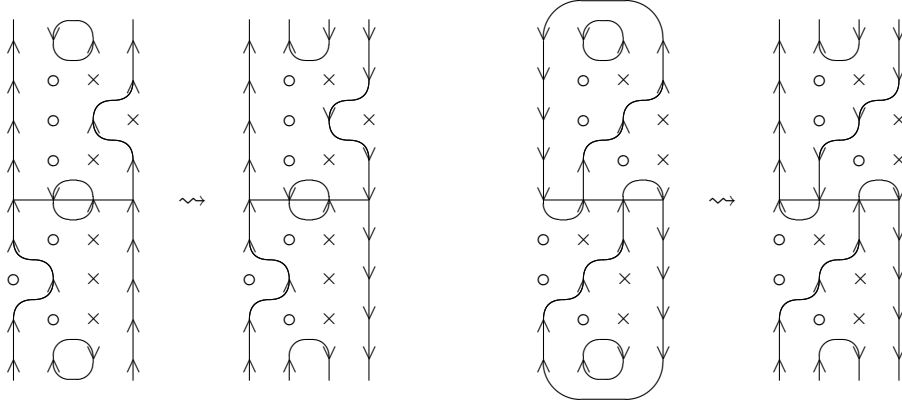
Lemma 8.8. *There is a bijection*

$$\left\{ \begin{array}{l} \textit{Good pairs } (\mathtt{S}, \mathtt{T}) \textit{ of} \\ R\textit{-tableaux of type } \zeta \end{array} \right\} \overset{\sim}{\to} \left\{ \begin{array}{l} \textit{Pairs } (\mathtt{S}', \mathtt{T}') \textit{ of } R\textit{-tableaux of type } \eta \\ \textit{with } \mathrm{sh}(\mathtt{S}') = \mathrm{sh}(\mathtt{T}') \textit{ and } k(\mathtt{S}', \mathtt{T}') \le k \end{array} \right\}.$$

It maps $(\mathtt{S}, \mathtt{T})$ *to* $(\mathtt{S}', \mathtt{T}')$ *defined as follows.*

(1) *Starting from the cup diagram* $\underline{\mathtt{S}}\overline{\mathtt{T}}$, *replace the cap diagram* $\overline{\zeta}$ *on the top by* $\overline{\eta}$ *and the cup diagram* $\underline{\zeta}$ *on the bottom by* $\underline{\eta}$.

(2) *Next adjust the labels of all the vertices lying on connected components passing through the top or bottom number lines so that the weight diagrams at the top and bottom are both equal to* η.

(3) *Then* $(\mathtt{S}', \mathtt{T}')$ *is the unique pair of R-tableaux of type* η *such that the resulting diagram is equal to* $\underline{\mathtt{S}}'\overline{\mathtt{T}}'$.

Proof. Construct a two-sided inverse. $\qquad\qquad\square$

Here are a couple of examples illustrating the definition in Lemma 8.8; actually we display the diagrams $\underline{\mathtt{S}}\overline{\mathtt{T}} \leadsto \underline{\mathtt{S}}'\overline{\mathtt{T}}'$ assuming $(\mathtt{S}, \mathtt{T}) \mapsto (\mathtt{S}', \mathtt{T}')$ under the bijection defined in the lemma. In the first example, $m = n = 1$ (so $k = 1$ and $\delta = 0$) and in the second example $m = n = 2$ (so $k = 2$ and $\delta = 0$).

Theorem 8.9. *There is an algebra isomorphism*

$$\Upsilon_R^{(k)} : \mathrm{End}_{K(m|n)}({}^*R\,L(\zeta))^{\mathrm{op}} \overset{\sim}{\to} B_R(\delta)/B_R(\delta)_{>k}$$

mapping $\bar{f}_{\mathtt{S},\mathtt{T}}$ *for a good pair* $(\mathtt{S}, \mathtt{T})$ *of R-tableaux of type* ζ *to the canonical image in* $B_R(\delta)/B_R(\delta)_{>k}$ *of* $f_{\mathtt{S}',\mathtt{T}'} \in B_R(\delta)$, *where* $(\mathtt{S}', \mathtt{T}')$ *is the image of* $(\mathtt{S}, \mathtt{T})$ *under the bijection from Lemma 8.8. Moreover the following diagrams commute.*

$$
\begin{array}{ccc}
\mathrm{End}_{K(m|n)}({}^*R\,L(\zeta))^{\mathrm{op}} & \overset{\iota_{R;i}^{RE}}{\longrightarrow} & \mathrm{End}_{K(m|n)}({}^*(RE)\,L(\zeta))^{\mathrm{op}} \\
\Upsilon_R^{(k)} \downarrow & & \downarrow \Upsilon_{RE}^{(k)} \\
B_R(\delta)/B_R(\delta)_{>k} & \overset{\iota_{R;i}^{RE}}{\longrightarrow} & B_{RE}(\delta)/B_{RE}(\delta)_{>k}
\end{array}
$$

$$\text{End}_{K(m|n)}(^{*}R\,L(\zeta))^{\text{op}} \xrightarrow{\ \iota_{R;i}^{RF}\ } \text{End}_{K(m|n)}(^{*}(RF)\,L(\zeta))^{\text{op}}$$

$$\Upsilon_{R}^{(k)} \Big\downarrow \qquad\qquad\qquad\qquad\qquad \Big\downarrow \Upsilon_{RF}^{(k)}$$

$$B_{R}(\delta)/B_{R}(\delta)_{>k} \xrightarrow{\ \iota_{R;i}^{RF}\ } \quad B_{RF}(\delta)/B_{RF}(\delta)_{>k}.$$

Proof. The map $\Upsilon_{R}^{(k)}$ is a vector space isomorphism by Lemma 8.8 and (6.19). It is multiplicative by the algorithmic descriptions of the multiplications in $\text{End}_{K(m|n)}(^{*}R\,L(\zeta))^{\text{op}})$ and $B_{R}(\delta)/B_{R}(\delta)_{>k}$. Finally, the commutativity of the diagrams follows by considering the algorithmic description of the maps $\iota_{R;i}^{RE}$ and $\iota_{R;i}^{RF}$ from the end of §5 and (6.20)–(6.21). $\qquad\square$

Morita equivalence. The following proves Theorem 1.1 from the introduction.

Theorem 8.10. *Fix $R \in \mathscr{R}_{r,s}$ and $k \gg 0$. Then there is an equivalence of categories*

$$\mathbf{g}_{R} : K_{r,s}(\delta)\text{-mod} \to B_{r,s}(\delta)\text{-mod}$$

defined by applying (the ungraded analogue of) the functor $\mathbf{f}_{R}$ from (6.11) followed by pull-back through the isomorphism $\Theta_{R}^{(k)}$. Moreover the following hold.

(1) *The equivalence commutes with i-restriction and i-induction, i.e. there are isomorphisms of functors*

$$\mathbf{g}_{R} \circ i\text{-res}_{r,s}^{r+1,s} \cong i\text{-res}_{r,s}^{r+1,s} \circ \mathbf{g}_{RE} : K_{r+1,s}(\delta)\text{-mod} \to B_{r,s}(\delta)\text{-mod},$$

$$\mathbf{g}_{R} \circ i\text{-res}_{r,s}^{r,s+1} \cong i\text{-res}_{r,s}^{r,s+1} \circ \mathbf{g}_{RF} : K_{r,s+1}(\delta)\text{-mod} \to B_{r,s}(\delta)\text{-mod},$$

$$\mathbf{g}_{RE} \circ i\text{-ind}_{r,s}^{r+1,s} \cong i\text{-ind}_{r,s}^{r+1,s} \circ \mathbf{g}_{R} : K_{r,s}(\delta)\text{-mod} \to B_{r+1,s}(\delta)\text{-mod},$$

$$\mathbf{g}_{RF} \circ i\text{-ind}_{r,s}^{r,s+1} \cong i\text{-ind}_{r,s}^{r,s+1} \circ \mathbf{g}_{R} : K_{r,s}(\delta)\text{-mod} \to B_{r,s+1}(\delta)\text{-mod}\,.$$

(2) *For any $\lambda \in \Lambda_{r,s}$, we have that $\mathbf{g}_{R}(V_{r,s}(\lambda)) \cong C_{r,s}(\lambda)$ as $B_{r,s}(\delta)$-modules.*

Proof. The opening statement follows from Theorem 6.2. Also (1) follows on combining the isomorphisms of functors in Theorem 6.2 with the commutativity of the diagrams in Theorem 8.1.

To prove (2) we proceed by induction on $r + s$, the base case $r + s = 0$ being trivial. For the induction step, suppose we know already that $\mathbf{g}_{R}(V_{r,s}(\lambda)) \cong C_{r,s}(\lambda)$ for all $\lambda \in \Lambda_{r,s}$. We need to show that $\mathbf{g}_{RE}(V_{r+1,s}(\mu)) \cong C_{r+1,s}(\mu)$ for all $\mu \in \Lambda_{r+1,s}$ and that $\mathbf{g}_{RF}(V_{r,s+1}(\mu)) \cong C_{r,s+1}(\mu)$ for all $\mu \in \Lambda_{r,s+1}$. We just verify the first of these statements, since the proof of the second is similar. For this we proceed by another induction, this time on $s - |\mu^{\text{R}}|$.

First consider the base case when $s - |\mu^{\text{R}}| = 0$. Then μ is minimal in $\Lambda_{r+1,s}$ with respect to the Bruhat order. So $V_{r+1,s}(\mu)$ is irreducible by Theorem 4.4(4). Let λ be obtained from μ by removing a box of content i from μ^{L}. We claim that $i\text{-ind}_{r,s}^{r+1,s}V_{r,s}(\lambda)$ has irreducible head isomorphic to $V_{r+1,s}(\mu)$. By Theorem 4.10(3), we either have that $i\text{-ind}_{r,s}^{r+1,s}V_{r,s}(\lambda) \cong V_{r+1,s}(\mu)$, when the claim is trivially true, or λ^{R} has a removable box of content $-i - \delta$. In the latter case, let $\nu \in \Lambda_{r+1,s}$ be obtained from λ by removing this box. Then $i\text{-ind}_{r,s}^{r+1,s}V_{r,s}(\lambda)$ has a two-step filtration with $V_{r+1,s}(\nu)\langle 1 \rangle$ at the bottom and $V_{r+1,s}(\mu)$ at the

top. It follows that the head of i-$\mathrm{ind}_{r,s}^{r+1,s}V_{r,s}(\lambda)$ contains $V_{r+1,s}(\mu)$ and possibly some other constituents which are also constituents of $L := \mathrm{head}\, V_{r+1,s}(\nu)\langle 1\rangle$. We need to show that there are actually no extra constituents. Suppose first that $\nu \in \dot{\Lambda}_{r+1,s}$, in which case $L = L_{r+1,s}(\nu)\langle 1\rangle$. The weight diagram of ν has ith and $(i+1)$th vertices labelled $\wedge$ and $\vee$, respectively, so by the last part of Theorem 3.5 we have that $E_i L(\nu) = \{0\}$. This implies using also Lemma 4.7 that i-$\mathrm{res}_{r,s}^{r+1,s}L_{r+1,s}(\nu) = i$-$\mathrm{res}_{r,s}^{r+1,s}(e_{r,s}L(\nu)) \cong e_{r,s}(E_i L(\nu)) = \{0\}$, hence

$$\dim \mathrm{Hom}_{K_{r+1,s}(\delta)}(i\text{-}\mathrm{ind}_{r,s}^{r+1,s}V_{r,s}(\lambda), L_{r+1,s}(\nu)) = $$
$$\dim \mathrm{Hom}_{K_{r,s}(\delta)}(V_{r,s}(\lambda), i\text{-}\mathrm{res}_{r,s}^{r+1,s}L_{r+1,s}(\nu)) = 0.$$

This proves that L does not appear in this case. It remains to treat the exceptional situation in which $\nu = (\varnothing, \varnothing)$, $\delta = 0$, $r = 0$ and $s = 1$. In this case Theorem 4.4(4) implies that $L \cong V_{r+1,s}(\mu)\langle 2\rangle$. As i-$\mathrm{res}_{r,s}^{r+1,s}V_{r+1,s}(\mu) \cong V_{r,s}(\lambda)$ by Theorem 4.10(1), we have that

$$\dim \mathrm{Hom}_{K_{r+1,s}(\delta)}(i\text{-}\mathrm{ind}_{r,s}^{r+1,s}V_{r,s}(\lambda), V_{r+1,s}(\mu)) = $$
$$\dim \mathrm{Hom}_{K_{r,s}(\delta)}(V_{r,s}(\lambda), i\text{-}\mathrm{res}_{r,s}^{r+1,s}V_{r+1,s}(\mu)) = 1,$$

which is good enough to complete the proof of the claim. Now to show that $\mathbf{g}_{RE}(V_{r+1,s}(\mu)) \cong C_{r+1,s}(\mu)$, we note by the definition (2.8) that $C_{r+1,s}(\mu)$ is irreducible too. Hence using the claim just established, it suffices to show that $C_{r+1,s}(\mu)$ appears in the head of $\mathbf{g}_{RE}(i\text{-}\mathrm{ind}_{r,s}^{r+1,s}V_{r,s}(\lambda))$. By (1) and induction, $\mathbf{g}_{RE}(i\text{-}\mathrm{ind}_{r,s}^{r+1,s}V_{r,s}(\lambda)) \cong i\text{-}\mathrm{ind}_{r,s}^{r+1,s}C_{r,s}(\lambda)$, which has $C_{r+1,s}(\mu)$ in its head by Theorem 2.5(3). This verifies the base case of the second induction.

For the induction step, we take $\mu \in \Lambda_{r+1,s}$ with $s - |\mu^{\mathrm{R}}| > 0$. Let λ be obtained from μ by adding a box of content $-i - \delta$ to μ^{R}. By induction we know that $\mathbf{g}_R(V_{r,s}(\lambda)) \cong C_{r,s}(\lambda)$. Now apply the functor i-$\mathrm{ind}_{r,s}^{r+1,s}$. If μ^{L} has no removable box of content i then we have that i-$\mathrm{ind}_{r,s}^{r+1,s}V_{r,s}(\lambda) \cong V_{r+1,s}(\mu)$ and i-$\mathrm{ind}_{r,s}^{r+1,s}C_{r,s}(\lambda) \cong C_{r+1,s}(\mu)$ by Theorems 4.10(3) and 2.5(3), respectively. Using also (1), this implies that $\mathbf{g}_{RE}(V_{r+1,s}(\mu)) \cong C_{r+1,s}(\mu)$ as required. If μ^{L} has a removable box of content i, let ν be obtained from λ by adding this box to λ^{L}. Then Theorem 2.5(3) asserts that $M := i\text{-}\mathrm{ind}_{r,s}^{r+1,s}C_{r,s}(\lambda)$ has a unique submodule M' such that $M/M' \cong C_{r+1,s}(\nu)$, and this submodule M' is isomorphic to $C_{r+1,s}(\mu)$. Also Theorem 4.10(3) and (1) show that $\mathbf{g}_{RE}(i\text{-}\mathrm{ind}_{r,s}^{r+1,s}V_{r,s}(\lambda)) \cong M$ has a submodule $M'' \cong \mathbf{g}_{RE}(V_{r+1,s}(\mu))$ such that $M/M'' \cong \mathbf{g}_{RE}(V_{r+1,s}(\nu))$. By the induction on $s - |\mu^{\mathrm{R}}|$ we know already that $\mathbf{g}_{RE}(V_{r+1,s}(\nu)) \cong C_{r+1,s}(\nu)$. Hence by the unicity of M' we deduce that $M'' \cong M'$. This shows that $C_{r+1,s}(\mu) \cong \mathbf{g}_{RE}(V_{r+1,s}(\mu))$. □

Corollary 8.11. *For $\lambda \in \Lambda_{r,s}$, the graded cell module $C_R(\lambda)$ is isomorphic to the ordinary cell module $C_{r,s}(\lambda)$ on forgetting the grading and viewing it as a $B_{r,s}(\delta)$-module via the isomorphism $\Theta_R^{(k)}$ from (8.5).*

Proof. Immediate from Theorem 8.10(2) and Lemma 6.5. □

Corollary 8.12. *For $\lambda \in \dot{\Lambda}_{r,s}$, $D_R(\lambda)$ is isomorphic to $D_{r,s}(\lambda)$ on forgetting the grading and viewing it as a $B_{r,s}(\delta)$-module via the isomorphism $\Theta_R^{(k)}$.*

Weight idempotents. Recall the Jucys-Murphy elements from (2.19)–(2.20) and the corresponding weight idempotents $e(\boldsymbol{i}) \in B_{r,s}(\delta)$; these depend implicitly on R. Recall also the idempotents $e(\boldsymbol{i}) \in B_R(\delta)$. We observed in Remark 6.11 that these are isotypic, hence so are their canonical images $\bar{e}(\boldsymbol{i}) \in B_R(\delta)/B_R(\delta)_{>k}$.

Lemma 8.13. *For $\boldsymbol{i} \in \mathbb{Z}^{r+s}$ and any $k \geq 0$, the map $\Theta_R^{(k)}$ from (8.1) sends $e(\boldsymbol{i}) \in B_{r,s}(\delta)$ to $\bar{e}(\boldsymbol{i}) \in B_R(\delta)/B_R(\delta)_{>k}$.*

Proof. We proceed by induction on $r + s$, the lemma being vacuous in the case $r + s = 0$. For the induction step assume we have already proved that $\Theta_R^{(k)}(e(\boldsymbol{i})) = \bar{e}(\boldsymbol{i})$. We need to show that $\Theta_{RE}^{(k)} : B_{r+1,s}(\delta) \to B_{RE}(\delta)/B_{RE}(\delta)_{>k}$ and $\Theta_{RF}^{(k)} : B_{r,s+1}(\delta) \to B_{RF}(\delta)/B_{RF}(\delta)_{>k}$ map $e(\boldsymbol{ii})$ to $\bar{e}(\boldsymbol{ii})$ for $i \in \mathbb{Z}$. We just explain the former. By the definition (2.19) we have in $B_{r+1,s}(\delta)$ that $\iota_{r,s}^{r+1,s}(e(\boldsymbol{i})) = \sum_{j \in \mathbb{Z}} e(\boldsymbol{ij})$. Recalling the idempotent $1_{r,s;i}^{r+1,s} \in B_{r+1,s}(\delta)$ defined just after (2.13), it follows in $B_{r+1,s}(\delta)$ that

$$e(\boldsymbol{ii}) = \iota_{r,s}^{r+1,s}(e(\boldsymbol{i}))1_{r,s;i}^{r+1,s} = \iota_{r,s;i}^{r+1,s}(e(\boldsymbol{i})).$$

Also by the observations after (5.10)–(5.11), we know that $\iota_{R;i}^{RE}(\bar{e}(\boldsymbol{i})) = \bar{e}(\boldsymbol{ii})$ in $B_{RE}(\delta)/B_{RE}(\delta)_{>k}$. Now we are done by the induction hypothesis and the commutativity of the first square in Theorem 8.1. $\qquad\square$

Lemma 8.14. *Fix $R \in \mathscr{R}_{r,s}$. For any $\boldsymbol{i} \in \mathbb{Z}^{r+s}$, the idempotent $e(\boldsymbol{i}) \in B_{r,s}(\delta)$ is non-zero if and only if there exists a restricted R-tableau $\mathtt{T}$ with $\boldsymbol{i}^{\mathtt{T}} = \boldsymbol{i}$. Assuming this is the case, let $\lambda := \mathrm{sh}(\mathtt{T}) \in \dot{\Lambda}_{r,s}$. Then $e(\boldsymbol{i})$ is isotypic in the sense that the projective module $B_{r,s}(\delta)e(\boldsymbol{i})$ is a direct sum of copies of the projective cover of $D_{r,s}(\lambda)$.*

Proof. Assuming $k \gg 0$ so that $\Theta_R^{(k)}$ is an isomorphism, this follows from Lemmas 6.9–6.10 together with Lemma 8.13 and Corollary 8.12. $\qquad\square$

Theorem 8.15. *If either $m = 0$ or $n = 0$, i.e. $k = 0$, then $\ker \Psi_{r,s}^{m,n}$ is generated by the idempotents $e(\boldsymbol{i})$ for all $\boldsymbol{i} \in \mathbb{Z}^{r+s}$ with $k(\boldsymbol{i}) > 0$.*

Proof. We know by Lemma 6.15 that the two-sided ideal of $B_R(\delta)$ generated by the idempotents $e(\boldsymbol{i})$ with $k(\boldsymbol{i}) > 0$ is of the same dimension as $B_R(\delta)_{>0}$. Applying Lemma 8.13 with $k \gg 0$, we deduce that the two-sided ideal of $B_{r,s}(\delta)$ generated by these idempotents has the same dimension as $B_R(\delta)_{>0}$.

Now assume that $k = 0$ as in the statement of the theorem. Combining the observation made in the previous paragraph with the opening statement of Corollary 8.2, we deduce that the two-sided ideal of $B_{r,s}(\delta)$ generated by the idempotents $e(\boldsymbol{i})$ with $k(\boldsymbol{i}) > 0$ has the same dimension as $\ker \Psi_{r,s}^{m,n}$. It remains to show that all these idempotents belong to $\ker \Psi_{r,s}^{m,n}$. Since $\ker \Psi_{r,s}^{m,n} = \ker \Theta_R^{(k)}$ this amounts to showing that $\Theta_R^{(k)}(e(\boldsymbol{i})) = 0$ when $k(\boldsymbol{i}) > 0$. This follows from Lemmas 8.13 and 6.15 $\qquad\square$

Cross bipartitions and irreducible representations. Following [CW], a bipartition $\lambda \in \Lambda$ is an *(m, n)-cross bipartition* if there exists some $1 \leq i \leq$

$m + 1$ with $\lambda_i^{\mathrm{L}} + \lambda_{m+2-i}^{\mathrm{R}} < n + 1$. If we represent λ by its picture obtained by rotating λ^{R} through $180°$ as in the following example, then λ is an (m, n)-cross bipartition if and only if its picture can be fitted into a cross of height m and width n.

$$\lambda = \big((3, 2), (2, 1^2)\big) \qquad \leftrightarrow \qquad m\big\{$$
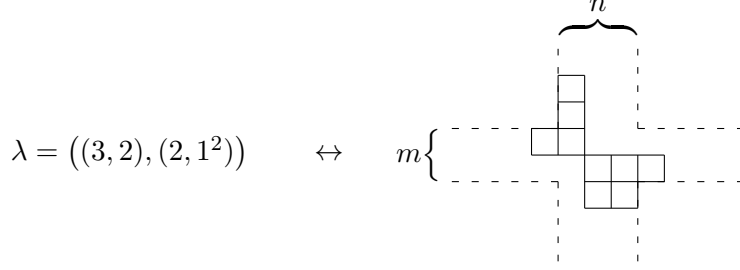
Recall the number $k(\lambda)$ defined from the weight diagram λ by (6.17), and recall that $k = \min(m, n)$.

Lemma 8.16. *Given $\lambda \in \Lambda$, we have that $k(\lambda) \leq k$ if and only if λ is an (m, n)-cross bipartition.*

Proof. We say λ is (m, n)-minimal if λ is not an (m, n)-cross bipartition but every bipartition μ obtained from λ by removing a box is an (m, n)-cross bipartition. We say that λ is k-minimal if $k(\lambda) > k$ but every bipartition μ obtained from λ by removing a box satisfies $k(\mu) \leq k$.

Suppose that μ is obtained from λ by removing a box. In terms of weight diagrams, this means that μ is obtained from λ by relabelling the ith and $(i+1)$th vertices according to one of the entries in the following table.

λ	$\circ\wedge$	$\vee\circ$	$\times\wedge$	$\vee\times$	$\vee\wedge$	$\vee\wedge$	$\circ\times$	$\times\circ$
μ	$\wedge\circ$	$\circ\vee$	$\wedge\times$	$\times\vee$	$\circ\times$	$\times\circ$	$\wedge\vee$	$\wedge\vee$

In all cases in this table, it is clear from (6.17) either that $k(\mu) = k(\lambda)$ or that $k(\mu) = k(\lambda) - 1$; the latter can hold only for the rightmost two entries in the table. In particular, we always have that $k(\mu) \leq k(\lambda)$. Hence if $k(\lambda) \leq k$ then $k(\mu) \leq k$. Also it is also obvious that if λ is an (m, n)-cross bipartition then so is μ. The last two observations reduce the proof of the lemma to showing that λ is (m, n)-minimal if and only if λ is k-minimal.

By the definition, λ is (m, n)-minimal if and only if $\lambda_i^{\mathrm{L}} + \lambda_{m+2-i}^{\mathrm{R}} = n + 1$ for each $i = 1, \ldots, m + 1$ and $\lambda_j^{\mathrm{L}} = \lambda_j^{\mathrm{R}} = 0$ for each $j \geq m + 2$.

Finally suppose that λ is k-minimal. Considering the cases in the above table, k-minimality implies that $k(\lambda) = k + 1$. Moreover the weight diagram λ has infinitely many $\wedge$'s to the left, infinitely many $\vee$'s to the right, and in between there are only $\circ$'s and $\times$'s. More precisely, there are exactly $(n + 1)$ vertices labelled $\circ$ and $(m + 1)$ vertices labelled $\times$. Now it follows from the weight dictionary that $\lambda_i^{\mathrm{L}} + \lambda_{m+2-i}^{\mathrm{R}} = n + 1$ for each $i = 1, \ldots, m + 1$ and $\lambda_j^{\mathrm{L}} = \lambda_j^{\mathrm{R}} = 0$ for each $j \geq m + 2$ just like in the previous paragraph. $\square$

Theorem 8.17. *If $\lambda \in \dot{\Lambda}_{r,s}$ is an (m, n)-cross bipartition then $\ker \Psi_{r,s}^{m,n}$ acts as zero on $D_{r,s}(\lambda)$, hence $D_{r,s}(\lambda)$ induces an irreducible $B_{r,s}(\delta)/\ker \Psi_{r,s}^{m,n}$-module. The modules*

$$\{D_{r,s}(\lambda) \,|\, \text{for all } (m, n)\text{-cross bipartitions } \lambda \in \dot{\Lambda}_{r,s}\}$$

*give a complete set of pairwise non-isomorphic irreducible $B_{r,s}(\delta)/\ker\Psi_{r,s}^{m,n}$-modules.*

Proof. The surjection $\Theta_R^{(k)}$ induces an isomorphism between $B_{r,s}(\delta)/\ker\Psi_{r,s}^{m,n}$ and $B_R(\delta)/B_R(\delta)_{>k}$. The pull-backs through this isomorphism of the irreducible $B_R(\delta)/B_R(\delta)_{>k}$-modules classified in Theorem 6.14 give us a complete set of pairwise non-isomorphic irreducible $B_{r,s}(\delta)/\ker\Psi_{r,s}^{m,n}$-modules. We do not know *a priori* that the labellings match up correctly, i.e. that the pull-back of $D_R(\lambda)$ is the same as the push-forward of $D_{r,s}(\lambda)$, but this follows from Lemmas 8.13–8.14. Finally the resulting irreducible modules are indexed by the (m,n)-cross bipartitions in $\dot{\Lambda}_{r,s}$ thanks to Lemma 8.16.   □

Indecomposable summands of mixed tensor space. In the final subsection we explain how to deduce the classification of indecomposable summands of the mixed tensor spaces $V^{\otimes r}\otimes W^{\otimes s}$ obtained recently by Comes and Wilson [CW]. Recall by Theorem 8.6 that the category $\mathscr{F}(m|n)$ of rational $\mathfrak{g}$-supermodules is equivalent to the category of finite dimensional $K(m|n)$-modules, and that the irreducible objects in either of these categories are indexed by the set Ξ from (8.11).

We begin by defining a map

$$\{(m,n)\text{-cross bipartitions}\}\to\Xi,\quad \lambda\mapsto\lambda^\dagger. \tag{8.17}$$

Let $\lambda\in\Lambda$ be an (m,n)-cross bipartition and define $k(\lambda)$ according to (6.17). Let α be the weight diagram obtained from the diagram η from (6.1) by switching the rightmost $k(\lambda)$ $\wedge$'s with the leftmost $k(\lambda)$ $\vee$'s. Let t be the crossingless matching obtained by drawing the cap diagram $\overline{\lambda}$ underneath the cup diagram $\underline{\alpha}$, then joining rays according to the unique order-preserving bijection that is the identity outside of some finite interval. Then take the oriented crossingless matching $\lambda t\alpha$, replace α with the weight diagram ζ from (8.14), and finally adjust the labels of λ that are at the bottoms of line segments, to obtain $\lambda^\dagger\in\Xi$ such that $\lambda^\dagger t\zeta$ is consistently oriented. Here is an example, taking $m=4, n=5$ and $\lambda=((5,3,2),(5,4,1^2))$, so $k=4$ and $k(\lambda)=3$:
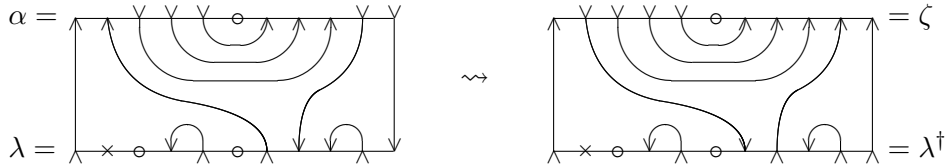
In general the map (8.17) is neither injective nor surjective.

Continuing with $\lambda,\lambda^\dagger$ and t as in the previous paragraph, let Γ be the block containing ζ, let Δ be the block containing $\lambda^\dagger$, and define

$$R(\lambda):=G_{\Delta\Gamma}^t L(\zeta)\in K(m|n)\text{-mod}_{lf}, \tag{8.18}$$

where $G_{\Delta\Gamma}^t$ is the indecomposable projective functor from (3.7). This is a finite dimensional graded $K(m|n)$-module. Theorem 3.5 (and also [BS2, Theorem 4.11]) gives detailed information about the structure of this module. In particular the last part of Theorem 3.5 implies the following lemma, recalling that the defect $\mathrm{def}(\lambda)$ is the number of caps in $\overline{\lambda}$.

Lemma 8.18. *For an (m, n)-cross bipartition λ, the graded $K(m|n)$-module $R(\lambda)$ is indecomposable with irreducible head isomorphic to $L(\lambda^\dagger)\langle -\operatorname{def}(\lambda)\rangle$ and irreducible socle isomorphic to $L(\lambda^\dagger)\langle\operatorname{def}(\lambda)\rangle$.*

For the final theorem, we transport the $K(m|n)$-module $R(\lambda)$ through the equivalence of categories from Theorem 8.6 to obtain a finite dimensional $\mathfrak{g}$-supermodule, which we also denote by $R(\lambda)$.

Theorem 8.19. *Up to isomorphism, the indecomposable summands of the mixed tensor space $V^{\otimes r} \otimes W^{\otimes s}$ are the $\mathfrak{g}$-supermodules*

$$\{R(\lambda) \mid \text{for all } (m, n)\text{-cross bipartitions } \lambda \in \dot\Lambda_{r,s}\}.$$

More precisely, given any primitive idempotent $e \in B_{r,s}(\delta)$, let $\lambda \in \dot\Lambda_{r,s}$ be the bipartition labelling the irreducible head of the projective indecomposable module $B_{r,s}(\delta)e$. Then e is non-zero on $V^{\otimes r} \otimes W^{\otimes s}$ if and only if λ is an (m, n)-cross bipartition, in which case $(V^{\otimes r} \otimes W^{\otimes s})e \cong R(\lambda)$ as $\mathfrak{g}$-supermodules.

Proof. The endomorphism algebra $\operatorname{End}_{\mathfrak{g}}(V^{\otimes r} \otimes W^{\otimes s})^{\mathrm{op}}$ is identified with the quotient algebra $B_{r,s}(\delta)/\ker \Psi^{m,n}_{r,s}$ thanks to Theorem 7.8. Hence the isomorphism classes of indecomposable summands of $V^{\otimes r} \otimes W^{\otimes s}$ are parametrised by the isomorphism classes of irreducible $B_{r,s}(\delta)/\ker \Psi^{m,n}_{r,s}$-modules, which we determined in Theorem 8.17. We deduce for a primitive idempotent $e \in B_{r,s}(\delta)$ of type $\lambda \in \dot\Lambda_{r,s}$ as in the statement of the theorem that $(V^{\otimes r} \otimes W^{\otimes s})e = \{0\}$ unless λ is an (m, n)-cross-bipartition, in which case $(V^{\otimes r} \otimes W^{\otimes s})e$ is a representative for the class of indecomposable summands parametrised by λ.

It remains to show that $(V^{\otimes r} \otimes W^{\otimes s})e \cong R(\lambda)$. Pick any $R \in \mathscr{R}_{r,s}$ and let $e(\boldsymbol{i}) \in B_{r,s}(\delta)$ denote the corresponding weight idempotents. Replacing e by a conjugate if necessary, we may assume that e is contained in $e(\boldsymbol{i})$ for some $\boldsymbol{i} \in \mathbb{Z}^{r+s}$, i.e. $e(\boldsymbol{i}) = e + f$ for some idempotent f orthogonal to e. By Lemma 8.14, we know that $e(\boldsymbol{i})$ is isotypic of type λ. Now we need to show that $(V^{\otimes r} \otimes W^{\otimes s})e(\boldsymbol{i})$ is isomorphic to a direct sum of copies of $R(\lambda)$. For this we observe from Theorem 8.5 that $(V^{\otimes r} \otimes W^{\otimes s})e(\boldsymbol{i}) \cong {}^*R_{\boldsymbol{i}}\mathbb{C}$ as $\mathfrak{g}$-supermodules. Transporting to $K(m|n)$ using Theorem 8.6 we are reduced to showing that all indecomposable summands of ${}^*R_{\boldsymbol{i}}L(\zeta)$ are isomorphic to $R(\lambda)$ (up to degree shift). Finally observe using Lemma 3.3 that ${}^*R_{\boldsymbol{i}}$ is a direct sum of copies of the indecomposable projective functor $G^t_{\Delta\Gamma}$ appearing in the definition (8.18). $\square$

In particular Theorem 8.19 implies that the indecomposable summands of the mixed tensor spaces $V^{\otimes r} \otimes W^{\otimes s}$ for all $r, s \geq 0$ are parametrised by the (m, n)-cross bipartitions, exactly as in [CW].

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