arXiv:1004.1168v1 [math.RT] 7 Apr 2010

BLOCKS OF THE CATEGORY OF CUSPIDAL $\mathfrak{sp}_{2n}$-MODULES

VOLODYMYR MAZORCHUK AND CATHARINA STROPPEL

ABSTRACT. We show that every block of the category of cuspidal generalized weight modules with finite dimensional generalized weight spaces over the Lie algebra $\mathfrak{sp}_{2n}(\mathbb{C})$ is equivalent to the category of finite dimensional $\mathbb{C}[[t_1, t_2, \ldots, t_n]]$-modules.

1. Introduction and description of the results

We fix the ground field to be the complex numbers. Fix $n \in \{2, 3, \ldots\}$ and consider the symplectic Lie algebra $\mathfrak{sp}_{2n} =: \mathfrak{g}$ with a fixed Cartan subalgebra $\mathfrak{h}$ and root space decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha,$$

where Δ denotes the corresponding set of roots. For a $\mathfrak{g}$-module V and $\lambda \in \mathfrak{h}^*$ set

$$V_\lambda := \{v \in V : h \cdot v = \lambda(h)v \text{ for any } h \in \mathfrak{h}\},$$
$$V^\lambda := \{v \in V : (h - \lambda(h))^k \cdot v = 0 \text{ for any } h \in \mathfrak{h} \text{ and } k \gg 0\}.$$

A $\mathfrak{g}$-module V is called

- *weight* provided that $V = \oplus_{\lambda \in \mathfrak{h}^*} V_\lambda$;
- *generalized weight* provided that $V = \oplus_{\lambda \in \mathfrak{h}^*} V^\lambda$;
- *cuspidal* provided that for any $\alpha \in \Delta$ the action of any nonzero element from $\mathfrak{g}_\alpha$ on V is bijective.

If V is a generalized weight module, then the set $\{\lambda \in \mathfrak{h}^* : V_\lambda \neq 0\}$ is called the *support* of V and is denoted by $\mathrm{supp}(V)$.

Denote by $\hat{\mathcal{C}}$ the full subcategory in $\mathfrak{g}$-mod which consist of all cuspidal generalized weight modules with finite-dimensional generalized weight spaces, and by $\mathcal{C}$ the full subcategory of $\hat{\mathcal{C}}$ consisting of all weight modules. Understanding the categories $\mathcal{C}$ and $\hat{\mathcal{C}}$ is a classical problem in the representation theory of Lie algebras. The first major step towards the solution of this problem was made in [Mat], where all simple objects in $\hat{\mathcal{C}}$ were classified. In [BKLM] it was shown that the category $\mathcal{C}$ is semi-simple, hence completely understood. The aim of the present note is to describe the category $\hat{\mathcal{C}}$.

Apart from $\mathfrak{sp}_{2n}$, cuspidal weight modules with finite dimensional weight spaces exist only for the Lie algebra $\mathfrak{sl}_n$ ([Fe]). In the latter case, simple objects in the corresponding category $\hat{\mathcal{C}}$ are classified in [Mat], the category $\mathcal{C}$ is described in [GS], see also [MS], and the category $\hat{\mathcal{C}}$ is described in [MS]. Taking all these results into account, the present paper completes the study of cuspidal generalized weight modules with finite dimensional generalized weight spaces over semi-simple finite-dimensional Lie algebras.

Let $U(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$ and $Z(\mathfrak{g})$ be the center of $U(\mathfrak{g})$. The action of $Z(\mathfrak{g})$ on any object from $\hat{\mathcal{C}}$ is locally finite. Using this and the

Date: April 8, 2010.

standard support arguments gives the following *block decomposition* of $\hat{\mathcal{C}}$:

$$\hat{\mathcal{C}} \cong \bigoplus_{\substack{\chi\,:\,Z(\mathfrak{g})\to\mathbb{C}\\ \xi\in\mathfrak{h}^*/\mathbb{Z}\Delta}} \hat{\mathcal{C}}_{\chi,\xi},$$

where $\hat{\mathcal{C}}_{\chi,\xi}$ consists of all V such that $\mathrm{Supp}(V)\subset\xi$ and $(z-\chi(z))^k\cdot v=0$ for all $v\in V$, $z\in Z(\mathfrak{g})$ and $k\gg 0$. Set $\mathcal{C}_{\chi,\xi}:=\mathcal{C}\cap\hat{\mathcal{C}}_{\chi,\xi}$. From [Mat, Section 9] it follows that each nontrivial $\hat{\mathcal{C}}_{\chi,\xi}$ contains a unique (up to isomorphism) simple object, in particular, $\hat{\mathcal{C}}_{\chi,\xi}$ is indecomposable, hence a block. From this and [BKLM] we thus get that every nontrivial block $\mathcal{C}_{\chi,\xi}$ is equivalent to the category of finite-dimensional $\mathbb{C}$-modules. Our main result is the following:

Theorem 1. *Every nontrivial block $\hat{\mathcal{C}}_{\chi,\xi}$ is equivalent to the category of finite dimensional $\mathbb{C}[[t_1,t_2,\dots,t_n]]$-modules.*

To prove Theorem 1 we use and further develop the technique of extension of the module structure from a Lie subalgebra, originally developed in [MS] for the study of categories of singular and non-integral cuspidal generalized weight $\mathfrak{sl}_n$-modules. The proof of Theorem 1 is given in Section 4. In Section 2 we recall the standard reduction to the case of the so-called simple *completely pointed* modules (i.e. simple weight cuspidal modules for which **all** nontrivial weight spaces are one-dimensional) and a realization of such modules using differential operators. In Section 3 we define a functor from the category of finite dimensional $\mathbb{C}[[t_1,t_2,\dots,t_n]]$-modules to any block $\hat{\mathcal{C}}_{\chi,\xi}$ containing a simple completely pointed module. In Section 4 we prove that this functor is an equivalence of categories. In Section 5 we present some consequences of our main result, in particular, we recover the main result from [BKLM] stated above.

Acknowledgments. The first author was partially supported by the Royal Swedish Academy of Sciences and the Swedish Research Council. The paper was written up, when the second author was visiting the Mathematical Sciences Research Institute (MSRI) in Berkeley. She deeply acknowledges the excellent working conditions and financial support.

2. Completely pointed simple cuspidal weight modules

A weight $\mathfrak{g}$-module V is called *pointed* provided that $\dim V_\lambda=1$ for some $\lambda\in\mathfrak{h}^*$. If V is a pointed simple cuspidal weight $\mathfrak{g}$-module, then, obviously, all nontrivial weight spaces of V are one-dimensional, in which case one says that V is *completely pointed* (see [BKLM]). It is enough to consider blocks with completely pointed simple modules because of the following:

Lemma 2. *All nontrivial blocks of $\hat{\mathcal{C}}$ are equivalent.*

Proof. In the case of the category $\mathcal{C}$ this is proved in [BKLM, Lemma 2]. The same argument works in the case of the category $\hat{\mathcal{C}}$ as well. □

Let us recall the explicit realization of completely pointed simple cuspidal modules from [BL]. Denote by W_n the n-th *Weyl algebra*, that is the algebra of differential operators with polynomial coefficients in variables $x_1,x_2,\dots,x_n$. The algebra W_n is generated by x_i and $\frac{\partial}{\partial x_i}$, $i=1,\dots,n$, which satisfy the relations $[\frac{\partial}{\partial x_i},x_j]=\delta_{i,j}$. Let $\varepsilon_1,\varepsilon_2,\dots,\varepsilon_n$ be the vectors of the standard basis in $\mathbb{C}^n$. Identify $\mathbb{C}^n$ with $\mathfrak{h}^*$ such that Δ becomes the following *standard root system* of type C_n:

$$\{\pm(\varepsilon_i\pm\varepsilon_j):1\le i<j\le n\}\cup\{\pm 2\varepsilon_i:1\le i\le n\}.$$

Then

$$\mathbf{H} = \mathbf{H}_n = \{2\varepsilon_1, \varepsilon_2 - \varepsilon_1, \varepsilon_3 - \varepsilon_2, \dots, \varepsilon_n - \varepsilon_{n-1}\}$$

is a basis of Δ. Fix a basis of $\mathfrak{g}$ of the form

$$\mathbf{C} := \{X_{\pm\varepsilon_i \pm \varepsilon_j} : 1 \leq i < j \leq n\} \cup \{X_{\pm 2\varepsilon_i} : i = 1, 2, \dots, n\} \cup \{H_\alpha : \alpha \in \mathbf{H}\}$$

such that the following map defines an injective Lie algebra homomorphism from $\mathfrak{g}$ to the Lie algebra associated with W_n:

$$(1)\qquad \begin{array}{lcll} X_{\varepsilon_i - \varepsilon_j} & \mapsto & x_i \frac{\partial}{\partial x_j}, & 1 \leq i \neq j \leq n; \\ X_{\varepsilon_i + \varepsilon_j} & \mapsto & x_i x_j, & i, j = 1, 2, \dots, n; \\ X_{-\varepsilon_i - \varepsilon_j} & \mapsto & \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j}, & i, j = 1, 2, \dots, n; \\ H_{\varepsilon_{i+1} - \varepsilon_i} & \mapsto & x_{i+1} \frac{\partial}{\partial x_{i+1}} - x_i \frac{\partial}{\partial x_i}, & i = 1, 2, \dots, n-1; \\ H_{2\varepsilon_1} & \mapsto & \frac{1}{2}\left(x_1 \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_1} x_1 \right). & \end{array}$$

Set

$$\mathbf{B} := \{(b_1, b_2, \dots, b_n) \in \mathbb{Z}^n : b_1 + b_2 + \cdots + b_n \in 2\mathbb{Z}\}.$$

For $\mathbf{a} = (a_1, a_2, \dots, a_n) \in \mathbb{C}^n$ define $N(\mathbf{a})$ to be the linear span of

$$\{\mathbf{x}^{\mathbf{b}} := x_1^{a_1 + b_1} x_2^{a_2 + b_2} \cdots x_n^{a_n + b_n} : \mathbf{b} \in \mathbf{B}\}.$$

We first define an action of the elements from $\mathbf{C}$ on $N(\mathbf{a})$ using the formulas from (1) as follows:

$$(2)\qquad \begin{array}{lcll} X_{\varepsilon_i - \varepsilon_j} \mathbf{x}^{\mathbf{b}} & = & (a_j + b_j) \mathbf{x}^{\mathbf{b} + \varepsilon_i - \varepsilon_j}, & 1 \leq i \neq j \leq n; \\ X_{\varepsilon_i + \varepsilon_j} \mathbf{x}^{\mathbf{b}} & = & \mathbf{x}^{\mathbf{b} + \varepsilon_i + \varepsilon_j}, & i, j = 1, 2, \dots, n; \\ X_{-\varepsilon_i - \varepsilon_j} \mathbf{x}^{\mathbf{b}} & = & (a_i + b_i)(a_j + b_j) \mathbf{x}^{\mathbf{b} - \varepsilon_i - \varepsilon_j}, & 1 \leq i \neq j \leq n; \\ X_{-2\varepsilon_i} \mathbf{x}^{\mathbf{b}} & = & (a_i + b_i)(a_i + b_i - 1) \mathbf{x}^{\mathbf{b} - 2\varepsilon_i}, & i = 1, 2, \dots, n; \\ H_{\varepsilon_{i+1} - \varepsilon_i} \mathbf{x}^{\mathbf{b}} & = & (a_{i+1} + b_{i+1} - a_i - b_i) \mathbf{x}^{\mathbf{b}}, & i = 1, 2, \dots, n-1; \\ H_{2\varepsilon_1} \mathbf{x}^{\mathbf{b}} & = & \frac{1}{2}(2a_1 + 2b_1 + 1) \mathbf{x}^{\mathbf{b}}. & \end{array}$$

Theorem 3 ([BL]). *(i) For every $\mathbf{a} \in \mathbb{C}^n$ formulae* (2) *define on $N(\mathbf{a})$ the structure of a completely pointed weight $\mathfrak{g}$-module.*

(ii) If $a_i \notin \mathbb{Z}$ for all $i = 1, \dots, n$, then the module $N(\mathbf{a})$ is simple and cuspidal.

(iii) Every completely pointed simple cuspidal $\mathfrak{g}$-module is isomorphic to $N(\mathbf{a})$ for some $\mathbf{a} \in \mathbb{C}^n$ such that $a_i \notin \mathbb{Z}$, $i = 1, \dots, n$.

3. The functor F

This section is similar to [MS, Subsection 3.1]. Fix $\mathbf{a} \in \mathbb{C}^n$ such that $a_i \notin \mathbb{Z}$, $i = 1, \dots, n$. Let $\hat{\mathcal{C}}_{\mathbf{a}}$ denote the block of $\hat{\mathcal{C}}$ containing $N(\mathbf{a})$. The category $\hat{\mathcal{C}}_{\mathbf{a}}$ is closed under extensions. Denote by $\mathbb{C}[[t_1, t_2, \dots, t_n]]$-mod the category of finite dimensional $\mathbb{C}[[t_1, t_2, \dots, t_n]]$-modules. For $V \in \mathbb{C}[[t_1, t_2, \dots, t_n]]$-mod denote by T_i the linear operator describing the action of t_i on V. Set $\mathbf{0} = (0, 0, \dots, 0) \in \mathbf{B}$.

For $\mathbf{b} \in \mathbf{B}$ consider a copy $V^{\mathbf{b}}$ of V. Define

$$\mathrm{F}V := \bigoplus_{\mathbf{b} \in \mathbf{B}} V^{\mathbf{b}}.$$

Define the action of elements from $\mathbf{C}$ on the vector space $\mathrm{F}V$ in the following way: for $v \in V^{\mathbf{b}}$ set

$$(3)\qquad \begin{array}{lcllcl} X_{\varepsilon_i - \varepsilon_j} v & = & (T_j + (a_j + b_j)\mathrm{Id}_V) v & \in & V^{\mathbf{b} + \varepsilon_i - \varepsilon_j}; \\ X_{\varepsilon_i + \varepsilon_j} v & = & v & \in & V^{\mathbf{b} + \varepsilon_i + \varepsilon_j}; \\ X_{-\varepsilon_i - \varepsilon_j} v & = & (T_i + (a_i + b_i)\mathrm{Id}_V)(T_j + (a_j + b_j)\mathrm{Id}_V) v & \in & V^{\mathbf{b} - \varepsilon_i - \varepsilon_j}; \\ X_{2\varepsilon_i} v & = & (T_i + (a_i + b_i)\mathrm{Id}_V)(T_i + (a_i + b_i - 1)\mathrm{Id}_V) v & \in & V^{\mathbf{b} - 2\varepsilon_i}; \\ H_{\varepsilon_{i+1} - \varepsilon_i} v & = & (T_{i+1} - T_i + (a_{i+1} + b_{i+1} - a_i - b_i)\mathrm{Id}_V) v & \in & V^{\mathbf{b}}; \\ H_{2\varepsilon_1} v & = & \frac{1}{2}(2T_1 + (2a_1 + 2b_1 + 1)\mathrm{Id}_V) v & \in & V^{\mathbf{b}}; \end{array}$$

where i and j are as in the respective row of (2). For a homomorphism $f : V \to W$ of $\mathbb{C}[[t_1, t_2, \dots, t_n]]$-modules denote by $\mathrm{F}f$ the diagonally extended linear map from $\mathrm{F}V$ to $\mathrm{F}W$, i.e. for every $\mathbf{b} \in \mathbf{B}$ and $v \in V^{\mathbf{b}}$ set

$$\mathrm{F}f(v) = f(v) \in W^{\mathbf{b}}. \tag{4}$$

Proposition 4. *(i) Formulae* (3) *define on* $\mathrm{F}V$ *the structure of a* $\mathfrak{g}$*-module.*

(ii) Every $V^{\mathbf{b}}$ *is a generalized weight space of* $\mathrm{F}V$*. Moreover, for* $\mathbf{b} \neq \mathbf{b}'$ *the weights of* $V^{\mathbf{b}}$ *and* $V^{\mathbf{b}'}$ *are different.*

(iii) The module $\mathrm{F}V$ *belongs to* $\hat{\mathcal{C}}_{\mathbf{a}}$*.*

(iv) Formulas (3) *and* (4) *turn* F *into a functor*

$$\mathrm{F} : \mathbb{C}[[t_1, t_2, \dots, t_n]]\text{-mod} \to \hat{\mathcal{C}}_{\mathbf{a}}.$$

(v) The functor F *is exact, faithful and full.*

Proof. Consider the $\mathfrak{g}$-module $N(\mathbf{a})$ for $\mathbf{a}$ as above. Then, for every $\mathbf{b}$ the defining relations of $\mathfrak{g}$ (in terms of elements from $\mathbf{C}$), applied to $\mathbf{x}^{\mathbf{b}}$, can be written as some polynomial equations in the a_i's. Since (2) defines a $\mathfrak{g}$-module for any $\mathbf{a}$ (Theorem 3(i)), these equations hold for any $\mathbf{a}$, that is they are actually formal identities in the a_i's. Write now $T_j + (a_j + b_j)\mathrm{Id}_V = A_j + B_j$, a sum of matrices, where $A_j = T_j + a_j\mathrm{Id}_V$ and $B_j = b_j\mathrm{Id}_V$. Note that all A_i and B_j commute with each other and with all T_l's. For a fixed $\mathbf{b}$, the defining relations for $\mathfrak{g}$ on $\mathrm{F}V$ reduce to our formal identities (in the A_i's) and hence are satisfied. This proves claim (i). Claim (ii) follows from the the last two lines in (3) and the fact that all T_i's are nilpotent (hence zero is the only eigenvalue).

As f commutes with all T_i, the map $\mathrm{F}f$ commutes with the action of all elements from $\mathbf{C}$ and hence defines a homomorphism of $\mathfrak{g}$-modules. By construction we also have $\mathrm{F}(f \circ f') = \mathrm{F}f \circ \mathrm{F}f'$, which implies claim (iv).

By construction, F is exact and faithful. It sends the simple one-dimensional $\mathbb{C}[[t_1, t_2, \dots, t_n]]$-module to $N(\mathbf{a})$ (as in this case all $T_i = 0$ and hence (3) gives (2)), which is an object of the category $\hat{\mathcal{C}}_{\mathbf{a}}$ closed under extensions. Claim (iii) follows.

To complete the proof of claim (v) we are left to show that F is full. Let $\varphi : \mathrm{F}V \to \mathrm{F}W$ be a $\mathfrak{g}$-homomorphism. Then φ commutes with the action of all elements from $\mathfrak{h}$. Using claim (ii), we get that φ induces, by restriction, a linear map $f : V = V^{\mathbf{0}} \to W^{\mathbf{0}} = W$. As φ commutes with all $H_{\varepsilon_{i+1}-\varepsilon_i}$, the map f commutes with all operators $T_{i+1} - T_i$. As φ commutes with $H_{2\varepsilon_1}$, the map f commutes with T_1. It follows that f is a homomorphism of $\mathbb{C}[[t_1, t_2, \dots, t_n]]$-modules. This yields $\varphi = \mathrm{F}f$ and thus the functor F is full. This completes the proof of claim (v) and of the whole proposition. □

4. Proof of Theorem 1

Because of Lemma 2 it is enough to fix one particular block and show there that F is an equivalence. Thus, we may assume that $a_i + a_j \notin \mathbb{Z}$ for all i, j (in particular, $a_i \notin \mathbb{Z}$ for all i). After Proposition 4, we are only left to show that F is dense (i.e. essentially surjective). We establish density of F by induction on n. We first prove the induction step and then the basis of the induction, which is the case $n = 2$.

Denote by λ the weight of $\mathbf{x}^{\mathbf{0}} \in N(\mathbf{a})$ (see Proposition 4(ii)). Let $M \in \hat{\mathcal{C}}_{\mathbf{a}}$. Set $V := M_\lambda$ and denote by M' the $\mathfrak{a}$-module $U(\mathfrak{a})V$.

4.1. Reduction to the case $n = 2$.

The main result of this subsection is the following:

Proposition 5. *If the functor* F *is dense for* $n = 2$*, then it is dense for any* $n \geq 2$*.*

Proof. Assume that $n > 2$ and that the functor F is dense in the case of the algebra $\mathfrak{sp}_{2n-2}$. We realize $\mathfrak{sp}_{2n-2}$ as the subalgebra $\mathfrak{a}$ of $\mathfrak{g}$ corresponding to the subset $\mathbf{H}_{n-1} \subset \mathbf{H}$ of simple roots.

Let Y_1, Y_2,..., Y_n be the linear operators representing the action of the elements $H_{2\varepsilon_1}, H_{\varepsilon_2-\varepsilon_1}, H_{\varepsilon_3-\varepsilon_2}, \dots, H_{\varepsilon_n-\varepsilon_{n-1}}$ on V, respectively. Set

$$
\begin{array}{lcl}
T_1 & := & Y_1 - \frac{1}{2}(2a_1+1)\mathrm{Id}_V;\\
T_2 & := & Y_2 + T_1 - (a_2-a_1)\mathrm{Id}_V;\\
T_3 & := & Y_3 + T_2 - (a_3-a_2)\mathrm{Id}_V;\\
 & & \vdots\\
T_n & := & Y_n + T_{n-1} - (a_n-a_{n-1})\mathrm{Id}_V.
\end{array}
\tag{5}
$$

The T_i's are obviously pairwise commuting nilpotent linear operators.

The module M' is a cuspidal generalized weight $\mathfrak{a}$-module with finite-dimensional weight spaces. Moreover, as all composition subquotients of M are of the form $N(\mathbf{a})$, all composition subquotients of M' are of the form $N(\mathbf{a})'$, the latter being a completely pointed simple cuspidal $\mathfrak{a}$-module. By our inductive assumption, the functor F is dense in the case of the algebra $\mathfrak{a}$. Hence $M' \cong N' := \oplus_{\mathbf{b}} V^{\mathbf{b}}$, where $\mathbf{b} \in \mathbf{B}$ is such that $b_n = 0$, and the action of $\mathfrak{a}$ on N' is given by (3).

Lemma 6. *There is a unique (up to isomorphism) $\mathfrak{g}$-module $Q \in \hat{\mathcal{C}}_{\mathbf{a}}$ such that $Q' = N'$ and which gives the linear operator T_n when computed using* (5)*.*

Proof. The existence statement is clear, so we need only to show uniqueness. Assume that $Q \in \hat{\mathcal{C}}_{\mathbf{a}}$ is such that $Q' = N'$ and the formulae (5), applied to Q, produce the linear operator T_n. Since $a_n \notin \mathbb{Z}$, the endomorphism $T_n + (a_n + b_n)\mathrm{Id}_V$ is invertible for all $b_n \in \mathbb{Z}$. As the action of $X_{\varepsilon_n-\varepsilon_{n-1}}$ on Q is bijective, we can fix a weight basis in Q such that both the $\mathfrak{a}$-action on $Q' = N'$ and the action of $X_{\varepsilon_n-\varepsilon_{n-1}}$ on the whole Q is given by (3). As $n > 2$, the elements $X_{\pm 2\varepsilon_1}$ commute with $X_{\varepsilon_n-\varepsilon_{n-1}}$ and hence their action extends uniquely to the whole of Q using this commutativity. Similarly for all elements $X_{\pm(\varepsilon_i-\varepsilon_{i-1})}$, $i < n-1$, and for the element $X_{\varepsilon_{n-2}-\varepsilon_{n-1}}$. This leaves us with the elements $X_{\varepsilon_{n-1}-\varepsilon_{n-2}}$ and $X_{\varepsilon_{n-1}-\varepsilon_n}$. Note that the simple roots $\varepsilon_{n-1}-\varepsilon_{n-2}$ and $\varepsilon_n-\varepsilon_{n-1}$ corresponding to the elements $X_{\varepsilon_{n-1}-\varepsilon_{n-2}}$ and $X_{\varepsilon_n-\varepsilon_{n-1}}$ generate a root system of type A_2 (this corresponds to the algebra $\mathfrak{sl}_3$). Therefore the fact that the action of $X_{\varepsilon_{n-1}-\varepsilon_{n-2}}$ extends uniquely to Q is proved in [MS, Lemma 21], and the fact that the action of $X_{\varepsilon_{n-1}-\varepsilon_n}$ extends uniquely to Q is proved in [MS, Lemma 22]. This completes the proof. □

The module FV obviously satisfies $(\mathrm{F}V)' = N'$ and defines the linear operator T_n when computed using (5). Hence Lemma 6 implies $M \cong \mathrm{F}V$. Since $M \in \hat{\mathcal{C}}_{\mathbf{a}}$ was arbitrary, this shows that the functor F is dense, completing the proof. □

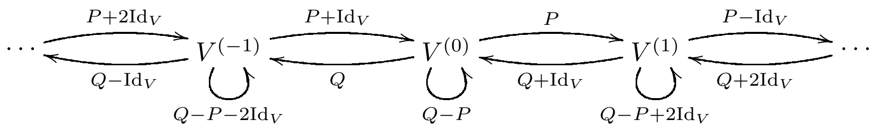
4.2. **Base of the induction: some $\mathfrak{sl}_2$-theory as preparation.** In this subsection we will recall (and slightly improve) some classical $\mathfrak{sl}_2$-theory. We refer the reader to [Maz] for more details. Consider the Lie algebra $\mathfrak{sl}_2 = \mathfrak{sl}_2(\mathbb{C})$ with standard basis

$$
\mathbf{e} := \left(\begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array}\right), \quad \mathbf{f} := \left(\begin{array}{cc} 0 & 0 \\ 1 & 0 \end{array}\right), \quad \mathbf{h} := \left(\begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array}\right).
$$

Let V be a finite-dimensional vector space and A and B be two commuting linear operators on V. For $i \in \mathbb{Z}$ denote by $V^{(i)}$ a copy of V and consider the vector space $\overline{V} := \oplus_{i\in\mathbb{Z}} V^{(i)}$ (a direct sum of copies of V indexed by i). Define the actions of $\mathbf{e}$, $\mathbf{f}$ and $\mathbf{h}$ on $\overline{V}$ as follows: for $v \in V^{(i)}$ set

$$
\begin{array}{lclcl}
\mathbf{e}v & := & (P - i\mathrm{Id}_V)v & \in & V^{(i+1)}\\
\mathbf{f}v & := & (Q + i\mathrm{Id}_V)v & \in & V^{(i-1)}\\
\mathbf{h}v & := & (Q - P + 2i\mathrm{Id}_V)v & \in & V^{(i)}.
\end{array}
\tag{6}
$$

This can be depicted as follows (here right arrows represent the action of $\mathbf{e}$, left arrows represent the action of $\mathbf{f}$ and loops represent the action of $\mathbf{h}$):



Proposition 7. *(i) Formulae (6) define on $\overline{V}$ the structure of a generalized weight $\mathfrak{sl}_2$-module with finite dimensional generalized weight spaces.*

(ii) Every cuspidal generalized weight $\mathfrak{sl}_2$-module with finite dimensional generalized weight spaces is isomorphic to $\overline{V}$ for some V with P and Q as above.

(iii) The action of the Casimir element $\mathbf{c} := (\mathbf{h}+1)^2 + 4\mathbf{fe}$ on $\overline{V}$ is given by the linear operator $(P+Q+\mathrm{Id}_V)^2$.

(iv) Let $\mathbb{C}^2$ denote the natural $\mathfrak{sl}_2$-module (the unique two-dimensional simple $\mathfrak{sl}_2$-module). Then the linear operator $(\mathbf{c}-(P+Q+2\mathrm{Id}_V)^2)(\mathbf{c}-(P+Q)^2)$ annihilates the $\mathfrak{sl}_2$-module $\mathbb{C}^2\otimes\overline{V}$.

(v) Let $\mathbb{C}^3$ denote the unique three-dimensional simple $\mathfrak{sl}_2$-module. Then the linear operator $(\mathbf{c}-(P+Q+3\mathrm{Id}_V)^2)(\mathbf{c}-(P+Q+\mathrm{Id}_V)^2)(\mathbf{c}-(P+Q-\mathrm{Id}_V)^2)$ annihilates the $\mathfrak{sl}_2$-module $\mathbb{C}^3\otimes\overline{V}$.

Proof. The fact that $\overline{V}$ is an $\mathfrak{sl}_2$-module is checked by a direct computation. That $\overline{V}$ is a generalized weight module follows from the fact that the action of $\mathbf{h}$ on $\overline{V}$ preserves (by (6)) each V^i and hence is locally finite. Since the category of generalized weight modules is closed under extensions, to prove that $\overline{V}$ has finite dimensional generalized weight spaces it is enough to consider the case when $\mathbf{h}$ has a unique eigenvalue on $V^{(0)}$, say λ. However, in this case $\mathbf{h}$ has a unique eigenvalue on V^i, namely $\lambda+2i$, which implies that $\overline{V}^{\lambda} = V$ is finite dimensional. Claim (i) follows. To prove Claim (iii) we observe that the action of $\mathbf{c}$ on V^i is given by:

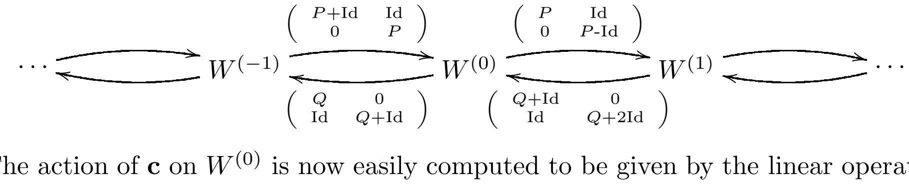
$$(Q-P+(2i+1)\mathrm{Id}_V)^2 + 4(Q+(i+1)\mathrm{Id}_V)(P-i\mathrm{Id}_V) \;=\; (P+Q+\mathrm{Id}_V)^2.$$

Claim (ii) can be found with all details in [Maz, Chapter 3].

To prove claim (iv) choose a basis $\{v_1,\dots,v_k\}$ in V, which gives rise to a basis $\{v_1^{(i)},\dots,v_k^{(i)}, i\in\mathbb{Z}\}$ in $\overline{V}$. Choose the standard basis $\{e_1,e_2\}$ in $\mathbb{C}^2$. Since $\mathbf{h}e_1 = e_1$, $\mathbf{h}e_2 = -e_2$ and $\mathbf{h}$ acts by $Q-P+2i\mathrm{Id}_V$ on $V^{(i)}$, we obtain that $\mathbf{h}$ acts by $Q-P+(2i+1)\mathrm{Id}_V$ on the vector space $W^{(i)}$ with basis

$$\{e_1\otimes v_1^{(i)},\dots,e_1\otimes v_1^{(i)}, e_2\otimes v_1^{(i+1)},\dots,e_2\otimes v_1^{(i+1)}\}.$$

We have $\mathbb{C}^2\otimes\overline{V}\cong\oplus_{i\in\mathbb{Z}}W^{(i)}$ and one easily computes that in the above basis the actions of $\mathbf{e}$ and $\mathbf{f}$ on $\mathbb{C}^2\otimes\overline{V}$ is given by the following picture:



The action of $\mathbf{c}$ on $W^{(0)}$ is now easily computed to be given by the linear operator

$$G := \begin{pmatrix} (Q-P+2\mathrm{Id})^2 + 4(Q+\mathrm{Id})P & 4(Q+\mathrm{Id}) \\ 4P & (Q-P+2\mathrm{Id})^2 + 4(Q+2\mathrm{Id})(P-\mathrm{Id}) + 4\mathrm{Id} \end{pmatrix}.$$

The characteristic polynomial of G is

$$\chi_G(\lambda) = (\lambda-(P+Q+2\mathrm{Id})^2)(\lambda-(P+Q)^2).$$

Claim (iv) now follows from the Cayley-Hamilton theorem.

We have an isomorphism of $\mathfrak{sl}_2$-modules as follows: $\mathbb{C}^2\otimes\mathbb{C}^2\cong\mathbb{C}^3\oplus\mathbb{C}$ (here $\mathbb{C}$ is the trivial module), and hence claim (v) follows applying claim (iv) twice.

Alternatively, one could do a direct calculation (similar to the proof of (iii)). The proposition follows. □

We note that the statement of Proposition 7(ii) is a special case of a more general result of Gabriel and Drozd describing blocks of the category of (generalized) weight $\mathfrak{sl}_2$-modules, in particular, simple weight $\mathfrak{sl}_2$-modules (see [Di, 7.8.16] and [Dr]). The statements of Proposition 7(iv) and (v) are $\mathfrak{sl}_2$-refinements of a theorem of Kostant describing possible (generalized) central characters of the tensor product of a finite dimensional module with an infinite dimensional module ([Ko, Theorem 5.1]).

4.3. **The case $n = 2$.** Assume now that $n = 2$. We have $a_1, a_2, a_1 + a_2 \notin \mathbb{Z}$. Let $\mathfrak{a}$ denote the Lie subalgebra of $\mathfrak{g}$ generated by $X_{\pm(\varepsilon_2-\varepsilon_1)}$. The algebra $\mathfrak{a}$ is isomorphic to $\mathfrak{sl}_2$.

Let $M \in \hat{\mathcal{C}}_{\mathbf{a}}$. Denote by λ the weight of $\mathbf{x^0} \in N(\mathbf{a})$ and set $V := M_\lambda$. Let Y_1 and Y_2 be the linear operators representing the actions of the elements $H_{\varepsilon_2-\varepsilon_1}$ and $C := (H_{\varepsilon_2-\varepsilon_1} + 1)^2 + 4X_{\varepsilon_1-\varepsilon_2}X_{\varepsilon_2-\varepsilon_1}$ on V. The element C is a Casimir element for $\mathfrak{a}$, in particular, the operators Y_1 and Y_2 commute. Our first observation is the following:

Lemma 8. *The action of C on V is invertible and hence has a square root.*

Proof. From (2) we have that C acts on $\mathbf{x^0}$ by

$$(a_2 - a_1 + 1)^2 + 4(a_2 + 1)a_1 = (a_1 + a_2 + 1)^2.$$

Since $a_1 + a_2 \notin \mathbb{Z}$ by our assumptions, $\mathbf{x^0}$ is an eigenvector of C with a nonzero eigenvalue. As the module M has a composition series with subquotients isomorphic to $N(\mathbf{a})$, the complex number $(a_1 + a_2 + 1)^2 \neq 0$ is the only eigenvalue of C on V. The claim follows. □

Consider the $\mathfrak{a}$-module $M' := U(\mathfrak{a})M_\lambda$. Let Y_2' denote any square root of Y_2, which is a polynomial in Y_2 (it exists by Lemma 8). Then Y_2' commutes with Y_1. Set

$$T_1 := \frac{Y_2' - Y_1 - \mathrm{Id}_V}{2} - a_1\mathrm{Id}_V, \quad T_2 := \frac{Y_2' + Y_1 - \mathrm{Id}_V}{2} - a_2\mathrm{Id}_V.$$

Then T_1 and T_2 are two commuting nilpotent linear operators (it is easy to check that 0 is the unique eigenvalue for both T_1 and T_2), hence define on V the structure of a $\mathbb{C}[[t_1, t_2]]$-module. The aim of this subsection is to establish an isomorphism $\mathrm{F}V \cong M$, which would complete the proof of Theorem 1.

Set $R' := U(\mathfrak{a})(\mathrm{F}V)_\lambda$. A direct computation (using (3)) shows that $H_{\varepsilon_2-\varepsilon_1}$ and C act on $(\mathrm{F}V)_\lambda = V^{\mathbf{0}}$ as the linear operators Y_1 and Y_2, respectively. As any cuspidal generalized weight $\mathfrak{a}$-module is uniquely determined by the actions of $H_{\varepsilon_2-\varepsilon_1}$ and C (see [Dr] or [Maz, 3.7] for full details), it follows that $M' \cong R'$. The isomorphism $\mathrm{F}V \cong M$ now follows from the following statement:

Proposition 9. *There is at most one (up to isomorphism) $\mathfrak{g}$-module $R \in \hat{\mathcal{C}}_{\mathbf{a}}$ such that $U(\mathfrak{a})R_\lambda = R'$.*

Proof. Let $R \in \hat{\mathcal{C}}_{\mathbf{a}}$ be such that $U(\mathfrak{a})R_\lambda = R'$. We choose a weight basis in R such that the action of $\mathfrak{a}$ on R' and the action of $X_{2\varepsilon_1}$ on R is given by (3) (in other words these actions coincide with the corresponding actions on $\mathrm{F}V$). Since $X_{\varepsilon_1-\varepsilon_2}$ commutes with $X_{2\varepsilon_1}$, it follows that the action of $X_{\varepsilon_1-\varepsilon_2}$ on R is also given by (3).

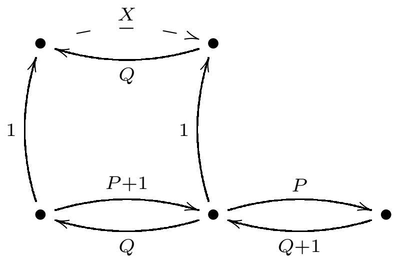
It is left to show that the action of $X_{\varepsilon_2-\varepsilon_1}$ extends uniquely from R' to R and then that there is a unique way to define the action of $X_{-2\varepsilon_1}$. This will be done in the Lemmata 10 and 11 below. □

Lemma 10. *There is a unique way to extend the action of $X_{\varepsilon_2-\varepsilon_1}$ from R' to R.*

Proof. Let us first show that for every $k \in \{1, 2, \dots\}$ the action of $X_{\varepsilon_2-\varepsilon_1}$ extends uniquely from $X_{2\varepsilon_1}^{k-1}R'$ to $X_{2\varepsilon_1}^{k}R'$ (here $X_{2\varepsilon_1}^{0}R' = R'$).

Consider the following picture:

(7)



Here bullets are weight spaces with some fixed bases. The lower row is a part of $X_{2\varepsilon_1}^{k-1}R'$ where the $\mathfrak{a}$-action is already known by induction. The bases in the weight spaces in the lower row are chosen such that the action of $\mathfrak{a}$ in the lower row is given by (3). The upper row is a part of $X_{2\varepsilon_1}^{k}R'$ where the $\mathfrak{a}$-action is to be determined. Arrows pointing up indicate the action of $X_{2\varepsilon_1}$. The bases of the weight spaces in the upper row are chosen such that the action of $X_{2\varepsilon_1}$ is given by the operator Id_V (as in (3)). Left arrows indicate the action of $X_{\varepsilon_1-\varepsilon_2}$. The latter commutes with the action of $X_{2\varepsilon_1}$ and hence is given by the same linear operator in each column. Right arrows indicate the action of $X_{\varepsilon_2-\varepsilon_1}$ (which is known for $X_{2\varepsilon_1}^{k-1}R'$ and is to be determined for $X_{2\varepsilon_1}^{k}R'$). The part to be determined is given by the dashed arrow. Labels P and Q represent coefficients (which are linear operators on V) appearing in the corresponding parts of formulae (3). Note that P and Q commute. The action of $X_{\varepsilon_2-\varepsilon_1}$ on $X_{2\varepsilon_1}^{k}R'$ which is to be determined is given by some unknown linear operators X.

From $H_{\varepsilon_2-\varepsilon_1} = [X_{\varepsilon_2-\varepsilon_1}, X_{\varepsilon_1-\varepsilon_2}]$ we compute that the action of $H_{\varepsilon_2-\varepsilon_1}$ on the middle weight space in the lower row is given by $Q - P$. Using $[H_{\varepsilon_2-\varepsilon_1}, X_{2\varepsilon_1}] = -2X_{2\varepsilon_1}$ we get that $H_{\varepsilon_2-\varepsilon_1}$ acts on the right dot of the upper row via $Q - P - 2$. Using $[H_{\varepsilon_2-\varepsilon_1}, X_{\varepsilon_1-\varepsilon_2}] = -2X_{\varepsilon_1-\varepsilon_2}$ we get that $H_{\varepsilon_2-\varepsilon_1}$ acts on the left dot of the upper row via $Q - P - 4$. Hence the action of C on the upper row is given by $(Q-P-3)^2 + 4XQ$. The action of C on the lower row is given by $(Q-P-1)^2 + 4(P+1)Q = (Q+P+1)^2$.

The elements $X_{2\varepsilon_1}$, $X_{2\varepsilon_2}$ and $X_{\varepsilon_1+\varepsilon_1}$ form a weight basis of a simple three-dimensional $\mathfrak{a}$-module $\mathbb{C}^3$ with respect to the adjoint action of $\mathfrak{a}$. Hence the upper row of our picture is a subquotient of the tensor product of the lower row and $\mathbb{C}^3$. Therefore, from Proposition 7(v) we obtain that the linear operator

$$(C - (Q+P-1)^2)(C - (Q+P+1)^2)(C - (Q+P+3)^2)$$

annihilates the upper row. A direct computation using (3) shows that the action of the operators $C - (Q+P-1)^2$ and $C - (Q+P+1)^2$ on the part $X_{2\varepsilon_1}^{k}N(\mathbf{a})'$ of the module $N(\mathbf{a})$ is invertible. As the $\mathfrak{g}$-module we are working with must have a composition series with subquotients $N(\mathbf{a})$, it follows that the action of both $C - (Q+P-1)^2$ and $C - (Q+P+1)^2$ on $X_{2\varepsilon_1}^{k}R'$ is invertible. Hence $C - (Q+P+3)^2$ annihilates $X_{2\varepsilon_1}^{k}R'$, which gives us the equation

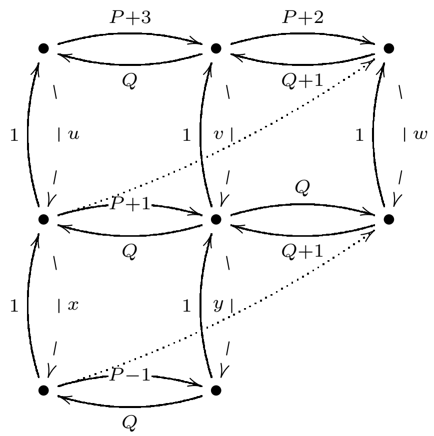
$$(Q-P-3)^2 + 4XQ = (Q+P+3)^2.$$

This equation has a unique solution, namely $X = Q + 3$, which gives the required extension.

Similarly one shows that for $k \in \{-1, -2, \dots\}$ the action of $X_{\varepsilon_2-\varepsilon_1}$ extends uniquely from $X_{2\varepsilon_1}^{k+1}R'$ to $X_{2\varepsilon_1}^{k}R'$ (here again $X_{2\varepsilon_1}^{0}R' = R'$). This completes the proof of our lemma. □

Lemma 11. *There is a unique way to define the action of $X_{-2\varepsilon_1}$ on N.*

Proof. To determine this action of $X_{-2\varepsilon_1}$ on N we consider the following extension of the picture (7) with the same notation as in the proof of Lemma (10):



Here all right arrows, representing the action of $X_{\varepsilon_2-\varepsilon_1}$, are now determined by Lemma 10 and we have to figure out the down arrows, representing the action of $X_{-2\varepsilon_1}$. The two dotted arrows will be used later on in the proof.

Consider the $\mathfrak{sl}_2$-subalgebra $\mathfrak{c}$ of $\mathfrak{g}$ generated by $e := X_{2\varepsilon_1}$ and $f := X_{-2\varepsilon_1}$. Set $h := [e, f]$. Denote by Z the action of h in the leftmost weight space of the middle row. Then $Z = x-u$. The element h commutes with both h and $H_{\varepsilon_2-\varepsilon_1}$. Therefore, by (3), the operator Z commutes with both T_1 and T_2 and hence with both P and Q.

The algebra algebra $\mathfrak{c}$ has the quadratic Casimir element $C_{\mathfrak{c}}$, whose action on the $\mathfrak{c}$-module given by the leftmost column of our picture is given by $x + f(Z)$, where f is some polynomial of degree two. From (3) it follows that the unique eigenvalue of this action is nonzero, in particular, $x + f(Z)$ is invertible. Let x' be a fixed square root $x + f(Z)$, which is a polynomial in $x + f(Z)$.

The elements $X_{\varepsilon_2-\varepsilon_1}$ and $X_{\varepsilon_2+\varepsilon_1}$ form a basis of a simple two-dimensional $\mathfrak{c}$-module with respect to the adjoint action. Using Proposition 7(iv) and arguments similar to those used in the proof of Lemma 10, we get that $C_{\mathfrak{c}} - (x' + 1)^2$ or $C_{\mathfrak{c}} - (x' - 1)^2$ annihilates the middle column (the sign depends on the original choice of x'). Note that the middle column equals $X_{\varepsilon_2-\varepsilon_1}$ applied to the leftmost column.

Similarly, the elements $X_{\varepsilon_1-\varepsilon_2}$ and $X_{-\varepsilon_2-\varepsilon_1}$ form a basis of a simple two-dimensional $\mathfrak{c}$-module with respect to the adjoint action. Applying the same arguments as in the previous paragraph we get that $C_{\mathfrak{c}} - (x')^2$ annihilates any vector of the form $X_{\varepsilon_1-\varepsilon_2}X_{\varepsilon_2-\varepsilon_1}\mathtt{v}$, where $\mathtt{v}$ is from the leftmost column. This implies that the actions of $C_{\mathfrak{c}}$ and $X_{\varepsilon_1-\varepsilon_2}X_{\varepsilon_2-\varepsilon_1}$ and thus the actions of $C_{\mathfrak{c}}$ and C on the leftmost column commute. As the action of H commutes with the action of C, we thus obtain that x commutes with the action of C. This implies that x commutes with $T_1 + T_2$. As it obviously commutes with $T_1 - T_2$, we get that x commutes with both T_1 and T_2 and hence with both P and Q.

Similarly one shows that y, u, v and w commute with both P and Q. From the commutativity of $X_{\varepsilon_2-\varepsilon_1}$ and $X_{-2\varepsilon_1}$ we get the following conditions:

$$y(P + 1) = (P - 1)x,\ v(P + 3) = (P + 1)u,\ w(P + 2)(P + 3) = P(P + 1)u.$$

Here everything commutes by the above and $P + 1$, $P + 2$ and $P + 3$ are invertible (as $X_{\varepsilon_2-\varepsilon_1}$ acts bijectively). Therefore

$$y = (P-1)(P+1)^{-1}x,\ v = (P+1)(P+3)^{-1}u,\ w = P(P+1)(P+3)^{-1}(P+2)^{-1}u.$$

This implies that y, v and w are uniquely determined by x and u.

Since the actions of both $X_{\varepsilon_2-\varepsilon_1}$ and $X_{2\varepsilon_1}$ are completely determined, we can compute the action of $X_{2\varepsilon_2}$ and see that it is given (similarly to the action of $X_{2\varepsilon_1}$) by Id_V (this is depicted by the dotted arrows in the picture). As $X_{-2\varepsilon_2}$ and $X_{2\varepsilon_2}$ commute, we obtain that $w = x$, that is

$$x = P(P+1)(P+3)^{-1}(P+2)^{-1}u. \tag{8}$$

Therefore the only parameter left for now is u.

On the one hand, the action of the element h on the middle dot of the second row is given by $y - v = (P-1)(P+1)^{-1}x - (P+1)(P+3)^{-1}u$. On the other hand, from $[h, X_{\varepsilon_2-\varepsilon_1}] = 4X_{\varepsilon_2-\varepsilon_1}$ we have that this action equals $Z+4 = x-u+4$. This gives us the equation

$$(P-1)(P+1)^{-1}x - (P+1)(P+3)^{-1}u = x-u+4. \tag{9}$$

Using (9) and (8) we get the equation

$$\frac{P(P-1)}{(P+2)(P+3)}u + \frac{P+1}{P+3}u = \frac{P(P+1)}{(P+2)(P+3)}u - u + 4.$$

This is a linear equation with nonzero coefficients and thus it has a unique solution, namely $u = (P+3)(P+2)$. Hence u is uniquely defined. The claim of the lemma follows. □

5. Consequences

Corollary 12. *Let $\mathbf{a} \in \mathbb{C}^n$ be such that $a_i \notin \mathbb{Z}$ and $a_i + a_j \notin \mathbb{Z}$ for all i and j. Let $M \in \hat{\mathcal{C}}$ and $\lambda \in \mathrm{supp}(M)$. Denote by U_0 the centralizer of $\mathfrak{h}$ in $U(\mathfrak{g})$. Then for any $A, B \in U_0$ the actions of A and B on M_λ commute.*

Proof. By Proposition 4, we may assume that $M \cong \mathrm{F}V$. For the module $\mathrm{F}V$ the claim follows from the formulae (3). □

Corollary 13. *For any simple weight cuspidal $\mathfrak{g}$-module L with finite dimensional weight spaces we have $\dim \mathrm{Ext}^1_{\mathfrak{g}}(L, L) = n$.*

Proof. This follows from Theorem 1 and the observation that a similar equality is true for the unique simple $\mathbb{C}[[t_1, t_2, \dots, t_n]]$-module. □

We also recover the main result of [BKLM]:

Corollary 14 ([BKLM])**.** *The category of all weight cuspidal $\mathfrak{g}$-modules is semi-simple.*

Proof. By [BKLM, Lemma 2], all blocks of the category of weight cuspidal $\mathfrak{g}$-modules are equivalent. Hence it is enough to prove the claim for the block containing $N(\mathbf{a})$ for some $\mathbf{a} \in \mathbb{C}^n$ such that $a_i + a_j \notin \mathbb{Z}$ for all i, j. From (3) it follows that the module $\mathrm{F}V$ is weight if and only if all operators T_i are semi-simple, hence zero. Therefore from Theorem 1 we get that the block of the category of weight cuspidal modules is equivalent to the category of finite dimensional modules over $\mathbb{C}[[t_1, t_2, \dots, t_n]]/(t_1 - 0, t_2 - 0, \dots, t_n - 0) \cong \mathbb{C}$. The claim follows. □

References

[BKLM] D. Britten, O. Khomenko, F. Lemire, V. Mazorchuk; Complete reducibility of torsion free C_n-modules of finite degree. J. Algebra **276** (2004), no. 1, 129–142.

[BL] D. Britten, F. Lemire; A classification of simple Lie modules having a 1-dimensional weight space. Trans. Amer. Math. Soc. **299** (1987), no. 2, 683–697.

[Di] J. Dixmier; Enveloping algebras. Graduate Studies in Mathematics, **11**. American Mathematical Society, Providence, RI, 1996.

[Dr] Yu. Drozd; Representations of Lie algebra $\mathfrak{sl}(2)$. Visnyk Kyiv. Univ. Ser. Mat. Mekh. No. **25** (1983), 70–77.

[Fe] S. Fernando; Lie algebra modules with finite-dimensional weight spaces. I. Trans. Amer. Math. Soc. **322** (1990), no. 2, 757–781.

[GS] D. Grantcharov, V. Serganova; Cuspidal representations of $\mathfrak{sl}(n+1)$, Adv. Math doi:10.1016/j.aim.2009.12.024

[Ko] B. Kostant; On the tensor product of a finite and an infinite dimensional representation. J. Functional Analysis **20** (1975), no. 4, 257–285.

[Mat] O. Mathieu; Classification of irreducible weight modules. Ann. Inst. Fourier (Grenoble) **50** (2000), no. 2, 537–592.

[Maz] V. Mazorchuk; Lectures on $\mathfrak{sl}_2(\mathbb{C})$-modules. Imperial College Press, 2009.

[MS] V. Mazorchuk, C. Stroppel; Cuspidal $\mathfrak{sl}_n$-modules and deformations of certain Brauer tree algebras, Preprint arXiv:1001.2633.

V.M.: Department of Mathematics, Uppsala University, SE 471 06, Uppsala, SWEDEN, e-mail: mazor@math.uu.se

C.S.: Mathematik Zentrum, Universität Bonn, Endenicher Allee 60, D-53115, Bonn, GERMANY, e-mail: stroppel@uni-bonn.de