NAZAROV-WENZL ALGEBRAS, COIDEAL SUBALGEBRAS AND CATEGORIFIED SKEW HOWE DUALITY

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Abstract. We describe how certain cyclotomic Nazarov-Wenzl algebras occur as endomorphism rings of projective modules in a parabolic version of BGG category $\mathcal{O}$ of type $D$. Furthermore we study a family of subalgebras of these endomorphism rings which exhibit similar behaviour to the family of Brauer algebras even when they are not semisimple. The translation functors on this parabolic category $\mathcal{O}$ are studied and proven to yield a categorification of a coideal subalgebra of the general linear Lie algebra. Finally this is put into the context of categorifying skew Howe duality for these subalgebras.

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Introduction

In [BK09] a remarkable connection between the representation theory of Hecke algebras and Lusztig’s canonical bases was established by showing that cyclotomic Hecke algebras are isomorphic to cyclotomic quotients of quiver Hecke algebras introduced in [KL09], [KL08], where also a connection between the representation theory of these algebras and Lusztig’s geometric construction of canonical bases was predicted. This prediction

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was verified, [VV11], and so therefore a connection between representations of cyclotomic Hecke algebras and canonical bases was established. In this way, cyclotomic Hecke algebras inherit an interesting grading which in type $A$ can also be obtained from the graded versions of parabolic category $O$’s and hence be described in terms of type $A$ Kazhdan-Lusztig polynomials, see [BS11b], [HM11]. In other types however, Schur-Weyl duality connects the Lie algebra with a centralizer algebra (Brauer algebra) different from the group algebra of any Weyl group. In this paper we investigate relations between parabolic category $\hat{O}(\mathfrak{so}_{2n})$, Brauer algebras and their degenerate affine versions $\mathcal{W}_d = \mathcal{W}_d(\Xi)$, depending on a parameter set $\Xi$, and their cyclotomic quotients, [AMR06]. The algebras $\mathcal{W}_d(\Xi)$ were introduced in [Naz96] we call them VW-algebras.\footnote{Keeping in mind that VW can be seen as a degeneration of BMW, the affine Birman-Murakami-Wenzl algebras, [DRV11].} These families are the Brauer algebra analogues of the cyclotomic Hecke algebras, but in contrast to them not well understood and so far slightly neglected; maybe also because of the lack of a good combinatorial description and geometric realization. The main goal of the paper is to connect these algebras to category $O$ and its Kazhdan-Lusztig combinatorics and in this way obtain canonical bases of representations for coideal subalgebras in quantum groups.

We start with a type $D$ analogue of the Arakawa-Suzuki theorem:

**Theorem A.** Let $M$ be a highest weight module in $O(\mathfrak{so}_{2n})$. With an appropriate choice of $\Xi$ there is an algebra homomorphism $$\Psi_M : \mathcal{W}_d \rightarrow \text{End}_q(M \otimes (\mathbb{C}^{2n})^{\otimes d})^{\text{opp}}.$$  

In general this morphism is not surjective and it is hard to fully describe the kernel. We study in detail the case where $M$ is a parabolic Verma module $M^P(\lambda)$ for a maximal parabolic subalgebra of type $A$ inside type $D$. We show that cyclotomic quotients $\mathcal{W}_d(\alpha, \beta)$ of level 2 occur as endomorphism rings if we choose $\lambda = \delta \omega_0$ as an appropriate multiple of a fundamental weight:

**Theorem B** (see Theorem 3.1). If $n \geq 2d$ and $\delta \in \mathbb{Z}$ then $$\mathcal{W}_d(\alpha, \beta) \cong \text{End}_q(M^P(\delta) \otimes (\mathbb{C}^{2n})^{\otimes d})^{\text{opp}}.$$  

We deduce that the $\mathcal{W}_d(\alpha, \beta)$’s inherit a positive Koszul grading from the graded version $\hat{O}$, of category $O$, hence a geometric interpretation; in terms of first perverse sheaves on isotropic Grassmannians, [ES13b], and second topological Springer fibres, [ES12], via the Khovanov algebra of type $D$. This Khovanov algebra also allows us, [ES13a], to mimic (with some effort) the approach from [BS12] to construct a graded version of the Brauer algebra $\text{Br}(\delta)$ for an arbitrary integral parameter $\delta$. Since this requires passing to orthosymplectic Lie superalgebras, we just identify here a subalgebra $z_d \mathcal{W}_d(\alpha, \beta) z_d$ of the cyclotomic quotient, given by an idempotent $z_d$ and show that this algebra shares properties of the Brauer algebra:
Theorem C (see Proposition 4.4 and Theorem 4.5).

1. The algebra $z_d W_d(\alpha, \beta) z_d$ has dimension $(2d - 1)!$.
2. The algebra $z_d W_d(\alpha, \beta) z_d$ is semisimple if and only if $\delta \neq 0$ and $\delta \geq d - 1$ or $\delta = 0$ and $d = 1, 3, 5$.

These connections give a conceptual explanation for the fact, [CDVM09b] and [CDVM09a], that the decomposition numbers of Brauer algebras are given by the combinatorics of Weyl groups of type $D$.

Theorems A and B and [ES13a] rely on a good understanding of (a graded version of) tensoring with the natural representation $\mathbb{C}^{2n}$ on category $O^p$. Since $\mathbb{C}^{2n}$ is self-dual and hence $\_ \otimes \mathbb{C}^{2n}$ self-adjoint, we do not get an action of a quantum group action on our categories, but rather of certain coideal subalgebras $H$ and $H'$ of $\mathcal{U}_q(\mathfrak{g}l_Z)$ so-called quantum symmetric pairs in [Let03], [Let02], [Kol12]. These are quantum group analogues of the fixed point subalgebra $\mathfrak{g}l_K \times \mathfrak{g}l_{-K}$ inside $\mathfrak{g}l_Z$. Since the integral weights of $\mathfrak{so}_{2n}$ can be partitioned into integer or half-integer weights in the standard $\epsilon$-basis, the integral part, $O^p(\mathfrak{so}_{2n})$, of parabolic category $O$ can be decomposed into two subcategories $O^p_1(\mathfrak{so}_{2n})$ and $O^p(\mathfrak{so}_{2n})$ stable under tensoring with the natural representation. We obtain two categorifications.

Theorem D (see Proposition 6.25).

1. The $H$-module $\Lambda^n \mathbb{C}^Z$ is categorified by $\hat{O}^p_1(\mathfrak{so}_{2n})$.
2. The $H'$-module $\Lambda^n \mathbb{C}^{Z+\frac{1}{2}}$ is categorified by $\hat{O}^p(\mathfrak{so}_{2n})$.

The classes of Verma modules correspond hereby to the standard basis. The involved categories have a contravariant duality, hence the categorification equips the modules on the left with a bar-involution, see Proposition 6.30. Mimicking Lusztig’s approach for quantum groups, we define canonical bases on the above modules and show that the classes of simple modules correspond to the canonical basis. Theorem D gives a new instance of based categorifications in the context of category $O$, but now connecting canonical bases of Hecke algebras with canonical bases of quantum symmetric pairs instead of quantum groups as for instance in [BS10], [FKS06], [Sar13], [Web10], see [Maz12] for an overview. The base change matrix is here given by parabolic type $D$ Kazhdan-Lusztig polynomials; see [LS13] for explicit formulas in the Grassmannian case.

Finally we investigate generalizations of these modules from the viewpoint of skew Howe duality, [How92], and its categorification. For this we consider more general parabolic category $O$’s and their block decompositions for Levi subalgebras isomorphic to products of $\mathfrak{g}l_k$’s. Denote the sum of these blocks by $\bigoplus \Gamma \bigotimes \Gamma(n)$. Considering analogous projective functors in this setup we categorify the two actions of $\mathfrak{g}l_m \times \mathfrak{g}l_m$ and $\mathfrak{g}l_r \times \mathfrak{g}l_r$ on the vector space $\Lambda(n, m, r) := \Lambda^n(\mathbb{C}^m \otimes \mathbb{C}^2 \otimes \mathbb{C}^r)$ separately and then show they are graded derived equivalent via Koszul duality. Under this identification we can realize both commuting actions on the same category. The projective functors turn into derived Zuckerman functors under this identification.
**Theorem E** (see Proposition 7.6 and Theorem 7.14). The bimodule $\bigwedge(n,m,r)$ is categorified by $\bigoplus \mathcal{O}_\Gamma(n)$. Furthermore, the two categorifications for the left and right action are Koszul dual to each other.

This skew Howe duality can be squeezed (and quantized) in between two type $A$ skew Howe dualities as follows. For an integer $m$ let $V_{2m}$ be the natural $\mathfrak{gl}_m$-module. Fix integers $n, m, r$ and consider the $\mathfrak{gl}_m \times \mathfrak{gl}_r$-module $\bigwedge^n(V_m \otimes V_r)$. By skew Howe duality, [How92], the weight spaces of the $\mathfrak{gl}_m$-action are representations of $\mathfrak{gl}_r$ and vice versa; and the decomposition is the following

$$\bigoplus_{\alpha} \bigwedge V_m \otimes \cdots \bigwedge V_m \cong \bigwedge^n(V_m \otimes V_r) \cong \bigoplus_{\beta} \bigwedge V_r \otimes \cdots \bigwedge V_r$$

where the first isomorphism is as $\mathfrak{gl}_m$-module and the second as $\mathfrak{gl}_r$-module. The sums run over all compositions (possibly with zero parts) of $n$ with $r$ respectively $m$ parts. Now consider the restriction $V_{2m} = V_m \otimes V_m$ to $\mathfrak{gl}_m$, where $\mathfrak{gl}_m$ is embedded diagonally into $\mathfrak{gl}_m \times \mathfrak{gl}_m \subset \mathfrak{gl}_{2m}$. Theorem E gives a categorification of the middle piece of diagram

$$\mathfrak{gl}_{2m} \sim \bigoplus_{\alpha} \bigwedge V_{2m} \otimes \bigwedge V_{2m} \otimes \cdots \otimes \bigwedge V_{2m} \subset \mathfrak{gl}_r$$

$$\mathfrak{gl}_m \times \mathfrak{gl}_m \sim \bigwedge(n,m,r) \subset \mathfrak{gl}_r \times \mathfrak{gl}_r$$

$$\mathfrak{gl}_m \sim \bigoplus_{\gamma} \bigwedge V_m \otimes \bigwedge V_m \otimes \cdots \otimes \bigwedge V_m \subset \mathfrak{gl}_{2r}$$

In fact, the top and bottom parts can be quantized, [CKM12], and were already categorified using parabolic-singular category $\mathcal{O}$'s in [MS09] and fit into the framework of [LQR12]. In our categorification of the middle part the commuting actions of the two Lie algebras, viewed as specializations of the coideal subalgebras mentioned above, are given by translation and derived Zuckerman functors respectively. Since this construction has a graded lift using $\hat{\mathcal{O}}$ there is a quantized version of skew Howe duality with the commuting actions of the coideal subalgebras.

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1. Basics

Throughout this paper let $\mathfrak{g}$ denote the Lie algebra $\mathfrak{so}_{2n}$ defined over the complex numbers. We denote by $U(\mathfrak{g})$ its enveloping algebra and by $Z(U(\mathfrak{g}))$ the center of $U(\mathfrak{g})$. In Section 2 we will fix a specific presentation for $\mathfrak{so}_{2n}$.

We denote by $e_i$ the standard basis of the dual of the Cartan $\mathfrak{h}^*$ with simple roots $\alpha_i = \epsilon_{i+1} - \epsilon_i$ for $1 \leq i \leq n - 1$ and $\alpha_0 = \epsilon_1 + \epsilon_2$. By abuse of notation we denote by $\mathcal{O}$ the sum of all integral blocks of the BGG-category $\mathcal{O}$ of $\mathfrak{g}$. We fix the maximum standard parabolic $\mathfrak{p}$ of type $A$ corresponding to the roots $\alpha_1, \ldots, \alpha_{n-1}$ and denote the associated parabolic subcategory of $\mathcal{O}$ by $\mathcal{O}^p(n)$, i.e. the subcategory of $\mathcal{O}$ of all $\mathfrak{p}$-finite modules. The combinatorics of this parabolic subcategory was studied in [ES13b] and we recall some of the notations used therein.

A weight $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathfrak{h}^*$ of $\mathfrak{so}_{2n}$ is integral if it is in the $\mathbb{Z}$-span or the $(\mathbb{Z} + \frac{1}{2})$-span of the $\epsilon_i$’s. In the former case we say the weight is supported on the integers, in the latter that it is supported on the half-integers.

For $\lambda \in \mathfrak{h}^*$ let $M^p(\lambda)$ denote the parabolic Verma module of highest weight $\lambda$, that is the maximal quotient of the ordinary Verma module $M(\lambda)$ which is locally $\mathfrak{p}$-finite. Explicitly, $M^p(\lambda) = 0$ if $\lambda$ is not $\mathfrak{p}$-dominant and otherwise

$$M^p(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} E(\lambda) \quad (1)$$

where $E(\lambda)$ denotes the finite dimensional $\mathfrak{l}$-module of highest weight $\lambda$ for the Levi subalgebra $\mathfrak{l}$ of $\mathfrak{p}$ inflated trivially to a $\mathfrak{p}$-module. The set of $\mathfrak{p}$-dominant weights will be denoted by

$$\Lambda = \left\{ \lambda \in \mathfrak{h}^* \text{ integral} \mid M^p(\lambda) \neq 0 \right\}$$

$$= \left\{ \lambda \in \mathfrak{h}^* \text{ integral} \mid \lambda = \sum_{i=1}^n \lambda_i \epsilon_i \text{ where } \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \right\}$$

$$= \left\{ \lambda \in \mathfrak{h}^* \text{ integral} \mid \lambda + \rho = \sum_{i=1}^n \lambda'_i \epsilon_i \text{ where } \lambda'_1 < \lambda'_2 < \cdots < \lambda'_n \right\} \quad (2)$$

where $\rho$ denotes the half-sum of positive roots, $\rho = \sum_{i=1}^n (i - 1) \epsilon_i$ or $\rho = (0, 1, 2, \ldots, n - 1)$ in the $\epsilon$-basis. It decomposes into a disjoint union $\Lambda' \cup \Lambda''$, the weights supported on the integers and those supported on half-integers. Since up to the last section we only work with parabolic Verma modules, we just call them by abuse of language for short Verma modules.

The action of the center $U(\mathfrak{g})$ decomposes each module into generalized eigenspaces, which induces a decomposition $\mathcal{O} = \bigoplus_{\lambda} \mathcal{O}_\lambda$ of the category $\mathcal{O}$ for $\mathfrak{g}$ indexed (via the Harish-Chandra isomorphism) by orbits of integral weights under the dot action $w \cdot \lambda = w(\lambda + \rho) - \rho$ of the Weyl group $W_n$ of $\mathfrak{g}$. The decomposition of $\mathcal{O}$ induces a decomposition of the subcategory $\mathcal{O}^p(n)$.

We denote by $\mathcal{O}^p(n)$ the summand containing $M^p(\lambda)$, i.e. the summand corresponding to the orbit through $\lambda + \rho$. Note that the ordinary action as well as the dot-action of the Weyl group on integral weights preserve the
lattices of weights supported on integers resp. on half-integers. Hence, we have a decomposition

$$\mathcal{O}_P(n) = \mathcal{O}_\lambda^p(n) \oplus \mathcal{O}_\lambda^p(n),$$

(3)

where $\mathcal{O}_\lambda^p(n)$ (resp. $\mathcal{O}_\lambda^p(n)$) is the direct sum of all $\mathcal{O}_\lambda^p(n)$, such that $\lambda$ is supported on the integers (resp. half-integers).

Our goal is to equip the Grothendieck groups of $\mathcal{O}_\lambda^p(n)$ and $\mathcal{O}_\lambda^p(n)$ with the structure of a representation of a certain coideal subalgebra induced by the action of translation functors, see Section 6. To make this explicit it is helpful to use the combinatorics from [ES13b, Section 2.2] to identify $p$-dominant weights with diagrammatic weights, allowing us to describe the blocks of category $\mathcal{O}_P(n)$ combinatorially. Since we want to distinguish the combinatorics for the two subcategories in (3) we slightly modify the indexing sets for diagrammatic weights from [ES13b].

Definition 1.1. We denote by $\mathbb{X}_n$ the set of sequences $a = (a_i)_{i \in \mathbb{Z}_{\geq 0}}$ such that $a_i \in \{\lambda, \nu, x, \circ, \diamond\}$, $a_i \neq \circ$ for $i \neq 0$, $a_0 \in \{\circ, \diamond\}$, and

$$\#\{a_i \mid a_i \in \{\lambda, \nu, \circ\}\} + 2\#\{a_i \mid a_i = x\} = n.$$  

Elements of $\mathbb{X}_n$ are called diagrammatic weights supported on the integers.

Remark 1.2. The translation from a diagrammatic weight $a \in \mathbb{X}$ to a diagrammatic weight $\tilde{a}$ in the sense of [ES13b] is done by setting $\tilde{a}_i = a_{i-1}$, except for $i = 1$ and $a_0 = \circ$ where we choose $\tilde{a}_1 \in \{\lambda, \nu\}$ such that the total number of $\nu$’s is even. In the language of [ES13b] we have thus always fixed the even parity for these blocks where $a_0 = \circ$.

Similarly we have a set-up for half-integers.

Definition 1.3. We denote by $\mathbb{X}_n'$ the set of sequences $a = (a_i)_{i \in \mathbb{Z}_{\geq 0} + \frac{1}{2}}$ such that $a_i \in \{\lambda, \nu, x, \circ\}$ and $\#\{a_i \mid a_i \in \{\lambda, \nu\}\} + 2\#\{a_i \mid a_i = x\} = n$. We call these elements diagrammatic weights supported on the half-integers.

Remark 1.4. Translating a diagrammatic weight $a \in \mathbb{X}'$ to a diagrammatic weight $\tilde{a}$ in the sense of [ES13b] is done by putting $\tilde{a}_i = a_{i-\frac{1}{2}}$.

Obviously a diagrammatic weight $a$ is uniquely determined by the sets

$$P_\ast(a) = \{i \mid a_i = \ast\},$$

for $\ast \in \{\lambda, \nu, x, \circ, \diamond\}$. Given a $p$-dominant weight $\lambda = (\lambda_1, \ldots, \lambda_n)$, denote $\lambda' = (\lambda'_1, \ldots, \lambda'_n) = \lambda + \rho$ and let $a_\lambda$ be the diagrammatic weight defined by

$$P_\ast(\lambda) = \{\lambda'_i \mid \lambda'_i > 0 \text{ and } -\lambda'_i \text{ does not appear in } \lambda'\},$$

$$P_\lambda(\lambda) = \{-\lambda'_i \mid \lambda'_i < 0 \text{ and } -\lambda'_i \text{ does not appear in } \lambda'\},$$

$$P_\circ(\lambda) = \{\lambda'_i \mid \lambda'_i > 0 \text{ and } -\lambda'_i \text{ appears in } \lambda'\},$$

$$P_\circ(\lambda) = \{\lambda'_i \mid \lambda'_i = 0\},$$

$$P_\circ(\lambda) = \begin{cases} \mathbb{Z}_{\geq 0} \setminus (P_\ast \cup P_\lambda \cup P_\ast \cup P_\circ) & \text{if } \lambda \text{ is supported on integers,} \\ \mathbb{Z}_{\geq 0} + \frac{1}{2} \setminus (P_\ast \cup P_\lambda \cup P_\ast) & \text{if } \lambda \text{ is supported on half-integers.} \end{cases}$$
The assignment $\lambda \mapsto a_\lambda$ defines a bijection between $p$-dominant weights and $X_n \cup X_n'$. In the following we will not distinguish between a weight and a diagrammatic weight and denote both by $\lambda$. We use the following identifications

$$K_0 \left( \mathcal{O}_\bullet^p(n) \right) \cong \langle X_n \rangle_Q \quad \text{and} \quad K_0 \left( \mathcal{O}_\bullet^p(n) \right) \cong \langle X'_n \rangle_Q,$$

between the $\mathbb{Q}$-vector spaces on basis $X_n$ resp. $X'_n$ and the Grothendieck group scalar extended to $\mathbb{Q}$ by sending the class of a parabolic Verma module of highest weight $\lambda$ to $a_\lambda$.

Our weight dictionary induces an action of the Weyl group $W_n$ on diagrammatic weights corresponding to the dot-action on weights for $g$. Two diagrammatic weights are in the same orbit (and thus the corresponding Verma modules have the same central character) if and only if one is obtained from the other by a finite sequence of changes of the following form: swapping a $\lor$ with an $\land$ or replacing two $\lor$'s with two $\land$'s or two $\land$'s with two $\lor$'s (keeping all $\ast$'s and $\circ$'s untouched), see [ES13b]. Orbits of diagrammatic weights, called diagrammatic blocks are given by fixing the positions of the $\ast$'s and $\circ$'s (called block diagram in [ES13b, Section 2.2]) and the parity of $\# \lor + \# \land$ of any of its weights. Moving $\lor$'s to the left or turning two $\land$'s into two $\lor$'s makes the weight bigger (with respect to the standard ordering on weights). Note that diagrammatic blocks correspond precisely to blocks of $\mathcal{O}_\bullet^p(n)$ by sending $\Gamma$ to the summand $\mathcal{O}_\bullet^p(n)_\Gamma$ containing all Verma modules with highest weight in $\Gamma$ which is indeed a block of the category.

2. Brauer and VW-algebra

2.1. VW-algebras and translated highest weight modules. The main purpose of this section is a generalization of the Arakawa-Suzuki action, [AS98], to the Lie algebra of type $D_n$. The replacement of the degenerate affine Hecke algebra is the affine Nazarov algebra $\mathcal{W}_d(\Xi)$.

**Definition 2.1.** Let $d \in \mathbb{N}$. We fix a set $\Xi$ of parameters $w_k \in \mathbb{C}$, $k \geq 0$. Then the affine Nazarov algebra $\mathcal{W}_d = \mathcal{W}_d(\Xi)$, short VW-algebra, is generated by

$$s_i, e_i, y_j \quad 1 \leq i \leq d - 1, 1 \leq i \leq d, k \in \mathbb{N},$$

subject to the relations (for $1 \leq a, b \leq d - 1, 1 \leq c < d - 1$, and $1 \leq i, j \leq d$):

(VW.1) $s_a^2 = 1$

(VW.2) (a) $s_a s_b = s_b s_a$ for $|a - b| > 1$
(b) $s_c s_{c+1} s_c = s_{c+1} s_c s_{c+1}$
(c) $s_a y_i = y_i s_a$ for $i \notin \{a, a + 1\}$

(VW.3) $e_a^2 = w_0 e_a$

(VW.4) $e_1 y_1^k e_1 = w_k e_1$ for $k \in \mathbb{N}$

(VW.5) (a) $s_a e_b = e_b s_a$ and $e_a e_b = e_b e_a$ for $|a - b| > 1$
(b) $e_a y_i = y_i e_a$ for $i \notin \{a, a + 1\}$
(c) $y_i y_j = y_j y_i$

(VW.6) (a) $e_a s_a = e_a = s_a e_a$
(b) \( s_c e_{c+1} e_c = s_{c+1} e_c \) and \( e_c e_{c+1} s_c = e_c s_{c+1} \)
(c) \( e_{c+1} e_c s_{c+1} = e_{c+1} s_c \) and \( s_{c+1} e_{c+1} = s_c e_{c+1} \)
(d) \( e_{c+1} e_c e_{c+1} = e_{c+1} \) and \( e_c e_{c+1} e_c = e_c \)

(VW.7) \( s_a y_a - y_a s_a = e_a - 1 \) and \( y_a s_a - s_a y_a = e_a - 1 \)

(VW.8) \( (a) e_a(y_a + y_a + 1) = 0 \)
(b) \( (y_a + y_a + 1)e_a = 0 \)

**Remark 2.2.** Relations (6b), (6c), (6d), (7) come in pairs and it is in fact sufficient to either require the first set of relations or the second, the other is then satisfied automatically. All relations are symmetric or come in symmetric pairs, thus we have a canonical isomorphism \( \mathcal{W}(\Xi) \cong \mathcal{W}(\Xi)^{\text{opp}} \).

From now on we fix a natural number \( n \geq 4 \) and set \( N = 2n \). Let \( I^+ \coloneqq \{1, \ldots, n\} \) and \( I \coloneqq I^+ \cup -I^+ \). We denote by \( V \) the vector space with basis \( \{v_i \mid i \in I\} \) and by \( \mathfrak{gl}(I) \) its corresponding Lie algebra of endomorphisms, viewed as the matrices with respect to the chosen basis. Let \( J \) be the matrix such that \( J_{kl} = \delta_{k,-l} \) for \( k, l \in I \) with respect to the chosen basis. If we order columns and rows decreasing from top to bottom and left to right this is the matrix with ones on the anti-diagonal and zeros elsewhere.

**Definition 2.3.** The Lie algebra \( \mathfrak{g} = \mathfrak{so}_{2n} \) is the Lie subalgebra of \( \mathfrak{gl}(I) \) of all matrices \( A \) satisfying \( JA + A^T J = 0 \); that is all matrices which are skew-symmetric with respect to the anti-diagonal, \( A_{ij} = -A_{-i,j} \). In terms of the bilinear form \( \langle -,- \rangle \) on \( V \) we thus have \( \langle XV, w \rangle + \langle v, Xw \rangle = 0 \).

Fix the Cartan subalgebra \( \mathfrak{h} \subset \mathfrak{g} \) given by all diagonal matrices and a basis \( \{\epsilon_i \mid i \in I^+\} \) for \( \mathfrak{h}^* \) such that the weight of \( v_i \) is \( \epsilon_i \) if \( i \in I^+ \) and the weight of \( v_i \) is \( -\epsilon_i \) if \( i \in I^- \). For any \( \alpha \in R_n = R(\mathfrak{so}_{2n}) = \{\pm \epsilon_i \pm \epsilon_j \mid i \neq j\} \) fix a root vector \( X_\alpha \) of weight \( \alpha \) and for \( i \in I^+ \) let \( X_i \) be the element in \( \mathfrak{h} \) dual to \( \epsilon_i \). Then \( \langle X_{\gamma} \mid \gamma \in B_n \rangle \) with \( B_n = R_n \cup I^+ \) form a basis of \( \mathfrak{so}_{2n} \). We set \( n^+ = \{X_{\epsilon_i \pm \epsilon_j} \mid i > j\} \) and \( n^- = \{X_{-(\epsilon_i \pm \epsilon_j)} \mid i > j\} \) and fix the Borel subalgebra \( \mathfrak{b} = n^+ \oplus \mathfrak{h} \). In this notation the natural representation \( V \) is the irreducible representation \( L(\epsilon_n) \) with highest weight \( \epsilon_n \), the fundamental weight corresponding to \( \alpha_{n-1} = \epsilon_n - \epsilon_{n-1} \). Furthermore, the \( X_{\pm(\epsilon_i \pm \epsilon_j)} \)'s for \( i > j \) together with \( \mathfrak{h} \) form a Levi subalgebra \( \mathfrak{l} \) isomorphic to \( \mathfrak{gl}_n \) with corresponding standard parabolic subalgebra \( \mathfrak{p} = I^+ n^+ \) from Section 1.

For \( i \in I \) denote by \( v_i^* = v_{-i} \) the basis element dual to \( v_i \) with respect to \( \langle -,- \rangle \) and for \( X_\gamma \) denote by \( X_\gamma^* \) the element dual to \( X_\gamma \) with respect to the Killing form of \( \mathfrak{so}_{2n} \).

**Definition 2.4.** Let \( M \) be a \( \mathfrak{g} \)-module. For \( d \geq 0 \) consider \( M \otimes V^{\otimes d} \). The linear endomorphisms \( \tau, \sigma : V \otimes V \to V \otimes V \) defined as
\[
\tau : \ v \otimes w \mapsto \langle v, w \rangle \sum_{i \in I} v_i \otimes v_i^* ,
\]
\[
\sigma : \ v \otimes w \mapsto w \otimes v .
\]
induce the following endomorphisms \( s_i, e_i \) of \( M \otimes V^{\otimes d} \) for \( 1 \leq i \leq d - 1 \)
\[
s_i = \text{Id} \otimes \text{Id}^{\otimes (i-1)} \otimes \sigma \otimes \text{Id}^{\otimes (d-i-3)} ,
\]
where $X$ by definition of $\Omega$ it is clear that $y$ acts on a module of highest weight

where $\Delta$ denotes the comultiplication of $g$.

Definition 2.5. The pseudo Casimir element in $U(g) \otimes U(g)$ is defined as

$$\Omega = \sum_{\gamma \in B_n} X_\gamma \otimes X_\gamma^*.$$  

It is connected to the ordinary Casimir element $C = \sum_{\gamma \in B_n} X_\gamma X_\gamma^*$ by the formula

$$\Omega = \frac{1}{2} (\Delta(C) - C \otimes 1 - 1 \otimes C),$$

where $\Delta$ denotes the comultiplication of $g$. Denote by $c_\lambda$ the value by which $C$ acts on a module of highest weight $\lambda$. For $0 \leq i < j \leq d$ we define

$$\Omega_{ij} = \sum_{\gamma \in B_n} 1 \otimes \cdots \otimes X_\gamma \otimes 1 \otimes \cdots \otimes 1 \otimes X_\gamma^* \otimes 1 \otimes \cdots \otimes 1,$$

where $X_\gamma$ is at position $i$ and $X_\gamma^*$ is at position $j$. Multiplication with $\Omega_{ij}$ defines an element $\Omega_{i,j} \in \text{End}_g(M \otimes V^{\otimes d})$ and we finally set for $i \in I$

$$y_i = \sum_{0 \leq k < i} \Omega_{ki} + \left(\frac{2n-1}{2} - 1\right).$$

By definition of $\Omega$ it is clear that $y_i$ is a $g$-endomorphism on $M \otimes V^{\otimes d}$.

Remark 2.6. The representation $V \otimes V$ decomposes as a $g$-module into the irreducible representations $L(0)$, $L(2\epsilon_n)$, and $L(\epsilon_n + \epsilon_{n-1})$. A small computation shows that on $V \otimes V$ the following equation holds

$$\Omega = -\text{pr}_{L(0)}(\epsilon_n \text{id} + \sigma) + \sigma.$$

Note that $\tau$ is a quasi-projection from $V \otimes V$ onto the copy of the trivial representation $L(0)$ inside $V \otimes V$. A small computation using the explicit form of $\Omega$ given above shows that multiplication with $\Omega$ on $V \otimes V$ is equal to the morphism $\sigma - \tau$.

Recall that a highest weight module for $g$ is a $g$-module $M$ which is generated by a non-zero vector $m \in M$ satisfying $n^+ m = 0$ and $\frak{h} m \subseteq \mathbb{C} m$. Note that it satisfies $\text{End}_g(M) = \mathbb{C}$. Associated with $\lambda \in \frak{h}^*$ and the corresponding 1-dimensional module $\mathbb{C}_\lambda$ we have the (ordinary) Verma module $M(\lambda) = U(g) \otimes U(\frak{b}) \mathbb{C}_\lambda$ of highest weight $\lambda$ and its irreducible quotient $L(\lambda)$.

Lemma 2.7. If $M$ is a highest weight module then for each $k \in \mathbb{N}$, there exist $a_k(M) \in \mathbb{C}$ such that $e_1 y_1^k e_1 = a_k(M) e_1$ as elements in $\text{End}_g(M \otimes V^{\otimes d})$ with $a_0(M) = N$.

Proof. The first equality follows directly from the definitions. Recalling Remark 2.6, we first let $d = 2$ and consider the composition

$$f : M = M \otimes L(0) \longrightarrow M \otimes V \otimes V \xrightarrow{y_1^k} M \otimes V \otimes V \xrightarrow{e_1} M \otimes L(0) = M,$$
where the first map is the canonical inclusion. Now \( f \) is an endomorphism of \( M \), hence must be a multiple, say \( a_k(M) \), of the identity. By pre-composing with \( e_1 \) we obtain \( e_1 y^k e_1 = a_k(M)e_1 \). This identity also holds for \( d > 2 \) since we just add a couple of identities on the following tensor factors.

\[ \square \]

**Theorem 2.8.** Let \( M \) be a highest weight module for \( g = \mathfrak{so}_{2n} \) and \( \Xi_M = \{a_k(M) \mid k \geq 0\} \) as in Lemma 2.7. Then there is a well-defined right action of \( \mathcal{W}_d(2n) \) on \( M \otimes V^{\otimes d} \) defined by

\[ p.s_i = s_i(p), \quad p.e_i = e_i(p), \quad p.y_j = y_j(p), \quad p.w_k = a_k(M)p \]

for \( p \in M \otimes V^{\otimes d} \), \( 1 \leq i \leq r-1, 1 \leq j \leq r \) and \( k \in \mathbb{Z} \). In particular, we get an algebra homomorphism

\[ \Psi_M = \Psi_M^{d,n} : \mathcal{W}_d(\Xi_M) \rightarrow \text{End}_g(M \otimes V^{\otimes d})^{\text{opp}}. \]

**Proof.** To prove the statement we need to show that the assignment respects the relations of the \( \mathcal{W} \)-algebra. This will be done in separate Lemmas in the Appendix. Relation (1) is obvious, as are relations (2a) and (2b), while (2c) follows from Lemma 8.1. Relation (3) is obvious as well and relation (4) follows from Lemma 8.2. The relations (6a)-(6d) follow from Lemma 8.2, Relation (5a) is trivial as well, while (5b) follows from Lemma 8.4 and (5c) from Lemma 8.5. Finally relation (7) follows from Lemma 8.3 and relations (8a)-(8b) from Lemma 8.6.

As one can see from the proof of Theorem 2.8 the parameters of the \( \mathcal{W} \)-algebra depend on \( n \) and the highest weight of \( M \). This is different from the type \( A \) situation. There, the degenerate affine Hecke algebra \( H_d \) acts on the endofunctor \( \mathcal{E}^r := \otimes V^{\otimes r} \) of \( \mathcal{O}(\mathfrak{gl}_n) \) for any \( n \), with \( V \) being the natural representation of \( \mathfrak{gl}_n \). This property plays an important role in the context of categorification of modules over quantum groups, [KL09], [Rou08], [BK08], [BS11b]. To achieve a similar situation one could enlarge the algebra to the algebra generated by \( e_i, s_i 1 \leq i \leq d-1 \) plus an extra central generator \( w_0 \) subject to the same relations as before except that we have to drop relation (4). Similar actions are described in [DRV13, 2.2] and for Birman-Murakawi-Wenzl algebras in [OR07].

**Remark 2.9.** The action defined here can also be modified to give an action of \( \mathcal{W}_d(\Xi_M) \) for a highest weight module \( M \) for a Lie algebra of types \( B \) or \( C \) and the respective defining representation as \( V \).

2.2. **Brauer algebras.** Let \( r \in \mathbb{Z}_{\geq 0} \). A Brauer diagram on \( 2r \) vertices is a partitioning \( b \) of the set \( \{1, 2, \ldots, r, 1^*, 2^*, \ldots, r^*\} \) into \( r \) subsets of cardinality 2. Let \( B[r] \) be the set of such Brauer diagrams. A Brauer diagram can be displayed graphically by arranging \( 2r \) vertices in two rows 1, 2, \ldots, \( r \) and 1, 2, \ldots, \( r^* \), such that each vertex is linked to precisely one other vertex. Two such diagrams are considered to be the same if they link the same \( r \) pairs of points. Special Brauer diagrams are the "unit" \( 1 = \{(1, 1^*), (2, 2^*), \ldots, (r, r^*)\} \) connecting always \( j \) with \( j^* \) for all \( 1 \leq j \leq r \),
and for $1 \leq i \leq r - 1$ the $\tilde{s}_i$ (respectively $\tilde{c}_i$) which connects $j$ with $j^*$ except of the new pairs $\{i,(i+1)^*\}, \{i+1,i^*\}$ (respectively $\{i,i+1\}, \{i^*,(i+1)^*\}$) involving the numbers $i$ and $i+1$.

**Example 2.10.**

$$
\begin{array}{cccc}
\cdots & \times & \cdots & \\
i & i+1 & & \\
\end{array}
\quad
\begin{array}{cccc}
\cdots & \cup & \cdots & \\
i & i+1 & & \\
\end{array}
$$

(12)

Given two Brauer diagrams $b$ and $b'$, their concatenation $b \circ b'$ is obtained by identifying vertex $i^*$ in $b$ with vertex $i$ in $b'$ and removing all the internal loops. Let $c(b,b')$ be the number of loops removed. Brauer introduced the following algebra $[\text{Bra37}].$

**Definition 2.11.** Let $d \in \mathbb{Z}_{\geq 0}$ and $\delta \in \mathbb{C}$. The *Brauer algebra* $\text{Br}_d(\delta)$ is the $\mathbb{C}$-algebra with basis $b$, $b \in \mathcal{B}[d]$ and multiplication $bb' = \delta^{c(b,b')}b \circ b'$.

The algebra is associative with unit 1 and generated by the elements $\tilde{s}_i$, $\tilde{c}_i$, $1 \leq i \leq d - 1$ modulo the relations from Lemma 8.2 together with $\tilde{c}_i^2 = \delta$ for $1 \leq i \leq n - 1$, hence the defining relations are

$$
\begin{align*}
\tilde{s}_a^2 &= 1, \quad \tilde{s}_a \tilde{s}_b = \tilde{s}_b \tilde{s}_a, \quad \tilde{s}_c \tilde{s}_{c+1} = \tilde{s}_{c+1} \tilde{s}_c, \\
(e_a)^2 &= \delta e_a, \quad \tilde{c}_c e_{c+1} e_c = \tilde{c}_{c+1} e_c e_{c+1} = \tilde{c}_c \\
\tilde{s}_a e_a &= e_a \tilde{s}_a, \quad \tilde{s}_c e_{c+1} e_c = \tilde{s}_{c+1} e_c e_{c+1} = \tilde{s}_c e_{c+1}
\end{align*}
$$

(13)

Note that $\text{Br}_d(\delta)$ is of dimension $d!!$, where for any natural number $m$ we set $m!! = 1 \cdot 3 \cdot 5 \cdot 7 \cdots m'$ where $m' = m$ if $m$ odd and $m' = m - 1$ if $m$ even.

**Remark 2.12.** If $M = L(0)$ is the trivial representation, then the action from Theorem 2.8 factors through the quotient $B_d(N) = \mathcal{W}_d(\Xi_{L(0)})/(y_1 - N^{-1})$ which is canonically isomorphic to the Brauer algebra $\text{Br}_d(N)$, see [Naz96, (2.2)]. In this case $w_a = N^{(N-1)/2}$ for $a \geq 0$. In case $N \geq r$, Theorem 2.8 turns then into the classical action of the Brauer algebra on tensor space, see e.g. [GW09], in particular $\text{Br}_d(N)$ is semisimple.

The algebra $\text{Br}_d(\delta)$ is generically semisimple, [Wen88]; for $\delta \geq 0$ it is semisimple in precisely the following cases, see [Bro56] or [Rui05]:

$$
\delta \neq 0, \text{ and } \delta \geq d - 1 \quad \text{or} \quad \delta = 0, \text{ and } d = 1, 3, 5.
$$

(14)

We are interested to relate the non-semisimple Brauer algebras $\text{Br}_d(\delta)$ for $\delta \geq 0$ to VW-algebras.

### 2.3. Cyclotomic quotients and admissibility.

Recall from [AMR06, Definition 2.10] that the parameters $w_a$, $a \geq 0$ are *admissible* if they satisfy the following admissibility condition

$$
w_{2a+1} + \frac{1}{2} w_{2a} - \frac{1}{2} \sum_{b=1}^{2a} (-1)^{b-1} w_{2a-b} w_{2a-b+1} = 0
$$

(15)
Example 2.13. The values $w_a := N(\frac{N-1}{2})^a$ for $a \geq 0$ from Remark 2.12 form an admissible sequence.

Admissibility ensures the existence of a nice basis of $\mathcal{W}_d$ as follows: Fix a Brauer diagram $b \in B[d]$. By Remark 2.12 we can write it as a product of generators $s_i, e_i$. Fix such an expression and consider the corresponding expression $B$ in the VW-algebra. More generally, given $\gamma, \eta \in \mathbb{Z}_{\geq 0}$ and $b \in B[d]$ we have the monomial $y_1^{\gamma_1} y_2^{\gamma_2} \cdots y_d^{\gamma_d} B y_1^{\eta_1} y_2^{\eta_2} \cdots y_d^{\eta_d} \in \mathcal{W}_d$. A monomial of this form is regular if $\gamma_i \neq 0$ implies $i$ is the left endpoint of a horizontal arc in $b$, and $\eta_i = 0$ if $i^*$ is the left endpoint of a horizontal arc in $b$, see (27). These monomials form a basis for $\mathcal{W}_d$ by [Naz96, Theorem 4.6] in case the parameters are admissible.

Given $u = (u_1, u_2, \ldots, u_l) \in \mathbb{C}^l$ we denote by $\mathcal{W}_d(\Xi; u)$ the quotient

$$\mathcal{W}_d(\Xi; u) = \mathcal{W}_d(\Xi)/\prod_{i=1}^{l} (y_i - u_i) \quad (16)$$

and call it the cyclotomic VW-algebra of level $l$ with parameters $u$. Under some additional admissibility condition, [AMR06, Theorem A, Prop. 2.15], the above basis is compatible with the cyclotomic quotients:

Proposition 2.14. If the $w_a$, $a \geq 0$ are $u$-admissible then $\mathcal{W}_d(u)$ has dimension $l^d(2d-1)!$. The regular monomials with $0 \leq \gamma_i, \eta_i < l$ for $1 \leq i \leq r$ form a basis $B(\mathcal{W}_d(u))$.

For the definition of $u$-admissibility see [AMR06, Def. 3.6]. We only need the special example from [AMR06, Lemma 3.5]:

Example 2.15. Assume the entries of $u$ are pairwise distinct and non-zero. Then the $w_a = \sum_{i=1}^{l} (2u_i - (-1)^l) u_i^a \prod_{1 \leq j \leq l} \frac{u_i + u_j}{u_i - u_j}$ for $a \geq 0$ form a $u$-admissible sequence.

2.4. The special case $M = M^p(\delta)$. In the following we study Theorem 2.8 in detail in the special case where $M$ is a specific type of parabolic Verma module. Recall our choice of parabolic $\mathfrak{p}$ from Section 1.

To $\delta \in \mathbb{Z}$ we associate the weight $\delta = \delta \omega_0 = \sum_{i=1}^{n} \delta_i \epsilon_i$, where $\omega_0 = \sum_{i=1}^{n} \epsilon_i$ denotes the fundamental weight of $\mathfrak{g}$ corresponding to $\alpha_0$. In particular, $\delta \in \Lambda$ and we have the Verma module $M^p(\delta)$.

Lemma 2.16. For $\lambda \in \Lambda$, $M^p(\lambda) \otimes V$ has a filtration with sections isomorphic to $M^p(\lambda \pm \epsilon_j)$ for all $j \in I^+$ such that $\lambda \pm \epsilon_j \in \Lambda$ and each of these Verma modules appearing exactly once.

Proof. This is a standard consequence of the definition (1) and the tensor identity; see e.g. [Hum08, Theorem 3.6].

Applying Lemma 2.16 iteratively, we obtain a bijection between Verma modules $M^p(\mu)$ appearing as subquotients in a Verma filtration of $M^p(\delta) \otimes V^\otimes d$ and $d$-admissible weight sequences ending at $\mu$ where the latter is defined as follows: For fixed $d \geq 1$ and $\delta \geq 0$ a weight $\mu \in \Lambda$ is called $d$-admissible
for δ if there is a sequence

\[ \delta = \lambda^1 \to \lambda^2 \to \cdots \to \lambda^d = \mu \] (17)

of length d, starting at \( \delta \) and ending at \( \mu \), of weights in \( \Lambda \) such that \( \lambda^{i+1} \) differs from \( \lambda^i \) by adding precisely one weight of \( V \), i.e. there exists \( j \in I^\ast \) such that \( \lambda^{i+1} = \lambda^i + \epsilon_j \). For instance, there are eight 2-admissible weight sequences for \( \delta \).

\[ \delta \to \delta - \epsilon_1 \to \delta - 2\epsilon_1, \quad \delta \to \delta - \epsilon_1 \to \delta - \epsilon_1 - \epsilon_2, \]

\[ \delta \to \delta - \epsilon_1 \to \delta + \epsilon_n, \quad \delta \to \delta - \epsilon_1 \to \delta + \epsilon_n + \epsilon_{n-1}, \]

\[ \delta \to \delta + \epsilon_n \to \delta. \]

\[ \delta \to \delta + \epsilon_n \to \delta. \]

Proposition 2.17. There is an isomorphism of \( g \)-modules \( M(\delta) \otimes V \cong M(\delta - \epsilon_1) \oplus M(\delta + \epsilon_n) \). This is an eigenspace decomposition for the action of \( y_1 \). The eigenvalues are \( \alpha = \frac{1}{2}(1 - \delta) \) and \( \beta = \frac{1}{2}(\delta + N - 1) \).

Proof. By Lemma 2.16, \( M^p(\delta) \otimes V \) has a Verma flag of length two with the asserted Verma modules appearing. The filtration obviously splits since they have different central character. The Casimir \( C = \sum_{\gamma \in B_\gamma} X_{\gamma}X_{\gamma}^\ast \) acts on a highest weight module with highest weight \( \lambda \) by \( c_\lambda = \langle \lambda, \lambda + 2p \rangle \), see e.g. [Mus12, Lemma 8.5.3] and on the tensor product \( M^p(\delta) \otimes V \) as \( \Delta(C) = C \otimes 1 + 1 \otimes C + \Omega_{0,1} \). Hence \( y_1 - \frac{N-1}{2} \) acts on the summands \( M^p(\delta + \nu) \), \( \nu = -\epsilon_1, \epsilon_n \) of \( M^p(\delta) \otimes V \) by

\[ \frac{1}{2} (\langle \delta + \nu, \delta + \nu + 2p \rangle - \langle \delta, \delta + 2p \rangle - \langle \epsilon_n, \epsilon_n + 2p \rangle) = \begin{cases} \frac{\delta}{2} - (n-1) & \text{if } \nu = -\epsilon_1 \\ \frac{\delta}{2} & \text{if } \nu = \epsilon_n \end{cases} \]

The statement follows now from the definition of \( \alpha \) and \( \beta \). \( \square \)

Corollary 2.18. For \( M = M^p(\delta) \), the action from Theorem 2.8 factors through the cyclotomic quotient with parameters \((\alpha, \beta)\) inducing an algebra homomorphism

\[ \Psi_{M(\delta)}^{d,n} : \bigwedge_d (\Xi_{M(\delta)}; \alpha, \beta) \to \text{End}_g(M^p(\delta) \otimes V^{\otimes d})^{opp}. \] (18)

Definition 2.19. From now on set \( \alpha = \frac{1}{2}(1 - \delta) \) and \( \beta = \frac{1}{2}(\delta + N - 1) \) and abbreviate \( \Psi_d = \Psi_d(\Xi_{M(\delta)} ; \alpha, \beta) \) and \( \Psi_d(\alpha, \beta) = \Psi_d(\Xi_{M(\delta)} ; \alpha, \beta) \).

Lemma 2.20. The elements \( w_a, a \geq 0 \) in \( \Psi_d \) satisfy the recursion formula \( w_0 = N, w_1 = N \frac{N-1}{2} \), and for \( a \geq 2 \)

\[ w_a = (\alpha + \beta)w_{a-1} - \alpha \beta w_{a-2}. \] (19)

Proof. By definition we have \( e_1 y_1 e_1 = e_1^2 = Ne_1 \), hence \( w_0 = N \). On the other hand for any \( j \in I^\ast \) we have

\[ e_1 y_1 e_1 (m \otimes v_j \otimes v_j^\ast) = \sum_{k\in I} e_1 y_1 (m \otimes v_k \otimes v_k^\ast) \]
Recalling (9), (11) this is equal to

\[
e_1 \sum_{k \in I^*} \left( \sum_{i \in I^*} X_i^* m \otimes X_i v_k \otimes v_k^* + \sum_{\alpha \in R_\alpha} X_\alpha m \otimes X_\alpha^* v_k \otimes v_k^* \right)
+ \frac{N(N-1)}{2} e_1 m \otimes v_j \otimes v_j^* =
\]

\[
e_1 \sum_{k \in I^*} \left( \sum_{i \in I^*} X_i^* m \otimes \epsilon_k(X_i) v_k \otimes v_k^* - \sum_{k \in I^-} X_i^* m \otimes \epsilon_k(X_i) v_k \otimes v_k^* \right)
+ \frac{N(N-1)}{2} e_1 m \otimes v_j \otimes v_j^*.
\]

We obtain \( w_1 = \frac{1}{2}(N-1)N \). Finally, \( y_1^2 = (\alpha + \beta)y_1 - \alpha \beta \) by Proposition 2.17, and hence \( e_1 y_1^2 e_1 = (\alpha + \beta)e_1 y_1^{(n-1)}e_1 - \alpha \beta e_1 y_1^{n-2}e_1 \).

**Lemma 2.21.** The \( w_a \) from Lemma 2.20 are explicitly given as

\[
w_a = N \sum_{k=0}^{a} \alpha^{a-k} \left( \frac{N}{2} - \alpha \right)^k - \frac{N}{2} \sum_{k=0}^{a-1} \alpha^{a-1-k} \left( \frac{N}{2} - \alpha \right)^k \tag{20}
\]

*Proof.* For \( a = 0, 1 \) the formula is obviously correct. Note that the recursion formula (19) has the general solution \( w_a = Aa^a + B\beta^a \) with boundary conditions \( A + B = N \) and \( A\alpha + B\beta = \frac{N}{2}(N-1) \), see e.g. [LP98, Theorem 33.10]. Hence

\[
A = \frac{1}{\alpha - \beta} \left( \frac{N}{2}(N-1) - N\beta \right) = \frac{1}{\alpha - \beta} \left( N\alpha - \frac{N}{2} \right)
\]

and therefore

\[
w_a = N(\alpha - \frac{1}{2}) \frac{\alpha^a - \beta^a}{\alpha - \beta} + N\beta^a = N(\alpha - \frac{1}{2}) \sum_{k=0}^{a-1} \alpha^{a-1-k} \beta^k + N\beta^a. \tag{21}
\]

The Lemma follows then by plugging in \( \beta = \frac{N}{2} - \alpha \). \( \square \)

For convenience we give a direct proof of the following result (which could alternatively be deduced from Lemma 2.21 using [AMR06, Corollary 3.9]).

**Lemma 2.22.** The \( w_a \) from Lemma 2.21 are admissible, i.e. satisfy (15).

*Proof.* Set \( Q(a) := N \sum_{k=0}^{a} \alpha^{a-k} \left( \frac{N}{2} - \alpha \right)^k \). Then \( w_a = Q(a) - \frac{1}{2}Q(a-1) \) and the admissibility condition (15) is for \( m = 2a + 1 \) equivalent to

\[
0 = 2Q(m) - \frac{1}{2}Q(m-2) + R
- \sum_{b=1}^{m-1} (-1)^{b-1} \left( Q(b-1)Q(m-b) + \frac{1}{4}Q(b-2)Q(m-b - 1) \right) \tag{22}
\]

where \( R = \sum_{b=1}^{m-1} (-1)^{b-1}(Q(b-2)Q(m-b) + Q(b-1)Q(m-b-1)) = 0 \), since \( Q(-1) = 0 \) by definition and then

\[
R = \sum_{b=0}^{m-1} (-1)^b Q(b-1)Q(m-b-1) + \sum_{b=1}^{m} (-1)^{b-1} Q(b-1)Q(m-b-1)
\]
Now by definition
\[ (m-1)Q(m-1)Q(-1) + Q(-1)Q(m-1) = 0. \]

By the right hand side of (22), it is enough to show that \( S(t) := 2Q(t) - \sum_{b=1}^{t} (-1)^{b-1}Q(b-1)Q(t-b) = 0 \) for \( t = m, m-1 \). For this we consider \( S(t) \) as a polynomial in \( N \) and show that all the coefficients vanish. First note that
\[
2Q(t) = 2N \sum_{k=0}^{t} \alpha^{t-k} \sum_{r=0}^{k} \binom{k}{r} \frac{1}{2^r} N^r (-\alpha)^{k-r} = \sum_{k=0}^{t} \sum_{r=0}^{k} \binom{k}{r} \frac{1}{2^r-1} (-1)^{k-r} \alpha^{t-r} N^{r+1}
\]

Hence its coefficient in front of \( N^{s+1} \) equals
\[
c_{s+1} = \frac{1}{2^{s-1}} (-1)^s \alpha^{t-s} \sum_{k=0}^{t} (-1)^k \binom{k}{s}. \tag{23}
\]

On the other hand
\[
\sum_{b=1}^{t} (-1)^{b-1} Q(b-1) Q(t-b) = N^2 \sum_{b=1}^{t} \alpha^{b-1} \sum_{r=0}^{b} \sum_{j=0}^{b-i-r} (\frac{N}{2} - \alpha)^{r+j}
\]

hence its coefficient in front of \( N^{s+1} \) equals
\[
d_{s+1} = \frac{1}{2^{s-1}} (-1)^s \alpha^{t-s} \sum_{b=1}^{t} \sum_{r=0}^{b} \sum_{j=0}^{b-r+j} (\frac{N}{2} - \alpha)^{r+j}. \tag{24}
\]

Clearly \( c_0 = 0 = d_0 \) and then \( c_{s+1} = d_{s+1} \) for all \( s \geq 0 \) by the Lemma 2.23 below.

**Lemma 2.23.** Let \( m \geq 0 \) be odd and \( s \geq 1 \). Then
\[
\sum_{k=0}^{m} (-1)^k \binom{k}{s} = \sum_{b=1}^{m} \sum_{r=0}^{m-b} \sum_{j=0}^{r-b} (-1)^{r+b+j} \binom{r+b}{s-1}. \tag{25}
\]

**Proof.** For \( m = 1 \) the statement is clear. Let \( L(m) \) and \( R(m) \) be the left and right side of (25). We assume \( L(m) = R(m) \) and want to deduce \( L(m+2) = R(m+2) \) for which it is enough to show \( R(m+2) - R(m) = L(m+2) - L(m) \).

Since \( m \) is odd, the latter is equivalent to verifying
\[
\binom{m+1}{s} - \binom{m+2}{s} = R(m+2) - R(m) \tag{26}
\]

Now by definition \( R(m+2) - R(m) \) equals
\[
\sum_{r=0}^{m+1} \sum_{j=0}^{r+b+j} \frac{r+b}{s-1} \binom{r+b}{s-1} + \sum_{r=0}^{m} \sum_{j=0}^{m+r} \binom{m+r+1}{s-1} \binom{r+b}{s-1} + \sum_{b=1}^{m} \sum_{r=0}^{m-b} (-1)^{m+r} \binom{r+m+2-b}{s-1}
\]
\[
= (-1)^{2m+1} \binom{m+1}{s-1} + \sum_{r=0}^{m} \sum_{j=0}^{m+r+2} \binom{r+1}{s-1} + \sum_{r=0}^{m-1} (-1)^{m+r+1} \binom{r+1}{s-1} + \sum_{r=0}^{m-1} (-1)^{m+r} \binom{m+1}{s-1}
\]
\[
\begin{align*}
&= -\binom{m+1}{s-1} + \binom{m+1}{s-1} - \binom{m+1}{s-1} - \binom{m+1}{s-1} \sum_{b=1}^{m-1} (-1)^b = -\binom{m+1}{s-1} \\
&= \binom{m+1}{s} - \binom{m+2}{s}.
\end{align*}
\]

\[\square\]

Lemma 2.24. The sequence \(w_a, a \geq 0\) from Lemma 2.21 is \(u\)-admissible for \(u = (\alpha, \beta)\).

Proof. Substituting \(N\) in formula (21) we obtain
\[
w_a = \frac{\alpha + \beta}{\alpha - \beta} (2\alpha^{a+1} - 2\beta^{a+1} - \alpha^a + \beta^a)
\]
which is easy to see to agree with the formula for \(w_a\) in Example 2.15. \[\square\]

Corollary 2.25. The level \(l = 2\) cyclotomic quotients \(\mathbb{V}_d(\alpha, \beta)\) are of dimension \(2^d(2d-1)!!\) with basis \(\mathcal{B}_d = \mathcal{B}(\mathbb{V}_d(\alpha, \beta))\) given by the regular monomials from Proposition 2.14.

We display basis elements diagrammatically by drawing the Brauer diagram \(B\) with small decorations indicating the \(y_i\)'s. For instance
\[
\begin{array}{c}
\uparrow \\
\times \\
\downarrow \\
\end{array}
\]
stands for \(y_4 y_1 y_{10} y_{11} \in \mathcal{B}_d\) where \(b\) is the Brauer algebra element without the decorations. Sliding a decoration through an arc produces a linear combination of basis vectors according to Definition 2.1, (VW.7), (VW.8).

3. ISOMORPHISM THEOREM

Theorem 3.1. If \(n \geq 2d\) and \(\delta \in \mathbb{Z}\) then the map \(\Psi_{M(\delta)}^{d,n}\) from Theorem 2.8 induces an isomorphism of algebras
\[
\Psi(\delta) : \mathbb{V}_d(\alpha, \beta) \longrightarrow \text{End}_d(M^p(\delta) \otimes V^{\otimes d})^{\text{opp}}
\]

Proof. By definition it is an algebra homomorphism. It is injective by Proposition 3.2 below and surjective since the dimensions agree, Corollary 3.6. \[\square\]

3.1. Injectivity. We start with the proof for the injectivity.

Proposition 3.2. If \(n \geq 2d\) and \(\delta \in \mathbb{Z}\) then \(\Psi_{M(\delta)}^{d,n}\) from Theorem 2.8 induces an injective map of algebras
\[
\Psi(\delta) : \mathbb{V}_d(\alpha, \beta) \longrightarrow \text{End}_d(M^p(\delta) \otimes V^{\otimes d})^{\text{opp}}
\]
Proof. By Corollary 2.25 it is enough to show that the regular monomials
\( y_1^\eta_1 y_2^\eta_2 \cdots y_d^\eta_d B y_1^{\gamma_1} y_2^{\gamma_2} y_d^{\gamma_d} \) for \( b \in \mathcal{B}_d \) and \( 1 \leq \gamma_1, \eta_i, \nu_i < l \) are mapped to linearly
independent morphisms. By Remark 2.6 and (9) we have \( y_i = \Omega_{0,i} + x \), where
\( x \) is some Brauer algebra element. Let now \( m \in M^p(\hat{\delta}) \) be a highest weight
vector. Then \( X_\gamma m = 0 \) for \( X_\gamma \in \mathfrak{n}^+ \) and furthermore \( X_{-(\epsilon_i + \epsilon_j)}m = 0 \) for all
\( i > j \) because of our specific choice of \( \hat{\delta} \). (Note that \( E(\hat{\delta}) \) is one dimensional
in the description of \( M^p(\hat{\delta}) \) from (1).) Applying (9) we obtain for any \( a \in I^+ \)
\[
\Omega_{0,1}m \otimes v_a = \frac{\delta}{2} m \otimes v_a, \quad \Omega_{0,1}m \otimes v_{-a} = \frac{\delta}{2} m \otimes v_a + \sum_{i \neq a} a_i X_{-(\epsilon_i + \epsilon_j)}m \otimes v_i
\]
for some \( a_i \in \mathbb{C}^* \). The \( pm \otimes v_{i_1} \otimes v_{i_2} \otimes \cdots v_{i_{d-1}} \otimes v_{i_d} \), where \( p \) runs through
all monomials in the variables \( X_{-(\epsilon_i + \epsilon_j)} \) for \( i, j \in I^+ \) with \( i > j \) form a basis
\( \mathcal{B}(M^p(\hat{\delta}) \otimes V^\otimes d) \) of \( M^p(\hat{\delta}) \otimes V^\otimes d \). (Note that the \( X_{-(\epsilon_i + \epsilon_j)} \)’s commute due
to the structure of the root system). In particular,
\[
y_pm \otimes v_a = \begin{cases} 
\sum_{i \neq a} a_i X_{-(\epsilon_i + \epsilon_j)}v_{i_1} \otimes \cdots \otimes v_{i_{r-1}} \otimes v_1 \otimes v_{i_{r+1}} \otimes v_{i_d} + (\dagger) & \text{if } -a \in I^+ \\
(\dagger) & \text{if } a \in I^+
\end{cases}
\]
where \( (\dagger) \) stands for some linear combination of basis vectors where the
above mentioned monomials have degree zero.

Now consider a standard basis vector \( \gamma \in \mathcal{B}_d \). Take its diagram (27) and
label the vertices with numbers from \( \{i, -i \mid 1 \leq i \leq 2d\} \) as follows:
Label all vertices at the top from left to right by \( d + 1 \) to \( 2d \) and at the
bottom from left to right by \( -1 \) to \( -d \). In case there is a vertical arc
without decoration connecting two vertices replace the label at the top
endpoint with the label at the bottom endpoint. If there is a horizontal arc
without decoration change the label at the right endpoint to the neg-
ative of the left endpoint. In the resulting diagram the labels, say \( a \) and
\( b \), at the endpoints of arcs satisfy the following: \( |a| \neq |b| \) and negative at the
bottom and positive at the top if the arc is decorated; \( a = b < 0 \) if the arc is
undecorated and vertical; and \( a = -b \) if the arc is undecorated and horizontal. Moreover, no number appears more than twice. In the example (27) we get \( 12, -3, -2, 15, -5, -15, -6, -8, 20, 21, 22 \) at the top and
\( -1, -2, -3, -4, -5, -6, -7, -8, -9, 9, -11 \) at the bottom.

Let now \( S := \sum_{\gamma \in \mathcal{B}_d} c_\gamma \Psi(\hat{\delta})(\gamma) \) with \( c_\gamma \in \mathbb{C} \). Pick \( \gamma \in \mathcal{B}_d \) corresponding
to a diagram with all arcs decorated. and let \( (-a_i, b_i) \), \( 1 \leq i \leq d \), \( a_i, b_i > 0 \)
be the pairs of labels attached to each arc. Then, by (30) the coefficient
of \( X_{-(\epsilon_a + \epsilon_b)} \cdots X_{-(\epsilon_a + \epsilon_b)}m \otimes v_1 \otimes \cdots \otimes v_d \) when expressing \( Sm \otimes v_{-1} \otimes \cdots \otimes v_{-d} \) in our basis is precisely \( c_\gamma \), hence \( c_\gamma = 0 \). Repeating this argument
gives \( c_\gamma = 0 \) for all diagrams with all strands decorated. Next pick \( b \in \mathcal{B}_d \)
which corresponds to a diagram with all arcs except one decorated and let
\( j_1, \ldots, j_d \) and \( j_1', \ldots, j_d' \) be the associated labels at the bottom respectively
top of the diagram read from left to right. Let \( (-a_i, b_i) \), \( 1 \leq i \leq d - 1 \),
\( a_i, b_i > 0 \) be the pairs of labels attached to each arc. By (30), the coefficient
of $X_{-(e_{a_1}+e_{b_1})}\cdots X_{-(e_{a_d}+e_{b_d})}m \otimes v_{i_1} \otimes \cdots v_{i_d}$ when expressing $Sm \otimes v_{j_1} \otimes \cdots v_{j_d}$ in our basis is then precisely $c_\gamma$. Hence $c_\gamma = 0$ for all diagrams with only one undecorated arc. Proceeding like this gives finally $c_\gamma = 0$ for all $\gamma$. Hence the linearly independence follows.

3.2. **Weight diagrams and bipartitions.** Based on [AMR06] we introduce a labelling set for a basis in the cyclotomic quotients and connect it with the counting of Verma modules from Lemma 2.16 via the diagrammatic weights.

Fix integers $d$ and $t$ with $0 \leq t \leq \lfloor \frac{d}{2} \rfloor$. A *partition* of $d$ is a sequence $\lambda = (\lambda_1, \lambda_2, \ldots)$ of weakly decreasing non-negative integers $\lambda_i$ which sum up to $d$, i.e. $|\lambda| := \sum_{i \geq 1} \lambda_i = d$. As usual we identify a partition with its Ferres or Young diagram containing $\lambda_i$ boxes in the $i$-th row. Its top left vertex is called the origin. A *bipartition* is an ordered pair $\lambda = (\lambda^1, \lambda^2)$ of partitions $\lambda^1$, $\lambda^2$. We denote by $P^2$ (resp. $P^2(d)$) the set of bipartitions (of $d$) and by $P^1$ (resp. $P^1(d)$) the set of partitions (of $d$). An *up-down-bitableau* is a sequence $(\emptyset, \emptyset) = \lambda(0), \lambda(1), \ldots, \lambda(d)$ of bipartitions starting from the empty bipartition and such that $\lambda(i+1)$ differs from $\lambda(i)$ by removing or adding exactly one box. If the length of the sequence is $d+1$ we call it also an up-down-$d$-bitableau. The set of all up-down-bitableaux is denoted $T^2$ with the subset $T^2_d$ of all up-down-$d$-bitableaux and the subset $T^2_d(\lambda)$ of all $\lambda$-up-down-tableaux. i.e. all those up-down-$d$-bitableau ending at a fixed bipartition $\lambda$. Similarly we define the sets $T^1$, $T^1_d$, $T^1_d(\lambda)$ of all up-down-tableaux, all up-down-$d$-tableaux and $\lambda$-up-down-$d$-tableaux for a partition $\lambda$. From [AMR06] it follows in particular that for $l = 1, 2$

$$\sum_\lambda |T^l_d(\lambda)|^2 = l^d(2d-1)!! \quad (30)$$

As in Section 1 we associate a diagrammatic weight to each weight in $\Lambda$, for simplicity denoted by the same letter. In light of Corollary 3.6 we stick here to the case $\delta \geq 0$. Then the corresponding diagrammatic weights are the following. (The first and third are in $X_n$, the second in $X'_n$).

$$\begin{cases}
0 & \delta = 0 \text{ even}, \\
\frac{\delta}{2} & \delta \geq 0 \text{ even}, \\
\frac{\delta}{2} & \delta \geq 0 \text{ odd}, \\
0 & \delta = 0.
\end{cases}$$

Suppose we are given a $d$-admissible weight sequence as in (17)$$\delta = \lambda^1 \rightarrow \lambda^2 \rightarrow \cdots \rightarrow \lambda^d.$$
We are going to represent this diagrammatically by what we call a Verma path of length \( d \) (depending on the parameter \( \delta \in \mathbb{Z}_{\geq 0} \), similar to the diagrams in [BS11b], [BS12]. Note that, since the weights \( \lambda^i \) and \( \lambda^{i+1} \) differ only by some \( \pm \epsilon_j \) and let \( r_i = |\lambda^i_j| \) and \( s_i = |\lambda^{i+1}_j| \), the associated weight diagrams differ precisely at the coordinates \( r_i \) and \( s_i \). We first draw the weight sequence as a sequence of weight diagrams from bottom to top. Then for each \( i = 1, \ldots, d-1 \), we insert vertical line segments connecting all coordinates strictly smaller or strictly larger than \( r_i \) and \( s_i \) that are labelled \( \lor \) or \( \land \) in \( \lambda^i \) and \( \lambda^{i+1} \). Then connect the remaining coordinates \( r_i \) and \( s_i \) of \( \lambda^i \) and \( \lambda^{i+1} \) as in the appropriate one of the following pictures:

\[
\begin{array}{ccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{array}
\]

The result is a Verma path of length \( d \). Note that in the third row all the given connections might be decorated with a \( \bullet \) in a similar way to the fourth row. For this replace the symbol \( \diamond \) with either \( \land \) or \( \lor \) such that the weight diagram has an even number of \( \lor \)'s. If the resulting picture does not look like one of the pictures in the first or second row, i.e. the strand is not oriented then it must be decorated with a \( \bullet \).

**Example 3.3.** The first eight pictures in Figure 31 display the Verma paths for \( d = 2 \) and \( \delta > 1 \), representing the eight Verma modules appearing in a Verma filtration of \( M^P(\delta) \otimes V^{\otimes d} \). In case \( \delta = 1 \), the first picture has to be replaced by the last one. Note the dot in the last Verma path that indicates that the symbol \( \diamond \) at position 0 for \( \lambda^1 \) would need to be substituted with a \( \land \) to obtain an even number of \( \lor \)'s.

To any weight \( \lambda \) appearing in a Verma path of length \( d \leq n \) we assign a bipartition \( \varphi(\lambda) = (\lambda^{(1)}, \lambda^{(2)}) \) with \( \lambda^{(1)}_i = |m_i| \) and \( \lambda^{(2)}_i = m_{s+i} \), where the \( m_i \) are defined by \( \lambda - \delta = \sum_{i=1}^n m_i \epsilon_i \) with \( m_i \leq 0 \) for \( 1 \leq i \leq s \) and \( m_i > 0 \) for \( i > s \). It is convenient to draw here the bipartition \( \varphi(\lambda) \in T^2(d), \ d \leq n \), by taking the first diagram transposed (i.e. reflected in the diagonal \( D \) through the origin) and the second diagram reflected in the line \( D' \) through the origin orthogonal to \( D \) and then arrange them in the plane such that reflected
origins are positioned at the coordinates \((0, 0)\) and \((0, n)\) respectively. The \(i\)th column contains then \(m_i\) boxes arranged below the \(x\)-axis if \(m_i < 0\) and above if \(m_i > 0\). Let \((\lambda^t, \mu^t)\) denote the resulting diagram. For example, the bipartition \((3, 2, 1, 1), (2, 2, 1)\) is displayed as follows:

\[
\begin{array}{cccc}
\lambda = \begin{array}{c}
\begin{array}{|c|c|c|c|c|}
\hline
& & & & \\
\hline
& & & & \\
\hline
& & & & \\
\hline
& & & & \\
\hline
& & & & \\
\hline
\end{array}
\end{array}, & \mu = \begin{array}{c}
\begin{array}{|c|c|c|}
\hline
& & \\
\hline
& & \\
\hline
\end{array}
\end{array}
\end{array}
\]

We label its columns from left to right by 0, 1, 2, \ldots and the rows increasing from top to bottom such that the \(\lambda^t\) has rows 1, 2, \ldots and \(\mu^t\) has rows \(-1, -2, \ldots\) (with a "ground state line" counted 0 between the two reflected partitions). The content \(c(b)\) of a box \(b\) is its column number minus its row number, \(c(b) = \text{col}(b) - \text{row}(b)\) and the content with charge \(\delta\) is defined as \(c(b) = \frac{\delta}{2} + c(b)\); in our example we have

\[
\begin{array}{cccc}
\delta - 3 & \delta - 1 & 1 & 2 \\
\frac{\delta}{2} + n - 2 & \frac{\delta}{2} + n - 1 & \frac{\delta}{2} + n & \frac{\delta}{2} + n + 1
\end{array}
\]

Then we can read off the corresponding diagrammatic weight \(\lambda \in \Lambda\) by setting \(I(\lambda) = \{\text{cont}_b(\lambda, \mu)\} = \{c_b(b_i) \mid 1 \leq i \leq n\}\), where the \(b_i\) are the boundary boxes of the columns. In the above example (with the relevant boxes shaded grey) we obtain \(I(\lambda) = \left\{\frac{\delta}{2} + \{-3, -1, 1, 2, \ldots, n-4, n-2, n, n+1\}\right\}\). In particular, \(\frac{\delta}{2}\) corresponds to the bipartition \((\varnothing, \varnothing)\). In this way, any \(t \in T_d\) defines a sequence \(\text{cont}(t)\) of diagrammatic weights.
Lemma 3.4. Let $n, \delta \in \mathbb{Z}_{\geq 0}$. Assume $n \geq \delta$. Then the assignment $t \mapsto \text{cont}(t)$ defines a bijection, with inverse map $\lambda \mapsto \varphi(\lambda)$,

$$T_d = \{ \text{up-down bitableaux of length } d \} \Downarrow$$

$$P_d(\delta) := \{ \text{Verma paths of length } d \text{ starting at } \delta \}$$

inducing also a bijection between $\bigcup_{0 \leq k \leq \frac{d}{2}} \mathbb{P}^2(d - 2k)$ and the set of weights appearing at the end of a path in $P_d(\delta)$.

Proof. The empty bipartition $(\emptyset, \emptyset)$ corresponds to the weight $\delta + \rho$. Now adding (resp. removing) a box in the $j$th column of the first partition corresponds to subtracting $\epsilon_j$ from (resp. adding $\epsilon_j$ to) the weight. On the other hand, adding (resp. removing) a box in the $(n - j)$th column of the second partition corresponds to adding $\epsilon_j$ to (resp. subtracting $\epsilon_{n-j}$ from) the weight. In any case, the result is an allowed weight if and only if adding (resp. removing) the box produces a new bipartition. Then the lemma follows from the definitions.

For $M^p(\lambda) \in \mathcal{O}^p(n)$ we denote by $L(\lambda)$ its irreducible quotient and by $P^p(\lambda) \in \mathcal{O}^p(n)$ its projective cover. For a module $M$ in category $\mathcal{O}^p(n)$ with Verma flag (i.e. with a filtration with subquotients isomorphic to Verma modules) we denote by $\lambda_M = [M:M^p(\lambda)]$ the multiplicity of $M^p(\lambda)$ in such a flag. By BGG reciprocity we have $[P(\mu):M^p(\nu)] = [M^p(\nu):L(\mu)]$ where the latter denotes the Jordan-Hoelder multiplicity of $L(\mu)$ in $M^p(\nu)$.

Lemma 3.5. The following holds in $\mathcal{O}^p(n)$:

1. $\dim \text{Hom}_g(P^p(\lambda), P^p(\mu)) = \sum_{\nu \in \Lambda} \nu_{P(\lambda)} \nu_{P(\mu)}$, the number of Verma modules which appear at the same time in $P(\lambda)$ and in $P(\mu)$.

2. If $P$ is a projective object then $\dim \text{End}_g(P) = \sum_{\nu \in \Lambda} (\nu_P)^2$.

Proof. Let $N = \{ \nu \in \Lambda \mid \nu_{P(\mu)} \neq 0, \nu_{P(\lambda)} \neq 0 \}$. Then $\dim \text{Hom}_g(P(\lambda), P(\mu)) = [P(\mu) : L(\lambda)] = \sum_{\nu \in \Lambda} [P(\mu) : M(\nu)] [M(\nu) : L(\lambda)] = \sum_{\nu \in N} \nu_{P(\mu)} \nu_{P(\lambda)} = \sum_{\nu \in N} 1 \cdot 1$, where for the last equality we used that the multiplicities are at most 1. This follows for regular weights for instance from [LS13, Theorem 2.1] and for singular weights from Lemma 3.10 below. Part (1) follows. Recall that $P$ decomposes into a direct sum of indecomposable projective modules $P = \bigoplus_{\mu} a_\mu P(\mu)$, where $a_\mu$ is the multiplicity of $P(\mu)$. Then

$$\dim \text{End}_g(P) = \sum_{\lambda, \mu \in \Lambda} a_\lambda a_\mu \dim \text{Hom}_g(P(\lambda), P(\mu))$$

$$= \sum_{\lambda, \mu \in \Lambda} a_\lambda a_\mu \sum_{\nu \in \Lambda} \nu_{P(\mu)} \nu_{P(\lambda)} = \sum_{\lambda, \mu \in \Lambda} a_\lambda a_\mu \nu_\lambda \nu_\mu$$

$$= \sum_{\nu \in \Lambda} \left( \sum_{\lambda, \mu \in \Lambda} a_\lambda \nu_\mu \sum_{\mu \in \Lambda} a_\mu \nu_{P(\lambda)} \right) = \sum_{\nu \in \Lambda} \nu_\lambda \nu_\mu = \sum_{\nu} (\nu_P)^2.$$

$\square$
Corollary 3.6. We have \( \dim \text{End}_g(M^p(\delta) \otimes V^{\otimes d})^{\text{opp}} = 2^d(2d - 1)!! \). Moreover, \( \text{End}_g(M^p(\delta) \otimes V^{\otimes d})^{\text{opp}} \cong \text{End}_g(M^p(-\delta) \otimes V^{\otimes d}) \) as algebras.

Proof. Let first \( \delta \geq 0 \). Since then \( M^p(\delta) \) is projective in \( O^n \), so is \( M^p(\delta) \otimes V^{\otimes d} \) and therefore it has a Verma filtration. By Lemma 2.16 and the paragraph afterwards \( V_{M^p(\delta) \otimes V^{\otimes d}} \) equals the number of Verma paths of length \( d \) ending at \( \nu + \rho \). Then the statement follows directly from Lemmas 3.4, Lemma 3.5 and the equality (30). If \( \delta < 0 \) then \( M^p(-\delta) \) is a tilting module and the algebra isomorphism follows from Ringel self-duality [MS08, Proposition 4.4] of \( O^n \) and we are done. \( \square \)

3.3. Special projective functors. In analogy to [BS11b] we introduce special translation functors which will later categorify the actions of the coideal subalgebras \( \mathcal{H} \) and \( \mathcal{H}' \) in Section 6.

For this let \( \Gamma \) be a diagrammatic block, i.e. given by a block sequence and a parity for the number of \( \lor \)'s and \( \times \)'s. Furthermore fix \( i \in \mathbb{Z} \geq 0 + \frac{1}{2} \) if \( \Gamma \) is supported on the integers and \( i \in \mathbb{Z}_{\geq 0} \) if \( \Gamma \) is supported on half-integers.

Case \( i \geq 1 \): We denote by \( \Gamma_{i,+} \) and \( \Gamma_{i,-} \) the blocks, whose corresponding block diagram differs from \( \Gamma \) diagrammatically only at the vertices \( i \frac{1}{2} \) and \( i + \frac{1}{2} \) as displayed in the following list (note that it has the same type of integrality and the same parity):

\[
\begin{array}{c|c|c|c|c|c}
\Gamma_{i,+} & \otimes & \bullet & \bullet & \circ & \circ \\
\Gamma_{i,-} & \circ & \bullet & \bullet & \circ & \circ \\
\Gamma & \bullet & \otimes & \bullet & \circ & \circ \\
\end{array}
\] (34)

Case \( i = \frac{1}{2} \): We denote by \( \Gamma_{\frac{1}{2},+} \) and \( \Gamma_{\frac{1}{2},-} \) the blocks or in one instance the sum of two blocks, whose corresponding block diagram differs from \( \Gamma \) diagrammatically only at the vertices \( 0 \) and \( 1 \) as displayed in the following list (note that it has the same type of integrality):

\[
\begin{array}{c|c|c|c|c|c}
\Gamma_{\frac{1}{2},+} & \otimes & \bullet & \bullet & \circ & \circ \\
\Gamma_{\frac{1}{2},-} & \circ & \bullet & \bullet & \circ & \circ \\
\Gamma & \bullet & \otimes & \bullet & \circ & \circ \\
\end{array}
\] (35)

Here the entry \( \circ \bullet + \circ \circ \) means that we take the direct sum of the two blocks with the same underlying block diagram but the two possible choices of parity.

Case \( i = 0 \): In this case the block \( \Gamma_0 \) has the same block diagram as \( \Gamma \) but the opposite parity. Let \( \text{pr}_1^\Gamma : O^p_\rho(n) \to O^p_\rho(n) \Gamma \) and \( \text{pr}_2^\Gamma : O^p_\rho(n) \to O^p_\rho(n) \Gamma \), denote the projections onto the summand \( O^p_\rho(n) \Gamma \), resp. \( O^p_\rho(n) \Gamma \), depending on whether \( \Gamma \) is supported on the integers or half-integers.
Definition 3.7. We have the special projective functors $F_{i,+}$, $F_{i,-}$ for $i \in \mathbb{Z}_{\geq 0} + \frac{1}{2}$ defined by

$$
F_{i,+} := \bigoplus_{\Gamma} \text{pr}^{1}_{\Gamma,i,+} \circ (\otimes V) \circ \text{pr}^{1}_{\Gamma} : \mathcal{O}^{p}_{1}(n) \to \mathcal{O}^{q}_{1}(n),
$$

$$
F_{i,-} := \bigoplus_{\Gamma} \text{pr}^{1}_{\Gamma,i,-} \circ (\otimes V) \circ \text{pr}^{1}_{\Gamma} : \mathcal{O}^{p}_{1}(n) \to \mathcal{O}^{q}_{1}(n),
$$

and $F_{i,+}, F_{i,-}$ for $i \in \mathbb{Z}_{\geq 0}$ and $F_{0}$ defined by

$$
F_{i,+} := \bigoplus_{\Gamma} \text{pr}^{1}_{\Gamma,i,+} \circ (\otimes V) \circ \text{pr}^{1}_{\Gamma} : \mathcal{O}^{p}_{1}(n) \to \mathcal{O}^{q}_{1}(n),
$$

$$
F_{i,-} := \bigoplus_{\Gamma} \text{pr}^{1}_{\Gamma,i,-} \circ (\otimes V) \circ \text{pr}^{1}_{\Gamma} : \mathcal{O}^{p}_{1}(n) \to \mathcal{O}^{q}_{1}(n),
$$

$$
F_{0} := \bigoplus_{\Gamma} \text{pr}^{1}_{\Gamma,0} \circ (\otimes V) \circ \text{pr}^{1}_{\Gamma} : \mathcal{O}^{p}_{1}(n) \to \mathcal{O}^{q}_{1}(n).
$$

In each case the direct sums are over all blocks $\Gamma$, where $\Gamma_{i,-}$, resp. $\Gamma_{i,+}$, resp. $\Gamma_{0}$ are defined.

Note that $F_{i,-}$ and $F_{i,+}$ are biadjoint and $F_{0}$ is selfadjoint, since $V$ is self-dual. Hence the functors are exact and send projectives to projectives. The following Lemma, whose proof can be found in the appendix, describes the effect of taking tensor product with $V$ on projective and simple modules. At the end of this section we give a diagrammatic interpretation of the action on indecomposable projective modules. The symbols $(i)$ in the lemma refer to a grading shift which should be ignored. It only makes sense in the graded setup of Lemma 5.3.

Lemma 3.8. 

(1) There are isomorphisms of functors

$$
\mathcal{F} := (\otimes V) \cong \bigoplus_{i \in \mathbb{Z}_{\geq 0} + \frac{1}{2}} (F_{i,+} \otimes F_{i,-}) : \mathcal{O}^{p}_{1}(n) \to \mathcal{O}^{q}_{1}(n)
$$

and

$$
\mathcal{F}^i := (\otimes V) \cong \bigoplus_{i \in \mathbb{Z}_{\geq 0}} (F_{i,+} \otimes F_{i,-}) \otimes F_{0} : \mathcal{O}^{p}_{1}(n) \to \mathcal{O}^{q}_{1}(n).
$$

(2) Let $\lambda$ be a diagrammatic weight and $i \in \mathbb{Z}_{\geq 0} + \frac{1}{2}$ if $\lambda \in \mathbb{K}_n$ and $i \in \mathbb{Z}_{\geq 0}$ if $\lambda \in \mathbb{K}_{n}^{i}$. Furthermore assume that $i \geq 1$ in both cases. For symbols $x, y \in \{0, \lambda, v, \times\}$ we write $\lambda_{xy}$ for the diagrammatic weight obtained from $\lambda$ by relabelling the $(i-\frac{1}{2})$th entry to $x$ and the $(i+\frac{1}{2})$th entry to $y$.

(i) If $\lambda = \lambda_{oo}$ then $F_{i,+}P(\lambda) \cong P(\lambda_{oo}), \ F_{i,-}M^{p}(\lambda) \cong M^{p}(\lambda_{oo}), \ F_{i,+}L(\lambda) \cong L(\lambda_{oo}).$

(ii) If $\lambda = \lambda_{oo}$ then $F_{i,-}P(\lambda) \cong P(\lambda_{oo}), \ F_{i,-}M^{p}(\lambda) \cong M^{p}(\lambda_{oo}), \ F_{i,-}L(\lambda) \cong L(\lambda_{oo}).$

(iii) If $\lambda = \lambda_{vv}$ then $F_{i,+}P(\lambda) \cong P(\lambda_{vv}), \ F_{i,-}M^{p}(\lambda) \cong M^{p}(\lambda_{vv}), \ F_{i,-}L(\lambda) \cong L(\lambda_{vv}).$

(iv) If $\lambda = \lambda_{xx}$ then $F_{i,-}P(\lambda) \cong P(\lambda_{xx}), \ F_{i,-}M^{p}(\lambda) \cong M^{p}(\lambda_{xx}), \ F_{i,-}L(\lambda) \cong L(\lambda_{xx}).$
(v) If $\lambda = \lambda_{\alpha}$ then $\mathcal{F}_{i,-}\!P(\lambda) \cong P(\lambda_{\alpha})$ and $\mathcal{F}_{i,-}\!MP(\lambda) \cong MP(\lambda_{\alpha})$.

(vi) If $\lambda = \lambda_{\alpha}$ then $\mathcal{F}_{i,-}\!MP(\lambda) \cong MP(\lambda_{\alpha}) \{1\}$ and $\mathcal{F}_{i,-}\!L(\lambda) = \{0\}$.

(vii) If $\lambda = \lambda_{\alpha}$ then:
(a) $\mathcal{F}_{i,-}\!P(\lambda) \cong P(\lambda_{\alpha})$;
(b) there is a short exact sequence

$$0 \to MP(\lambda_{\alpha}) \to \mathcal{F}_{i,-}\!MP(\lambda) \to MP(\lambda_{\alpha})(1) \to 0;$$

(c) $[\mathcal{F}_{i,-}L(\lambda) : L(\lambda_{\alpha})] = 2$ and all other composition factors are of the form $L(\mu)$ for $\mu = \mu_{\alpha}$, $\mu = \mu_{\alpha}$ or $\mu = \mu_{\alpha}$.

(d) $\mathcal{F}_{i,-}\!L(\lambda)$ has irreducible socle and head isomorphic to $L(\lambda_{\alpha})$.

(viii) If $\lambda = \lambda_{\alpha}$ then $\mathcal{F}_{i,-}\!MP(\lambda) = \mathcal{F}_{i,-}\!L(\lambda) = \{0\}$.

(ix) If $\lambda = \lambda_{\alpha}$ then $\mathcal{F}_{i,-}\!MP(\lambda) = \mathcal{F}_{i,-}\!L(\lambda) = \{0\}$.

(x) For the dual statement about $\mathcal{F}_{i,+}$, interchange all occurrences of $\diamond$ and $\times$.

(3) Let $\lambda \in \mathfrak{X}_{n}$. For symbols $x, y \in \{\diamond, \land, \lor, \times, \circ\}$ we write $\lambda_{xy}$ for the diagrammatic weight obtained from $\lambda$ by relabelling the 0th entry to $x$ and the 1st entry to $y$.

(xi) If $\lambda = \lambda_{\alpha\beta}$ then $\mathcal{F}_{\frac{1}{2},-}\!P(\lambda) \cong P(\lambda_{\alpha\beta})$ and $\mathcal{F}_{\frac{1}{2},-}\!MP(\lambda) \cong MP(\lambda_{\alpha\beta}) \{1\}$.

(xii) If $\lambda = \lambda_{\alpha\beta}$ then $\mathcal{F}_{\frac{1}{2},-}\!P(\lambda) \cong P(\lambda_{\alpha\beta})$, $\mathcal{F}_{\frac{1}{2},-}\!MP(\lambda) \cong MP(\lambda_{\alpha\beta})$.

(xiii) If $\lambda = \lambda_{\alpha\beta}$ then $\mathcal{F}_{\frac{1}{2},-}\!P(\lambda) \cong P(\lambda_{\alpha\beta}) \oplus P(\lambda_{\alpha\beta})$, $\mathcal{F}_{\frac{1}{2},-}\!L(\lambda) \cong L(\lambda_{\alpha\beta}) \oplus L(\lambda_{\alpha\beta})$.

(xiv) If $\lambda = \lambda_{\alpha\beta}$ then we have $\mathcal{F}_{\frac{1}{2},-}P(\lambda) \cong P(\lambda_{\alpha\beta})$, $\mathcal{F}_{\frac{1}{2},-}MP(\lambda) \cong MP(\lambda_{\alpha\beta})$.

(xv) If $\lambda = \lambda_{\alpha\beta}$ then $\mathcal{F}_{\frac{1}{2},-}P(\lambda) \cong P(\lambda_{\alpha\beta})$, $\mathcal{F}_{\frac{1}{2},-}MP(\lambda) \cong MP(\lambda_{\alpha\beta})$.

(xvi) If $\lambda = \lambda_{\alpha\beta}$ then:
(a) $\mathcal{F}_{\frac{1}{2},-}P(\lambda) \cong P(\lambda_{\alpha\beta})$;
(b) there is a short exact sequence

$$0 \to MP(\lambda_{\alpha\beta}) \to \mathcal{F}_{\frac{1}{2},-}\!MP(\lambda) \to MP(\lambda_{\alpha\beta})(1) \to 0;$$

(c) $[\mathcal{F}_{\frac{1}{2},-}L(\lambda) : L(\lambda_{\alpha\beta})] = 2$ and all other composition factors are of the form $L(\mu)$ for $\mu = \mu_{\alpha\beta}$.

(xvii) If $\lambda = \lambda_{\alpha\beta}$ then we have $\mathcal{F}_{\frac{1}{2},-}P(\lambda) = \{0\}$, $\mathcal{F}_{\frac{1}{2},-}MP(\lambda) = \{0\}$, $\mathcal{F}_{\frac{1}{2},-}L(\lambda) = \{0\}$.

(xviii) If $\lambda = \lambda_{\alpha\beta}$ then we have $\mathcal{F}_{\frac{1}{2},-}P(\lambda) = \{0\}$, $\mathcal{F}_{\frac{1}{2},-}MP(\lambda) = \{0\}$, $\mathcal{F}_{\frac{1}{2},-}L(\lambda) = \{0\}$. 

(4) Finally assume $\lambda \in \mathbb{X}_n^i$. For symbols $x \in \{\circ, \wedge, \vee, \times\}$ we write $\lambda_x$ for the diagrammatic weight obtained from $\lambda$ by relabelling the $\frac{1}{2}$th entry to $x$. Then we have

\[\begin{align*}
(xiv) & \text{ If } \lambda = \lambda_\wedge \text{ then } F_0 P(\lambda) \cong P(\lambda_\vee), F_0 M^p(\lambda) \cong M^p(\lambda_\vee), F_0 L(\lambda) \cong L(\lambda_\vee), \\
(xiv) & \text{ If } \lambda = \lambda_\vee \text{ then } F_0 P(\lambda) \cong P(\lambda_\wedge), F_0 M^p(\lambda) \cong M^p(\lambda_\wedge), F_0 L(\lambda) \cong L(\lambda_\wedge).
\end{align*}\]

Remark 3.9. Note that each of the $F_i$, $F_{i,+}$, $F_{i,-}$, $F_0$ is a translation functor in the sense of [Hum08]. It is either an equivalence of categories, a translation functor to the wall, a translation functor out of the wall or, in the special case of $F_{\frac{1}{2},-}$ and the choice of a specific block diagram a direct sum of two translation functors out of the wall. In contrast to the type $A$ situation from [BS11b] not all translations to walls appear however as summands.

3.4. Projective modules and cup diagrams. In this section we indicate how to make explicit calculations related to projective modules using decorated cup diagrams following [LS13]. We briefly recall from [ES13b, Definition 3.5] how one associates a (decorated) cup diagram, denoted $\Lambda$, to a diagrammatic weight $\lambda$. For a more detailed construction we refer to [ES13b].

First draw the diagrammatic weight as weight diagram by putting the symbol $\lambda_i$ at coordinate $i$ on the real positive line. Start by connecting neighboured coordinates in the weight diagram labelled $\vee$ and $\wedge$ successively by an arc (ignoring already joint coordinates and coordinates not having the symbols $\wedge$ or $\vee$) as long as possible and attach to each remaining $\vee$ a vertical ray. Then connect from left to right pairs of two neighboured $\wedge$’s by a dotted arc, i.e. an arc decorated with a $\bullet$. If a single $\wedge$ remains, attach a vertical ray decorated with a $\bullet$ to it. Finally forget the labelling at the vertices other then $\circ$ and $\times$. Putting a diagrammatic weight $\mu$ on top of a cup diagram $\Lambda$ we call the resulting diagram oriented if the labels $\circ$ and $\times$ in $\mu$ and $\Lambda$ match precisely, undecorated rays are labelled $\vee$ and decorated rays are labelled $\wedge$ and every undecorated cup has exactly one $\wedge$ and one $\vee$ whereas any decorated cup has either two $\vee$’s or two $\wedge$’s at its endpoints.

The multiplicities $\mu P(\lambda)$ are then easily calculated:

**Lemma 3.10.** \[
\mu P(\lambda) = \begin{cases} 
1 & \text{if } \mu \lambda \text{ is oriented} \\
0 & \text{if } \mu \lambda \text{ is not oriented.}
\end{cases}
\]

**Proof.** For regular weights, i.e. weights which do not contain $\times$’s or $\circ$’s this was proved is [LS13, Theorem 2.1]. (Lemma 4.3 therein and the paragraph afterwards makes the translation into our setup.) For singular weights observe that diagrammatically the numbers stay the same if we remove the $\times$’s and $\circ$’s. This corresponds Lie theoretically to the Enright-Shelton equivalence [ES87] between singular blocks and regular blocks for smaller rank Lie algebras, and hence the multiplicities agree. \qed
The action of the special projective functors on projectives can be calculated easily in terms of yet another diagram calculus inside the Temperley-Lieb category of type $D$, [Gre98]. Let $\mathcal{C}$ be the free abelian group generated by $\lambda, \lambda \in \Lambda$. To any indecomposable projective object $P(\lambda)$ we assign the cup diagram $\lambda \in \mathcal{C}$ and extend linearly to direct sums. To the functor $F := (\otimes V)$ we assign the formal sum $F^{\text{diag}}$ of decorated tangles:

Each of these tangle diagrams induces a group homomorphism of $\mathcal{C}$. On a cup diagram $\lambda$ the action produces a multiple of a cup diagram which is computed as follows: first concatenate and remove any pair of dots on the same line or internal circle. If the labels $\circ, \times$ and $\blacklozenge$ do not match or the result contains a circle with one dot or a cap without dots then it acts by zero. Otherwise remove each occurring (undotted) circle and (dotted) cap. The resulting diagram is again a cup diagram $\lambda$ (where we always consider two diagrams as the same if they differ only by an isotopy in the plane and moving decorations, i.e. moving $\bullet$’s, along lines) and the result of the action is $2c\lambda$, where $c$ is the number of circles removed. We extend this linearly to an action of $F^{\text{diag}}$ on $\mathcal{C}$.

**Lemma 3.11.** Fix the isomorphism of free abelian groups $\Phi : K_0(O^p(n)) \cong \mathcal{C}$, which sends the class of $P(\lambda)$ to $\lambda$. Then $F^{\text{diag}}(\Phi[P]) = [P \otimes V]$ for any projective module $P \in O^p(n)$.

**Proof.** In the case of columns 4-7 in (34), the group homomorphism $K_0(F_{i,-}) : K_0(O^p(n)) \to K_0(O^p(n)_{i,-})$ induced by $F_{i,-}$ is an isomorphism when expressed in the basis of the isomorphism classes of Verma modules; it has an inverse, the morphism induced by $F_{i,+}$. They both agree with the morphism induced from the restriction of $F$ to these blocks. Hence indecomposable projectives are sent to the corresponding indecomposable projectives. The corresponding statement on the diagram side is obvious, involving the first four diagrams from (36) only. In the case of column 2 or 3 in (34) we can apply to a cup diagram $\lambda$ exactly one of the first two diagrams from the second line of (36). In each case the result is $\lambda$ with an extra cup and we are done by Proposition 3.8 (vii). Column 1 is the most involved situation. In this case $\lambda$ looks locally as one of the following 18 diagrams $D_i, 1 \leq i \leq 18$ drawn...
in thick black lines or the same picture with \(\circ\) and \(\times\) swapped. Together with the dashed lines they indicate the result after applying the diagrammatic functor (the first two in the second line give zero).

\[
\begin{align*}
P(\lambda_{ox}) & \quad P(\lambda_{ox\lambda}) \quad P(\lambda_{o\lambda\lambda}) \quad P(\lambda_{vox}) \quad P(\lambda_{o\lambda x}) \quad P(\lambda_{v o x}) \quad P(\lambda_{vox}) \\
P(\lambda_{\lambda\lambda\lambda}) & \quad P(\lambda_{\lambda\lambda\lambda}) \quad P(\lambda_{\lambda\lambda\lambda}) \quad P(\lambda_{\lambda\lambda\lambda}) \quad P(\lambda_{\lambda\lambda\lambda}) \quad P(\lambda_{\lambda\lambda\lambda}) \quad P(\lambda_{\lambda\lambda\lambda}) \\
0 & \quad 0 \quad P(\circ\times)^{\oplus 2} \quad P(\lambda_{ovox}) \quad P(\lambda_{o\lambda x\lambda}) \quad P(\lambda_{o\lambda x\lambda}) \quad P(\lambda_{v o x\lambda}) \quad P(\lambda_{vox}) \\
P(\lambda_{\lambda\lambda\lambda}) \quad P(\lambda_{\lambda\lambda\lambda}) \quad P(\lambda_{\lambda\lambda\lambda}) \quad P(\lambda_{\lambda\lambda\lambda}) \quad P(\lambda_{\lambda\lambda\lambda}) & \quad P(\lambda_{\lambda\lambda\lambda}) \quad P(\lambda_{\lambda\lambda\lambda}) \\
P(\lambda_{o\lambda\lambda\lambda\lambda}) & \quad P(\lambda_{v\lambda\lambda\lambda\lambda}) \quad P(\lambda_{o\lambda\lambda\lambda\lambda}) \quad P(\lambda_{v\lambda\lambda\lambda\lambda}) \quad P(\lambda_{o\lambda\lambda\lambda\lambda}) \quad P(\lambda_{v\lambda\lambda\lambda\lambda}) \quad P(\lambda_{o\lambda\lambda\lambda\lambda}) \\
P(\lambda_{v\lambda\lambda\lambda\lambda}) \quad P(\lambda_{v\lambda\lambda\lambda\lambda}) \quad P(\lambda_{v\lambda\lambda\lambda\lambda}) & \quad P(\lambda_{v\lambda\lambda\lambda\lambda}) \quad P(\lambda_{v\lambda\lambda\lambda\lambda}) \quad P(\lambda_{v\lambda\lambda\lambda\lambda}) \
\end{align*}
\]

In the first seven non-zero pictures, we have only one possible orientation for the displayed resulting diagram. On the other hand, there are at most two orientations for each \(D_i\) of which exactly one corresponds to a Verma module which is not annihilated by \(\mathcal{F}\) with the image in our chosen blocks except of the last and penultimate diagram where none of the two is annihilated. Similar arguments can be used for the functors \(\mathcal{F}^{1+}, \mathcal{F}^{1-}, \mathcal{F}_0\). Comparing with Proposition 3.8 using Lemma 3.10, we see that \(\mathcal{F}^{\text{diag}}(\Phi[P]) = [P \otimes V]\) in the basis of Verma modules.

As a special case we obtain the following:

**Corollary 3.12.** Assume we are in case (vi) of Proposition 3.8 and assume there is no \(\vee\) to the left of our fixed \(\wedge\vee\) pair then \(\mathcal{F}_{i,-}P^p(\lambda) \cong P^p(\lambda_{o\vee})\).

**Proof.** By assumption and construction, the cup diagram \(\lambda\) looks locally at our two fixed vertices as displayed in one of the pictures in the top line of the following diagram. Applying \(\mathcal{F}^{\text{diag}}\) gives for each diagram precisely one new diagram in the required block as displayed. Hence the statement follows by applying \(\varphi^{-1}\) and the definition of \(\mathcal{F}_{i,-}\). \(\square\)
For this whole section we will always assume \( n \geq 2d \). This is important for all statements that involve the idempotent \( z_d \) introduced in Definition 4.3.

The action of the commuting \( y \)’s decomposes \( M^P(\hat{\delta}) \otimes V^{\otimes d} \) and \( \mathbb{W}_d(\alpha, \beta) \) into generalized eigenspaces. Fix the sets \( J = \frac{\delta}{2} + \mathbb{Z} \) and \( J_c = \frac{\delta}{2} + \mathbb{Z}_{<n-1} \).

**Definition 4.1.** We identify \( \mathbb{W}_d(\alpha, \beta) \) with \( \text{End}_d(M^P(\hat{\delta}) \otimes V^{\otimes d}) \) via Theorem 3.1. Define the orthogonal **weight idempotents** \( e(\tilde{i}) \), \( \tilde{i} \in J^d \) characterized by the property that
\[
e(\tilde{i})(y_r - i_r)^m = (y_r - i_r)^m e(\tilde{i}) = 0
\]
for each \( 1 \leq r \leq d \) and \( m \gg 0 \) (for instance \( m \geq d \) is enough).

Then we have the generalized eigenspace decompositions
\[
M^P(\hat{\delta}) \otimes V^{\otimes d} = \sum_{\tilde{i} \in J^d} M^P(\hat{\delta}) \otimes V^{\otimes d} e(\tilde{i}),
\]
\[
\mathbb{W}_d(\alpha, \beta) = \sum_{\tilde{i} \in J^d} \mathbb{W}_d(\alpha, \beta) e(\tilde{i}) \quad \text{and} \quad \mathbb{W}_d(\alpha, \beta) = \sum_{\tilde{i} \in J^d} e(\tilde{i}) \mathbb{W}_d(\alpha, \beta)
\]
as right respectively left modules. The quasi-idempotents \( e_k \) only acts between certain generalized eigenspaces:

**Lemma 4.2.** Let \( 1 \leq k \leq d - 1 \). If \( p := \sum_{\tilde{i} \in J^d, i_{k+1} = -i_k} e(\tilde{i}) \) then we have \( e_k = p e_k = e_k p = p e_k p \). In particular \( e(\tilde{i}) e_k = 0 = e_k e(\tilde{i}) \) if \( i_{k+1} \neq i_k \).

**Proof.** Let \( m \in \mathbb{W}_d(\alpha, \beta) \) or \( M^P(\hat{\delta}) \otimes V^{\otimes d} \) be contained in the generalized eigenspace for \( 1 \), hence \( m e(\tilde{i}) \neq 0 \). We claim that if \( m e_k \neq 0 \) then \( i_{r+1} + i_r = 0 \).

First note that for \( (y_k + y_{k+1} - i_k - i_{k+1})^r m = \sum_{a=0}^r (y_k - i_k)^a (y_{k+1} - i_{k+1})^{r-a} m = 0 \) for \( r \gg 0 \). Hence \( m \) is in the \( \mu := i_{k+1} + i_k \)-generalized eigenspace for \( y_{k+1} + y_k \).

Then \( 0 = m(y_k + y_{k+1} - \mu)^r e_k = m e_k(-\mu)^k \) by (8a). Hence \( \mu = 0 \) and the claim follows. Since the modules have a generalized eigenspace decomposition \( p e_k = 0 = e_k p \). The rest follows analogously. \( \square \)

**4.1. Semisimplicity.**

**Definition 4.3.** The **Brauer algebra idempotent** is defined as
\[
z_d = \sum_{\tilde{i} \in J_c^d} e(\tilde{i}). \tag{38}
\]

Multiplication by \( e(\tilde{i}) \) projects any \( \mathbb{W}_d(\alpha, \beta) \)-module onto its \( \tilde{i} \)-**weight space**, that is, the simultaneous generalised eigenspace for the commuting operators \( y_1, \ldots, y_r \) with respective eigenvalues \( i_1, \ldots, i_r \). By (33), Lemma 3.4 and our assumption on \( n \) being large, the element \( z_d \) is just the projection onto the blocks containing Verma modules indexed by bipartitions of the form \( (\lambda, \varnothing) \) where \( \lambda \) is an up-down-tableau. (The corresponding weights are precisely those \( \lambda \) which satisfy \( I(\lambda) \subset [-(\frac{\delta}{2} + n - 1), \frac{\delta}{2} + n - 1] \).)

**Proposition 4.4.** The algebra \( z_d \mathbb{W}_d(\alpha, \beta) z_d \) has dimension \( (2d - 1)! \).
Proof. Using Lemma 3.4 and the fact that the algebra has a basis given by Verma paths that correspond to bipartitions of the form \((\lambda, \emptyset)\) gives the result. \(\Box\)

Theorem 4.5.

(1) The algebra \(\mathcal{W}_d(\alpha, \beta)\) is semisimple if and only if \(\delta \geq d - 1\).

(2) The algebra \(z_d\mathcal{W}_d(\alpha, \beta)z_d\) is semisimple if and only if \(\delta \neq 0\) and \(\delta \geq d - 1\) or \(\delta = 0\) and \(d = 1, 3, 5\).

Proof. Note that \(\text{End}_q(M^p(\delta) \otimes V^\otimes d)\) is semisimple if and only if \(M^p(\delta) \otimes V^\otimes d\) is a direct sum of projective Verma modules. Or equivalently if all occurring Verma modules have highest weight \(\lambda\) such that \(\lambda + \rho\) is dominant. This is equivalent to the statement that the corresponding diagrammatic weights do not contain two \(^\wedge\)'s any pair \(\lor, \land\) (in this order) or a pair \(\lor \land\). Starting from the diagrammatic weight \(\delta\) we need precisely \(\delta\) steps to create an \(\land\) left to a \(\lor\), hence \(\delta + 2\) steps to create a \(\lor, \land\); we also need \(2(\delta + 1)\) steps to create a pair of two \(\land\)'s and \(\delta + 2\) steps to create \(\lor, \land\). Hence both algebras are semisimple at least if \(d < \delta + 2\). Now in case (1), we always can add \(\epsilon_n\) without changing those pairs, an hence the algebra is not semisimple for all \(d \geq \delta + 2\). In case (2) with \(\delta \neq 0, 1\) we can always find some \(\circ\) at some position < \(n\) and hence add or subtract some appropriate \(\epsilon_j\) without changing the pair. That means the truncated algebra is not semisimple for \(d \geq \delta + 2\). For \(\delta = 1\) we need 3 steps to create a weight starting with \(\circ \land \lor\). Hence the algebra is not semisimple and stays not semisimple, since we can always repeatedly swap the \(\circ\) with the \(\lor\) without changing the \(\circ \land\) pair. Hence (2) holds for any \(\delta > 0\). In case \(\delta = 0\) one can calculate directly that it is semisimple in the cases \(d = 1, d = 3, d = 5\), but not in the cases \(d = 2, 4, 6\) and for \(d = 7\) we obtain a weight starting with \(\circ \land \lor\). Since we can change the last two symbols \(\land \lor\) into \(\circ \times\) and back again without loosing the \(\lor \land\)-pair, the algebra stays not semisimple for \(d \geq 7\) and \(\delta = 0\). The theorem follows. \(\Box\)

4.2. The basic algebra underlying \(\text{End}_q(M^p(\delta) \otimes V^\otimes d)\). Our next goal is to determine for \(\delta \geq 2\) the projective modules appearing in \(M^p(\delta) \otimes V^\otimes d\) and hence the basic algebra underlying \(\text{End}_q(M^p(\delta) \otimes V^\otimes d)\) for \(\delta \in \mathbb{Z}\). Our main tool here is the notion of \(ht_\delta\) which measures the minimal length of a Verma path needed to create a Verma module of a given weight.

Definition 4.6. Let \(\delta, d \in \mathbb{Z}\), \(\delta \geq 0\), \(d \geq 1\) and \(\mu \in \Lambda\). The \(\delta\)-height of \(\mu\), \(ht_\delta(\mu)\) is defined as \(ht_\delta(\mu) = \sum_{i=1}^\delta |(\delta + \rho)_i - (\mu + \rho)_i| = \sum_{i=1}^\delta |\delta_i - \mu_i|\).

Note that, by Lemma 2.16, the highest weights \(\lambda\) occurring in a Verma filtration of \(M^p(\delta) \otimes V^\otimes d\) satisfy \(ht_\delta(\lambda) \leq d\) and \(ht_\delta(\lambda)\) is precisely the number of boxes in \(\varphi(\lambda)\), see (32), (33).

Proposition 4.7. Let \(\delta, d \in \mathbb{Z}\), \(\delta \geq 0\), \(d \geq 1\). Then \(M^p(\delta) \otimes V^\otimes d \cong \bigoplus_{\mu \in \Lambda} a_\mu P(\mu)\) with multiplicities \(a_\mu\) and \(a_\mu \neq 0\) if and only if \(ht_\delta \leq d\) and \(d - ht_\delta\) is even.
Proof. Starting with \( d = 0 \) there is only one weight with \( \delta \)-height zero, namely \( \delta \) and the statement is clear. For \( d = 1 \) there are two weights, \( \delta - e_1 \) and \( \delta + e_n \). Again the statement is clear, since the corresponding two Verma modules are projective. Now we want to prove the lemma for arbitrary \( d \) assuming it for all smaller ones. Note that \( a_\mu \neq 0 \) implies that \( M^P(\mu) \) appears in a Verma flag of \( M^P(\delta) \otimes V^{\otimes d} \), and hence \( \text{ht}_\delta(\mu) \leq d \). Moreover, the highest weights of the occurring Verma modules change in each step by \( \delta \) which changes the \( \delta \)-height either by 1 or \(-1\). By induction, \( d - \text{ht}_\delta(\mu) \) is even.

Conversely, assume \( \mu \) is a weight with \( \text{ht}_\delta(\mu) = d - 2k \) for some \( k \geq 0 \). If \( k > 0 \) then by induction hypothesis \( P(\mu) \) appears in \( M(\delta) \otimes V^{\otimes d-2} \). The weight \( \mu \) contains at least one neighboured \( \wedge, \vee, \) or \( \times \) pair (since only finitely many vertices are labelled \( \wedge, \vee \) or \( \times \)). Hence we are in one of the cases (i), (ii) or (vii) of Proposition 3.8 so \( P(\mu) \) is a summand of \( F_i, F_{i-1} - P(\mu) \) and therefore also a summand in \( M^P(\delta) \otimes V^{\otimes d} \). Now we have to proceed via a case-by-case argument for \( k = 0 \):

\begin{enumerate}
  \item Assume first that there exist \( i \in I(\mu), -i \notin I(\mu) \) with \( i := (\mu + \rho)_a > 0 \) and \( \delta_a - \mu_a > 0 \) for some \( a \). (In the diagram picture this means that the \( a \)-th \( \vee \) in \( \delta \) was moved to the left and gives a \( \vee \) in \( \mu \) at position \( i \)). We assume that \( a \) was chosen to be maximal amongst these (that is the rightmost \( \vee \) which got moved to the left to create some \( \vee \)). Locally at the positions \( i \) and \( i + 1 \) the diagram for \( \mu \) could look as follows
    \begin{enumerate}
      \item In this case set \( \nu = \circ \vee \)
      \item In this case set \( \nu = \circ \times \)
      \item In this case set \( \nu = \times \vee \)
      \item Since \( a \) was maximal, this means that the \( \vee \) at position \( i + 1 \) forces \( i + 1 = (\mu + \rho)_{a+1} \) and at the same time \( \delta_{a+1} - \mu_{a+1} \leq 0 \) which is a contradiction.
    \end{enumerate}
  
  The diagram for \( \mu \) could look as follows

\begin{enumerate}
  \item Assume \( (1) \) does not hold and moreover that if \( i := (\mu + \rho)_a > 0 \) for some \( a \) and \( -i \notin I \) then \( \delta_a - \mu_a = 0 \). (Diagrammatically this means that \( \vee \)‘s did either not move or turned into \( \wedge \)‘s, creating possibly some \( \times \)‘s). Choose, if it exists, \( -j \in I \) maximal such that \( -j < 0 \). (Diagrammatically we look at the \( \wedge \)‘s and \( \times \)‘s and choose the leftmost. It sits at position \( j \)). Locally at the positions \( j - 1 \) and \( j \), for \( j > \frac{1}{2} \), the diagram for \( \mu \) could look as follows
      \begin{enumerate}
        \item In this case set \( \nu = \circ \wedge \).
        \item In this case set \( \nu = \times \circ \).
        \item In this case set \( \nu = \circ \circ \).
        \item In this case set \( \nu = \wedge \circ \).
        \item In this case set \( \nu = \vee \circ \).
      \end{enumerate}
\end{enumerate}
(e) $\circ \times$. In this case set $\nu = \land \lor$. The $\lor$ in $\delta$ at place $j-1$ got moved, hence turned into an $\land$ or created a $\times$. But then the same happened to all the $\lor$'s further to the left. In particular, $\nu$ has no $\lor$ left of the $\land$ at position $j$ and hence we can apply Lemma 3.12 to $P(\nu)$.

Again, in each of the last four cases $P(\mu)$ appears as a summand in $P(\nu) \otimes V$ by Proposition 3.8 and by induction hypothesis $P(\nu)$ appears as summand in $M(\delta) \otimes V^{\otimes d-1}$. In case no $\land$ or $\times$ exists we either have $\mu = \delta$ and there is nothing to do or $\mu$ has a $\circ \lor$ or $\lor \circ$ pair which we swap to create a new weight $\nu$ and argue as above using Proposition 3.8 (i) or (iii) with (x).

(3) Assume finally that neither (1) nor (2) holds. The arguments are completely analogous to (2) except for the case (e') $\circ \times$. In case there is no $\lor$ to the left of the cross we set $\nu = \land \lor$ and argue as before using Lemma 3.12. Otherwise such a $\lor$ did not get moved or got moved to the right, since we excluded case (1). Since it is separated from our $\times$ at position $j$ by a $\circ$, our $\times$ is created from a $\lor$ which got moved to the right. Hence changing $\circ \times$ to $\land \lor$ decreases the height by 1. Then we can argue again using Proposition 3.8, this time part (v).

\end{proof}

Lemma 4.8. Let $\delta, d \in \mathbb{Z}$, $d \geq 0$, $d \geq 1$. The set

$$\mathcal{S} = \mathcal{S}(\delta, d) = \{ \mu \in \Lambda \mid \text{ht}_\delta(\mu) \leq d, \text{ht}_\delta(\mu) \equiv d \mod 2 \} \quad (39)$$

is saturated in the following sense: if $\mu \in \mathcal{S}$ and $\nu \in \Lambda$ such that $\mu$ and $\nu$ are in the same block and $\nu \geq \lambda$ then $\nu \in \mathcal{S}$.

\begin{proof}
Assume $\mu$ is in the same block as $\nu$, $\mu < \nu$. Then the diagrammatic weight for $\nu$ is obtained from $\mu$ by applying a finite sequence of changes from $\land \lor$ to $\lor \lor$ or from $\land \lor$ to $\lor \land$ at positions only separated by $\circ'$ and $\times$'s. Hence it is enough to consider these basic changes.

Change from $\land \lor$ to $\lor \land$: Let $0 \leq i \leq i+j$ be the positions of the two symbols. In the $e$-basis $\mu + \rho$ and $\nu + \rho$ are then of the form

$$(-x_1, \ldots, -x_a, -(i+j), -y_1, \ldots, -y_b, z_1, \ldots, z_c, u_1, \ldots, u_r, i, v_1, \ldots, v_s, w_1, \ldots, w_t)$$

$$(-x_1, \ldots, -x_a, -y_1, \ldots, -y_b, -i z_1, \ldots, -z_c, u_1, \ldots, u_r, v_1, \ldots, v_s, i+j, w_1, \ldots, w_t)$$

where $a, b, c, r, t \geq 0$ with the $x_k > 0$ indicating the $\land$'s to the right of position $i + j$, the $y_k > 0$ indicating the $\land$'s as part of the crosses between positions $i$ and $i + j$, the $z_k > 0$ indicating the $\land$'s to the left of position $i$, the $u_k > 0$ indicating the $\lor$'s to the left of position $i$, the $v_k > 0$ indicating the $\lor$'s as part of the crosses between positions $i$ and $i + j$ and the $w_k > 0$ indicating the $\lor$'s to the right of position $i + j$. Hence to determine $\text{ht}_\delta(\mu) - \text{ht}_\delta(\nu)$ we could without loss of generality assume that the $x$'s, $y$'s, $z$'s, $u$'s and $w$'s are
zero. Recalling that the vector $\delta + \rho$ is an increasing sequence of consecutive numbers, we are left with

$$\text{ht}_\delta(\mu) - \text{ht}_\delta(\nu) = (|m + i + j| + \sum_{k=1}^{b} |m + k + y_k| - \sum_{k=1}^{b} |m + k - 1 + y_k| - |m + b + i|)$$

$$+ (|p - i| + \sum_{k=1}^{b} |p + k - v_k| - \sum_{k=1}^{b} |p + k - 1 - v_k| - |p + b - i - j|)$$

for some non-negative $m, p$ which are integers in case $i, j$ are integers and half-integers in case $i, j$ are half-integers. The absolute values in the first summand can be removed and hence gives $j$ in total. Set $v_0 = i$ and $v_{b+1} = i + j$, and choose $-1 \leq k_0 \leq b+1$ such that $p + k - v_k > 0$ for $0 \leq k \leq k_0$ and $p + k - v_k \leq 0$ for $b+1 \geq k > k_0$.

$$\text{ht}_\delta(\mu) - \text{ht}_\delta(\nu) = j + |p - v_0| - |p + b - v_{b+1}| + \sum_{1 \leq k \leq k_0, b} ((p + k - v_k) - (p + k - 1 - v_k))$$

$$+ \sum_{k_0+1 \leq k \leq b} ((v_k - p - k) - (v_k - p - k + 1))$$

$$= j + |p - v_0| - |p + b - v_{b+1}| + k_0 - (b - k_0) = \begin{cases} 2j & \text{if } k_0 = b + 1, \\ 2(p - i + k_0) & \text{if } 0 \leq k_0 \leq b, \\ 0 & \text{if } k_0 = -1. \end{cases}$$

In any case $\text{ht}_\delta(\mu) \geq \text{ht}_\delta(\nu)$ and their difference is even, since $j, k_0$ and $p - i$ are integral. Change from $\wedge \wedge$ to $\lor \lor$: It is easy to check that the difference of the heights is even. We claim that if $\mu'$ and $\nu'$ are weights such that $\nu'$ is obtained from $\mu'$ just by changing an $\wedge$ at some position $i$ which has no other $\wedge$ or $\lor$ to its left into a $\lor$ then $\text{ht}_\delta(nu') \leq \text{ht}_\delta(\mu)$. Together with the previous paragraph this claim proves the lemma. In the $\epsilon$-basis, the weights $\mu' + \rho$ and $\nu' + \rho$ have the following form

$$\begin{pmatrix} -x_1, \ldots, -x_a, y_1, \ldots, -y_b, z_1, \ldots, z_b, i, w_1, \ldots, w_c \end{pmatrix}$$

$$\begin{pmatrix} -x_1, \ldots, -x_a, -i, -y_1, \ldots, -y_b, z_1, \ldots, z_b, w_1, \ldots, w_c \end{pmatrix}$$

where $a, b, c \geq 0$ with the $x_k > 0$ indicating the $\wedge$’s to the right of position $i$, the $y_k > 0$ indicating the $\wedge$’s as part of the crosses to the left of $i$, the $z_k > 0$ indicating the $\lor$’s as part of the crosses to the left of $i$ and the $w_k > 0$ indicating the $\lor$’s to the right of position $i$. Again, the $x$’s and $w$’s are irrelevant for the difference of the heights. Note that the number of $y$’s equals the number of $z$’s. With the arguments from above we get

$$\text{ht}_\delta(\mu') - \text{ht}_\delta(\nu') = (|m + i| + \sum_{k=1}^{b} (m + k + y_k) - \sum_{k=1}^{b} (m + k - 1 + y_k))$$
with \( p = m + b + 1 \). The first sum equals \( b \), the second \( \geq -b \) and hence \( \text{ht}_\delta(\mu') - \text{ht}_\delta(\nu') \geq m + i - \lfloor m + b + 1 - i \rfloor \). This is obviously positive if \( m + b + 1 - i < 0 \) and equals \(-2b + 2i - 1\) otherwise. On the other hand, \( b < i \) and our claim follows. \( \square \)

4.3. **Quasi-hereditaryness.** The category \( \mathcal{O}^p(n) \) is a highest weight category in the sense of [CPS88], i.e. it is a category \( \mathcal{C} \) with a poset \((\mathcal{S}, \leq)\) satisfying

- \( \mathcal{C} \) is an \( \mathbb{C} \)-linear Artinian category equipped with a duality, that is, a contravariant involutive equivalence of categories \( \oplus : \mathcal{C} \to \mathcal{C} \);
- for each \( \lambda \in \mathcal{S} \) there is a given object \( L(\lambda) \cong L(\lambda)^\oplus \in \mathcal{C} \) such that \( \{ L(\lambda) | \lambda \in \mathcal{S} \} \) is a complete set of representatives for the isomorphism classes of irreducible objects in \( \mathcal{C} \);
- each \( L(\lambda) \) has a projective cover \( P(\lambda) \in \mathcal{C} \) such that all composition factors of \( P(\lambda) \) are isomorphic to \( L(\mu) \)'s for \( \mu \in \mathcal{S} \);
- writing \( V(\lambda) \) for the largest quotient of \( P(\lambda) \) with the property that all composition factors of its radical are isomorphic to \( L(\mu) \)'s for \( \mu > \lambda \), the object \( P(\lambda) \) has a filtration with \( V(\lambda) \) at the top and all other factors isomorphic to \( V(\nu) \)'s for \( \nu < \lambda \).

The poset \( \mathcal{S} \) is in our example the set of \( \Lambda \) highest weight vectors for irreducible modules with the reversed Bruhat ordering \( \leq_{r\text{Bruhat}} \); \( V(\lambda) \) is the Verma module of highest weight \( \lambda \) and the duality is the standard duality from \( \mathcal{O} \), [Hum08].

The category \( \Psi_d(\alpha, \beta)^{\text{opp}} \) - mod of finite dimensional (right) \( \Psi_d(\alpha, \beta) \)-modules is by [AMR06, Corollary 8.6] also a highest weight category, with poset the set of bipartitions of \( d - 2t \) for \( 1 \leq t \leq \lfloor \frac{d}{2} \rfloor \) equipped with the dominance ordering defined below, \( V(\lambda) \) the cell module from [AMR06, Section 6], and \( \oplus \) the ordinary vector space duality using Remark 2.2.

The category of finite dimensional Brauer\((\delta)\)-modules with \( \delta \neq 0 \) is by [KX98, Theorem 1.3], [CDVM09a, Corollary 2.3] a highest weight category, with poset the set of partitions of \( d - 2t \) for \( 1 \leq t \leq \lfloor \frac{d}{2} \rfloor \) with the dominance ordering, \( V(\lambda) \) the cell module from [CDVM09a, Lemma 2.4] and \( \oplus \), the ordinary vector space duality using the analogue of Remark 2.2 for the Brauer algebra. In case \( \delta = 0 \) it is a highest weight category if and only if \( \delta \neq 0 \) or \( \delta = 0 \) and \( d \) odd.

**Theorem 4.9.** For fixed \( \delta, d \) and \( \mathcal{S} = \mathcal{S}(\delta, d) \) as in (39) let \( \mathcal{O}^p(n, \delta, d) \) be the full subcategory of \( \mathcal{O}^p(n) \) given by all modules with composition factors \( L(\mu) \) with \( \mu \in \mathcal{S} \). Then

1. \( \mathcal{O}^p(n, \delta, d) \) inherits the structure of a highest weight category from \( \mathcal{O}^p(n) \) with poset \((\mathcal{S}, \leq_{r\text{Bruhat}})\) and the \( V(\lambda) \) for \( \lambda \in \mathcal{S} \). The duality on \( \mathcal{O}^p(n) \) restricts to the duality on the subcategory.
(2) The functor $\mathcal{E} = \text{Hom}_g(M^p(\delta) \otimes V^\otimes d, ?)$ defines an equivalence
\[ \mathcal{E} : \mathcal{O}^p(n, \delta, d) \cong \bigvee_d(\alpha, \beta)^{\text{opp}} - \text{mod} \]
of highest weight categories, i.e. $\mathcal{E}P(\lambda) \cong P(\varphi(\lambda))$, $\mathcal{E}V(\lambda) \cong V(\varphi(\lambda))$, $\mathcal{E}L(\lambda) \cong L(\varphi(\lambda))$ for any $\lambda \in S(\delta, \lambda)$ and $\mathcal{E}$ interchanges the dualities.

**Proof.** The first statement follows directly from the fact, Lemma 4.8, that $\mathcal{S}$ is saturated, see [Don98, Appendix]. The functor $\mathcal{E}$ is an equivalence of categories, hence induces a highest weight structure on $\mathcal{O}_d(\alpha, \beta)^{\text{opp}} - \text{mod}$ and it is enough to show that it agrees with the one defined in [AMR06]. Using Remark 2.2 we can find an isomorphism $\mathcal{W}_d(\alpha, \beta)^{\text{opp}} \cong \mathcal{W}_d(\alpha, \beta)$ which we fix. Theorem 3.1 induces an isomorphism $\text{End}_g(M^p(\delta) \otimes V^\otimes d)^{\text{opp}} \cong \text{End}_g(M^p(\delta) \otimes V^\otimes d)$. Under the identifications $\mathcal{O}_d(n, \delta, d) = \text{End}_g(M^p(\delta) \otimes V^\otimes d)^{\text{opp}} - \text{mod} = \text{End}_g(M^p(\delta) \otimes V^\otimes d) - \text{mod}$, the duality on $\mathcal{O}_d(n, \delta, d)$ corresponds to the ordinary duality $\text{Hom}_{\text{End}_g(M^p(\delta) \otimes V^\otimes d)}(?, \mathbb{C})$ and hence our equivalence interchanges the dualities. Since $\phi$ induces a bijection between the labelling sets of the two highest weight structures, it is enough to show that one ordering refines the other which is done in Lemma 4.12 below. □

**Remark 4.10.** Due to the fact that Verma path always only include diagrammatic weights with the same integrality as $\delta$, $\mathcal{O}_d(n, \delta, d)$ is either a subcategory of $\mathcal{O}^p_1(n)$ or $\mathcal{O}^p_1(n)$ depending on whether $\delta$ is even or odd.

**Definition 4.11.** The dominance ordering on $\mathcal{P}^1$ is defined by
\[ \lambda \succeq \mu \text{ if } |\lambda| > |\mu| \text{ or } |\lambda| = |\mu| \text{ and } \sum_{j=1}^{k} \lambda_j \geq \sum_{j=1}^{k} \mu_j \text{ for any } k \geq 1; \]
and on $\mathcal{P}^2$ by $\lambda = (\lambda^1, \lambda^2) \succeq \mu = (\mu^1, \mu^2)$ if $|\lambda^{(1)}| + |\lambda^{(2)}| > |\mu^{(1)}| + |\mu^{(2)}|$ or $|\lambda^{(1)}| + |\lambda^{(2)}| = |\mu^{(1)}| + |\mu^{(2)}|$, $\lambda^{(1)} \succeq \mu^{(1)}$ and $|\lambda^{(1)}| + \sum_{j=1}^{k} \lambda_j^{(2)} \geq |\mu^{(1)}| + \sum_{j=1}^{k} \mu_j^{(2)}$ for all $k \geq 0$.

The reversed partial ordering on $\mathcal{S}$ is weaker than the dominance ordering on bipartitions:

**Lemma 4.12.** For any $\lambda, \mu \in \mathcal{S}(\delta, d)$ we have
\[ \lambda \leq_{\text{Bruhat}} \mu \Rightarrow \varphi(\lambda) \succeq \varphi(\mu). \]

**Proof.** Recall that the Bruhat ordering is generated by basic swaps, turning two symbols $\wedge \vee$ (in this order with only $\circ$’s and $\times$’s between them) into a $\vee \wedge$ or two $\wedge$’s into two $\vee$’s making the weight smaller. In the first case, the $\wedge$ moves to the right and a $\vee$ moves to the left. The $\wedge$ increases the number of boxes in the first partition, whereas the $\vee$ increases the number of boxes in the first partition or decreases the number of boxes in the second partition. In any case however, the total number of boxes removed is at most the total number of boxes added. Similarly, in case two if two $\wedge$’s turn into two $\vee$’s, the total number of boxes increases or stays the same, but the
number of boxes in the first partition increases. In both cases, the newly created bipartition is larger in the dominance order.

Due to our assumptions on \( n \geq 2d \) and thus the simple structure of the idempotent \( z_d \), the category \( z_d \text{End}_g(M^\delta(\delta) \otimes V^{\otimes d})^{\text{opp}} \mod \) is equivalent to a quotient category of \( \mathcal{O}^p(n, \delta, d) \).

For \( \mathcal{O}_1^p(n) \), resp. \( \mathcal{O}_1^p(n) \), we denote by \( p_\delta \) the projection onto those blocks whose block diagram \( \Theta \) satisfies \( P_\bullet(\Theta) \cup P_\bullet(\Theta) \subset [0, \frac{d}{2} + n - 1] \).

For even \( d \) we put \( \mathcal{F} = p_\delta F p_\delta \), otherwise we put \( \mathcal{F} = p_\delta F p_\delta \). Furthermore let \( P_d = \mathcal{F}^d(M^\delta(\delta)) \). Then \( \text{End}_g(P_d) = z_d \text{End}_g(M^\delta(\delta) \otimes V^{\otimes d})z_d \) and we have the quotient functor

\[
\text{Hom}_g(P_d, ?) : \mathcal{O}(n, \delta, d) \rightarrow z_d \text{End}_g(M^\delta(\delta) \otimes V^{\otimes d})^{\text{opp}} \mod.
\]

We will denote the corresponding quotient category by \( \mathcal{O}^p(n, \delta, d) \).

**Theorem 4.13.** The highest weight structure on \( \mathcal{O}^p(n, \delta, d) \) induces a highest weight structure on \( \mathcal{O}^p(n, \delta, d) \) iff \( \delta \neq 0 \) or \( \delta = 0 \) and \( d \) odd or \( d = 0 \).

**Proof.** A Verma path is compatible with the truncation if any two steps differ by some \( \pm \varepsilon_j \) for \( j < n \). Let \( \mathcal{S}' \) be the set of weights \( \lambda \in \mathcal{S}(\delta, d) \) which appear as endpoints of a compatible Verma path of length \( d \). The paths correspond to up-down-\( d \)-bitableaux where the second component is always the empty partition. Ignoring the second component we identify these paths with up-down-\( d \)-tableaux. For the algebra to be quasi-hereditary is is enough that all indecomposable projective modules indexed by these up-down-\( d \)-tableaux appear as summands in \( P_d \). For \( d = 0 \) there is nothing to check and for \( d = 1 \) the partition with one box corresponds to \( P(\lambda_{\text{max}}) = M^\delta(\delta - \varepsilon_1) \). Now assume the claim is true for \( d - 2 \). Hence all partitions with \( d - 2k \) boxes arise as labels of indecomposable projectives in \( P_{d-2} \). Let now \( \lambda \) be such a partition. If \( \delta > 0 \) then the associated weight diagram has at least one \( \circ \) and hence at least one of the configurations displayed in the bottom line of the following table at positions \( i, i + 1 \) smaller than \( \frac{\delta}{2} + n \):

\[
\begin{array}{ccccccc}
\mu & \wedge & \wedge & \vee & \vee & \wedge & \wedge \\
\lambda & \wedge & \wedge & \vee & \vee & \wedge & \wedge \\
\end{array}
\]

In each line we choose the weight \( \mu \) to be equal to \( \lambda \) except at the positions \( i, i + 1 \) where it is as shown in the table. Then \( \mathcal{F}P(\lambda) \) contains \( P(\mu) \) as direct summand and \( \mathcal{F}P(\mu) \) contains \( P(\lambda) \) as summand. Hence \( P(\lambda) \) appears as a summand in \( P_d \). Now assume \( \lambda \) is a weight corresponding to a partition with \( d \) boxes and \( \delta > 0 \). If \( d > 0 \) then there is at least one box which can be removed from the partition, hence \( \lambda \) contains at least one of the configurations displayed in the top line of the following table at positions \( i, i + 1 \) smaller than \( \frac{\delta}{2} + n \):

\[
\begin{array}{ccccccc}
\lambda & \wedge & \wedge & \vee & \vee & \wedge & \wedge \\
\mu & \wedge & \wedge & \vee & \vee & \wedge & \wedge \\
\end{array}
\]

at \( \frac{\delta}{2} + n \)

\[
\begin{array}{ccccccc}
\lambda & \wedge & \wedge & \vee & \vee & \wedge & \wedge \\
\mu & \wedge & \wedge & \vee & \vee & \wedge & \wedge \\
\end{array}
\]

\[
\begin{array}{ccccccc}
\lambda & \wedge & \wedge & \vee & \vee & \wedge & \wedge \\
\mu & \wedge & \wedge & \vee & \vee & \wedge & \wedge \\
\end{array}
\]
In this case we create by Lemma 3.11, (37) and Corollary 3.12 a weight \( \mu \) whose partition has \( d - 1 \) boxes such that \( \mathcal{F}P(\mu) \) contains \( P(\lambda) \) as direct summand and we are done by induction.

It remains to consider the case \( \delta = 0 \). For \( d = 0, d = 1 \) the assertion is clear. For \( d = 2 \) we have \( P_d = P(a) \oplus P(b) \), where \( a \) and \( b \) correspond to the partitions (2) and (1,1) respectively and \( \text{End}_g(P_2) \cong \mathbb{C}[x] / (x^2) \oplus \mathbb{C} \) which has infinite global dimension and hence is not quasi-hereditary. In case \( \delta = 0 \) and \( d = 0 \) then every weight contains at least one \( \circ \) and we can argue as in the case \( \delta > 0 \). For \( d \) odd observe that the occurring weights \( \lambda \) always contain at least one \( \wedge \) or \( \times \), and \( P_d \) contains a summand isomorphic to \( P_2 \). Hence the algebra is not quasi-hereditary. □

**Remark 4.14.** The truncated algebra \( z_d \text{End}_g(M \otimes V \otimes d)_{\operatorname{opp}} z_d \) inherits always a cellular algebra structure in the sense of [GL96] from \( \text{End}_g(M \otimes V \otimes d)_{\operatorname{opp}} \), see [KX98]. By Theorem 4.13, \( z_d \text{End}_g(M \otimes V \otimes d)_{\operatorname{opp}} z_d \) is quasi-hereditary if and only if \( \delta \neq 0 \) or \( \delta = 0 \) and \( d \) odd.

5. Koszulness and Gradings

The following theorem shows the existence of a hidden grading on the cyclotomic VW-algebras. Recall that any block \( \mathcal{B} \) of ordinary or parabolic category \( \mathcal{O} \) for \( g \) is equivalent to the category of modules over the endomorphism ring \( A = \text{End}_g P_B \) of a minimal projective generator \( P \). By [BGS96], \( A \) has a natural Koszul grading which we fix. The category of finite dimensional graded \( A \)-modules is called the graded version \( \hat{\mathcal{B}} \). For a block of \( \mathcal{O}(n) \), the algebra \( A \) is precisely the Khovanov algebra attached to the block in [ES13b].

**Theorem 5.1.** Let \( d, \delta \in \mathbb{Z}_{\geq 0} \).

1. The algebra \( \Psi_d(\alpha, \beta) \) is Morita equivalent to a Koszul algebra.
2. The algebra \( z_d \Psi_d(\alpha, \beta) z_d \) is Morita equivalent to a Koszul algebra if and only if \( \delta \neq 0 \) or \( \delta = 0 \) and \( d \) odd or \( d = 0 \).

**Proof.** The category \( \mathcal{O}^P(n) \) is a highest weight category and each block is equivalent to a category of finite dimensional modules over a Koszul algebra \( A = \text{End}_g(P) \), [BGS96]. Then \( A \) is standard Koszul by [ÁDL03, Corollary 3.8]. Since the set \( \mathcal{S} \) is saturated, the algebra \( \Psi_d(\alpha, \beta) \) is again standard Koszul by [ÁDL03, Proposition 1.11]. Since it is standard Koszul and quasi-hereditary it is Koszul by [ÁDL03, Theorem 1]. The second statement follows by the same arguments using Theorem 4.13. □

Any indecomposable projective, Verma or simple module in \( \mathcal{B} \) as above has a graded lift in \( \hat{\mathcal{B}} \) and this lift is unique up to isomorphism and overall shift in the grading, [Str03]. We choose a lift such that the head is concentrated in degree zero and denote it by the same symbol as the original module. For an explicit description of these modules for \( \mathcal{O}(n) \) see [ES13b].
Remark 5.2. The Koszul grading on the basic algebra underlying $\mathcal{W}_d(\alpha, \beta)$ or $z_i^k \mathcal{W}_d(\alpha, \beta)$ is unique up to isomorphism. Lemma 3.10 refines then to graded decomposition numbers given by parabolic Kazhdan-Lusztig polynomials of type $(D_n, A_{n-1})$, see [LS13] for explicit formulas and [ES13b] for an explicit description of the underlying basic Koszul algebra.

Lemma 5.3. The special projective functors from Definition 3.7 lift to the graded functors $\hat{F} := (\otimes V) \cong \bigoplus_{i \in \mathbb{Z} \cap [0, \frac{1}{2})} (\hat{F}_{i,-} \otimes \hat{F}_{i,+}) : \hat{O}_\lambda^i(n) \to \hat{O}_\lambda^{i+1}(n)$ and $\hat{F}^i := (\otimes V) \cong \bigoplus_{i \in \mathbb{Z} \cap (\frac{1}{2}, \infty]} (\hat{F}_{i,-} \otimes \hat{F}_{i,+}) \otimes \hat{F}_0 : \hat{O}_\lambda^{i-1}(n) \to \hat{O}_\lambda^{i}(n)$ satisfying now the properties of Lemma 3.8 by replacing the $\mathcal{F}$ by $\hat{F}$. Then $(i)$, $i \in \mathbb{Z}$ denotes the grading shift up by $i$.

Proof. By Remark 3.9 each of these functors is a translation functor to or out of a wall or an equivalence of categories, hence the existence of graded lifts is given by [Str03]. To deduce the formulas note that our normalization is slightly different here, since we shift the translation functor to the wall in [Str03] by $(1)$ and the translation functor out of the wall by $(-1)$. Alternatively one could also mimic the arguments in [BS11b] for the Khovanov algebra of type $D$ from [ES13b].

Note that $A = \text{End}_q(P)$ for a fixed block $O_\lambda(n)$ surjects onto the endomorphism ring $A^q = \text{End}_q(P^q)$ for any parabolic $q$, where $P^q \in O_\lambda^q$ is the maximal locally $q$-finite quotient of $P$. For $q_1 \leq q_2$ we have the inclusion functor $i_{q_1}^q : O_\lambda^{q_2} \to O_\lambda^{q_1}$ and the right exact Zuckermann functor projection functor $(Z_\lambda)^{q_2} : O_\lambda^{q_2} \to O_\lambda^{q_2}$ of taking the maximal quotient in $O_\lambda^{q_2}$. Note that $i$ is just the inclusion of a Serre subcategory and the bimodule $\text{Hom}_O(P^{q_2}, (Z_\lambda)^{q_2} P^q)$ is isomorphic to the bimodule $\text{Hom}_O(P, P)$ where we factor out all morphisms factoring through a projective whose head is not in $O_\lambda^{q_2}$. In particular, with the arguments from [Str03], there are graded versions

$$i_{q_1}^q : O_\lambda^{q_2} \longrightarrow O_\lambda^{q_1}, \quad (Z_\lambda)^{q_2} : O_\lambda^{q_1} \longrightarrow O_\lambda^{q_2}. \quad (40)$$

Given a graded abelian category $A$, the Grothendieck group is a $\mathbb{Z}[q, q^{-1}]$-module where $q^r$ acts by shifting the grading by $+r$. The $\mathbb{Q}(q)$-vector space obtained by extension of scalars is denoted $K_0(A)$. In particular, we have $K_0(O^q(n))$ with the induced action of the graded special projective functors for any parabolic $q$. We describe this now in detail.

6. Coideal subalgebras

6.1. Quantum groups. The following constructions fall again into two families, the even/half-integer family and the odd/integer family. To be able to treat these cases simultaneously as much as possible we introduce the following indexing sets:

$I = \mathbb{Z} + \frac{1}{2}, \quad I^{++} = \{i \in I \mid i \geq \frac{3}{2}\}$, \quad $J = \mathbb{Z}, \quad J^+ = \{j \in J \mid j \geq 1\}$,

$I^i = \mathbb{Z}, \quad I^{i,++} = \{i \in I \mid i \geq 1\}$, \quad $J^i = \mathbb{Z} + \frac{1}{2}, \quad J^{i,+} = \{j \in J \mid j \geq \frac{1}{2}\}$. 


Definition 6.1. Let $U = U_q(\mathfrak{gl}_2)$ be the (generic) quantized universal enveloping algebra for $\mathfrak{gl}_2$. That means $U$ is the associative unital algebra over the field of rational functions $\mathbb{Q}(q)$ in an indeterminate $q$ with generators $D_j^{\pm 1}$ for $j \in J$ and $E_i, F_i$ for $i \in I$ subject to the following relations:

(i) all $D_j^{\pm 1}$ commute and $D_j D_j^{-1} = 1$ for $j \in J$,

(ii) For $i \in I$ and $j \in J$:

\[
D_j E_i D_j^{-1} = \begin{cases} 
q E_i & \text{for } j = i - \frac{1}{2} \\
q^{-1} E_i & \text{for } j = i + \frac{1}{2} \\
E_i & \text{otherwise}
\end{cases} \quad \quad D_j F_i D_j^{-1} = \begin{cases} 
q F_i & \text{for } j = i + \frac{1}{2} \\
q^{-1} F_i & \text{for } j = i - \frac{1}{2} \\
F_i & \text{otherwise}
\end{cases}
\]

(iii) $E_i$ and $F_{i'}$ commute unless $i = i'$ and in this case

\[
E_i F_i - F_i E_i = \frac{D_{i - \frac{1}{2}} D_{i + \frac{1}{2}}^{-1} - D_{i + \frac{1}{2}} D_{i - \frac{1}{2}}^{-1}}{q - q^{-1}}.
\]

(iv) If $|i - i'| = 1$ then

\[
E_i^2 E_{i'} - (q + q^{-1}) E_i E_{i'} E_i + E_{i'} E_i^2 = 0
\]

in all other cases $E_i E_{i'} = E_{i'} E_i$.

(v) Analogously, if $|i - i'| = 1$ then

\[
F_i^2 F_{i'} - (q + q^{-1}) F_i F_{i'} F_i + F_{i'} F_i^2 = 0
\]

in all other cases $F_i F_{i'} = F_{i'} F_i$.

The quantized enveloping algebra $U^i = U_q(\mathfrak{gl}_2)$ is defined in exactly the same way by replacing $J$ with $I'$ and $J$ with $J'$ in the definitions.

In $U$, resp. $U^i$, let $K_i := D_{i - \frac{1}{2}} D_{i + \frac{1}{2}}^{-1}$ for $i \in I$, resp. $i \in I'$. From this definition it follows immediately that

\[
K_i E_i K_i^{-1} = q^2 E_i \quad \text{and} \quad K_i F_i K_i^{-1} = q^{-2} F_i.
\]

Both $U$ and $U^i$ are Hopf algebras with comultiplication $\Delta$ defined on generators by

\[
\Delta(E_i) = K_i \otimes E_i + E_i \otimes 1, \quad \Delta(F_i) = F_i \otimes K_i^{-1} + 1 \otimes F_i,
\]

\[
\Delta(D_j^{\pm 1}) = D_j^{\pm 1} \otimes D_j^{\pm 1}.
\]

Definition 6.2. Let $V$ be the vector space on basis $\{v_l \mid l \in J\}$. The natural representation of $U$ is the representation on $V$ given by the rules

\[
E_i v_l = \delta_{i+\frac{1}{2},l} v_{l-1}, \quad F_i v_l = \delta_{i-\frac{1}{2},l} v_{l+1}, \quad D_j^{\pm 1} v_l = q^{\pm \delta_{i,j}} v_l
\]

for $i \in I$ and $j \in J$. The natural representation $V^i$ of $U^i$ is defined analogously by using $J'$ as the indexing set for the basis; all formulas stay the same.

Remark 6.3. The action on $V$ can be illustrated as follows, showing how the $F_i$’s and $E_i$’s move between weight spaces and on which weight space $D_j$ acts by multiplication with $q$, for $V$.
The fundamental representations of \( \mathfrak{g} \) are given by the definitions:

**Definition 6.4.** The \( k \)-th quantum exterior power \( \wedge^k \mathcal{V} \) is the \( \mathcal{U} \)-submodule of \( \mathfrak{g} \wedge \mathcal{V} \) with basis given by the vectors

\[
v_{i_1} \wedge \cdots \wedge v_{i_k} := \sum_{w \in S_k} (-q)^{\ell(w)} v_{w(1)} \otimes \cdots \otimes v_{w(k)} \tag{41}
\]

for all \( i_1 < \cdots < i_k \) from the index set \( J \). Here, \( \ell(w) \) denotes the usual length of a permutation \( w \) in the symmetric group \( S_k \).

The corresponding module \( \wedge^k \mathcal{V} \) for \( \mathcal{U}^j \) is defined in the same way by again just changing the indexing set for the basis.

**Definition 6.5.** The quantized universal enveloping algebra \( \mathcal{U}_q(\mathfrak{g}_{\mathbb{N}}) \) is defined as the associative unital algebra over the field of rational functions \( \mathbb{Q}(q^{1/2}) \) with the sets of generators

\[
\{ E_i \mid i \in I^{++} \} \cup \{ F_i \mid i \in I^{++} \} \cup \{ D^{+1}_j \mid j \in J^+ \},
\]

which satisfy the same relations as the generators of \( \mathcal{U} \) by identifying \( X_i \in \mathcal{U}_q(\mathfrak{g}_{\mathbb{N}}) \) with \( X_i \) in \( \mathcal{U} \) for \( X \in \{ E, F, D \} \).

Let \( \mathcal{W} \) denote the natural representation of \( \mathcal{U}_q(\mathfrak{g}_{\mathbb{N}}) \) with basis \( \{ w_j \mid j \in J^+ \} \) and consider the \( \mathcal{U}_q(\mathfrak{g}_{\mathbb{N}}) \)-module \( \wedge^r \mathcal{W} \otimes \wedge^k \mathcal{W} \). Each factor has the same type of basis as \( \mathcal{V} \) above. Hence the tensor product has the obvious monomial basis

\[
\left\{ (w_{i_1} \wedge \cdots \wedge w_{i_r}) \otimes (w_{j_1} \wedge \cdots \wedge w_{j_k}) \mid i_1, \ldots, i_r, j_1, \ldots, j_{k-r} \in J^+, i_1 < \cdots < i_r, j_1 < \cdots < j_{k-r} \right\}. \tag{42}
\]

As before we can change the indexing set for the generators of \( \mathcal{U}_q(\mathfrak{g}_{\mathbb{N}}) \) to \( I^{++,+} \) and \( J^{++,+} \) and would obtain an isomorphic algebra with corresponding natural representation \( \mathcal{W}^+ \).

**Remark 6.6.** Although the above definition of \( \mathcal{U}_q(\mathfrak{g}_{\mathbb{N}}) \) gives a natural embedding of Hopf algebras into both \( \mathcal{U} \) and \( \mathcal{U}^j \), we will be interested in another...
copy of $\mathcal{U}_q(\mathfrak{gl}_n)$ inside $\mathcal{U}$ and $\mathcal{U}^i$ given by certain involution invariant generators.

6.2. Coideal subalgebra. We are interested in the quantization of the fixed point Lie algebra $\mathfrak{g}^\theta$ for a certain involution $\theta$. Unfortunately, the obvious construction $\mathcal{U}_q(\mathfrak{g}^\theta)$ does not naturally embed into $\mathcal{U}_q(\mathfrak{g})$. A way around this is to study coideal subalgebras (instead of Hopf subalgebras) of $\mathcal{U}_q(\mathfrak{g})$ which specialize to $\mathfrak{g}^\theta$. This was studied in [Let03] and [Let02].

**Lemma 6.7.** The assignment

$$
E_i \mapsto qK_{-i}F_{-i}, \quad F_i \mapsto q^{-1}E_{-i}K_{-i}^{-1} \\
D_j \mapsto D_j^{-\frac{1}{2}}, \quad K_i \mapsto K_{-i}, \quad q \mapsto q^{-1}
$$

(43)

defines a $q$-antilinear involution $\theta$ on both $\mathcal{U}$ and $\mathcal{U}^i$.

**Proof.** We first note that $\theta^2(X) = X$ for any generator $X$, e.g.

$$
\theta^2(E_i) = \theta(qK_{-i}F_{-i}) = q^{-1}K_iq^{-1}E_iK_i^{-1} = E_i.
$$

To see that it is well-defined we have to verify the relations. They fall into three families of which we check explicitly one each and omit the analogous calculations. Commuting relations:

$$
\theta(D_{i+\frac{1}{2}}E_{i}D_{i-\frac{1}{2}}^{-1}) = D_{i-\frac{1}{2}}D_{i+\frac{1}{2}}^{-1}qK_{-i}F_{-i}D_{i-\frac{1}{2}}^{-1} = q^2K_{-i}F_{-i} = \theta(q^{-1}E_i).
$$

Commutator relations:

$$
\theta(E_iF_i - F_iE_i) = K_{-i}F_{-i}E_{-i}K_{-i}^{-1} - E_{-i}F_{-i}
$$

$$
= F_{-i}E_{-i} - E_{-i}F_{-i} = -\frac{D_{i-\frac{1}{2}}D_{i+\frac{1}{2}}^{-1} - D_{i-\frac{1}{2}}D_{i+\frac{1}{2}}^{-1}}{q - q^{-1}}
$$

$$
= \theta \left( \frac{D_{i+\frac{1}{2}}D_{i-\frac{1}{2}}^{-1} - D_{i+\frac{1}{2}}D_{i-\frac{1}{2}}^{-1}}{q^{-1} - q} \right) \theta \left( \frac{D_{i+\frac{1}{2}}D_{i-\frac{1}{2}}^{-1} - D_{i+\frac{1}{2}}D_{i-\frac{1}{2}}^{-1}}{q - q^{-1}} \right),
$$

Quantum Serre relations:

$$
\theta \left( E_{i+1}^2E_{i+1} - (q + q^{-1})E_iE_{i+1}E_i + E_{i+1}E_i^2 \right)
$$

$$
= q^3K_{-i}F_{-i}K_{-i}K_{-i}F_{-i-1} - (q + q^{-1})q^3K_{-i}F_{-i}K_{-i}F_{-i-1}K_{-i}F_{-i}
$$

$$
+ q^3Q_{-i}F_{-i}K_{-i}F_{-i}K_{-i}F_{-i} = q^3K_{-i}F_{-i} \left( F_{-i}^2F_{-i-1} - (q + q^{-1})F_{-i}F_{-i-1}F_{-i} + F_{-i-1}F_{-i}^2 \right) = 0
$$

The claim follows. 

To study the fixed points of our involution $\theta$, we enlarge the quantum groups $\mathcal{U}$ and $\mathcal{U}^i$ by also including the elements $d_j^\pm_1$ for $j \in J$, resp. $j \in J^i$, satisfying $d_j^2 = D_j$ and corresponding elements $k_i = d_{i+\frac{1}{2}}^{-1}d_{i+\frac{1}{2}}^\pm$. The relations for these elements are the same as for their squares just dividing the powers of $q$ by two, hence we enlarge the base field to $\mathbb{Q}(q^{\frac{1}{2}})$. Denote these algebras by $\tilde{\mathcal{U}}$ and $\tilde{\mathcal{U}}^i$ respectively.
Lemma 6.8. The involution $\theta : \overline{U} \to \overline{U}$ fixes the following elements $\overline{B_i} \in \overline{U}$

\[
\begin{align*}
\overline{B_i} &= q^{-\frac{i}{2}} E_i k_i^{-1} k^{-1}_i + q^{-\frac{i}{2}} F_i k_i^{-1} k^{-1}_i \quad &\text{for } i \neq -\frac{1}{2}, \\
\overline{B_{-\frac{1}{2}}} &= E_{-\frac{1}{2}} k_{-\frac{1}{2}}^{-1} k_{-\frac{1}{2}}^{-1} + q^{-1} F_{\frac{1}{2}} k_{\frac{1}{2}}^{-1} k_{-\frac{1}{2}}^{-1}.
\end{align*}
\]

The involution $\theta$ fixes the following elements $B_i \in \overline{U}^i$

\[
\begin{align*}
\overline{B_i} &= q^{-\frac{i}{2}} E_i k_i^{-1} k^{-1}_i + q^{-\frac{i}{2}} F_i k_i^{-1} k^{-1}_i \quad &\text{for } i \neq 0, \\
\overline{B_0} &= q^{-1} E_0 K_0^{-1} + F_0.
\end{align*}
\]

Proof. This is just a straight-forward calculation. For example one checks $\theta \left( q^{-\frac{i}{2}} E_i k_i^{-1} k^{-1}_i \right) = q^3 K_i F_i k_i^{-1} k^{-1}_i = q^{-\frac{i}{2}} F_i k_i^{-1} k^{-1}_i$. \qed

Note that the definition of the generators of the coideal subalgebras vary slightly in the literature, see [Let03] and [Let02], although the coideals are classified. For our purposes a slight renormalisation of the generators is more suited.

Definition 6.9. We fix the following elements in $U$

\[
\begin{align*}
B_i := q^{\frac{i}{2}} B_i k_i^{-1} k^{-1}_i &= E_i K_i^{-1} + F_i \quad &\text{for } i \neq \pm \frac{1}{2}, \\
B_{\frac{1}{2}} := q^{\frac{1}{2}} B_{\frac{1}{2}} k_{\frac{1}{2}}^{-1} k^{-1}_{\frac{1}{2}} &= E_{\frac{1}{2}} K_{\frac{1}{2}}^{-1} + F_{-\frac{1}{2}}, \\
B_{-\frac{1}{2}} := q B_{-\frac{1}{2}} k_{-\frac{1}{2}}^{-1} k^{-1}_{\frac{1}{2}} &= q E_{-\frac{1}{2}} K_{\frac{1}{2}}^{-1} + F_{\frac{1}{2}}.
\end{align*}
\]

and also the following elements in $U^i$

\[
\begin{align*}
B_i := q^{\frac{i}{2}} B_i k_i^{-1} k^{-1}_i &= E_i K_i^{-1} + F_i \quad &\text{for } i \neq 0, \\
B_0 := \overline{B_0} &= q^{-1} E_0 K_0^{-1} + F_0.
\end{align*}
\]

Definition 6.10. Let $R$ denote the commutative subalgebra of $U$ generated by the elements of the form $(D_j D_{-j})^{\pm 1}$ for all $j \in J$. Analogously we have $R^i$ the commutative subalgebra of $U^i$ generated by all $(D_j D_{-j})^{\pm 1}$ for all $j \in J^i$.

Even though the generators are not invariant under $\theta$ it follows from the definition that they are mapped to their inverse via $\theta$ and thus $R$ and $R^i$ themselves are stable under $\theta$.

Remark 6.11. The subalgebras $R$ and $R^i$ contain the elements of the form $K_i K_i^{-1}$. They are slightly larger than the commutative subalgebra of the coideal subalgebra appearing in [Let03], since we are considering $ \mathfrak{g} \mathfrak{l}$ instead of $\mathfrak{g} \mathfrak{t}$.

Definition 6.12. Let $H$ be the subalgebra of $U$ generated by the $B_i$ for $i \in I$ and the subalgebra $R$. In analogy, let $H^i$ be the subalgebra of $U^i$ generated by the $B_i$ for $i \in I^i$ and the subalgebra $R^i$.

The following lemma is straight-forward and checked on generators.

Lemma 6.13. We have $\Delta(H) \subset H \otimes U$, resp. $\Delta(H^i) \subset H^i \otimes U^i$, hence $H$, resp. $H^i$, is a right coideal subalgebra of $U$, resp. $U^i$. 

Remark 6.14. If we truncate our indexing set at \( m \), we obtain the coideal subalgebra of type \( AIII \) discussed in [Let03, Section 7] specializing to the Lie subalgebra \( \mathfrak{gl}_m \times \mathfrak{gl}_m \subset \mathfrak{gl}_{2m} \) (resp. \( \mathfrak{gl}_m \times \mathfrak{g}l_{m+1} \)). Our renormalisation does not change this fact as we only multiplied with elements of the form \( k_i k_j^{-1} \). See also Section 7 for fixed point Lie algebras.

Although \( \mathcal{H} \) and \( \mathcal{H}^i \) are not quantum groups, they each contain a copy of the quantum group \( \mathcal{U}_q(\mathfrak{gl}_m) \):

Lemma 6.15. The assignment

\[
E_i \mapsto B_i, \quad F_i \mapsto B_{-i}, \quad D_j^{\pm 1} \mapsto (D_j D_{-j})^{\pm 1},
\]

for \( i \in I^+ \) and \( j \in J^+ \) extends to an injective map of algebras from \( \mathcal{U}_q(\mathfrak{g}l_m) \) to \( \mathcal{H} \). We denote the image of this embedding by \( \tilde{\mathcal{U}} \).

Proof. Obviously the images of the \( D_j^{\pm 1} \)'s are pairwise inverse elements that form a commutative subalgebra. Furthermore one checks that

\[
D_j D_{-j} B_i (D_j D_{-j})^{-1} = \begin{cases} q B_i & \text{if } j = i - \frac{1}{2} \\ q^{-1} B_i & \text{if } j = i + \frac{1}{2} \\ B_i & \text{otherwise} \end{cases}
\]

and that the analogous statement for \( B_{-i} \) with inverse \( q \)'s holds as well. Computing the commutator we obtain

\[
[B_i, B_{-i}] = [E_i K_i^{-1}, F_i] + [F_{-i}, E_{-i} K_i^{-1}] = [E_i, F_i] K_i^{-1} + [F_{-i}, E_{-i}] K_i^{-1} = K_i K_{-i}^{-1} - K_{-i} K_i^{-1} \frac{q - q^{-1}}{q - q^{-1}}.
\]

and \([B_i, B_{-i'}] = 0 \) if \( i \neq i' \). It remains to verify the quantum Serre relations, which is a long, but straight-forward calculation and therefore omitted. □

Remark 6.16. By using the indexing set \( I^{++} \) and \( J^{++} \) for \( \mathcal{U}_q(\mathfrak{g}l_{2m}) \) we obtain the same embedding for \( \mathcal{H}^i \) and denote its image by \( \tilde{\mathcal{U}}^i \).

The coideal subalgebras have a Serre type presentation very similar to the quantum groups except that the Serre relations involving the generators \( B_{\pm \frac{1}{2}} \) and \( B_0 \) are slightly modified, see [Let03, Theorem 7.1] for a general statement. In our case we have:

Proposition 6.17. The coideal subalgebra \( \mathcal{H} \) is isomorphic to the \( \mathbb{Q}(q) \)-algebra \( \tilde{\mathcal{H}} \) with generators

\[
\{ \tilde{E}_i, \tilde{F}_i | i \in I^{++} \} \cup \{ \tilde{B}_+, \tilde{B}_- \} \cup \{ \tilde{D}_j^{\pm 1} | j \in J^+ \} \cup \{ \tilde{D}_0^{\pm 1} \},
\]

subject to the following relations

1. The \( \tilde{D}_j^{\pm 1} \) generate a subalgebra isomorphic to \( \tilde{\mathcal{R}} \).
2. The \( \tilde{E}_i, \tilde{F}_i, \) and \( \tilde{D}_j^{\pm 1} \) for \( i \in I^{++} \) and \( j \in J^+ \) generate a subalgebra isomorphic to \( \mathcal{U}_q(\mathfrak{g}l_{2m}) \).
3. \( \tilde{D}_0 \) commutes with \( \tilde{E}_i \) and \( \tilde{F}_i \) for all \( i \in I^{++} \), while \( \tilde{B}_+ \) and \( \tilde{B}_- \) commute with all other generators except for the following relations.
subject to the following relations

Proposition 6.18. The coideal subalgebra

(a) Commutation relations:

\[ \tilde{D}_1 \tilde{B}_+ \tilde{D}_1^{-1} = q^{-1} \tilde{B}_+ , \quad \tilde{D}_1 \tilde{B}_- \tilde{D}_1^{-1} = q \tilde{B}_- \]

\[ \tilde{D}_0 \tilde{B}_+ \tilde{D}_0^{-1} = q^2 \tilde{B}_+ , \quad \tilde{D}_0 \tilde{B}_- \tilde{D}_0^{-1} = q^{-2} \tilde{B}_- \]

(b) Quantum Serre relations:

\[ \tilde{B}_2^2 \tilde{E}_2 = (q + q^{-1}) \tilde{B}_+ \tilde{E}_2 + \tilde{E}_2 \tilde{B}_2^2 = 0, \]

\[ \tilde{B}_2^2 \tilde{F}_2 = (q + q^{-1}) \tilde{B}_- \tilde{F}_2 + \tilde{F}_2 \tilde{B}_2^2 = 0, \]

\[ \tilde{E}_2^2 \tilde{B}_+ = (q + q^{-1}) \tilde{E}_2 \tilde{B}_+ + \tilde{B}_+ \tilde{E}_2^2 = 0, \]

\[ \tilde{F}_2^2 \tilde{B}_- = (q + q^{-1}) \tilde{F}_2 \tilde{B}_- + \tilde{B}_- \tilde{F}_2^2 = 0, \]

(c) Modified quantum Serre relations:

\[ \tilde{B}_2^2 \tilde{B}_- = (q + q^{-1}) \tilde{B}_- \tilde{B}_+ + \tilde{B}_+ \tilde{B}_2^2 = 0, \]

\[ = -(q + q^{-1}) \tilde{B}_- (q \tilde{D}_1^{-1} \tilde{D}_0 + q^{-1} \tilde{D}_1 \tilde{D}_0^{-1}), \]

\[ \tilde{B}_2^2 \tilde{B}_+ = (q + q^{-1}) \tilde{B}_+ \tilde{B}_- + \tilde{B}_- \tilde{B}_2^2 = 0, \]

\[ = -(q + q^{-1}) \tilde{B}_+ (q^2 \tilde{D}_1 \tilde{D}_0^{-1} + q^{-2} \tilde{D}_1^{-1} \tilde{D}_0). \]

Proof. Up to renormalization these are the relations from [Let03] for the coideal subalgebra of type AIII. The isomorphism is then given by \( \tilde{E}_i \mapsto B_i, \tilde{F}_i \mapsto B_+, \tilde{D}_j \mapsto (D_j D_{-j})^{+1} \), and \( \tilde{B}_+ \mapsto B_+, \tilde{B}_- \mapsto B_- \), \( \tilde{D}_0 \mapsto D_0 \).

An analogous statement also holds for \( \mathcal{H}' \).

Proposition 6.18. The coideal subalgebra \( \mathcal{H}' \) is isomorphic to the \( \mathbb{Q}(q) \)-algebra \( \mathcal{H}' \) with generators

\[ \{ \tilde{E}_i, \tilde{F}_i | i \in I^{i+} \} \cup \{ \tilde{B} \} \cup \{ \tilde{D}_j^{\pm 1} | j \in J^{i+} \}, \]

subject to the following relations

1. The \( \tilde{D}_j^{\pm 1} \) generate a subalgebra isomorphic to \( \mathcal{R}' \).
2. The \( \tilde{E}_i, \tilde{F}_i \), and \( \tilde{D}_j^{\pm 1} \) for \( i \in I^{i+} \) and \( j \in J^{i+} \) generate a subalgebra isomorphic to \( \mathcal{U}_q(\mathfrak{gl}_n) \).
3. The generator \( \tilde{B} \) commutes with all other generators except for the following relations:

\[ \tilde{E}_1^2 \tilde{B} - (q + q^{-1}) \tilde{E}_1 \tilde{B} \tilde{E}_1 + \tilde{B} \tilde{E}_1^2 = 0, \]

\[ \tilde{F}_1^2 \tilde{B} - (q + q^{-1}) \tilde{F}_1 \tilde{B} \tilde{F}_1 + \tilde{B} \tilde{F}_1^2 = 0, \]

\[ \tilde{B} \tilde{E}_1 - (q + q^{-1}) \tilde{B} \tilde{E}_1 + \tilde{E}_1 \tilde{B}^2 = \tilde{E}_1, \]

\[ \tilde{B}^2 \tilde{F}_1 - (q + q^{-1}) \tilde{B} \tilde{F}_1 + \tilde{F}_1 \tilde{B}^2 = \tilde{F}_1. \]
6.3. Action on $\wedge^n V$. In [BS11a], a calculus of diagrammatic weights and diagrammatic algebras was introduced to describe maximal parabolic versions of the BGG-category $O$ in type $A$. In particular, the Grothendieck group of the maximal parabolic category $O$ was identified with the $\mathbb{Q}(q)$-vector space over the basis of diagrammatic weights which in turn was identified with a module for the quantum group $U_q(\mathfrak{g}l_n)$. Here we like to develop an analogous picture (based on [ES13b]). The first step is to decompose specific $U$-modules with respect to $\tilde{U}$.

For a sequence $i = (i_1 < \ldots < i_r)$ of strictly decreasing integers we denote by $i_+$ the subsequence of strictly positive numbers, by $i_-$ the subsequence of strictly negative numbers and by $i_0$ the subsequence consisting of the entry equal to 0 if they exists, otherwise they are empty. Furthermore we denote by $-i$ the sequence $(-i_r < \ldots < -i_1)$.

**Lemma 6.19.** There is an isomorphism of $U_q(\mathfrak{g}l_n)$-modules

$$\Phi : \bigwedge^n V \longrightarrow \bigoplus_{r=0}^n \left( \bigwedge^r \mathbb{W} \otimes \bigwedge^{n-r} \mathbb{W} \right) \oplus \bigoplus_{r=0}^{n-1} \left( \bigwedge^r \mathbb{W} \otimes \mathbb{Q}(q) \otimes \bigwedge^{n-1-r} \mathbb{W} \right)$$

where $U_q(\mathfrak{g}l_n)$ acts on the left via the isomorphism with $\tilde{U}$ and $\mathbb{Q}(q)$ is the trivial representation with basis $\{w_0\}$. The map is given by

$$\Phi(v_i) = \begin{cases} w_{-i_+} \otimes w_{i_0} \otimes w_{i_-} & \text{if } i \neq \emptyset, \\ w_{-i_+} \otimes w_{i_-} & \text{otherwise.} \end{cases}$$

Similarly, there is an isomorphism of $U_q(\mathfrak{g}l_n)$-modules

$$\Phi^i : \bigwedge^i V \longrightarrow \bigoplus_{r=0}^n \left( \bigwedge^r \mathbb{W}^i \otimes \bigwedge^{n-r} \mathbb{W}^i \right),$$

where $U_q(\mathfrak{g}l_n)$ acts on the left via the isomorphism with $\tilde{U}^i$ and the map is given by $\Phi^i(v_i) = w_{-i_+} \otimes w_{i_-}$.

**Proof.** We will omit the proof since the second isomorphism is a special case of Lemma 6.20 and the first completely analogous. \qed

The following more general decomposition will be used in Section 7.

**Lemma 6.20.** Let $r \in \mathbb{Z}_{\geq 1}$ and $k_1, \ldots, k_r \in \mathbb{Z}_{\geq 1}$. We have an isomorphism of $U_q(\mathfrak{g}l_n)$-modules

$$\Phi^i : \bigwedge^{k_1} \mathbb{W}^i \otimes \ldots \otimes \bigwedge^{k_r} \mathbb{W}^i \longrightarrow \bigoplus_{0 \leq s_1 \leq k_1} \left( \bigwedge^{k_1-s_1} \mathbb{W}^i \otimes \ldots \otimes \bigwedge^{k_r-s_r} \mathbb{W}^i \otimes \bigwedge^{s_1} \mathbb{W}^i \otimes \ldots \otimes \bigwedge^{s_r} \mathbb{W}^i \right),$$

where $U_q(\mathfrak{g}l_n)$ acts on the left via the isomorphism with $\tilde{U}^i$ followed by the comultiplication of $\tilde{U}^i$ and the map is given by

$$\Phi^i(v_{k_1} \otimes \ldots \otimes v_{k_r}) = w_{-i_r-} \otimes \ldots \otimes w_{-i_+} \otimes w_{i_-} \otimes \ldots \otimes w_{i_+}.$$
since the arguments for the others are parallel. On the left hand side, $E_a$ acts via the isomorphism, which maps $E_a$ to $B_a$ followed by successive application of the comultiplication. Hence $E_a$ acts by multiplication with

$$\sum_{l=0}^{r-1} (K_aK_a^{-1})^{(l)} \otimes E_aK_a^{-1} \otimes (K_a^{-1})^{(r-1-l)} + \sum_{l=0}^{r-1} 1^{(l)} \otimes F_{-a} \otimes (K_a^{-1})^{(r-1-l)}.$$ 

If we denote by $d(1,a)$ and $d(1,-a)$ the integers such that

$$K_av_1 = q^{d(1,a)}v_1, \quad K_{-a}v_1 = q^{-d(1,-a)}v_1,$$

for a vector $v = v_1 \otimes \ldots \otimes v_r$, then we get

$$E_av = \sum_{l=0}^{r-1} 1^{(l)} \otimes E_a \otimes 1^{(r-l)} \cdot v \cdot q^{d(1,-a)+\ldots+d(1,-a)}q^{d(1,a)+\ldots+d(i,a)}$$

$$+ \sum_{l=0}^{r-1} 1^{(l)} \otimes F_{-a} \otimes 1^{(r-l)} \cdot v \cdot q^{d(1,-a)+\ldots+d(1,-a)}.$$ 

On the other hand $w := \Phi^i(v) = w_{-i_1} \otimes \ldots \otimes w_{-i_r} \otimes w_{i_1} \otimes \ldots \otimes w_{i_r}$. Since

$$K_aw_{i_1} = q^{d(1,a)}w_{i_1}, \quad K_aw_{-i_1} = q^{d(1,-a)}w_{-i_1},$$

for $K_a \in \mathcal{U}_q(gl_n)$ and any sequence $i$ we obtain

$$E_aw = \sum_{l=0}^{r-1} 1^{(l)} \otimes E_a \otimes 1^{(2r-l)} \cdot w \cdot q^{d(-i_1,-a)+\ldots+d(-i_{r+1},-a)}$$

$$+ \sum_{l=r}^{2r-1} 1^{(l)} \otimes E_a \otimes 1^{(2r-l)} \cdot w \cdot q^{d(1,-a)+\ldots+d(1,a)}q^{d(1,a)+\ldots+d(i,-a)}$$

$$= \sum_{l=0}^{r-1} 1^{(r-l)} \otimes E_a \otimes 1^{(r+l)} \cdot w \cdot q^{d(1,-a)+\ldots+d(1,-a)}q^{d(1,a)+\ldots+d(i,a)}$$

$$+ \sum_{l=0}^{r-1} 1^{(r+l)} \otimes E_a \otimes 1^{(r-l)} \cdot w \cdot q^{d(1,-a)+\ldots+d(1,-a)}q^{d(1,a)+\ldots+d(i,a)}.$$ 

In the last step we just reordered the first sum and shifted the indexing of the second. In particular, $\Phi^i$ commutes with the action of $E_a$. Similar calculations for the other generators imply that $\Phi^i$ is a $\mathcal{U}_q(gl_n)$-module homomorphism. 

Using the identification of $K_0(\hat{O}_p^n(n))$, resp. $K_0(\hat{O}_p^n(n))$, with the $\mathbb{Q}(q)$-vector spaces having $X_n$, resp. $X'_n$, as basis we can identify our modules with these spaces. 

**Integer case:** Consider the $\mathcal{U}_q(gl_n)$-module $\bigwedge^n V$. To a standard basis vector $v_i$ we can assign a diagrammatic weight in $X_n$ via defining the sets $P_+(\hat{i}) = \{ij \mid i_j > 0\}$, $P_-(\hat{i}) = \{-ij \mid i_j < 0\}$ and $P_0(\hat{i}) = \{i_j \mid i_j = 0\}$. The map

$$\Gamma: \bigwedge^n V \rightarrow K_0(\hat{O}_p^n(n)),$$
defined on bases vectors by

\[
\Gamma(v_i)_l := \begin{cases} 
\lor l \in P_i(\lambda) \setminus P_n(\lambda), \\
\land l \in P_n(\lambda) \setminus P_i(\lambda), \\
\circ l \in P_i(\lambda), \\
\times l \in P_n(\lambda) \setminus P_i(\lambda), \\
\circ l \notin P_i(\lambda) \cup P_n(\lambda) \cup P_i(\lambda).
\end{cases}
\]

yields an isomorphism of vector spaces. For a diagrammatic weight \(\lambda\) we denote the corresponding standard basis vector by \(v_\lambda\).

**Half-integer case:** Consider the \(U_q(\mathfrak{gl}_n)\)-module \(\bigwedge^n \mathcal{V}^i\). To a standard basis vector \(v_\lambda\) we again assign a diagrammatic weight, this time in \(x, y\) symbols. Let \(\text{Lemma 6.22}\). Defined on basis vectors by \(U\) of \(\mathcal{H}\) of the categories underlying these vector spaces, but we will see that the basis vector \(v\) yields an isomorphism of vector spaces. Again we denote the preimage of a diagrammatic weight \(\lambda\) by \(v_\lambda\).

Using these two isomorphisms \(\Gamma\) and \(\Gamma'\) we can transfer the action of \(U\) and \(U'\) to the vector spaces of diagrammatic weights. In general the generators of \(U\) and \(U'\) will not act in a way that is compatible with the structure of the categories underlying these vector spaces, but we will see that the generators of \(\mathcal{H}\) and \(\mathcal{H}'\) act in a way that can be interpreted categorically as the induced action of translation functors. The following Lemmas are all proved by a straightforward case by case calculation.

**Lemma 6.21.** Let \(\lambda\) be a diagrammatic weight, then

\[
D_j D_{-j} \lambda = \begin{cases} 
\lambda & \text{if } \lambda_j = \circ, \\
q \lambda & \text{if } \lambda_j \in \{\lor, \land\}, \\
q^2 \lambda & \text{if } \lambda_j = \times.
\end{cases}
\]

**Lemma 6.22.** Let \(\lambda\) be a diagrammatic weight in \(\mathcal{X}_n\) and \(i \in \Gamma^+\). For symbols \(x, y \in \{\circ, \land, \lor, \times\}\) we write \(\lambda_{xy}\) for the diagrammatic weight obtained from \(\lambda\) by relabelling the \((i - \frac{1}{2})\)th entry to \(x\) and the \((i + \frac{1}{2})\)th entry to \(y\).

1. We first consider the action of \(B_i\):
   (i) If \(\lambda = \lambda_{xy}\), then \(B_i \lambda = \lambda_{x0}\).
   (ii) If \(\lambda = \lambda_{ox}\), then \(B_i \lambda = \lambda_{o0}\).
   (iii) If \(\lambda = \lambda_{xx}\), then \(B_i \lambda = \lambda_{x0}\).
   (iv) If \(\lambda = \lambda_{vx}\), then \(B_i \lambda = \lambda_{x0}\).
   (v) If \(\lambda = \lambda_{xx}\), then \(B_i \lambda = q \lambda_{x0}\).
   (vi) If \(\lambda = \lambda_{ox}\), then \(B_i \lambda = q^{-1} \lambda_{ox} + \lambda_{vx}\).
   (vii) If \(\lambda = \lambda_{ox}\), then \(B_i \lambda = q^{-1} \lambda_{ox} + \lambda_{vx}\).


\begin{proof}

This follows by comparing the action as described in Lemma 6.22. Proposition 6.25.

On the action of translation functors we obtain the following result.

Lemma 6.24. (1) The element $1$ and $\widehat{1}$ at diagrammatic weights in $X$ is

\begin{itemize}
  \item[(i)] If $\lambda = \lambda_\wedge$, then $B_{-i}.\lambda = \lambda_\wedge$.
  \item[(ii)] If $\lambda = \lambda_\vee$, then $B_{-i}.\lambda = \lambda_\vee$.
  \item[(iii)] If $\lambda = \lambda_\wedge$, then $B_{-i}.\lambda = \lambda_\wedge$.
  \item[(iv)] If $\lambda = \lambda_\vee$, then $B_{-i}.\lambda = q\lambda_\vee$.
  \item[(v)] If $\lambda = \lambda_\wedge$, then $B_{-i}.\lambda = \lambda_\wedge$.
  \item[(vi)] If $\lambda = \lambda_\vee$, then $B_{-i}.\lambda = q^{-1}\lambda_\vee + \lambda_\wedge$.
  \item[(vii)] For all other $\lambda$ we have $B_{-i}.\lambda = 0$.
\end{itemize}

By comparing the actions of $B_i$ on diagrammatic weights with the induced action of translation functors we obtain the following result.

Proposition 6.25. On $K_0\left(\hat{\mathcal{O}}_\ell^p(n)\right)$, the induced action of $B_{\pm i}$ and $B_{\pm \frac{1}{2}}$ on $K_0$ via $\Gamma$ coincides with the action of $[\mathcal{F}_{i,\pm}]$ and $[\mathcal{F}_{\pm \frac{1}{2},n}]$ respectively. On $K_0\left(\hat{\mathcal{O}}_\ell^p(n)\right)$, the induced action of $B_{\pm i}$ and $B_0$ on $K_0$ via $\Gamma^i$ coincides with the action of $[\mathcal{F}_{i,\pm}]$ and $[\mathcal{F}_{\pm \frac{1}{2},\pm}]$ respectively.

Proof. This follows by comparing the action as described in Lemma 6.22 and Lemma 6.24 with the description of the action of the special translation functors $\mathcal{F}_{i,\pm}$ on Verma modules in Lemma 3.8. \hfill \qed

\end{proof}
6.4. Bar involution and dual canonical basis. We start by defining an involution on $\mathcal{H}$ and $\mathcal{H}^t$. In generalizes Lusztig’s bar involution to coideals.

**Lemma 6.26.** There is a unique $\mathbb{Q}(q)$-antilinear bar-involution $-\colon \mathcal{H} \to \mathcal{H}$ such that $B_i = B_i$ and $D_j D^{-1}_j = D^{-1}_j D^{-1}_j$; similar for $\mathcal{H}^t$.

**Proof.** Using Proposition 6.17 one checks easily that such an assignment extends to an algebra involution. \hfill $\square$

**Definition 6.27.** By a compatible bar-involution on an $\mathcal{H}$-module $M$ we mean an anti-linear involution $-\colon M \to M$ such that $uv = vu$ for each $u \in \mathcal{H}, v \in M$ and analogously for a $\mathcal{H}^t$-module.

Note that as a $\mathcal{U}$-module $V$ has one-dimensional weight spaces with all standard basis vectors being weight vectors. The next lemma shows that the module $\Lambda^n V$ possesses a compatible bar-involution. As our identification of the action of $\mathcal{H}$ and the translation functors already indicates, we will need to focus on vectors $w_{\lambda}$ that correspond to a diagrammatic weight $\lambda$. Recall that for two diagrammatic weights in the same block $\Theta$ we have the Bruhat order which gives a unique minimal element in each block. As usual, we start with the case of diagrammatic weights supported on the integers.

**Lemma 6.28.** There is a unique compatible bar-involution on the $\mathcal{H}$-module $\Lambda^n V$ such that $w_{\lambda} = w_{\lambda}$ for each diagrammatic weight $\lambda \in \mathcal{X}$ that is minimal in its block with respect to the Bruhat order. Moreover:

$$\overline{w_{\lambda}} \in w_{\lambda} + \sum_{\mu < \lambda} q^{-1} \mathbb{Z}[q^{-1}] w_{\mu},$$

for any $\lambda \in \mathcal{X}$.

**Proof.** We prove uniqueness, the existence follows below from Lemma 6.30. For any diagrammatic weight $\lambda$ let $l(\lambda)$ denote the height of the weight in the reversed Bruhat order, i.e. the minimal number of changes that need to be made to obtain the minimal element as described in [ES13b, Lemma 2.3].

Enough to show: $\overline{w_{\lambda}}$ is uniquely determined by the properties. For $\lambda$ minimal $\overline{w_{\lambda}} = w_{\lambda}$, hence it is determined. So let us fix $\lambda$ non-minimal and assume that $\overline{w_{\lambda}}$ is uniquely determined for all $\mu$ with $l(\mu) < l(\lambda)$. Note that this includes diagrammatic weights in other blocks.

**Case I:** Assume there exists $i < j$ such that $\lambda_i = \land$ and $\lambda_j = \lor$ and $\lambda_s \in \{\circ, \times\}$ for $i < s < j$. Let $\lambda'$ be the diagrammatic weight that coincides with $\lambda$ except for $\lambda'_{i+1} = \lor$ and $\lambda'_{s} = \lambda_{s-1}$ for $i+1 < s \leq j$, i.e. we have moved the $\lor$ at position $j$ next to the $\land$ at position $i$. By Lemma 6.22 it follows that there exists $B \in \mathcal{H}$ such that $Bw_{\lambda'} = w_{\lambda}$. Furthermore let $\mu$ be the diagrammatic weight that coincides with $\lambda'$ except the positions $i$ and $i+1$ are swapped, then $l(\mu) < l(\lambda') = l(\lambda)$. Then it follows again by Lemma 6.22 that

$$B_{-i - \frac{1}{2}} B_{i + \frac{1}{2}} w_{\mu} = q^{-1} w_{\mu} + w_{\lambda'}.$$
This implies 
\[ \overline{w_\lambda} = BB_{i-1}B_i w_\mu - qBw_\mu. \]
Hence \( \overline{w_\lambda} \) is uniquely determined.

**Case II:** Assume there is no pair \( i < j \) as in Case I, then there exists \( i < j \) such that \( \lambda_i \in \{\vee, \circ\} \) and \( \lambda_j = \vee \) and choose \( i \) and \( j \) minimal with this property. As before we can always assume that \( j = i + 1 \) since there can be no \( \wedge \)'s between \( i \) and \( j \) by assumption.

If \( i = 0 \) let \( \mu \) be the weight sequence coinciding with \( \lambda \) except \( \mu_1 = \wedge \). Then by Lemma 2
\[ B_{1/2}B_{-1/2}(w_\mu) = q^{-1}w_\mu + w_\lambda. \]
Again this implies
\[ \overline{w_\lambda} = B_{1/2}B_{-1/2}(\overline{w_\mu}) - q\overline{w_\mu}. \]

If on the other hand \( i \neq 0 \) then move the symbols at position \( i \) and \( i+1 \) to position 0 and 1, denote the corresponding weight by \( \eta \). By the calculation above we already know
\[ \overline{w_\eta} = B_{1/2}B_{-1/2}(\overline{w_\mu}) - q\overline{w_\mu}, \]
where \( \mu \) coincides with \( \eta \) except that \( \mu_1 = \wedge \). We can now use the element \( B' \in \mathcal{H} \) that first moves the symbols at position 1 back to position \( i+1 \) and afterwards the symbol at position 0 back to position \( i \) and obtain
\[ B'\overline{w_\eta} = B' B_{1/2}B_{-1/2}(\overline{w_\mu}) - qB'\overline{w_\mu}, \]
since \( l(\mu) < l(\lambda) \) the right hand side is known, but
\[ B'\overline{w_\eta} = \overline{w_\lambda} + \overline{w_\lambda'}, \]
with \( \lambda' \) coinciding with \( \lambda \) except \( \lambda'_{i+1} = \wedge \). Since \( l(\lambda') = l(\lambda) \) we can use Case I to deduce that \( \overline{w_\lambda'} \) is already determined and hence \( \overline{w_\lambda} \) is uniquely determined. \( \square \)

**Lemma 6.29.** There is a unique compatible bar-involution on the \( \mathcal{H} \)-module \( \wedge^n V^i \) such that \( \overline{w_\lambda} = w_\lambda \) for each diagrammatic weight \( \lambda \in X^i \) that is minimal in its block with respect to the Bruhat order. Moreover:
\[ \overline{w_\lambda} \in w_\lambda + \sum_{\mu < \lambda} q^{-1}Z[q^{-1}]w_\mu, \]
for any \( \lambda \in X^i \).

**Proof.** The uniqueness of a bar-involution on the \( \mathcal{H} \)-module \( \wedge^n V^i \) follows with similar arguments to Lemma 6.28, although they are slightly simpler because the special generator \( B_0 \) behaves nicer. The details are left to the reader. \( \square \)

For the existence of a bar-involution with these properties we use that \( \wedge^n V \cong K_0(\mathcal{O}_1^p(n)) \) and that there is a graded duality on \( \mathcal{O}_1^p(n) \) satisfying these properties.
Proposition 6.30. The graded duality \( d \) on \( \hat{O}^p(n) \), sending a module \( M \) to \( M^\circ \), induces a compatible bar-involution on \( \wedge^n V \), satisfying the properties of Lemma 6.28. The same also holds for the graded duality on \( \hat{O}^p(n) \) and the induced involution on \( \wedge^n V^i \).

Proof. Let \( M(\lambda) \) be a parabolic Verma module corresponding to a diagrammatic weight being minimal in its block. Then \( M(\lambda) = L(\lambda) \) and hence
\[
d M(\lambda) = d L(\lambda) = L(\lambda) = M(\lambda).
\]
If on the other hand \( M(\lambda) \) is a Verma module corresponding to an arbitrary diagrammatic weight, we have that \( d M(\lambda) = M(\lambda)^\circ = \nabla(\lambda) \), the dual parabolic Verma module by definition. It is known, see [Str03] that
\[
\nabla(\lambda) \in [M(\lambda)] + \sum_{\mu < \lambda} q^{-1} Z[q^{-1}][M(\mu)].
\]

It remains to show that \( d \) is compatible with the action of \( \mathcal{H} \).

Since the simple modules form a basis of the Grothendieck group it is enough to check the compatibility on those, i.e. we need to show that for any simple \( \mathcal{H} \)-module \( L(\lambda) \) we have
\[
[F L(\lambda)] = [F d L(\lambda)] = [d F L(\lambda)],
\]
where the first equality is due to the fact that simple modules are self-dual.

Thus we need to show that \([F L(\lambda)]\) is self-dual as well. We will focus on \( F_{i,+} \) and leave the other cases to the reader. Using Lemma 3.8 we know that
\[
F_{i,+} L(\lambda_{\circ v}) = L(\lambda_{v0}) \quad F_{i,+} L(\lambda_{0\circ}) = 0,
\]
\[
F_{i,+} L(\lambda_{\wedge x}) = L(\lambda_{x\circ}) \quad F_{i,+} L(\lambda_{x\circ}) = 0,
\]
\[
F_{i,+} L(\lambda_{\circ v}) = 0 \quad F_{i,+} L(\lambda_{v\circ}) = L(\lambda_{x0}).
\]

All of these are obviously self-dual and we are left with the case where \( F_{i,+} \) is applied to a weight \( \lambda = \lambda_{0\circ} \). This is done by showing that
\[
[F_{i,+} L(\lambda_{0\circ})] = q [L(\lambda_{v\circ})] + [L(\lambda_{x\circ})] + q^{-1} [L(\lambda_{v\circ})],
\]
which is self-dual. Let \( Y \) be a set of diagrammatic weights and \( r_\mu \) and \( s_\mu \) integers such that
\[
[L(\lambda_{0\circ})] = \sum_{\mu \in Y} (-1)^{r_\mu} q^{s_\mu}[M(\mu)],
\]
then for all these \( \mu \) we have \( \mu = \mu_{0\circ} \). Applying Lemma 6.22 we obtain
\[
[F_{i,+} L(\lambda_{0\circ})] = \sum_{\mu \in Y} (-1)^{r_\mu} q^{s_\mu-1}[M(\mu_{v\circ})] + (-1)^{r_\mu} q^{s_\mu}[M(\mu_{x\circ})]
\]
\[
= q^{-1} \sum_{\mu \in Y} (-1)^{r_\mu} q^{s_\mu}[M(\mu_{v\circ})]
\]
\[
+ \sum_{\mu \in Y} (-1)^{r_\mu} q^{s_\mu}[M(\mu_{x\circ})] - q^{s_\mu+1}[M(\mu_{v\circ})]
\]
\[
+ q \sum_{\mu \in Y} (-1)^{r_\mu} q^{s_\mu}[M(\mu_{v\circ})].
\]
Expressing $[L(\lambda^\lor)]$ and $[L(\lambda^\land)]$ in terms of Verma modules one checks that this is equal to $(q + q^{-1})[L(\lambda^\lor)] + [L(\lambda^\land)]$.

Note that the isomorphism classes of simple modules concentrated in degree zero form a canonical basis of $\bigwedge^n \mathbb{V} \cong K_0(\overline{\mathcal{O}}^F(n))$, that is they are bar-invariant and the transformation matrix to the standard basis is lower diagonal with 1’s in the diagonal and elements in $\mathbb{Z}[q]$ below the diagonal. These entries are the parabolic Kazhdan-Lusztig polynomials of type $(D_n, A_{n-1})$.

7. Skew Howe duality

In the following we want to look at more general versions of parabolic category $\mathcal{O}$ of type $D$ and correspondingly categorifications of tensor products as in Lemma 6.20 for $\mathcal{U}(\mathfrak{gl}_2)$ and $\mathcal{H}$. Since the quantum parameter in these cases is very technical we will work first in the classical case and consider categorifications of $\mathfrak{gl}_r \times \mathfrak{gl}_r$-modules, instead of $\mathcal{H}$-modules.

7.1. The module $\bigwedge(n,m,r)$ in the classical case. For a positive integer $t$ we denote by $\mathbb{W}_t$ the vector space of dimension $t$ with basis $v_1, \ldots, v_{t-\frac{1}{2}}$, with basis elements labelled by half-integers. Similarly we denote by $\mathbb{V}_{2k}$ the vector space of dimension $2t$ with basis $v_{-t+\frac{1}{2}}, \ldots, v_{t-\frac{1}{2}}$. Finally denote by $\mathbb{M}$ the 2-dimensional vector space with basis $v_+$ and $v_-$. All of these will be considered as complex vector spaces.

In the following fix $m$ and $r$ positive integers. We want to consider module structures on the vector space $\bigwedge(n,m,r) := \bigwedge^n (\mathbb{W}_m \otimes \mathbb{M} \otimes \mathbb{W}_r)$.

Since one can identify $\mathbb{W}_n \otimes \mathbb{M} \cong \mathbb{V}_{2m}$ and $\mathbb{M} \otimes \mathbb{W}_n \cong \mathbb{V}_{2r}$ we can view $\bigwedge(n,m,r)$ naturally both as a $(\mathfrak{gl}_{2m}, \mathfrak{gl}_r)$-bimodule and a $(\mathfrak{gl}_m, \mathfrak{gl}_{2r})$-bimodule.

7.2. Fixed point subalgebra. We first identify the fixed point subalgebra of $\mathfrak{gl}_{2m}$ which is the non-quantized analogue of $\mathcal{H}$. The classical analogue of the involution from Lemma 6.7 is $\Theta : \mathfrak{gl}_{2m} \to \mathfrak{gl}_{2m}$ defined as $A \mapsto \bar{J}A\bar{J}$, where $\bar{J} = \begin{pmatrix} 0 & J \\ J & 0 \end{pmatrix}$ with $J$ as in Section 2. i.e has 1’s on the anti-diagonal and zeros elsewhere. Then

$$\mathfrak{g}^\Theta = \{ X \in \mathfrak{gl}_{2m} \mid \bar{J}X\bar{J} = X \} \text{ with } \bar{J} = \begin{pmatrix} 0 & J \\ J & 0 \end{pmatrix},$$

We have $\mathfrak{g}^\Theta \cong \mathfrak{gl}_m \times \mathfrak{gl}_m$, since it is the image of the involution

$$T_A : \mathfrak{gl}_{2m} \longrightarrow \mathfrak{gl}_{2m}, X \mapsto AXA^{-1} \text{ with } A = \begin{pmatrix} 1 & J \\ J & -1 \end{pmatrix},$$
where $1$ is the $m \times m$ identity matrix. Restricted to $\mathfrak{gl}_m \times \mathfrak{gl}_m$, let $\mathfrak{gl}_m^+$ and $\mathfrak{gl}_m^-$ denote the images of the first respectively second factor. One easily verifies, using $\tilde{J} = A \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $A \tilde{J} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} A$, that

$$\mathfrak{gl}_m^+ = \left\{ \begin{pmatrix} X & XJ \\ JX & JXJ \end{pmatrix} \middle| X \in \mathfrak{gl}_m \right\} \quad \text{and}$$

$$\mathfrak{gl}_m^- = \left\{ \begin{pmatrix} X & -XJ \\ -JX & JXJ \end{pmatrix} \middle| X \in \mathfrak{gl}_m \right\}.$$  

Fix the identification $\mathcal{W}_m^i \otimes \mathcal{M} = \mathcal{V}_m^i$, $v_i \otimes v_+ \mapsto v_i$ and $v_i \otimes v_- \mapsto v_i$ as vector spaces. Twisting with $T_A$ gives the decomposition

$$\mathcal{V}_{2m}^i = \mathcal{W}_m^i \otimes (v_+ + v_-) \oplus \mathcal{W}_m^i \otimes (v_+ - v_-),$$  

as a $\mathfrak{g}^\Theta = \mathfrak{gl}_m^+ \times \mathfrak{gl}_m^-$-module. One should note that $\mathfrak{gl}_m^+$ acts as zero on the second summand, while $\mathfrak{gl}_m^-$ acts as zero on the first summand. The \textit{diagonally embedded} $\mathfrak{gl}_m$ in $\mathfrak{gl}_m \times \mathfrak{gl}_m$ with its Chevalley generators normalized by $\frac{1}{2}$ maps via $T_A$ to an isomorphic subalgebra of $\mathfrak{gl}_{2m}$, with generators $E_i := E_{i+1,1} + E_{-i-1,-i}$ and $F_i := E_{i+1,i} + E_{-i-1,-i}$ for $0 < i < m$. These are exactly the specialization of the generators $B_i$ and $B_{-i}$ from Definition 6.9. Together with the specialization of $\mathcal{R}$ they generate the \textit{diagonally embedded} $\mathfrak{gl}_m$ in $\mathfrak{gl}_m^+ \times \mathfrak{gl}_m^-$. The special generator $B_0$ specializes to the element $G = E_{1,-1} + E_{-1,1}$ and generates together with the diagonal embedded Lie algebra the whole fixed point Lie algebra $\mathfrak{g}^\Theta$.

7.3. The classical skew Howe duality. All the identifications made above can of course also be done for $\mathfrak{gl}_2$, and we will use the same notations for them. Using the decomposition from (47) we obtain

$$\bigwedge^n (n, m, r) = \bigwedge^n \left( \mathcal{W}_m^i \otimes (v_+ + v_-) \otimes \mathcal{W}_r^j \otimes \mathcal{W}_m^i \otimes (v_+ - v_-) \otimes \mathcal{W}_r^j \right)$$

$$\approx \bigoplus_{i=0}^{n} \bigwedge^{n-i} \left( \mathcal{W}_m^i \otimes (v_+ + v_-) \otimes \mathcal{W}_r^j \right) \otimes \bigwedge^{n-i} \left( \mathcal{W}_m^i \otimes (v_+ - v_-) \otimes \mathcal{W}_r^j \right),$$

where $\otimes$ means that we regard the first factor as a $(\mathfrak{gl}_m^+, \mathfrak{gl}_m^-)$-bimodule and the second as a $(\mathfrak{gl}_m^+, \mathfrak{gl}_m^-)$-bimodule and take the outer tensor product of these two bimodules in each summand. From the usual skew Howe duality of $\mathfrak{gl}_m$ and $\mathfrak{gl}_r$ on $\bigwedge^n (\mathcal{W}_m^i \otimes \mathcal{W}_r^j)$ we can deduce that on each summand the $\mathfrak{gl}_m^+ \times \mathfrak{gl}_m^-$-endomorphisms are generated by $\mathfrak{gl}_m^+ \times \mathfrak{gl}_m^-$ and that there are no such morphisms between different summands. We obtain

\textbf{Theorem 7.1.} The actions of $U(\mathfrak{gl}_m^+ \times \mathfrak{gl}_m^-)$ and $U(\mathfrak{gl}_m^+ \times \mathfrak{gl}_m^-)$ on $\bigwedge^n (n, m, r)$ commute and generate each others commutant.
7.4. The categorified classical skew Howe duality. To categorify the bimodule $\mathbin{\bigwedge}(n,m,r)$ we first use classical skew Howe duality for the pair $(\mathfrak{gl}_{2m},\mathfrak{gl}_r)$ to obtain a decomposition as $\mathfrak{gl}_{2m}$-module

$$\mathbin{\bigwedge}(n,m,r) \cong \bigoplus_{k \in C(n,r)} \mathbin{\bigwedge}^k \mathbb{V}_m^i,$$

where $C(n,r)$ denotes the set of composition of $n$ with $r$ parts and $\mathbin{\bigwedge}^k \mathbb{V}_m^i = \mathbin{\bigwedge}^{k_1} \mathbb{V}_m^i \otimes \ldots \otimes \mathbin{\bigwedge}^{k_r} \mathbb{V}_m^i$. Naturally each summand is also a module for the subalgebra $\mathfrak{gl}_m^+ \times \mathfrak{gl}_m^\prime$. To identify each summand with the Grothendieck group of certain blocks of parabolic category $\mathcal{O}$ we denote by $q_k$ the standard parabolic of $\mathfrak{so}_{2n}$ corresponding to the root subsystem where we omit the simple roots

$$\alpha_0, \alpha_{k_1}, \alpha_{k_1+k_2}, \alpha_{k_1+k_2+k_3}, \ldots, \alpha_{k_1+\ldots+k_{r-1}},$$

for each choice of $k \in C(n,r)$. Our original parabolic is then $p = q_{(n)}$.

As in Section 1 we have the corresponding parabolic category $\mathcal{O}^{q_k}(n)$ inside the full category $\mathcal{O}$. For the basis of the corresponding Grothendieck groups, we have to look at $q_k$-dominant weights.

**Definition 7.2.** An integral weight $\lambda$ is $q_k$-dominant if it is of the form

$$\lambda = (\lambda_1 \leq \ldots \leq \lambda_{k_1}, \lambda_{k_1+1} \leq \ldots \leq \lambda_{k_1+k_2}, \ldots, \lambda_{k_1+\ldots+k_{r-1}+1} \leq \ldots \leq \lambda_n).$$

Let $\Lambda_k^\pm$ denote the set of all $q_k$-dominant weights, decomposing as usual into $\Lambda_k = \Lambda_k^1 \cup \Lambda_k^\prime$, depending on whether the $\lambda_i$’s are integers or half-integers.

Consider the subcategory $\mathcal{O}^{q_k}_{\leq m}(n)$ generated by simple modules with highest weight in

$$\Lambda_k^{\leq m} = \left\{ \lambda \in \Lambda_k^1 \left| \lambda_i + \rho_i \leq m - \frac{1}{2} \right. \right\}.$$

This is obviously a finite set and stable under the dot-action of the Weyl group, thus $\mathcal{O}^{q_k}_{\leq m}(n)$ is a finite sum of certain blocks in $\mathcal{O}^{q_k}_{\leq m}(n)$. In complete analogy to Section 6, we associate to $\lambda \in \Lambda_k^{\leq m}$ a vector $v_\lambda \in \mathbin{\bigwedge}^k \mathbb{V}_m^i$ as follows. First denote $\lambda' = \lambda + \rho$, then consider the sequence

$$\tilde{\lambda}' = (\lambda'_{k_1} + \ldots + \lambda'_{j}, \lambda'_{k_1} + \ldots + \lambda'_{j-1} + \ldots, \lambda'_{k_1} + \ldots + \lambda'_{k_{r-1}+1}),$$

and define the corresponding standard basis vector as $v_\lambda = v_{\lambda_1} \otimes \ldots \otimes v_{\lambda_n}$. Then $[M^{q_k}(\lambda)] \mapsto v_\lambda$ defines an isomorphism of vector spaces

$$\Gamma_k : K_0 \left( \mathcal{O}^{q_k}_{\leq m}(n) \right) \longrightarrow \mathbin{\bigwedge}^k \mathbb{V}_m^i.$$ 

Taking (direct) sums we obtain

$$\Gamma_m : K_0 \left( \bigoplus_{k \in C(n,r)} \mathcal{O}^{q_k}_{\leq m}(n) \right) \longrightarrow \mathbin{\bigwedge}(n,m,r). \quad (49)$$
Alternatively we could look at the decomposition as \( \mathfrak{gl}_2 \)-module and get
\[
\Gamma_r : K_0 \left( \bigoplus_{k \in \mathbb{C}(n,m)} O^k_{\leq m}(n) \right) \to \wedge(n, m, r).
\]

Next, we decompose the categories \( O^k_{\leq m}(n) \). Although we will refer to this as a block decomposition, this is not completely correct as the subcategories might decompose further, but it is a decomposition by central character.

**Remark 7.3.** Note that two \( q \)-dominant weights give rise to parabolic Verma modules with the same central character iff they are in the same Weyl group orbit with respect to the dot-action. After adding \( \rho \) this corresponds to usual Weyl group orbits. Then two weights lie in the same orbit iff the multiplicity for each half-integer \( \frac{1}{2}, \ldots, m - \frac{1}{2} \) up to sign, agrees in the two weights and the multiplicities of negative integers have the same parity.

**Definition 7.4.** Let \( (k, \mu, \epsilon) \) be two compositions of \( n, k \) with at most \( r \) non-trivial parts and \( \mu \) with at most \( m \) non-trivial parts and \( \epsilon \in \{0, 1\} \) a parity. Then \( O_{(k, \mu, \epsilon)}(n) \), will denote the blocks of \( O^k_{\leq m}(n) \) containing the parabolic Verma modules of highest weights \( \lambda \) such that
\[
\# \{ j \mid \lambda_j + \rho_j = i - \frac{1}{2} \} = \mu_i \quad \text{and} \quad \# \{ j \mid \lambda_j + \rho_j < 0 \} = \epsilon.
\]
Such a triple \( (k, \mu, \epsilon) \) is called a **block triple**.

We denote by \( \text{Bl}(n, r, m) \) the set of all triples and define for a block triple \( \Gamma \) the projection into the corresponding block
\[
\text{pr}_\Gamma : O^k_{\leq m} \to O^\Gamma(n).
\]

For \( 1 \leq i \leq m - 1 \) and a block triple \( \Gamma = (k, \mu, \epsilon) \) we define the triples
\[
\Gamma_{i,+} = (k, \mu', \epsilon) \quad \text{via} \quad \mu'_l = \begin{cases} 
\mu_l - 1 & \text{if } l = i + 1 \\
\mu_l + 1 & \text{if } l = i \\
\mu_l & \text{otherwise},
\end{cases}
\]
\[
\Gamma_{i,-} = (k, \mu', \epsilon) \quad \text{via} \quad \mu'_l = \begin{cases} 
\mu_l + 1 & \text{if } l = i + 1 \\
\mu_l - 1 & \text{if } l = i \\
\mu_l & \text{otherwise},
\end{cases}
\]
and
\[
\Gamma_0 = (k, \mu, \epsilon + 1).
\]

Generalizing Definition 3.7 we can now define certain translation functors.

**Definition 7.5.** We have the **special projective functors** \( \mathcal{F}_{i,+}, \mathcal{F}_{i,-} \) for \( i \in \mathbb{Z}_{\geq 0} \) and \( \mathcal{F}_0 \) defined by
\[
\mathcal{F}_{i,+} := \bigoplus_{\Gamma} \text{pr}_{\Gamma_{i,+}} \circ (? \otimes V) \circ \text{pr}_\Gamma : \bigoplus_{\Gamma} O^\Gamma(n) \to \bigoplus_{\Gamma} O^\Gamma(n),
\]
\[
\mathcal{F}_{i,-} := \bigoplus_{\Gamma} \text{pr}_{\Gamma_{i,-}} \circ (? \otimes V) \circ \text{pr}_\Gamma : \bigoplus_{\Gamma} O^\Gamma(n) \to \bigoplus_{\Gamma} O^\Gamma(n),
\]
\[
\mathcal{F}_0 := \bigoplus_{\Gamma} \text{pr}_{\Gamma_0} \circ (? \otimes V) \circ \text{pr}_\Gamma : \bigoplus_{\Gamma} O^\Gamma(n) \to \bigoplus_{\Gamma} O^\Gamma(n).
\]
\[ \mathcal{F}_{i,+} := \bigoplus_{\Gamma} \text{pr}_{\Gamma,+} \circ (\otimes V) \circ \text{pr}_{\Gamma} : \bigoplus_{\Gamma} \mathcal{O}_{\Gamma}(n) \to \bigoplus_{\Gamma} \mathcal{O}_{\Gamma}(n), \]

\[ \mathcal{F}_0 := \bigoplus_{\Gamma} \text{pr}_{\Gamma,0} \circ (\otimes V) \circ \text{pr}_{\Gamma} : \bigoplus_{\Gamma} \mathcal{O}_{\Gamma}(n) \to \bigoplus_{\Gamma} \mathcal{O}_{\Gamma}(n), \]

with all sums running over \( \Gamma \in \text{Bl}(n,r,m) \).

**Proposition 7.6.** Under the identification

\[ K_0 \left( \bigoplus_{\Gamma \in \text{Bl}(n,r,m)} \mathcal{O}_{\Gamma}(n) \right) = \bigwedge(n,m,r) \]

the action of [\( \mathcal{F}_{i,-} \)] coincides with the one of \( F_i \), the action of [\( \mathcal{F}_{i,+} \)] coincides with the one of \( E_i \) and the action of [\( \mathcal{F}_0 \)] coincides with the action of \( G \), hence gives a categorification of the action of \( \mathfrak{gl}_n^* \times \mathfrak{gl}_m^* \).

**Proof.** We show the claim for \( \mathcal{F}_{i,+} \) and leave the rest to the reader. Let \( \Gamma = (\mathbf{k}, \mathbf{l}, \mathbf{\epsilon}) \) be a block triple and \( \mathcal{M}_{\mathbf{k}}(\lambda) \) be a parabolic Verma module in the corresponding block. Then we can express \( \mathcal{M}_{\mathbf{k}}(\lambda) \otimes V \) in the Grothendieck group as

\[ [\mathcal{M}_{\mathbf{k}}(\lambda) \otimes V] = \sum_{l: \lambda + \epsilon_l \in \Lambda_+} [\mathcal{M}_{\mathbf{k}}(\lambda + \epsilon_l)] + \sum_{l: \lambda - \epsilon_l \in \Lambda_+} [\mathcal{M}_{\mathbf{k}}(\lambda - \epsilon_l)]. \]

Checking now when a parabolic Verma module of the form \( \mathcal{M}_{\mathbf{k}}(\lambda - \epsilon_l) \) lies in the block corresponding to \( \Gamma_{i,+} \) we see that this is the case if and only if \( \lambda_i + \rho_i = l + \frac{1}{2} \) and similarly with \( \mathcal{M}_{\mathbf{k}}(\lambda + \epsilon_l) \). Under our identification this coincides exactly with the way \( E_i \) acts on the Grothendieck group. \( \square \)

### 7.5. The module \( \bigwedge(n,m,r) \) in the quantum case.

From now on we will consider all vector spaces as defined over \( \mathbb{Q}(q) \), but use the same notations as before. Thus we consider truncations of the spaces \( \mathbb{W} \) and \( \mathbb{V} \) we had in Section 6. In general it is not obvious how to lift exterior powers to the quantum setting, see [BZ08] for a detailed treatment of this problem. In particular a quantization of (48) is non-trivial, since we had to deal with exterior powers of triple tensor products. A possible definition using Lemma 7.8 is technical and will appear in [ES13a]. Here we take an easier route and quantize our space as

\[ \bigwedge(n,m,r) = \bigoplus_{k \in C(n,r)} \bigwedge^k \mathbb{V}_m, \]

and define on this space explicitly two commuting coideal actions.

### 7.6. The commuting action of the Hecke algebra.

As before we restrict to the half-integer case. Consider the special case where \( k = (1,1,\ldots,1) \), hence \( \bigwedge^k \mathbb{V}_m = \mathbb{V}_m^r \).

**Definition 7.7.** The *Hecke algebra* \( \mathbb{H}_r \) corresponding to the Weyl group \( W_r = \langle s_{\alpha_i} \mid 0 \leq i \leq r \rangle \) of \( \mathfrak{so}_{2r} \) is the unital \( \mathbb{Q}(q) \)-algebra with generators \( H_i, 0 \leq i \leq r \) subject to the quadratic relation \( H_i^2 = 1 + (q^{-1} - q)H_i \), and the type
D braid relations $H_i H_j = H_j H_i$ if $s_{\alpha_i} s_{\alpha_j} = s_{\alpha_j} s_{\alpha_i}$ and $H_i H_j H_i = H_j H_i H_j$ if $s_{\alpha_i} s_{\alpha_j} s_{\alpha_i} = s_{\alpha_j} s_{\alpha_i} s_{\alpha_j}$ with $1 \leq i, j \leq r$.

**Lemma 7.8.** There is a right action of the Hecke algebra $H_r$ on $\mathcal{V}_m^r \otimes r$ commuting with the action of $\mathcal{H}$. The generator $H_i$ for $1 \leq i \leq r$ acts on the $i$th and $i+1$th tensor factor by the formula

$$v_a \otimes v_b.H_i = \begin{cases} v_b \otimes v_a & \text{if } a < b, \\ v_b \otimes v_a + (q^{-1} - q)v_a \otimes v_b & \text{if } a > b, \\ q^{-1}v_b \otimes v_a & \text{if } a = b. \end{cases} \tag{54}$$

and $H_0$ acts on the first and second tensor factor by

$$v_a \otimes v_b.H_0 = \begin{cases} v_{-b} \otimes v_{-a} & \text{if } a + b > 0, \\ v_{-b} \otimes v_{-a} + (q^{-1} - q)v_a \otimes v_b & \text{if } a + b < 0, \\ q^{-1}v_{-b} \otimes v_{-a} & \text{if } a = -b. \end{cases} \tag{55}$$

**Proof.** One can directly verify that the rules are compatible with the relations in the Hecke algebra, hence indeed define an action. The verification that it commutes with the action of the coideal subalgebra is a lengthy straight forward case by case calculation showing that $(B_j,v).H_i = B_j.(v.H_i)$ for any choice of $j, i$ and basis vector $v$. \hfill $\square$

Analogously we can identify $\Lambda(n,m,r)$ with $K_0\left(\bigoplus_{r \in B(n,m,r)} \mathcal{O}_r(n)\right)$ and with the analogous functors obtain an action of $\mathfrak{gl}_r^+ \times \mathfrak{gl}_r^-$. Consider the graded version $\hat{\mathcal{O}}^r_{\mathbb{S}_m^k}(n)$ of $\mathcal{O}^r_{\mathbb{S}_m}(n)$ as described in Section 5. We again have the corresponding identification of $\mathbb{Q}(q)$-vector spaces

$$\hat{\Gamma}_k^r : K_0\left(\hat{\mathcal{O}}^r_{\mathbb{S}_m^k}(n)\right) \rightarrow k \bigwedge^r \mathcal{V}_m. \tag{56}$$

by sending the class $[M^k_{\mathbb{S}_m^k}(\lambda)]$ of the standard graded lift of the parabolic Verma module $M^k_{\mathbb{S}_m^k}(\lambda)$ to $v_1$ with head concentrated in degree zero. (The multiplication with $q$ corresponds to the grading shift (1)).

Recall from [AL03], [AS03] the right exact *twisting endofunctors* $T_i := T_{s_{\alpha_i}}$ of the integral category $\mathcal{O}$ attached to $s_{\alpha_i}$. Their left derived functors define auto-equivalences of the derived category. The Hecke algebra action has a categorification given by graded lifts of these derived functors:

**Proposition 7.9.** Let $k = (1,1,\ldots,1)$. Then the derived functor $\mathcal{L}T_i$, $0 \leq i \leq n-1$ has a graded lift,

$$\hat{\mathcal{L}}T_i : D^b(\hat{\mathcal{O}}^r_{\mathbb{S}_m^k}(n)) \rightarrow D^b(\hat{\mathcal{O}}^r_{\mathbb{S}_m^k}(n)), \tag{57}$$

an automorphism of the bounded derived category $D^b(\hat{\mathcal{O}}^r_{\mathbb{S}_m^k}(n))$ such that the induced action of $\hat{\mathcal{L}}T_i$ on $K_0(D^b(\hat{\mathcal{O}}^r_{\mathbb{S}_m^k}(n))) = K_0(\hat{\mathcal{O}}^r_{\mathbb{S}_m^k}(n))$ agrees with the action of the Hecke algebra generator $H_i$ under the identification $\hat{\Gamma}_k^r$.

**Proof.** If we forget the grading this is [AL03, 2.1]. The graded version is [FKS06, Propositions 5.1 and 5.2]. \hfill $\square$
We like to extend the categorification from Proposition 7.6 to the quantum case. We first have to give an alternative description of the special functors. Recall from [BG80], [Hum08] the definition and classification of translation on and out of walls. Given integral weights $\lambda, \mu$ with corresponding block $O_\lambda(n)$ and $O_\mu(n)$ let $W_n(\lambda), W_n(\mu)$ be their stabilizer under the dot-action of the Weyl group. Assume $W_n(\lambda) \subset W_n(\mu)$ then we have the translation to and the translation out of the walls functors

$$
\theta^\mu_\lambda : O_\lambda(n) \rightarrow O_\mu(n),
\theta^\mu_\lambda : O_\mu(n) \rightarrow O_\lambda(n).
$$

We are in particular interested in the case, where $W_n(\mu)$ differs from $W_n(\lambda)$ by one additional simple reflection $s_\alpha$. We call the corresponding functors then translation to the $i$th wall and from the $i$th wall and denote them by $\theta_{on,i}$, and $\theta_{out,i}$ respectively. To make formulas easier we abuse notation and also write $\theta_{on,i}$ and $\theta_{out,i}$ in case $W_n(\mu) = W_n(\lambda)$ in which case the two translation functors are just inverse equivalences, [Hum08, 7.8].

For $0 \leq i \leq m - 1$ and a block triple $\Gamma = ((1, 1, \ldots, 1), \mu, \bar{\tau})$ we define $i' := \sum_{1 \leq j \leq i} \mu_j$ (with $i' = 0$ if $i = 0$) and the compositions $\mu^{i, i'}, \mu^{i', i}$, where

$$
\mu^{i, i'}_l = \begin{cases} 
\mu_{l-1} & \text{if } l > i + 1, \\
0 & \text{if } l = i + 1, \\
\mu_l & \text{if } l < i + 1,
\end{cases} \quad \mu^{i', i}_l = \begin{cases} 
\mu_{l+1} & \text{if } l \geq i + 1, \\
\mu_l & \text{if } l < i + 1,
\end{cases}
$$

and $\Gamma_{i, i} = ((1, 1, \ldots, 1), \mu^{i, i'}, \bar{\tau}), \Gamma_{i, i} = ((1, 1, \ldots, 1), \mu^{i', i}, \bar{\tau})$ and consider the composition of functors:

$$
\mathbb{B}_{i, +\Gamma} : O_{\Gamma_1}(n) \xrightarrow{\eta} O_{\Gamma_1}(n) \xrightarrow{\theta_{out,i+1}^{i', i'}} O_{\Gamma_2}(n) \xrightarrow{\theta_{on,i}^{i', i}} O_{\Gamma_3}(n) \xrightarrow{\eta} O_{\Gamma_4}(n), \quad (58)
$$

where $\Gamma_1 = \Gamma_{i, i}, \Gamma_2 = (\Gamma_1)_{i+1, i}, \Gamma_3 = (\Gamma_2)_{i, i},$ and $\Gamma_4 = (\Gamma_3)_{i, i} = (\Gamma_1)_{i, i}$, using Definition, (51). The $\eta$’s stand for the translation functors indicated by the blocks. Since by [Hum08, 7.8] these are equivalences of categories we just denote them by $\eta$. Note that these equivalences exists, since the blocks only depends on the singularity type of the corresponding highest weights. Let

$$
\mathbb{B}_{i, -\Gamma} : O_{\Gamma_4}(n) \xrightarrow{\eta^{-1}} O_{\Gamma_3} \xrightarrow{\theta_{out,i'}^{i, i'}} O_{\Gamma_2}(n) \xrightarrow{\theta_{on,i'}^{i, i}} O_{\Gamma_1}(n) \xrightarrow{\eta^{-1}} O_{\Gamma}(n), \quad (59)
$$

be the adjoint functor. We set in analogy to Definition 7.5

$$
\mathbb{B}_i = \oplus_{\Gamma} \mathbb{B}_{i, +\Gamma} \oplus \oplus_{\Gamma \text{ with } \epsilon = 0} \mathbb{B}_{i, -\Gamma} = \oplus_{\Gamma \text{ with } \epsilon = 0} \mathbb{B}_{0, +\Gamma} \oplus \oplus_{\Gamma \text{ with } \epsilon = 0} \mathbb{B}_{0, -\Gamma}.
$$

Note that by convention, the summands are equivalences for regular blocks just changing the parity.
Example 7.10. For instance let $\Gamma = (k = (1,1,\ldots,1), \mu = (2,2,2), \emptyset)$ and $i = 1$. In each step of $B_i$ the parabolic $k$ and parity is preserved, but the singularity $\mu$ is changed. The first equivalence changes to $\mu = (2,2,0,2)$, then translating out of the third wall followed by translating to the second wall goes via $\mu = (2,1,1,2)$ to $\mu = (3,1,0,2)$ and the final equivalence ends in $(\mu = (3,1,2))$. Hence we reduced the numbers of 2’s and increased the number of 1’s.

Lemma 7.11. There are isomorphisms $B_{\epsilon i} \cong F_{i,k}$ for $i > 0$ and $B_0 \cong F_0$.

Proof. Recall from [BG80] that projective functors, i.e. direct summands of endofunctors on $O(n)$ given by taking the tensor product with finite dimensional modules, are determined up to isomorphism already by their value on the Verma modules whose weight is maximal in their dot-orbit of $W_0$. Hence to prove the lemma it is enough to compare the value on these Verma modules. Since such a Verma module is a projective object it is enough to compare their values in the Grothendieck group. We explain this for $B_i$ and $B_0$, the case $B_{-i}$ is similar. Consider a block $\Gamma = ((1,1,\ldots,1), \mu, \tau)$ with corresponding maximal weight $\lambda$ such that

$$\lambda + \rho = \left(\frac{1}{2}, \ldots, \frac{1}{2}, n - \frac{1}{2}, \ldots, n - \frac{1}{2}\right)$$

$\mu_1 \ldots \mu_n$ in case $\epsilon = 0$ and with the first number negated if $\epsilon = 1$. For instance

$$\frac{1}{2}(3,3,3,5,5,7,7)$$

are examples. Now fix first $i > 0$ and consider $B_i$. Applying the equivalence $\eta$ sends $M(\lambda)$ to $M(\lambda^{1,i})$, where $\lambda^{1,i}$ is obtained from $\lambda$ by changing the numbers with absolute value $\geq (i + 1)$ by 1 if positive and by $-1$ if negative.

In our examples above we obtain for instance for $i = 2$ the highest weight $\lambda^{1,2}$ (up to the shift by $\rho$) of Verma modules as

$$\frac{1}{2}(3,3,3,7,7,9,9)$$

Translation out of the $(i + 1)$th wall gives a module which has a Verma flag with Verma subquotients precisely the $M(\nu - \rho)$ for $\nu$ obtained from $\lambda^{1,i}$ by changing exactly one $(i + 1) + \frac{1}{2}$ to $(i + 1) - \frac{1}{2}$, [Hum08, 7.12]. In our example we get for $\nu$ exactly the weights

$$\frac{1}{2}(3,3,3,7,5,9,9), \frac{1}{2}(3,3,3,7,5,7,9,9), \frac{1}{2}(3,3,3,5,7,7,9,9)$$

for $\epsilon = 0$ and

$$\frac{1}{2}(-3,3,3,7,5,9,9), \frac{1}{2}(-3,3,3,7,5,7,9,9), \frac{1}{2}(-3,3,3,5,7,7,9,9)$$

for $\epsilon = 1$. Translation to the $i$th wall maps each Verma module $M(\nu - \rho)$ to a Verma module $M(\nu' - \rho)$, where $\nu'$ is obtained from $\nu$ by changing the unique number with absolute value $i + \frac{1}{2}$ to absolute value $i - \frac{1}{2}$ and keeping
the sign, [Hum08, 7.6]. In our example we get the three highest weights (again without \( \rho \)-shift)
\[
\frac{1}{2}(3,3,3,7,7,3,9,9), \frac{1}{2}(3,3,3,7,7,9,9), \frac{1}{2}(3,3,3,7,7,9,9)
\]
in case \( \epsilon = 0 \) and analogously for \( \epsilon = 1 \). The final equivalence just rescales again and we obtain
\[
\frac{1}{2}(3,3,3,5,5,3,7,7), \frac{1}{2}(3,3,3,5,5,3,7,7), \frac{1}{2}(3,3,3,5,5,7,7)
\]
for \( \epsilon = 0 \) and with the different sign in the first entry for \( \epsilon = 1 \). Comparing with Proposition 7.6 we see this agrees with the values for \( F_{i,\pm} \), hence \( F_{i,+} \cong \mathbb{B}_i \).

By [BG80], the functors \( \hat{F} \) are contained in the stabilizer.) Applying the translation to the 0th respectively 1st wall if \( \epsilon = 0 \) and out of the 0th wall if \( \epsilon = 1 \). In our examples we obtain the three highest weights (up to \( \rho \)-shift) equal to \( \frac{1}{2}(3,3,1,5,5), \frac{1}{2}(3,1,3,5,5), \frac{1}{2}(1,3,3,5,5) \) respectively \( \frac{1}{2}(3,3,-1,5,5), \frac{1}{2}(3,-1,3,5,5), \frac{1}{2}(-1,3,3,5,5) \). (Note that we used here that all weights are supported on half-integers, so not both \( s_1 \) and \( s_0 \) are contained in the stabilizer.) Applying the translation to the 0th respectively 1st wall we obtain Verma of highest weights \( \frac{1}{2}(-1,1,1,5,5), \frac{1}{2}(1,-1,1,5,5), \frac{1}{2}(1,1,1,5,5), \frac{1}{2}(-1,-1,1,5,5), \frac{1}{2}(-1,1,1,5,5), \frac{1}{2}(-1,-1,1,5,5) \). Applying the last equivalence gives the desired result and hence \( \mathbb{B}_0 \cong F_0 \).

7.7. Graded translation functors. By [Str03], the translation functors to and out of a wall \( \theta_{\text{on},i} : \mathcal{O}_\lambda(n) \to \mathcal{O}_\mu(n) \), \( \theta_{\text{out},i} : \mathcal{O}_\lambda(n) \to \mathcal{O}_\mu(n) \) have graded lifts, that means can be lifted to graded functors\(^2\)
\[
\hat{\theta}_{\text{on},i} : \hat{\mathcal{O}}_\lambda(n) \to \hat{\mathcal{O}}_\mu(n), \quad \hat{\theta}_{\text{out},i} : \hat{\mathcal{O}}_\mu(n) \to \hat{\mathcal{O}}_\lambda(n).
\] (60)

By [BGS0], the functors \( \hat{\theta}_{\text{on},i} \) and \( \hat{\theta}_{\text{out},i} \) are indecomposable, hence a graded lift is unique up to isomorphism and overall shift in the grading, [Str03, Lemma 1.5]. We fix graded lifts \( \hat{\theta}_{\text{on},i} \), \( \hat{\theta}_{\text{out},i} \) such that the graded version of the Verma module with maximal weight in the orbit is sent to a projective module with head concentrated in degree zero. We define graded lifts of the functors in (58) and (59) by setting for \( i > 0 \):
\[
\hat{\mathbb{B}}_{i,+} = (\eta \circ \hat{\theta}_{\text{on},i} \circ \hat{\theta}_{\text{out},i+1} \circ \eta)(\mu_i - \mu_{i+1} + 1) \tag{61}
\]
\[
\hat{\mathbb{B}}_{i,-} = (\eta \circ \hat{\theta}_{\text{on},i} \circ \hat{\theta}_{\text{out},i+1} \circ \eta)(\mu_{i+1} - \mu_i + 1) \tag{62}
\]
and \( \hat{\mathbb{B}}_i = \oplus_{r} \hat{\mathbb{B}}_{i,+} \) and \( \hat{\mathbb{B}}_{-i} = \oplus_{r} \hat{\mathbb{B}}_{i,-} \), where \( \langle r \rangle \), \( r \in \mathbb{Z} \), means that the grading is shifted by adding \( r \) to the degree. For \( i = 0 \) we set
\[
\hat{\mathbb{B}}_0 := \bigoplus_{\Gamma \text{ with } \epsilon = 0} (\eta \circ \hat{\theta}_{\text{on},0} \circ \hat{\theta}_{\text{out},0} \circ \eta) \bigoplus_{\Gamma \text{ with } \epsilon = 1} (\eta \circ \hat{\theta}_{\text{on},0} \circ \hat{\theta}_{\text{out},1} \circ \eta) \tag{63}
\]

\(^2\)The arguments about the existence in [Str03] are only for the principal block, but generalize directly to arbitrary integral blocks by taking invariants of the coinvariants.
These functors induce functors on $\mathcal{O}_{\mathbb{Z}^d}(n)$ and then via the identification (56) $\mathbb{Q}(q)$-linear endomorphisms on the $\mathbb{Q}(q)$-vector space $\wedge(n,m,r)$. Our main result here is the following categorification theorem

**Theorem 7.12.** Under the identification

$$K_0\left(\bigoplus_{\Gamma \in \text{Bi}(n,m,r)} \hat{\mathcal{O}}_\Gamma(n)\right) = \wedge(n,m,r)$$

as $\mathbb{Q}(q)$-vector space, the action induced from the exact functor $\hat{\mathcal{B}}_i$ coincides with the action of $B_i \in \mathcal{H}^i$ for any $i \in \mathbb{Z}$.

We start with the following main insight:

**Lemma 7.13.** Theorem 7.12 is true when restricted to $V_m \otimes \mu^r$.

**Proof.** It is enough to verify that the action of $[\hat{\mathcal{B}}_i]$ agrees with the action of $B_i \in \mathcal{H}^i$ on the standard basis corresponding to isomorphism classes of graded lifts of Verma modules. Recall the commuting action of the Hecke algebra from Lemma 7.8 and note that twisting functors commute with translation functors, [AS03, Theorem 3.2]. Hence by Proposition 7.9 it is enough to compare the action for Verma modules whose highest weight $\lambda$ is maximal in their dot-orbit. Moreover by linearity we can restrict ourselves to the case of standard graded lifts $\hat{M}(\lambda)$. We follow the proof of Lemma 7.11. Consider first the functors $\hat{\mathcal{B}}_i$ for $i > 0$. The equivalence $\eta$ lifts to the graded setting and just renames the highest weight of $\hat{M}(\lambda)$. The translation out of the wall produces an indecomposable projective module $P$, [Hum08, 7.11] with a graded Verma flag. The Verma subquotients are, up to some grading shifts, the $\hat{M}(\nu - \rho)$ with the same weights $\nu$ as before. Amongst them let $\nu(r)$ be the weight where the $r$th $i + 2$ from the left has been changed. By our normalization, $\hat{M}(\nu(r_{\text{max}}) - \rho)$ with $r = r_{\text{max}}$ maximal occurs without grading shift and hence $P \cong \hat{P}(\nu(r_{\text{max}}) - \rho)$. By [BGS96, Theorem 3.11.4], the graded multiplicities are encoded in parabolic Kazhdan-Lusztig polynomials. Our case corresponds to the explicit formula [Soe97, Proposition 2.9] and we obtain that $P$ has a filtration with subquotients $\hat{M}(\nu(r) - \rho)(r_{\text{max}} - r)$.

In our example we get

$$[\hat{M}(\frac{1}{2}(3,3,3,7,7,5,9,9) - \rho)] + q[\hat{M}(\frac{1}{2}(3,3,3,7,7,9,9,9) - \rho)]$$

$$+ q^2[\hat{M}(\frac{1}{2}(3,3,3,5,7,7,9,9) - \rho)$$

in the Grothendieck group. Now translation to the wall just changes the numbers $i + 1$ appearing in the weights to $i$ without any extra grading shift by [Str03, Theorem 8.1]. The last step is a graded equivalence renaming the weights. In our example we obtain

$$[\hat{M}(\frac{1}{2}(3,3,3,5,5,5,7,7) - \rho)] + q[\hat{M}(\frac{1}{2}(3,3,3,5,5,5,7,7) - \rho)]$$

$$+ q^2[\hat{M}(\frac{1}{2}(3,3,3,5,5,5,5,7) - \rho)].$$
In any case, the result agrees with the action of $B_i$ up to an overall grading
shift by $\langle \mu_i - \mu_{i+1} + 1 \rangle$. In case $i = 0$ the arguments are completely parallel
except of the grading shift at the end. As above, the lowest grading shift
appearing for Verma modules is zero. Expressing $B_0 v_\lambda$ in the standard basis,
the smallest $q$ power appearing equals $(-1) + n_1 - n_{-1} + (n_{-1} - n_1 + 1) = 0$, where $n_j$ is the number of $j$'s appearing in the weight $\lambda$. Hence the claim of
the Lemma follows for $B_0$. For $B_{-i}$, $i > 0$ the arguments are analogous. □

Proof of Theorem 7.12. Let $i > 0$. Recall from (41) that the action of $B_i$ on
some standard basis vector $v = v_1 \otimes \ldots \otimes v_r \in \Lambda^{k_1} V^i \otimes \ldots \otimes \Lambda^{k_r} V^i$ from (42)
can be obtained by viewing $v$ as a vector inside $\bigotimes^{k_1} V^i \otimes \ldots \otimes \bigotimes^{k_r} V^i$ and
compute the action there. Let $[\hat{M}^q\hat{\lambda}(\mu)]$ be the class of the standard lift of
the parabolic Verma module attached to $v$. Assume first that the stabilizer
of $\mu$ is trivial under the dot-action and write $\mu = x \cdot \lambda$ with $\lambda$ maximal in the
same dot-orbit and $x \in W_n$. Recall that by definition the Levi subalgebra of
$q_k$ is of type $A$ with Weyl group isomorphic to the product $S_k$ of symmetric
groups. Then
\[
[\hat{M}^q\hat{\lambda}(\mu)] = [\hat{M}^q\hat{\lambda}(x \cdot \lambda)] = \sum_{y \in S_k} (-q)^{\ell(w)}[\hat{M}(yx \cdot \lambda)]
\] (64)
by the formula [Soe97, Proposition 3.4] for parabolic Kazhdan-Lusztig polynomials
altogether with [Str05, Corollary 2.5]. This however fits precisely
with the formula (41) and the claim follows. Assume now that the stabilizer
$W(\mu)$ of $\mu$ is not trivial under the dot-action. Then choose $\lambda'$ an integral
weight, maximal in its dot-orbit and with $W(\lambda')$ trivial. Following [Str03]
choose a graded lift $\hat{\theta}_\lambda^{\mu}$ of $\theta_\lambda^{\mu}$ such that $\hat{\theta}_\lambda^{\mu} \hat{M}(\lambda') \simeq \hat{M}(\lambda)$. We claim that
\[
\hat{\theta}_\lambda^{\mu} \hat{M}(x \cdot \lambda') \simeq \hat{M}(x \cdot \lambda)
\] (65)
for any shortest coset representative $x \in W_n/W(\lambda)$. Since this is true if we
forget the grading and $M(x \cdot \lambda)$ is indecomposable we only have to figure out
possible overall grading shifts, [Str03, Lemma 1.5]. If $x$ is the identity we are
done by convention. Otherwise we find a simple reflection $s_{\alpha}$ such that
$\ell(s_{\alpha} x) < \ell(x)$ and use again that twisting functors, even graded, commute
with translation functors. From Lemma 7.8 and Proposition 7.9 the result
follows then by induction on $\ell(x)$. This implies $\hat{\theta}_\lambda^{\mu} \hat{M}^q\hat{\lambda}(x \cdot \lambda') \simeq \hat{M}^q\hat{\lambda}(x \cdot \lambda)$,
since this is again true up to a possible overall shift in the grading which is
however zero by (65). Moreover, we can choose $\lambda'$ such that $yx$ is of minimal
length in its coset for any $y \in S_k$. (In practice we consider $\lambda + \rho$ and replace
its entries by a strictly increasing sequence of positive integers. We do this
by replacing first the lowest numbers from the left to right, then replace the
second lowest from the left to right etc.) Now we apply $[\hat{\theta}_\lambda^{\mu}]$ to (64) and
obtain with (65)
\[
[\hat{M}^q\hat{\lambda}(x \cdot \lambda)] = \sum_{y \in S_k} (-q)^{\ell(w)}[\hat{M}(yx \cdot \lambda)]
\] (66)
The claim follows then from (41) and Lemma 7.13. □
For the block triple $\Gamma = (k, \mu, \epsilon) \in \text{Bl}(n, r, m)$ we associate the transposed block triple $\Gamma^\vee = (\mu, k, \overline{\epsilon}) \in \text{Bl}(n, m, r)$. In other words, the parabolic type is flipped with the type of the singularity and the parity is kept if $n$ is even and swapped if $n$ is odd.

**Theorem 7.14.** Let $\Gamma \in \text{Bl}(n, r, m)$ then $\hat{O}_\Gamma(n)$ is Koszul dual to $\hat{O}_{\Gamma^\vee}(n)$.

**Proof.** We first recall the description of the Koszul dual block in general. Assume we have a block $O_J$ of category $O$ for any complex semisimple Lie algebra, where the parabolic is given by a subset $I$ of the simple roots and the stabilizer of the highest weights of the parabolic Verma modules are generated by simple reflections given by a subset $J$ of simple roots. Then the Koszul dual is the block $O_J$, where $I' = -w_0(I)$ with $w_0$ the longest element in the Weyl group, [Bac99, Theorem 1.1]. In case of type $D_n$, we have $-w_0(I) = I$ if $n$ is even, whereas if $n$ is odd $-w_0(I)$ is the set obtained from $I$ by applying the unique non-trivial diagram automorphism of the Dynkin diagram. Hence the Koszul dual of $\Gamma = (k, \mu, \overline{0})$ is the block $(\mu, k, \overline{0})$ if $n$ is even and equal to $(\mu, k, \overline{1})$ if $n$ is odd; and the Koszul dual of $\Gamma = (k, \mu, \overline{1})$ is the block $(\mu, k, \overline{1})$ if $n$ is even and equal to $(\mu, k, \overline{0})$ if $n$ is odd. $\square$

Koszul duality, [BGS96], [Bac99], defines an equivalence of categories

$$D^b(\hat{O}_\Gamma(n)) \cong D^b(\hat{O}_{\Gamma^\vee}(n)).$$

(67)

This duality and its connection to the quantum exterior powers can be nicely encoded in terms of box diagrams which we introduce next.

### 7.8. Combinatorics of box diagrams.

A $r \times m$ box diagram $\blacksquare$ is a filling of $r$ rows and $m$ columns of boxes such that each box is either empty, filled with one of the symbols $\text{+}, \text{-}$ or filled with both, i.e. with $\pm$; for an example see 7.16 below. We denote the box in row $i$ and column $j$ by $\blacksquare_{i,j}$. We say $\blacksquare$ is of type $\Gamma = (k, \mu, \epsilon)$ if there are a total of $k_i$ symbols in row $i$, a total of $\mu_j$ symbols in column $j$, and the parity of the number of $-$ is $\epsilon$.

**Definition 7.15.** We define for any box $\blacksquare_{i,j}$ as above the statistics

$$c^{+}_{i,j} (\blacksquare, \rightarrow) = \# \{ l \mid l > j : \blacksquare_{i,l} = + \text{ or } \blacksquare_{i,l} = \pm \},$$

i.e., the number of symbols $+$ to the right of the box $\blacksquare_{i,j}$; analogously for the arrows $\leftarrow$, $\uparrow$, and $\downarrow$ and for the symbol $-$. Similarly we define

$$c^{+}_{i,j} (\blacksquare, \leftrightarrow) = \# \{ l \mid \blacksquare_{i,l} = + \text{ or } \blacksquare_{i,l} = \pm \},$$

i.e., the number of symbols $+$ in row $i$; analogously for $\downarrow$ and the symbol $-$.

Given $\hat{M}^\blacksquare(x \cdot \mu)$ consider the weight $x \cdot \mu + \rho = (\mu_1, \mu_2, \ldots, \mu_n)$. This is a sequence of integers, strictly increasing in the $r$ parts given by $k$. Assign to this weight a $r \times m$ box diagram $\blacksquare$ where $\blacksquare_{i,j}$ is filled with:

- the symbol $+$ if $j - \frac{1}{2}$, but not $-(j - \frac{1}{2})$, occurs in the $i$th part of $k$,
- the symbol $-$ if $-(j - \frac{1}{2})$, but not $j - \frac{1}{2}$, occurs,
- both symbols, i.e. $\mp$ if both occur.
Obviously this will produce a box diagram of the same type as the block of
the weight and any such box diagram defines uniquely a corresponding
weight, by just reading the rows from top to bottom and ordering the en-
tries given by each row in increasing order. Let \( \hat{\omega} \) be the transposed of \( \omega \) with additionally all symbols (+ and −) swapped and let \((x \cdot \mu + \rho)^t\) be the corresponding weight.

**Example 7.16.** Here is an example for \( n = 9, r = 5, \) and \( m = 4 \) with the
empty boxes indicated by \( \circ \). Put \( k = (2, 2, 2, 2, 1) \) and look at \( w \cdot \mu + \rho = (-\frac{5}{2}, \frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{7}{2}, \frac{3}{2}, -\frac{1}{2}, \frac{5}{2}, \frac{3}{2}) \) then the corresponding box diagram and its
transpose are

\[
\begin{array}{c}
\text{\( \hat{\omega} = \)}
\begin{array}{c}
\circ + - \circ \\
- + 0 - \\
- - 0 + \\
0 + + 0
\end{array}
\end{array}
\begin{array}{c}
\text{\( \hat{\omega}^t = \)}
\begin{array}{c}
\circ + + - \\
- + - 0 \\
+ + 0 - \\
0 + + 0
\end{array}
\end{array}
\]

and thus
\( w^{-1} w_0 \cdot \lambda + \rho = (-\frac{3}{2}, \frac{3}{2}, -\frac{7}{2}, -\frac{5}{2}, -\frac{1}{2}, -\frac{7}{2}, \frac{1}{2}, \frac{5}{2}) \).

**Lemma 7.17.** Let \( \hat{\omega} \) be the box diagram corresponding to \( \hat{M}^\mu(\omega \cdot \mu) \) in the
block \( \hat{O}_r(n) \). Then \( \hat{M}^\mu(\omega^{-1} w_0 \cdot \lambda) \in \hat{O}_r(n) \) corresponds to the box diagram \( \hat{\omega}^t \). Here \( \mu \) and \( \lambda \) are maximal elements in their respective orbits for the dot
action of the Weyl group.

**Proof.** Transposing the box diagram corresponds to passing from \( x \) to \( x^{-1} \).
Swapping the signs corresponds to right multiplication with \( w_0 \). If \( n \) is even,
then this procedure keeps the parity of − symbols, whereas if \( n \) is odd then
the parity gets swapped. The claim follows. □

By encoding basis vectors by box diagrams we can give explicit formulae
for the actions of \( \mathcal{H}_m^r \) on \( \Lambda(n, m, r) \). For this let \( \hat{\omega} \) be a \( r \times m \) box diagram corresponding to a standard basis vector in \( \Lambda(n, m, r) \), then for \( 1 \leq j < m \)
\[
B_{j, \hat{\omega}} = \sum_{\hat{\omega}' \in \hat{\omega}} q^{m(\hat{\omega}')} \hat{\omega}', \quad B_{-j, \hat{\omega}} = \sum_{\hat{\omega}'' \in \hat{\omega}''} q^{m(\hat{\omega}'')} \hat{\omega}'', \quad B_{0, \hat{\omega}} = \sum_{\hat{\omega}''' \in \hat{\omega}'''} q^{m(\hat{\omega'''})} \hat{\omega}'''.
\]
The first sum runs over all box diagrams \( \hat{\omega}' \) obtained from \( \hat{\omega} \) by moving a
symbol from a box in column \( j + 1 \) to the box to its left in column \( j \). Let \( \hat{\omega}_{i,j+1} \) be the box with the symbol that is moved to the left, then the power
of \( q \) depends on the symbol moved as follows
\[
m(\hat{\omega}') = \{ c_{i,j}(\hat{\omega}, \uparrow) - c_{i,j+1}(\hat{\omega}, \uparrow) + c_{j}^-(\hat{\omega}, \downarrow) - c_{j+1}^-(\hat{\omega}, \downarrow) \} \text{ for } \uparrow \text{ and } -\cdot
\]
The second sum runs over all \( \hat{\omega}'' \) obtained from \( \hat{\omega} \) by moving a symbol from
column \( j \) to column \( j + 1 \) and the corresponding power of \( q \) for an occurring
box diagram is given by
\[
m(\hat{\omega}'') = \{ c_{i,j+1}(\hat{\omega}, \downarrow) - c_{i,j}^+(\hat{\omega}, \downarrow) \} \text{ for } +\quad \text{and } \quad m(\hat{\omega}') = \{ c_{i,j+1}(\hat{\omega}, \uparrow) - c_{i,j}^+(\hat{\omega}, \uparrow) + c_{j+1}^+(\hat{\omega}, \downarrow) - c_{j}^+(\hat{\omega}, \downarrow) \} \text{ for } -\cdot
The third sum runs over all $\mathfrak{m}'''$ obtained from $\mathfrak{m}$ by flipping a symbol in the first column. In this case the power of $q$ is independent of the symbol and only depends on the position

$$m(\mathfrak{m}''') = c_{i,1}^r(\mathfrak{m}, \downarrow) - c_{i,1}^r(\mathfrak{m}, \downarrow).$$

For the action of $\mathcal{H}_k$ on $\wedge(n, m, r)$ we need special morphisms between the summands of the form $\wedge^k \mathfrak{v}^i_m$. Let $k \in C(n, r)$ with $k_{i+1} > 0$ and define two further compositions $k' \in C(n, r + 1)$ and $k'' \in C(n, r)$ via

$$k_j' = \begin{cases} k_j & \text{if } j < i \\ 1 & \text{if } j = i + 1 \\ k_{i+1} - 1 & \text{if } j = i + 2 \\ k_{j-1} & \text{if } j > i + 2 \end{cases} \quad k_j'' = \begin{cases} k_j & \text{if } j < i \\ k_i + 1 & \text{if } j = i \\ k_{i+1} - 1 & \text{if } j = i + 1 \\ k_j & \text{if } j > i + 1. \end{cases}$$

Then let $\tilde{B}_{i,k}$ be the following composition of the natural $\mathcal{U}_q(\mathfrak{gl}_{2m})$-morphisms multiplied with a certain power of $-q$,

$$\tilde{B}_{i,k} = (-q)^{1-k_{i+1}} \left( k' \wedge \mathfrak{v}^i_m \rightarrow \wedge^k \mathfrak{v}^i_m \rightarrow k'' \wedge \mathfrak{v}^i_m \right). \quad (69)$$

For $k$ with $k_{i+1} = 0$ we put $\tilde{B}_{i,k} = 0$ and finally $\tilde{B}_i = \bigoplus_{k \in C(n, r)} \tilde{B}_{i,k}$.

**Remark 7.18.** The two maps in the definition of $\tilde{B}_{i,k}$ are generalizations of the inclusion map $j : \wedge^{l+1} \mathfrak{v}^i_m \rightarrow \wedge^l \mathfrak{v}^i_m \otimes \mathfrak{v}^i_m$, (41), defined on basis vectors $v_1 = v_{i_1} \wedge \ldots \wedge v_{i_{l+1}}$ as

$$j_l(v_1) = \sum_{j=1}^l (-q)^{l+1-j} v_{i_1} \wedge \ldots \wedge \widehat{v_{i_j}} \wedge \ldots \wedge v_{i_{l+1}} \otimes v_{i_j}$$

and of the projection map $\mathfrak{j}_l : \wedge^{l+1} \mathfrak{v}^i_m \otimes \mathfrak{v}^i_m \rightarrow \wedge^{l+1} \mathfrak{v}^i_m$, defined on basis vectors $v_1 = v_{i_1} \wedge \ldots \wedge v_{i_l}$ as

$$\mathfrak{j}_l(v_1 \otimes v_s) = (-q)^{\#(j \mid j > s)} v_{i(s)} ,$$

where $i(s)$ is the sequence obtained from $i$ by adding the entry $s$ and ordering everything in increasing order. Note that $v_{1(s)}$ is zero if an entry occurs twice in $i(s)$. These are special cases of the intertwiner maps seen in [Sus07] and [MS09].

Analogously we can define for $k \in C(n, r)$ with $k_i > 0$ two further compositions $k' \in C(n, r + 1)$ and $k'' \in C(n, r)$ via

$$k_j' = \begin{cases} k_j & \text{if } j < i \\ k_{i+1} - 1 & \text{if } j = i + 1 \\ 1 & \text{if } j = i + 2 \\ k_{j-1} & \text{if } j > i + 2 \end{cases} \quad k_j'' = \begin{cases} k_j & \text{if } j < i \\ k_{i+1} - 1 & \text{if } j = i + 1 \\ k_j & \text{if } j > i + 1. \end{cases}$$

Again this gives us a composition of the natural $\mathcal{U}_q(\mathfrak{gl}_{2m})$-morphisms

$$\tilde{B}_{-i,k} = (-q)^{1-k_i} \left( k' \wedge \mathfrak{v}^i_m \rightarrow \wedge^k \mathfrak{v}^i_m \rightarrow k'' \wedge \mathfrak{v}^i_m \right). \quad (70)$$
and for $k$ with $k_i = 0$ we put $\tilde{B}_{-i,k} = 0$ and finally $\tilde{B}_{-i} = \bigoplus_{k \in C(n,r)} \tilde{B}_{-i,k}$.

7.9. **Quantum skew Howe duality.** By construction the following is automatic for $1 \leq i < r$:

**Lemma 7.19.** The maps $\tilde{B}_i$ and $\tilde{B}_{-i}$ are $\mathcal{U}_q(\mathfrak{gl}_{2m})$-equivariant, hence also $\mathcal{H}^i_m$-equivariant.

The final map $\tilde{B}_0$ can of course not be constructed as a $\mathcal{U}_q(\mathfrak{gl}_{2m})$-equivariant morphism; we will later have to check by hand that it is $\mathcal{H}^i_m$-equivariant. For $k \in C(n,r)$ with $k_1 > 0$ define a composition $k' \in C(n, r+1)$ by setting

$$k'_1 = 1, \quad k'_2 = k_1 - 1 \quad \text{and} \quad k'_i = k_{i-1} \quad \text{for} \quad i > 2.$$

Then consider the composition

$$\tilde{B}_{0,k} = (-q)^{1-k_1} \left( \bigwedge \mathbb{V}_i^i \rightarrow \bigwedge k' \mathbb{V}_i^i \rightarrow \bigwedge k' \mathbb{V}_i^i \rightarrow \bigwedge k' \mathbb{V}_m \rightarrow \bigwedge k' \mathbb{V}_m \right),$$

(71)

where the first and third map are as before the natural maps, while the one in the middle changes the sign of the index in the very first tensor factor of $\bigwedge k' \mathbb{V}_m$, i.e. it send a vector $v_i \otimes (\text{Rest})$ to $v_{-i} \otimes (\text{Rest})$. As before, for $k$ with $k_1 = 0$ we put $\tilde{B}_{0,k} = 0$ and finally $\tilde{B}_0 = \bigoplus_{k \in C(n,r)} \tilde{B}_{0,k}$.

**Lemma 7.20.** The map $\tilde{B}_0$ is $\mathcal{H}^i_m$-equivariant.

*Proof.* We give the arguments for the generator $B_i$, $i \geq 0$, since the others are similar. Fix a box diagram $\blacksquare$. Since it is obvious that $\tilde{B}_0(B_i, \blacksquare)$ and $B_i \cdot \tilde{B}_0(\blacksquare)$ contain the same summands it remains to compare the powers of $q$ involved. Choose boxes $\blacksquare_{j,i+1}$ and $\blacksquare_{1,l}$ containing symbols that can be moved resp. flipped by the action of $B_i$ resp. $\tilde{B}_0$.

**Case** $l \notin \{i, i+1\}$ and $j \neq 1$: The powers of $q$ are independent of the action of the other operator and hence the results will be the same.

**Case** $l \in \{i, i+1\}$ and $j \neq 1$: The box modified by $\tilde{B}_0$ is in the same column as the one modified by $B_i$. The formulas for $B_i$ imply that a symbol + means we count the symbols above in column $i$ and $i+1$ either all as $q$ or $q^{-1}$. So it does not matter if we apply $\tilde{B}_0$ before or after, the power remains the same. In case of a $-$, the symbols above are not counted at all.

**Case** $l \notin \{i, i+1\}$ and $j = 1$: Then the box modified by $\tilde{B}_0$ is in the first row but far enough away from the one for $\tilde{B}_0$. In this case the powers of $q$ given by $B_i$ do not involve the box $\blacksquare_{1,l}$ at all; and the powers given by $\tilde{B}_0$ are obviously the same, no matter if we apply $B_i$ before or after.

**Case** $l \in \{i, i+1\}$ and $j = 1$: The box $\blacksquare_{1,l}$ is either directly to the left of $\blacksquare_{j,i+1}$ or they coincide. It is easy to see that for $\tilde{B}_0$ the $q$-powers are the same or the two compositions are zero anyway. Similarly for $B_i$, because the $q$-powers do not depend on the sign if our box is in the first row. □

Using the definition of the box diagrams and the intertwiner maps for the quantized exterior tensor product we obtain the following explicit formulas.
Lemma 7.21. Let $\mathfrak{m}$ be a box diagram of type $(k, \mu, \epsilon)$. Then
\[
\hat{B}_i(\mathfrak{m}) = \sum_{\mathfrak{m}'} (-q)^{m(\mathfrak{m}')} \mathfrak{m}', \quad \hat{B}_{-i}(\mathfrak{m}) = \sum_{\mathfrak{m}''} (-q)^{m(\mathfrak{m}'')} \mathfrak{m}'',
\]
\[
\hat{B}_0(\mathfrak{m}) = \sum_{\mathfrak{m}'''} (-q)^{m(\mathfrak{m}'')} \mathfrak{m}''',
\]
where the first sum runs over all box diagrams obtained from $\mathfrak{m}$ by moving a symbol from row $i+1$ to the box above, the second sum over those where a symbol in row $i$ is moved to the box immediately below and the third sum over those where a symbol in the first row is flipped.

If for $\mathfrak{m}'$ the symbol in $\mathfrak{m}_{i+1,j}$ was moved then the power of $-q$ is, depending on the symbol, equal to
\[
m(\mathfrak{m}') = \begin{cases} c_{i,j}^{+}(\mathfrak{m}, \rightarrow) - c_{i+1,j}(\mathfrak{m}, \rightarrow) & \text{for } + \\ c_{i+1,j}(\mathfrak{m}, \rightarrow) - c_{i,j}^{-}(\mathfrak{m}, \rightarrow) + k_i - k_{i+1} + 1 & \text{for } - . \end{cases}
\]
If for $\mathfrak{m}''$ the symbol in $\mathfrak{m}_{i,j}$ was moved then the power of $-q$ is equal to
\[
m(\mathfrak{m}'') = \begin{cases} c_{i,j}^{+}(\mathfrak{m}, \rightarrow) - c_{i+1,j}(\mathfrak{m}, \rightarrow) + k_{i+1} - k_i + 1 & \text{for } + \\ c_{i+1,j}(\mathfrak{m}, \rightarrow) - c_{i,j}^{-}(\mathfrak{m}, \rightarrow) & \text{for } - . \end{cases}
\]
If for $\mathfrak{m}'''$ the swapped symbol is in $\mathfrak{m}_{i,j}$ then the power of $-q$ is equal to
\[
m(\mathfrak{m}''') = c_{1,j}(\mathfrak{m}, \rightarrow) - c_{1,j}^{+}(\mathfrak{m}, \rightarrow).
\]

As a result we obtain a quantum skew Howe duality. For this denote by $\mathcal{H}_m^i$ the coideal subalgebra corresponding to the quantum group $U_q(\mathfrak{gl}_m)$, i.e. the truncation of $\mathcal{H}$ and consider the two actions on $\Lambda(n, m, r)$:

Theorem 7.22 (Quantum skew Howe duality). Lemmas 7.19 and 7.20 define commuting actions of $\mathcal{H}_m^i$ and $\mathcal{H}_r^i$ on $\Lambda(n, m, r)$.

We expect that a careful analysis of the occurring representations in (48) also shows that the two actions generate each others commutant.

7.10. Categorified quantum skew Howe duality and Koszul duality. The Koszul duality functor from [BGS96] defines a contravariant equivalence of triangulated categories
\[
\mathcal{K} : \bigoplus_{k \in \mathbb{C}(n, r)} D^b\left(\hat{\mathcal{O}}_{\mathbb{Z}^n}(n)\right) \rightarrow \bigoplus_{k \in \mathbb{C}(n, m)} D^b\left(\hat{\mathcal{O}}_{\mathbb{Z}^m}(n)\right).
\]

sending the standard lift of the parabolic Verma module of highest weight $\nu - \rho$ to the standard lift of the parabolic Verma module of highest weight $\nu' - \rho$, see [Bac99] with Lemma 7.17. The grading shift on the left corresponds to a homological and grading shift on the right, $\mathcal{K}(j) = (j)[j]\mathcal{K}$. Hence, on the level of Grothendieck groups $q$ is sent to $-q$.

Proposition 7.23. The following equalities of endomorphisms of the Grothendieck group hold
\[
\mathcal{K}^{-1} \circ \left[\hat{B}_1\right] \circ \mathcal{K} = \hat{B}_1, \quad \mathcal{K}^{-1} \circ \left[\hat{B}_{-1}\right] \circ \mathcal{K} = \hat{B}_{-1},
\]
Proposition 7.24. The functors (75) and (76) induce the action of the \(B_i\)'s on the Grothendieck group, i.e. \(B_i = [\hat{\mathbb{E}}^\vee_i]\) and \(B_{-i} = [\hat{\mathbb{E}}^\vee_{-i}]\) for \(i > 0\), and \(B_0 = [\hat{\mathbb{E}}^\vee_0]\).

Proof. The definition of the functors, (75) and (76) compared with the linear maps (69), (70) and (71) shows that it is enough to verify that the derived Zuckerman and the inclusion functors induce the canonical maps involved in the definition of these linear maps. We start with the identities \(\tau_i = \iota_\lambda\). In case an inclusion ends in a block of \(\hat{\mathcal{O}}^{\mathfrak{q}_3}(n)\) with \(k = (1, \ldots, 1)\) then the claim is exactly (66). In general, we have \(\iota^{q_3}_{q_1} = \iota^{q_3}_{q_1} \circ \iota^{q_2}_{q_1}\) for standard parabolic subalgebras \(q_3 \subseteq q_2 \subseteq q_1\). Now let \(q_3\) be of type \((1, \ldots, 1)\) and write the image of a parabolic Verma module under \(\iota^{q_1}_{q_3}\) in terms of the standard basis, \([\iota^{q_2}_{q_1}(M^{q_1}(x \cdot \lambda))] = \sum_w a_w [M^{q_2}(w \cdot \lambda)].\) If we apply \([\iota^{q_3}_{q_1}]\) to it...
and compare with the formula for \([q_{q_1}^q(M^{q_1}(x \cdot \lambda))]\) we obtain, by induction on the highest weights, that

\[
[q_{q_1}^q M^{q_1}(x \cdot \lambda)] = \sum_y (-q)^{\ell(w)}[\hat{M}(yx \cdot \lambda)]
\]

(77)

where \(y\) runs through all coset representatives of minimal length in \(W_{q_2} \backslash W_{q_1}\), where \(W_{q_1}\) denotes the Weyl group of the Levi subalgebra of \(q_1\). (Here and in the rest of the proof \(\lambda\) always denotes the maximal weight in its dot orbit and \(x\) is chosen of minimal length.) Now consider the derived Zuckerman functors \(\hat{Z} = \hat{Z}^{q_2}_{q_1}\). If we forget the grading then it is well-known that a parabolic Verma module, \(M = M^{q_1}(x \cdot \lambda)\), is sent to a parabolic Verma module, \(N = M^{q_2}(x \cdot \lambda)\), possibly shifted in homological degree, [ES87, Proposition 5.5]. More precisely \(L_iZ(M) \cong N\) for \(i = \ell(x)\) and \(L_{l'}ZM = 0\) for \(l' \neq l\). Note that the cohomology functor \(\check{L}_iZ\) of the graded lift is again graded and so the indecomposability of \(N\) implies \(\check{L}_i\hat{Z}M \cong \hat{N}(r)\) for some \(r \in \mathbb{Z}\) and \(\check{L}_{l'}\hat{Z}M = 0\) for \(l' \neq l\). Now \(\hat{Z}\) is obviously left adjoint to the inclusion \(\check{q}_{q_2}^{q_1}\), hence \(\mathrm{Hom}_O(\check{Z}[l](M), N(s)) \neq 0\) implies by (77) and an easy induction argument \(s = l\). Hence

\[
\hat{Z}\check{M}^{q_1}(x \cdot \lambda) \cong \check{M}^{q_2}(x \cdot \lambda)[\ell(x)][\ell(x)]
\]

. The claim follows now from the projection formula in Remark 7.18.

Combining the results we obtain the main theorem:

**Theorem 7.25.** Fix the equivalence given by Koszul duality,

\[
\mathcal{K}: \bigoplus_{k \in C(n,r)} D^b\left(\hat{\mathcal{O}}_{\mathbb{C}[n]}^{q_k}(n)\right) \cong \bigoplus_{k \in C(n,m)} D^b\left(\hat{\mathcal{O}}_{\mathbb{C}[n]}^{q_k}(n)\right).
\]

(78)

Then the coideal algebra actions from Theorem 7.12 on the two sides turn into commuting actions on the same space, one given by Theorem 7.12, the other by Proposition 7.24.

More concretely this means that on

\[
K_0 \left( \bigoplus_{k \in C(n,r)} D^b\left(\hat{\mathcal{O}}_{\mathbb{C}[n]}^{q_k}(n)\right) \right) = \bigwedge(N,m,r),
\]

the action of \([\mathbb{B}_{i,\pm,\Gamma}] := \mathcal{K}^{-1} \circ [\mathbb{B}_{i,\pm,\Gamma}] \circ \mathcal{K}\) and \([\mathbb{B}_{0,\Gamma}] := \mathcal{K}^{-1} \circ [\mathbb{B}_{0,\Gamma}] \circ \mathcal{K}\) commutes with the action of \([\mathbb{B}_{j,\pm,\Gamma}]\) and \([\mathbb{B}_{0,\Gamma}]\) for \(1 \leq i \leq r\) and \(1 \leq j \leq m\).

**Remark 7.26.**

- With some technical arguments generalizing [Kho05] one can actually show that the family of functors \(\hat{\mathbb{B}}_{\pm i, \pm j}\) naturally commute with the family of functors \(\check{\mathbb{B}}_{\pm i, \pm j}\).
- Analogous (and in fact much easier) arguments applied to the Lie algebra \(\mathfrak{gl}_n\) instead of \(\mathfrak{so}_n\) give rise to the quantum and categorified quantum skew Howe duality of type \(A\) mentioned in the introduction.
Note that the graded duality again defines a bar involution on $\bigwedge(n,m,r)$ and Proposition 6.30 generalizes. This aspect in connection with generalizations to type $B$ and $C$ will appear in [ES13a].

8. Appendix

8.1. Proof of 2.8. This subsection is devoted to the proof of Theorem 2.8.

Lemma 8.1. For $1 \leq i \leq d-1$ and $j \notin \{i, i+1\}$ we have $s_i y_j = y_j s_i$.

Proof. The case $j < i$ is obvious by definition. For the case $j > i + 1$ we have for $v \in M \otimes V^d$:

$$s_i y_j(v) = s_i \left( \sum_{0 \leq k < j} \Omega_{kj} + \frac{2n-1}{2} \right) v = \sum_{0 \leq k < j} s_i \left( \Omega_{kj} v \right) + \frac{2n-1}{2} s_i(v)$$

$$= \left( \sum_{0 \leq k < j} \Omega_{kj} \right) s_i(v) + \frac{2n-1}{2} s_i(v) = y_j s_i(v).$$

The claim follows. \qed

The following are the Brauer algebra relations, see [GW09, Lemma 10.1.5]:

Lemma 8.2. For $1 \leq i \leq d-2$, the relations $e_i s_i = e_i$, $s_i e_i$, $s_i e_i e_i = s_i e_i$, $e_{i+1} e_i s_{i+1} = e_i s_{i+1}$, and $e_{i+1} e_i e_{i+1} = e_{i+1}$ hold.

Proof. The first equality is obvious by definition of the map $\tau$. We will only prove the second one, the final two are shown analogously. To shorten the notations we will assume that our morphisms $s_i$, $e_i$, and $e_{i+1}$ only live on a threefold tensor product, with the first factor being the position $i$, the second $i + 1$, and the third $i + 2$. Let $a, b, c \in I$ and $v = v_a \otimes v_b \otimes v_c$, then

$$(s_{i+1} e_i) v = (v_a, v_b) \sum_{k \in I} v_k \otimes v_c \otimes v_k$$

and

$$(s_i e_{i+1} e_i) v = (s_i e_{i+1}) \left( (v_a, v_b) \sum_{k \in I} v_k \otimes v^*_k \otimes v_c \right) = (v_a, v_b) \sum_{k \in I} v_k \otimes v_c \otimes v^*_k.$$
Lemma 8.4. For $1 \leq i, j \leq d - 1$ and $j \neq \{i, i + 1\}$ we have $e_i y_j = y_j e_i$.

Proof. As in Lemma 8.1 the case $j < i$ is obvious by definition and we are left with the case $j > i + 1$. The essential part that we need to prove is

\[
(\Omega_{ij} + \Omega_{(i+1)j}) e_i = e_i (\Omega_{ij} + \Omega_{(i+1)j}),
\]

since $e_i$ naturally commutes with all other summands of $y_j$. Again, we pretend that $\Omega_{ij}$, $\Omega_{(i+1)j}$, and $e_i$ act only on a threefold tensor product, corresponding to the factors at position $i$, $i + 1$ and $j$.

Consider first the left hand side. Since we first apply $e_i$, the element $e_i y_j$ kills all basis vectors except the ones of the form $x = v_a \otimes v_b^* \otimes v_c$ for some $a, c \in I$, in which case $x e_i = \sum_{k \in I} v_k \otimes v_k^* \otimes v_c$ and applying $\Omega_{ij} + \Omega_{(i+1)j}$ gives

\[
\sum_{\alpha \in B_n} \sum_{k \in I} (X_\alpha v_k \otimes v_k^* + v_k^* \otimes X_\alpha v_k) \otimes X_\alpha^* v_c
\]

Thus \((\Omega_{ij} + \Omega_{(i+1)j}) e_i = 0\) on $M \otimes V^\otimes d$.

Consider the right hand side of (79), let $x = v_a \otimes v_b \otimes v_c$ for $a, b, c \in I$, then

\[
x(\Omega_{ij} + \Omega_{(i+1)j}) = \sum_{\alpha \in B_n} (X_\alpha v_a \otimes v_b + v_a \otimes X_\alpha v_b) \otimes X_\alpha^* v_c
\]

Applying $e_i$ kills, for fixed $\alpha$, all except one summand and we obtain

\[
\sum_{l \in I} (X_\alpha v_a, v_b) v_l \otimes v_l^* + (X_\alpha v_b, v_a) v_l \otimes v_l^*) \otimes X_\alpha^* v_c
\]

Thus, $e_i (\Omega_{ij} + \Omega_{(i+1)j}) = 0$ on $M \otimes V^\otimes d$ and the lemma follows. \hfill \square

Lemma 8.5. For $1 \leq i, j \leq d$ we have $y_i y_j = y_j y_i$.

Proof. We show this by using the following statements:

(i) $[\Omega_{ij}, \Omega_{kl}] = 0$ for pairwise different $i, j, k, l$,

(ii) $[\Omega_{ij} + \Omega_{jk}, \Omega_{ik}] = 0$ for pairwise different $i, j, k$.

Equality (i) is obvious. For (ii) we consider factors $i, j$, and $k$ to obtain

\[
[\Omega_{ij} + \Omega_{jk}, \Omega_{ik}] = \sum_{\alpha, \beta, \gamma \in B_n} [X_\alpha \otimes X_\alpha^* \otimes 1 + 1 \otimes X_\beta \otimes X_\beta^*, X_\gamma \otimes 1 \otimes X_\gamma^*]
\]

\[
= \sum_{\alpha, \beta, \gamma} [X_\alpha, X_\gamma] \otimes X_\alpha^* \otimes X_\gamma^* + X_\gamma \otimes X_\beta \otimes [X_\beta^*, X_\gamma^*]
\]
Now applying
\[
\alpha, \beta, \gamma, \mu
\]
\[
\sum_{\alpha, \beta, \gamma, \mu} \left( ([X_\alpha, X_\gamma], X_\mu) X_\mu \otimes X_\alpha \otimes X_\gamma + X_\gamma \otimes X_\beta \otimes ([X_\beta, X_\gamma^*], X_\mu) X_\mu^* \right)
\]
\[
= \sum_{\alpha, \beta, \gamma, \mu} \left( ([X_\alpha, X_\gamma], X_\mu) X_\mu \otimes X_\alpha \otimes X_\gamma^* + ([X_\alpha^*, X_\mu], X_\gamma) X_\mu \otimes X_\alpha \otimes X_\gamma^* \right)
\]
\[
= \sum_{\alpha, \beta, \gamma, \mu} \left( ([X_\alpha, X_\gamma], X_\mu^*) X_\mu \otimes X_\alpha \otimes X_\gamma^* - ([X_\alpha, X_\gamma], X_\mu^*) X_\mu \otimes X_\alpha \otimes X_\gamma^* \right) = 0
\]

Here equality (1) is relabelling the second summand and extracting a scalar factor, while equality (2) is due to the invariance of the Killing form, i.e.

\[
\sum_{\alpha \in B_n} ([X^*_\alpha, X^*_\mu], X_\gamma) X_\alpha = \sum_{\alpha \in B_n} (X^*_\alpha, [X^*_\mu, X_\gamma]) X_\alpha = \sum_{\alpha \in B_n} (X_\alpha, [X^*_\mu, X_\gamma]) X_\alpha^* = - \sum_{\alpha \in B_n} ([X_\alpha, X_\gamma], X^*_\mu) X^*_\alpha.
\]

Lemma 8.6. For \(1 \leq 1 < d\) we have
\[
\epsilon_i(y_i + y_{i+1}) = 0 \quad \text{and} \quad (y_i + y_{i+1})\epsilon_i = 0.
\]

Proof. We start with the first relation and expand the left hand side.
\[
\epsilon_i(y_i + y_{i+1}) = \epsilon_i \left( \sum_{0 \leq k < i} \left( \Omega_{ki} + \Omega_{k(i+1)} \right) + \Omega_{i(i+1)} + N - 1 \right)
\]
\[
= \epsilon_i \left( \sum_{0 \leq k < i} \left( \Omega_{ki} + \Omega_{k(i+1)} \right) + s_i - e_i + N - 1 \right)
\]
\[
= \epsilon_i \left( \sum_{0 \leq k < i} \left( \Omega_{ki} + \Omega_{k(i+1)} \right) \right) + \epsilon_i - Ne_i + (N - 1)e_i
\]
\[
= \epsilon_i \left( \sum_{0 \leq k < i} \left( \Omega_{ki} + \Omega_{k(i+1)} \right) \right)
\]

We have seen in the proof of Lemma 8.4 that \(\epsilon_i \left( \Omega_{ki} + \Omega_{k(i+1)} \right) \) acts as zero if \(k > 0\). Thus we are left to show that \(\epsilon_i \left( \Omega_{0i} + \Omega_{0(i+1)} \right) \) acts as zero. Let \(x = m \otimes v_a \otimes v_b\) for \(m \in M\) and \(a, b \in I\), again these should be thought of positions 0, i, and \(i + 1\). Applying \(\Omega_{0i} + \Omega_{0(i+1)} \) gives us
\[
x. \left( \Omega_{0i} + \Omega_{0(i+1)} \right)
\]
\[
= \sum_{\alpha \in B_n} X_{\alpha} m \otimes X^*_\alpha v_a \otimes v_b + X_{\alpha} m \otimes v_a \otimes X^*_\alpha v_b
\]
\[
= \sum_{\alpha \in B_n} \sum_{k \in I} \left( X^*_\alpha v_a, v_k \right) X_{\alpha} m \otimes v_k^* \otimes v_b + \left( X^*_\alpha v_b, v_k \right) X_{\alpha} m \otimes v_a \otimes v_k^*.
\]

Now applying \(\epsilon_i\) to each summand for \(\alpha \in B_n\) we obtain
\[
\sum_{l \in I} \left( X^*_\alpha v_a, v_b \right) X_{\alpha} m \otimes v_l \otimes v_l^* + \left( X^*_\alpha v_b, v_a \right) X_{\alpha} m \otimes v_l \otimes v_l^*
\]
\[
= \sum_{l \in I} \left( X^*_\alpha v_a, v_b \right) X_{\alpha} m \otimes v_l \otimes v_l^* - \left( X^*_\alpha v_a, v_b \right) X_{\alpha} m \otimes v_l \otimes v_l^* = 0.
\]
Thus we have $e_i(y_i + y_{i+1}) = 0$ on $M \otimes V^\otimes d$.

For the second equality with start with the same $x$ as above and apply $e_i$, then the result will be either zero or

$$
\sum_{i \in I} m \otimes v_i \otimes v_i^* = \sum_{i \in I} m \otimes v_i^* \otimes v_i.
$$

Applying $(\Omega_{0i} + \Omega_{0(i+1)})$ to this, we obtain

$$
\sum_{\alpha \in B_n} \sum_{l \in I} X_\alpha m \otimes X_\alpha^* v_l^* \otimes v_l + X_\alpha m \otimes v_l \otimes X_\alpha^* v_l^*
$$

$$
= \sum_{\alpha \in B_n} \sum_{k,l \in I} \langle X_\alpha^* v_l^*, v_k \rangle X_\alpha m \otimes v_k \otimes v_l + \langle X_\alpha^* v_l^*, v_k \rangle X_\alpha m \otimes v_l \otimes v_k - \langle X_\alpha^* v_k^*, v_l^* \rangle X_\alpha m \otimes v_l \otimes v_k = 0.
$$

For the last equality one just switches $k$ and $l$ in the second sum. Hence we also have $(y_i + y_{i+1})e_i = 0$ on $M \otimes V^\otimes d$. 

\[ \square \]


Proof. The statements about Verma modules follow directly from Lemma 2.16 and the definition of our functors. Moreover, the functors $\oplus_{i \in X_i}(F_{i,-} \oplus \mathcal{F}_{i,-}) \oplus F_i$ and $F := (\cdot \otimes V)$ applied to a Verma module produce the same number of Verma modules, hence part (1) of the Lemma follows.

We next check (vii) and (viii) for $L(\lambda)$. In all these cases, we know already that $\mathcal{F}_{i,-}M^\otimes(\lambda) = \{0\}$. As $L(\lambda)$ is a quotient of $M^\otimes(\lambda)$ and $\mathcal{F}_{i,-}$ is exact, it follows immediately that $\mathcal{F}_{i,-}L(\lambda) = \{0\}$.

The proofs of (i)-(iv) for $L(\lambda)$ are not much harder. For example, if $\lambda = \lambda_\nu$ as in (i), then $\mathcal{F}_{i,-}L(\lambda)$ is a quotient of $\mathcal{F}_{i,-}M^\otimes(\lambda) \cong M^\otimes(\lambda_\nu)$. Moreover it is self-dual. This is because $L(\lambda)$ is self-dual with respect to the duality on category $\mathcal{O}$, and $\mathcal{F}$ commutes with this duality and $\mathcal{F} = \mathcal{F}_{i,-}$ when applied to our modules. Hence we either have that $\mathcal{F}_{i,-}L(\lambda) = \{0\}$ or $\mathcal{F}_{i,-}L(\lambda) \cong L(\lambda_\nu)$, as these are the only self-dual quotients of $M^\otimes(\lambda_\nu)$. To rule out the possibility that it is zero, consider the group homomorphism $K_0(\mathcal{F}_{i,-}) : K_0(\mathcal{O}^\otimes(n, \delta)_{\Gamma}) \to K_0(\mathcal{O}^\otimes(n, \delta)_{\Gamma,-})$ induced by $\mathcal{F}_{i,-}$. Expressed in the basis of isomorphism classes of Verma modules, it has an inverse, the morphism induced by $\mathcal{F}_{i,+}$. Hence $\mathcal{F}_{i,-}$ is non-zero on every non-zero module. This proves (i) for $L(\lambda)$, and the proofs of (ii)-(iv) are similar.

Next we check (v) and (vi) for $L(\lambda)$, i.e. we show that $\mathcal{F}_{i,-}L(\lambda_\nu) \cong L(\lambda_\nu)$ and $\mathcal{F}_{i,-}L(\lambda_\nu) = \{0\}$. We know that $\mathcal{F}_{i,-}M^\otimes(\lambda_\nu) \cong \mathcal{F}_{i,-}M^\otimes(\lambda_\nu) \cong M^\otimes(\lambda_\nu)$. Again by self-duality, we either have that $\mathcal{F}_{i,-}L(\lambda_\nu) \cong L(\lambda_\nu)$ or $\mathcal{F}_{i,-}L(\lambda_\nu) = \{0\}$. Similarly, either $\mathcal{F}_{i,-}L(\lambda_\nu) \cong L(\lambda_\nu)$ or $\mathcal{F}_{i,-}L(\lambda_\nu) = \{0\}$. As $[\mathcal{F}_{i,-}M^\otimes(\lambda_\nu) : L(\lambda_\nu)] = 1$, there must be some composition factor $L(\mu)$ of $M^\otimes(\lambda_\nu)$ such that $[\mathcal{F}_{i,-}L(\mu) : L(\lambda_\nu)] = 1$. The facts proved so far imply either that $\mu = \lambda_\nu$ or that $\mu = \lambda_\nu$. But the latter case cannot occur as $\lambda_\nu$ is strictly bigger than $\lambda_\nu$ in the Bruhat ordering. Hence $\mu = \lambda_\nu$, and we have proved that $[\mathcal{F}_{i,-}L(\lambda_\nu) : L(\lambda_\nu)] = 1$. This gives $\mathcal{F}_{i,-}L(\lambda_\nu) \cong L(\lambda_\nu)$.
as required for (v). It remains for (vi) to show that \( \mathcal{F}_{i,-}L(\lambda_{\nu}) = \{0\} \).

Suppose for a contradiction that it is non-zero, hence \( \mathcal{F}_{i,-}L(\lambda_{\nu}) \cong L(\lambda_{\nu}) \).

By Lemma 3.10 and BGG-reciprocity we have \( M^P(\lambda_{\nu}) \) has both \( L(\lambda_{\nu}) \) and \( L(\lambda_{\nu}) \) as composition factors, so we deduce that \( [\mathcal{F}_{i}M^P(\lambda_{\nu}) : L(\lambda_{\nu})] \geq 2 \), which is the desired contradiction.

In this paragraph, we check (vii)(c). Take \( \lambda \) with \( \lambda = \lambda_\circ \). Let \( \Gamma \) be the block containing \( \lambda \). Note that \( \mathcal{F}_{i,-} \) maps the summand corresponding to the block \( \Gamma \) to \( \Gamma_- \) and then to \( \Gamma' = (\Gamma_-)_- \). We know for any \( \nu \in \Gamma \) that \( \mathcal{F}_{i,-}^2M^P(\nu) \cong M^P(\nu_\circ) \otimes M^P(\nu_\circ) \). Hence \( \mathcal{F}_{i,-}^2 \) induces a \( \mathbb{Z} \)-module isomorphism between \( K_0(O^P(n, \delta)_\Gamma) \) and \( 2[O^P(n, \delta)_{\Gamma'}] \). We deduce for any non-zero module \( M \in O^P(n, \delta)_\Gamma \) that \( \mathcal{F}_{i,-}^2M \) is non-zero and its class is divisible by two in \([O^P(n, \delta)_{\Gamma'}] \). In particular, \( \mathcal{F}_{i,-}^2L(\lambda) \) is a non-zero self-dual quotient of \( M^P(\lambda_{\circ}) \otimes M^P(\lambda_{\circ}) \) whose class is divisible by two. This implies that

\[
\mathcal{F}_{i,-}^2L(\lambda) \cong L(\lambda_{\circ}) \oplus L(\lambda_{\circ}).
\]

Now take any \( \mu \in \Gamma_- \). We know already that \( \mathcal{F}_{i,-}L(\mu) \cong L(\mu_{\circ}) \) if \( \mu = \mu_{\circ} \), and \( \mathcal{F}_{i,-}L(\mu) = \{0\} \) otherwise. Assuming now that \( \mu = \mu_{\circ} \), we deduce from this that \( [\mathcal{F}_{i,-}L(\lambda) : L(\mu)] = [\mathcal{F}_{i,-}^2L(\lambda) : L(\mu_{\circ})] \). Using (80), we conclude for \( \mu = \mu_{\circ} \) that \( [\mathcal{F}_{i,-}L(\lambda) : L(\mu)] = 0 \) unless \( \mu = \lambda_{\circ} \), and \( [\mathcal{F}_{i,-}L(\lambda) : L(\lambda_{\circ})] = 2 \). It is easy to see that the analogous statements hold for \( \mathcal{F}_{i,+} \).

Now we deduce all the statements (i)–(ix) for \( P(\lambda) \) by using the fact that \( (\mathcal{F}_{i,-}, \mathcal{F}_{i,+}) \) is an adjoint pair of functors. We just explain the argument in case (v), since the other cases are similar (actually, easier). As \( \mathcal{F}_{i,-} \) sends projectives to projectives, \( \mathcal{F}_{i,-}P(\lambda) \) is a direct sum of indecomposable projectives. To compute the multiplicity of \( P(\mu) \) in this decomposition we calculate

\[
\text{Hom}_g(\mathcal{F}_{i,-}P(\lambda), L(\mu)) \cong \text{Hom}_g(P(\lambda), \mathcal{F}_{i,-}L(\mu)) = [\mathcal{F}_{i,-}L(\mu) : L(\lambda)].
\]

By the analogue of (vi)(c) for \( \mathcal{F}_{i,+} \) this multiplicity is zero unless \( \mu = \lambda_{\circ} \), when it is two. Hence \( \mathcal{F}_{i,-}P(\lambda) \cong P(\lambda_{\circ}) \oplus P(\lambda_{\circ}) \).

For part (1) it just remains to deduce (vii)(d). By (vii)(a), (vii)(c) and exactness of \( \mathcal{F}_{i,-} \), we get that \( \mathcal{F}_{i,-}L(\lambda) \) is a non-zero quotient of \( P(\lambda_{\circ}) \), hence it has irreducible head isomorphic to \( L(\lambda_{\circ}) \). Since it is self-dual it also has irreducible socle isomorphic to \( L(\lambda_{\circ}) \). Now consider the case \( i = 0 \). The statement for the Verma modules is again clear and for the simple modules we argue as above. For the projective modules we consider again each case. Part (xiii) is proved as (i) or (ii), whereas (xv) is proved as the dual version (x) of (iii). Now for (xiv) and (xvii) each Verma module occurring in a filtration of the projective module has highest weight \( \nu \) and \( \nu \) equals \( \lambda \) at the places 0 and 1, hence the whole module get killed by the functor. Case (xvi) is proved as case (iv) whereas (xvii) is proved as the dual (x) of (vii). Hence it remains to show (xi) and (xii) which however can be proved with the arguments from (v). Finally (4) can be proved as the cases (i)-(iv). \qed
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